Chapter 12

H^∞ **Four-Block Problem** ($\|\mathcal{F}_\ell(\mathbb{D},\mathbb{Q})\| < \gamma$)

What is now proved was once only imagin'd.
— William Blake (1757–1827)

The H^∞ Four-Block Problem (H^∞ 4BP) is presented on p. 36. In Section 12.1 we solve the 4BP, in Section 12.3 we solve the corresponding frequency-domain problem, and in Section 12.2 we treat the discrete-time counterparts of these problems. The rest of this chapter consists of proofs and minor results.

Our main contributions are the solution of the continuous-time H^∞ problem in terms of two independent Riccati equations and a coupling condition (Theorems 12.1.4 and 12.1.5, under different assumptions) and the parametrization of all sub-optimal controllers (Theorem 12.1.8), as well as the corresponding discrete-time results (including Theorem 12.2.1). Another important result is the factorization solution of the same problem (Theorem 12.3.6 solves the frequency domain problem for MTIC systems, Theorem 12.3.7 (partially) for more general ones; Theorem 12.3.5 connects these solutions to the state-space solution).

In this chapter, $\widetilde{\mathcal{A}}$ stands for $\mathrm{MTIC}_{T\mathbf{Z}}$ or for some other suitable class:

Standing Hypothesis 12.0.1 *Throughout this chapter we assume that H,U,W,Y,Z are Hilbert spaces and the spaces $\widetilde{\mathcal{A}}(U\times W)$, $\widetilde{\mathcal{A}}(Y\times U)$ and $\widetilde{\mathcal{A}}(Y\times Z)$ satisfy Hypothesis 8.4.7 and that $\widetilde{\mathcal{A}}=\widetilde{\mathcal{A}}^{\mathrm{d}}$.*

(Cf. Theorem 8.4.9(c), Lemma 14.3.5 and Definition 6.2.4.)

Note also that Hypothesis 12.3.1 is assumed through Sections 12.3–12.4 and Hypothesis 12.1.1 through Sections 12.1, 12.2, 12.5 and 12.6.

In this chapter, we allow the controllers to have internal loops unless we use the term "well-posed controller"; cf. Figures 7.8 and 7.10 (or Figures 7.9 and 7.11)). We also often drop the prefix "DPF-", since we study no other controllers than DPF-controllers in this chapter (see Section 7.3).

12.1 The standard H^∞ problem (H^∞ 4BP)

Not every problem is necessarily due to the capitalist mode of production.

— Herbert Marcuse

We assume the dynamics of form (1.25) (see p. 37):

Standing Hypothesis 12.1.1 *Throughout Sections 12.1, 12.2, 12.5 and 12.6, we assume that* $\gamma > 0$ *and*

$$\Sigma = \left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D} \end{array}\right] = \left[\begin{array}{c|cc} \mathbb{A} & \mathbb{B}_1 & \mathbb{B}_2 \\ \hline \mathbb{C}_1 & \mathbb{D}_{11} & \mathbb{D}_{12} \\ \mathbb{C}_2 & \mathbb{D}_{21} & \mathbb{D}_{22} \end{array}\right] \in \mathrm{WPLS}(U \times W, H, Z \times Y). \qquad (12.1)$$

Definition 12.1.2 (Suboptimal controller) *A stabilizing DPF-controller for* Σ *is called* suboptimal *if it makes the norm of the map* $w \mapsto z$ *less that* γ.

See Definition 7.3.1 for stabilizing Dynamic Partial Feedback (DPF) controllers for Σ (recall that in this chapter the words "with internal loop" are usually omitted unlike in Chapter 7).

The map $w \mapsto z$ is usually denoted by $\mathcal{F}_\ell(\mathbb{D}, \mathbb{Q})$, where $\mathbb{Q}$ is the I/O-map of the controller; see Corollary 7.3.20 (or Lemma 12.3.2) for $\mathcal{F}_\ell(\mathbb{D}, \mathbb{Q})$. As noted on p. 321, we have $\mathcal{F}_\ell(\mathbb{D}, \mathbb{Q}) = \mathbb{D}_{12} + \mathbb{D}_{11}\mathbb{Q}(I - \mathbb{D}_{21}\mathbb{Q})\mathbb{D}_{22}$ for well-posed $\mathbb{Q}$. In the literature, often "w comes before u", hence the latter subindices should be interchanged in the formula for $\mathcal{F}_\ell$ for comparison.

We shall show in Theorems 12.1.4 and 12.1.5 that, under standard coercivity assumptions and certain regularity assumptions, the existence of such a controller is equivalent to the standard CARE, signature and coupling conditions. Moreover, in Theorem 12.1.8 we parametrize all such controllers and show that all of them are well-posed when $D_{21} = 0$ (as one usually assumes). Thus, for most readers it suffices to consider well-posed controllers only. We also make some less important remarks on the problem under different assumptions.

Recall from Definition 7.3.1 that $\widetilde{\Sigma}$ being an exponentially stabilizing DPF-controller means that the closed-loop system in Figure 7.9 (or 7.11) is exponentially stable, i.e., that all maps between the signals in this figure are exponentially stable. By Lemma 6.1.10, this is the case iff the corresponding closed-loop semigroup is exponentially stable.

Thus, our concept of a suboptimal exponentially stabilizing DPF-controller is a direct generalization of the standard concept (the "suboptimal admissible controller" of [ZDG], "γ-admissible controller" of [Keu], or "stabilizing γ-contracting controller" of [IOW]).

In most applications of Definition 12.1.2, we can show that Hypothesis 12.3.1 is satisfied. See Proposition 7.3.4 and Lemma 12.5.7 for $\mathcal{F}_\ell(\mathbb{D}, \mathbb{Q})$ in the general case.

We first note that the problem of this section contains that of Section 12.3:

Lemma 12.1.3 (exp-4BP $\Rightarrow$ I/O-4BP) *The I/O map of any suboptimal [exponentially] stabilizing DPF-controller for Σ is a suboptimal [exponentially] stabilizing DPF-controller for $\mathbb{D}$.* □

(This is trivial; see Definitions 12.3.3 and 7.3.1 (cf. Theorem 7.3.11(c1)).)

In the first form of our solution we shall assume "B_{w}^*-CARE" type regularity to obtain simpler Riccati equations and have the solution look exactly like its finite-dimensional counterparts (with the exception of weak Weiss extensions B_{w}^* and C_{w} in place of B^* and C):

Theorem 12.1.4 ($\mathbf{H}^\infty$ 4BP $\Leftrightarrow$ B_{w}^*-CAREs)

(A1) **(Regularity)** *Assume that at least one of (I)–(V) holds, where*

(I) **(Analytic $\mathbb{A}$)** *1. A generates an analytic semigroup on H, $B_1 \in \mathcal{B}(U,H_{\beta_1})$, $B_2 \in \mathcal{B}(W,H_{\beta_2})$, $C_1 \in \mathcal{B}(H_{\gamma_1},Z)$, $C_2 \in \mathcal{B}(H_{\gamma_2},Y)$, $D \in \mathcal{B}(U\times W,Z\times Y)$, $\beta_k,\gamma_k \in (-1/2,1/2)$ $(k=1,2)$; 2. $\gamma_1 < 1/4$ or $\gamma_1-\min\{\beta_1,\beta_2\} < 1/2$; and 3. $\beta_2 > -1/4$ or $\max\{\gamma_1,\gamma_2\}-\beta_2 < 1/2$;*

(II) B is bounded (i.e., $B \in \mathcal{B}(U\times W,H)$) and $\pi_{[0,1)}C_{\mathrm{w}}\mathbb{A} \in \mathrm{L}^1([0,1);\mathcal{B}(H,Z\times Y))$;

(III) $\pi_{[0,1)}\mathbb{A}B \in \mathrm{L}^1([0,1];\mathcal{B}(U\times W,H))$ and $C \in \mathcal{B}(H,Z\times Y)$;

(IV) $\pi_{[0,1)}\mathbb{A}Bv_0 \in \mathrm{L}^2([0,1];H)$, $\pi_{[0,1)}\mathbb{A}^*C^*t_0 \in \mathrm{L}^2([0,1];H)$, $\pi_{[0,1)}C_{\mathrm{w}}\mathbb{A}Bv_0 \in \mathrm{L}^2([0,1];Z\times Y)$, $\pi_{[0,1)}B_{\mathrm{w}}^*\mathbb{A}^*C^*t_0 \in \mathrm{L}^2([0,1];U\times W)$ for all $v_0 \in U\times W$, $t_0 \in Z\times Y$ (equivalently, $(\cdot-A)^{-1}B, (\cdot-A^*)^{-1}C^*$, $C_{\mathrm{w}}(\cdot-A)^{-1}B, B_{\mathrm{w}}^*(\cdot-A^*)^{-1}C^* \in \mathrm{H}^2_{\mathrm{strong}}(\mathbf{C}^+_\omega;\mathcal{B}(*,*))$ for some $\omega \in \mathbf{R}$);

(V) $\mathbb{A}$ is exponentially stable and $\widehat{\mathbb{D}}-D, \widehat{\mathbb{D}}(\bar{\cdot})^*-D^* \in \mathrm{H}^2_{\mathrm{strong}}(\mathbf{C}^+_\omega;\mathcal{B}(*,*))$ for some $\omega<0$.

(A2) **(Nonsingularity)** *Assume that $D_{11}^*D_{11} \gg 0$ and $D_{22}D_{22}^* \gg 0$, and that there is $\varepsilon>0$ s.t.*

$$(ir-A)x_0 = B_1u_0 \implies \|C_{1\mathrm{w}}x_0 + D_{11}u_0\|_Z \geq \varepsilon\|x_0\|_H \quad \text{and} \tag{12.2}$$

$$(ir-A^*)x_0 = C_2^*y_0 \implies \|B_{2\mathrm{w}}^*x_0 + D_{22}^*y_0\|_W \geq \varepsilon\|x_0\|_H \tag{12.3}$$

for all $x_0\in H$, $u_0\in U$, $y_0\in Y$, $r\in\mathbf{R}$.

Then there is a suboptimal exponentially stabilizing DPF-controller for Σ (possibly with internal loop) iff (1.)–(3.) hold:

(1.) **($\mathcal{P}_{\mathrm{X}}$-CARE)** *$D_{12}^*D_{12} - D_{12}^*D_{11}(D_{11}^*D_{11})^{-1}D_{11}^*D_{12} \ll \gamma^2 I$, and the ($B_{\mathrm{w}}^*$-)CARE*

$$\begin{cases} K_X^*S_XK_X = A^*\mathcal{P}_{\mathrm{X}} + \mathcal{P}_{\mathrm{X}}A + C_1^*C_1, \\ S_X = \begin{bmatrix} D_{11}^*D_{11} & D_{11}^*D_{12} \\ D_{12}^*D_{11} & D_{12}^*D_{12}-\gamma^2 I \end{bmatrix}, \\ K_X = -S_X^{-1}\left(\begin{bmatrix} D_{11}^* \\ D_{12}^* \end{bmatrix} C_1 + B_{\mathrm{w}}^*\mathcal{P}_{\mathrm{X}}\right), \end{cases} \tag{12.4}$$

has a solution $(\mathcal{P}_{\mathrm{X}},S_X,K_X) \in \mathcal{B}(H,\mathrm{Dom}(B_{\mathrm{w}}^*)) \times \mathcal{B}(U\times W)\times\mathcal{B}(H_1,U\times W)$ s.t. $\mathcal{P}_{\mathrm{X}} \geq 0$, and the semigroup generated by $A+BK_X$ is exponentially stable.

(2.) **($\mathcal{P}_{\mathrm{Y}}$-CARE)** $D_{12}D_{12}^* - D_{12}D_{22}^*(D_{22}D_{22}^*)^{-1}D_{22}D_{12}^* \ll \gamma^2 I$, *and the ($B_{\mathrm{w}}^*$-)CARE*

$$\begin{cases} K_Y^* S_Y K_Y = A\mathcal{P}_{\mathrm{Y}} + \mathcal{P}_{\mathrm{Y}}A^* + B_2B_2^*, \\ \qquad S_Y = \begin{bmatrix} D_{22}D_{22}^* & D_{22}D_{12}^* \\ D_{12}D_{22}^* & D_{12}D_{12}^* - \gamma^2 I \end{bmatrix}, \\ \qquad K_Y = -S_Y^{-1}\left(\begin{bmatrix} D_{22} \\ D_{12} \end{bmatrix} B_2^* + \begin{bmatrix} C_{2\mathrm{w}} \\ C_{1\mathrm{w}} \end{bmatrix} \mathcal{P}_{\mathrm{Y}} \right), \end{cases} \tag{12.5}$$

has a solution $(\mathcal{P}_{\mathrm{Y}}, S_Y, K_Y) \in \mathcal{B}(H, \mathrm{Dom}(\begin{bmatrix} C_{2\mathrm{w}} \\ C_{1\mathrm{w}} \end{bmatrix})) \times \mathcal{B}(Y \times Z) \times \mathcal{B}(H_1^*, Y \times Z)$ *s.t.* $\mathcal{P}_{\mathrm{Y}} \geq 0$, *and the semigroup generated by* $A^* + \begin{bmatrix} C_2^* & C_1^* \end{bmatrix} K_Y$ *is exponentially stable.*

(3.) **(Coupling condition)** $\rho(\mathcal{P}_{\mathrm{X}}\mathcal{P}_{\mathrm{Y}}) < \gamma^2$.

(We recall from Theorem 9.8.12(a) that any exponentially stabilizing solutions of Riccati equations are unique.)

Assume that (1.)–(3.) are satisfied. Then the following hold:

(a) All suboptimal exponentially stabilizing DPF-controllers for Σ *are the ones parametrized in Theorem 12.1.8 (for* $\widetilde{\mathcal{A}} = \mathrm{MTIC}^{\mathrm{L}^1}_{\mathrm{exp}}$ *in cases (I)–(IV) and* $\widetilde{\mathcal{A}}$*=Theorem 8.4.9(*γ'*) in case (V)); note that if* $D_{21} = 0$*, then all of them are well-posed. Moreover, also condition (4.) of Theorem 12.1.8 can be written as a* B_{w}^**-CARE, i.e., as follows:*

(4.) **($\mathcal{P}_{\mathrm{Z}}$-CARE)** *For some (equivalently, all)* $X \in \mathcal{GB}(U \times W)$ *s.t.* $X_{21} = 0$ *and* $S = X^* J_1 X$*, the CARE*

$$\begin{cases} K_Z^* S_Z K_Z = A_Z^* \mathcal{P}_{\mathrm{Z}} + \mathcal{P}_{\mathrm{Z}} A_Z + B_2 X_{22}^{-1} X_{22}^{-*} B_2^*, \\ \qquad S_Z = D_Z^* J_1 D_Z, \\ \qquad K_Z = -S_Z^{-1}\left(\begin{bmatrix} D_{22}X_{22}^{-1} \\ X_{12}X_{22}^{-1} \end{bmatrix} X_{22}^{-*} B_2^* + (B_Z^*)_{\mathrm{w}} \mathcal{P}_{\mathrm{Z}} \right), \end{cases} \tag{12.6}$$

has a solution $(\mathcal{P}_{\mathrm{Z}}, S_Z, K_Z) \in \mathcal{B}(H, \mathrm{Dom}((B_Z^*)_{\mathrm{w}})) \times \mathcal{B}(Y \times Z) \times \mathcal{B}(\mathrm{Dom}(A_Z), Y \times U)$ *s.t.* $\mathcal{P}_{\mathrm{Z}} \geq 0$, $S_{Z11} \gg 0$, $S_{Z22} - S_{Z21}S_{Z11}^{-1}S_{Z12} \ll 0$ *and the semigroup generated by* $A_Z + B_Z K_Z$ *is exponentially stable.*

(b) The systems Σ_{X}, Σ_{Y}, Σ_{Z}, $\Sigma_{\mathbb{E}^{\mathrm{d}}}$, $\Sigma_{\mathbb{T}}$ *and* Σ_{alt} *satisfy the assumptions of Lemma 6.8.5 for* $p = 1 = q$ *if any of (I)–(IV) holds (and for* $p = 2 = q$ *if (IV) holds). If (I) holds, then they also satisfy Hypothesis 9.5.1.*

(See Corollary 9.5.12(b) for additional smoothness for case (I). By Corollary 9.5.12(a), conditions “2.” and “3.” may be omitted from (I) if we write (1.)–(2.) to the form of Theorem 12.1.5.)

Under the normalizing conditions

$$D_{12} = 0, \qquad D_{11}^* \begin{bmatrix} C_1 & D_{11} \end{bmatrix} = \begin{bmatrix} 0 & I \end{bmatrix}, \tag{12.7}$$

condition (1.) can be written as follows:

$$((B_1^*)_{\mathrm{w}}\mathcal{P}_{\mathrm{X}})^*(B_1^*)_{\mathrm{w}}\mathcal{P}_{\mathrm{X}} - \gamma^{-2}((B_2^*)_{\mathrm{w}}\mathcal{P}_{\mathrm{X}})^*(B_2^*)_{\mathrm{w}}\mathcal{P}_{\mathrm{X}} = A^*\mathcal{P}_{\mathrm{X}} + \mathcal{P}_{\mathrm{X}}A + C_1^*C_1 \tag{12.8}$$

with the requirements that $\mathcal{P}_{\mathrm{X}} \in \mathcal{B}(H, \mathrm{Dom}(B_{\mathrm{w}}^*))$, $\mathcal{P}_{\mathrm{X}} \geq 0$, and $A + (\gamma^{-2} B_2 (B_2^*)_{\mathrm{w}} - B_1 (B_1^*)_{\mathrm{w}}) \mathcal{P}_{\mathrm{X}}$ is exponentially stable. (Note that now $S_X = J_\gamma := \left[\begin{smallmatrix} I & 0 \\ 0 & -\gamma^2 I \end{smallmatrix}\right]$ and $K_X = \left[\begin{smallmatrix} -(B_1^*)_{\mathrm{w}} \mathcal{P}_{\mathrm{X}} \\ \gamma^{-2} (B_2^*)_{\mathrm{w}} \mathcal{P}_{\mathrm{X}} \end{smallmatrix}\right] \in \mathcal{B}(H, U \times W)$.) If B is bounded, then (12.8) takes the classical form

$$\mathcal{P}_{\mathrm{X}} (B_1 B_1^* - \gamma^{-2} B_2 B_2^*) \mathcal{P}_{\mathrm{X}} = A^* \mathcal{P}_{\mathrm{X}} + \mathcal{P}_{\mathrm{X}} A + C_1^* C_1. \tag{12.9}$$

See p. 618 for further simplification and remarks. Analogous remarks apply to (2.) and (4.). We thus observe that the classical H^∞ CAREs become special cases of (1.) and (2.) (cf. p. 618).

Recall from Definition 9.8.1, that each of these $\mathcal{P}_{\mathrm{X}}$-CAREs is given on $\mathcal{B}(H_1, H_{-1}^*) := \mathcal{B}(\mathrm{Dom}(A), \mathrm{Dom}(A^*)^*)$; e.g., (12.8) holds iff

$$\begin{aligned} \langle (B_1^*)_{\mathrm{w}} \mathcal{P}_{\mathrm{X}} x_0, (B_1^*)_{\mathrm{w}} \mathcal{P}_{\mathrm{X}} x_1 \rangle - \gamma^{-2} \langle (B_2^*)_{\mathrm{w}} \mathcal{P}_{\mathrm{X}} x_0, (B_2^*)_{\mathrm{w}} \mathcal{P}_{\mathrm{X}} x_1 \rangle \\ = \langle A x_0, \mathcal{P}_{\mathrm{X}} x_1 \rangle + \langle \mathcal{P}_{\mathrm{X}} x_0, A x_1 \rangle + \langle C_1 x_0, C_1 x_1 \rangle \end{aligned} \tag{12.10}$$

for all $x_0, x_1 \in \mathrm{Dom}(A)$ (we can take $x_1 = x_0$ w.l.o.g., by Lemma A.3.1(g1)). Analogously, the $\mathcal{P}_{\mathrm{Y}}$-CAREs are given on $\mathrm{Dom}(A^*)$ and the $\mathcal{P}_{\mathrm{Z}}$-CAREs (see Theorem 12.1.8) on $\mathrm{Dom}(A_Z)$.

See the remark in the proof for weakening (A1) (e.g., by assuming that Hypothesis 9.2.1 is satisfied by certain systems with J_γ or J_1). See Remark 12.1.7 for several equivalent conditions for (A2).

One usually assumes that (A, B_1) is exponentially stabilizable and (A, C_2) is exponentially detectable. By the above, such assumptions are necessary but redundant under (A1)–(A2) (or somewhat weaker analogous assumptions, see Theorem 12.1.5 or Theorem 12.2.1) .

Note that in "$(\mathcal{P}_{\mathrm{X}}, S_X, K_X) \in \ldots$" only "$\mathcal{P}_{\mathrm{X}} \in \mathcal{B}(H, \mathrm{Dom}(B_{\mathrm{w}}^*))$" is a requirement; the other two conditions are automatically satisfied whenever $\mathcal{P}_{\mathrm{X}} \in \mathcal{B}(H, \mathrm{Dom}(B_{\mathrm{w}}^*))$ and S_X and K_X are determined by the second and third equations of (12.4). An analogous remark applies to (2.) and (4.) (and to any other B_{w}^*-CARE).

Proof of Theorem 12.1.4: 0.1° *Remarks on (I)–(V):* Conditions (I) says that Hypothesis 9.5.1 holds for Σ (hence for Σ_{X} and Σ_{Y} too) and that Hypothesis 9.5.7(2.) (without the D^*JD condition) is satisfied by Σ_{X} and Σ_{Y} (but not necessarily by Σ). The assumptions in (II)–(V) could be weakened analogously.

Even without the standing hypotheses, (I) implies that $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ generate an ULR WPLS, by Lemma 6.3.15. For (II) (resp. (III)), we must add the conditions (4.) (resp. (2.)) and (1.) of Lemma 6.3.15. Conditions (IV) and (V) are far from sufficient to guarantee the axioms of WPLSs.

The $\mathrm{H}^2_{\mathrm{strong}}$-condition in (V) could be rephrased as "$\mathbb{D} - D \in \mathcal{B}(U \times W, \mathrm{L}^2(\mathbf{R}_+; Z \times Y)) *$ and $\mathbb{D}^{\mathrm{d}} - D \in \mathcal{B}(Z \times Y, \mathrm{L}^2(\mathbf{R}_+; U \times W)) *$", i.e., as "$C_{\mathrm{w}} \mathbb{A} B v_0 \in \mathrm{L}^2(\mathbf{R}_+; Z \times Y)$ and $B_{\mathrm{w}}^* \mathbb{A}^* C^* \widetilde{y}_0 \in \mathrm{L}^2(\mathbf{R}_+; U \times W)) *$ for all $v_0 \in U \times W$ and $\widetilde{y}_0 \in Z \times Y$".

0.2° *Remark: alternatives for (A1):* In Theorem 12.1.4, an alternative for (A1) would be to assume that $(\Sigma_{\mathrm{X}}, J_\gamma)$ and $(\Sigma_{\mathrm{Y}}, J_\gamma)$ have the property of Remark 9.9.14(c).

Then (1.)–(3.) become necessary, by Lemma 12.1.13, but sufficiency would require a slightly stronger assumption to guarantee that also $\Sigma_{\mathbb{E}^{\mathrm{d}}}$ or Σ_{Z} is as smooth (whenever (1.) holds).

An even weaker assumption can also suffice: if $(\Sigma_{\rm X}, J_\gamma), (\Sigma_{\rm Y}, J_\gamma) \in$ coerciveCARE, then (1.)–(3.) are still necessary (possibly in a modified form, see Lemma 12.1.13 and Remark 12.1.6; also (A2) must possibly be altered if $\mathbb{D}$ is not ULR), but for sufficiency one needs several additional assumptions, cf. the remark in the proof of Lemma 12.1.12

1° *Case (V):* This is contained in Theorem 12.1.11 (let $\widetilde{\mathcal{A}}$ be the class of Theorem 8.4.9(γ'); note that Hypothesis 12.5.1 holds, by Remark 12.1.7(a)), since the two forms of (1.)–(2.) are equivalent, by Remark 12.1.6 (because $\Sigma_{\rm X}$ and $\Sigma_{\rm Y}$ satisfy Hypothesis 9.2.2(7.)). Consequently, we do not treat case (V) in 2°–3° below.

2° *Condition (A1) of Theorem 12.1.5 is satisfied:* For (II)–(IV) this is obvious, since $\mathbb{A} \in \mathcal{C}(\mathbf{R}_+; \mathcal{B}(H)) \subset \mathrm{L}^1_{\rm loc}(\mathbf{R}_+; \mathcal{B}(H))$; for (I) this follows from Lemma 9.5.2.

3° *The equivalence and all controllers:* We deduce from Theorem 12.1.5 that (1.)–(3.) of that theorem are equivalent to the existence of an exponentially stabilizing suboptimal DPF-controller for Σ (possibly with internal loop), and that if either holds, then all such controllers are parametrized by Theorem 12.1.8 (and that in this case the assumptions of Theorem 12.1.8 are satisfied).

It only remains to be shown that (1.)–(2.) can be written as in this theorem.

4° *(1.)–(3.) are equivalent to (1.)–(3.) of Theorem 12.1.5:* Note first that if either conditions hold, then $D_X^* J_\gamma D_X, D_Y^* J_\gamma D_Y \in \mathcal{GB}$, by (A2) and the signature conditions, hence the condition "$D^* J D \in \mathcal{GB}$" of Hypothesis 9.2.2(4.) (or of Hypothesis 9.5.7(2.)) is satisfied by both $\Sigma_{\rm X}$ and $\Sigma_{\rm Y}$.

Condition (A1) implies that Hypothesis 9.2.2 is satisfied by both $\Sigma_{\rm X}$ and $\Sigma_{\rm Y}$; indeed, (I) implies (2.) or (3.) (resp. (2.) or (3.)) of Hypothesis 9.2.2 for $\Sigma_{\rm X}$ (resp. for $\Sigma_{\rm Y}$), (II) implies (1.) (resp. (4.)), (III) implies (4.) (resp. (1.)), (IV) implies (5.) (resp. (5.)), (V) implies (7.) (resp. (7.)).

By Theorem 9.2.3, it follows that Hypothesis 9.2.1 is satisfied by $(\Sigma_{\rm X}, J_\gamma)$ and by $(\Sigma_{\rm Y}, J_\gamma)$, hence the two CAREs become equivalent to BwCAREs, by Theorem 9.2.9.

(a) The claim (a) is contained in Theorem 12.1.5 except for the fact that the $\mathcal{P}_{\rm Z}$-CARE can be written as (4.) above; equivalently, for the fact that $\Sigma_{\rm Z}$ and J_1 satisfy Hypothesis 9.2.1. This will be established below.

(a)&(b) We shall establish (b) and show that $\Sigma_{\rm Z}$ satisfies Hypothesis 9.2.2, so that $\Sigma_{\rm Z}$ and J_1 satisfy Hypothesis 9.2.1. This completes the proof of (a). It also implies that (4.) is satisfied (since "(4.)" of Theorem 12.1.8 is necessarily satisfied, by the theorem).

*a.*1° *The systems $\Sigma_{\rm X}$, $\Sigma_{\rm Y}$, $\Sigma_{\rm Z}$ and $\Sigma_{\mathbb{E}^{\rm d}}$ satisfy the assumptions of Lemma 6.8.5 for $p = 1 = q$ if any of (I)–(IV) holds (and for $p = 2 = q$ if (IV) holds):*

For $\Sigma_{\rm X}$ and $\Sigma_{\rm Y}$, this was noted in 1°. Since $B_{\rm w}^* \mathcal{P}_{\rm X} \in \mathcal{B}(H, U \times W)$, the operator K_X is of the form described in Lemma 6.8.5 (for $\Sigma_{\rm X}$, hence also for Σ), hence also $\Sigma_{\circlearrowleft}$ and $\Sigma_{\circlearrowleft}^{\rm d}$ satisfy the assumptions of the lemma (by the lemma). Consequently, so does $\Sigma_{\mathbb{E}^{\rm d}}$ (since it is a part of $\Sigma_{\circlearrowleft}^{\rm d}$, by Lemma 12.5.15), hence so does $\Sigma_{\rm Z}$, by Lemma 6.8.5(b).

*a.*2° *The systems $\Sigma_{\mathbb{T}}$ and $\Sigma_{\rm alt}$ satisfy the assumptions of Lemma 6.8.5 for*

$p = 1 = q$ if (4.) and any of (I)–(IV) hold (and for $p = 2 = q$ if (4.) and (IV) hold): Since K_Z in (4.) is of the form described in Lemma 6.8.5 (by $a.1°$) for Σ_Z, it follows that $\Sigma_{\rm alt}$ satisfies the assumptions of Lemma 6.8.5, hence so does $\Sigma_{\mathbb{T}}$ (because it is a part of $\Sigma_{\rm alt}$).

$a.3°$ *The rest of (a) and (b):* We show this separately under each assumption. Note that except for claim on Hypothesis 9.5.1, it suffices to be shown that Hypothesis 9.2.2 is satisfied (so that (4.) holds).

We shall use the fact that "(4.)" of Theorem 12.1.8 is satisfied and $S_Z = D_Z^* J_1 D_Z \in \mathcal{GB}$, as noted in the proof of Theorem 12.1.5. (N.B., even if we had not assumed (1.)–(3.), conditions (1.) and (4.) above would imply that also the corresponding CAREs were satisfied (with $S_Z = D_Z^* J D_Z$), by Theorem 9.2.9(iii)&(iv), hence then (1.)–(3.) would again hold, by Theorem 12.1.8 and the above.)

(II): Since C_Z is bounded, by (II) and (12.94), Hypothesis 9.2.2(4.) is satisfied by Σ_Z.

(III): Now C and K_X are bounded (since $C_X = \begin{bmatrix} C_1 \\ 0 \end{bmatrix}$ is), hence B_Z is is bounded, by (12.94). Thus, Hypothesis 9.2.2(1.) is satisfied by Σ_Z.

(IV): By $a.1°$, Hypothesis 9.2.2(5.) is satisfied by Σ_Z.

(I): Since $C_X = \begin{bmatrix} C_1 \\ 0 \end{bmatrix} \in \mathcal{B}(H_{\gamma_1}, Z \times W)$, we have $K_X \in \mathcal{B}(H_{\gamma_1}, U \times W)$, hence also Σ_Z is analytic, by Lemma 9.5.4.

It follows that $B_Z \in \mathcal{B}(Y \times U, (H_{\beta_Z})^*)$, where $\beta_Z := -\max\{\gamma_1, \gamma_2\}$, by (12.94). Moreover, $C_Z \in \mathcal{B}(H_{\gamma_Z}, W \times U)$, where $\gamma_Z = -\beta_2$, by (12.94).

We conclude from (I) that ($\beta_Z, \gamma_Z \in (-1/2, 1/2)$ and) $\gamma_Z < 1/4$ or $\gamma_Z - \beta_Z < 1/2$, hence (2.) or (3.) of Hypothesis 9.5.7 is satisfied by Σ_Z (recall that $D_Z^* J_1 D_Z \in \mathcal{GB}$), hence so is Hypothesis 9.2.2(2.). Consequently, the $\mathcal{P}_Z$-CARE becomes a $B_{\rm w}^*$-CARE.

It follows that $K_Z \in \mathcal{B}(H_{\gamma_Z}, Y \times U)$. We conclude from Proposition 12.5.19 and Lemma 9.5.4 that $\Sigma_{\rm alt}$ satisfies Hypothesis 9.5.1, hence so does $\Sigma_{\mathbb{T}}$. (Note that Hypothesis 9.5.1 is stronger than the assumptions of Lemma 6.8.5, by Lemma 9.5.2.) □

Next we shall allow for any "L^1 systems" instead of the "$B_{\rm w}^*$-CARE" type regularity above. Thus, assumptions (I)–(III) above become special cases of those in (A1) below. This leads to more general (read: more complicated) Riccati equations:

Theorem 12.1.5 (H^∞ 4BP ⇔ CAREs)

(A1) **(Regularity)** *Assume that* $\pi_{[0,1)} \mathbb{A} B \in \mathrm{L}^1([0,1); \mathcal{B}(U \times W, H))$, $\pi_{[0,1)} C_{\rm w} \mathbb{A} \in \mathrm{L}^1([0,1); \mathcal{B}(H, Z \times Y))$, *and* $\pi_{[0,1)} C_{\rm w} \mathbb{A} B \in \mathrm{L}^1([0,1); \mathcal{B}(U \times W, Z \times Y))$, *or that (IV) of Theorem 12.1.4 holds.*

(A2) **(Nonsingularity)** *Assume that* $D_{11}^* D_{11} \gg 0$ *and* $D_{22} D_{22}^* \gg 0$, *and that there is* $\varepsilon > 0$ *s.t.*

$$(ir - A)x_0 = B_1 u_0 \implies \|C_{1w} x_0 + D_{11} u_0\|_Z \geq \varepsilon \|x_0\|_H \quad \text{and} \tag{12.11}$$

$$(ir - A^*)x_0 = C_2^* y_0 \implies \|B_{2\rm w}^* x_0 + D_{22}^* y_0\|_W \geq \varepsilon \|x_0\|_H \tag{12.12}$$

for all $x_0 \in H,\ u_0 \in U,\ y_0 \in Y,\ r \in \mathbf{R}$.

Then there is a suboptimal exponentially stabilizing DPF-controller (possibly with internal loop) for Σ *iff (1.)–(3.) hold:*

(1.) **($\mathcal{P}_{\mathrm{X}}$-CARE)** $D_{12}^* D_{12} - D_{12}^* D_{11}(D_{11}^* D_{11})^{-1} D_{11}^* D_{12} \ll \gamma^2 I_W$, *and the CARE*

$$\begin{cases} K_X^* S_X K_X = A^* \mathcal{P}_{\mathrm{X}} + \mathcal{P}_{\mathrm{X}} A + C_1^* C_1, \\ S_X = \begin{bmatrix} D_{11}^* D_{11} & D_{11}^* D_{12} \\ D_{12}^* D_{11} & D_{12}^* D_{12} - \gamma^2 I \end{bmatrix}, \\ K_X = -S_X^{-1}\left(\begin{bmatrix} D_{11}^* \\ D_{12}^* \end{bmatrix} C_1 + B_{\mathrm{w}}^* \mathcal{P}_{\mathrm{X}} \right), \end{cases} \tag{12.13}$$

has a solution $(\mathcal{P}_{\mathrm{X}}, S_X, K_X) \in \mathcal{B}(H) \times \mathcal{B}(U \times W) \times \mathcal{B}(H_1, U \times W)$ *s.t.* $\mathcal{P}_{\mathrm{X}} \geq 0$, K_X *is exponentially stabilizing for* $\left(\begin{array}{c|c} A & B \end{array} \right)$, *and* $\lim_{s \to +\infty} B_{\mathrm{w}}^* \mathcal{P}_{\mathrm{X}} (s - A)^{-1} B = 0$.

(2.) **($\mathcal{P}_{\mathrm{Y}}$-CARE)** $D_{12} D_{12}^* - D_{12} D_{22}^* (D_{22} D_{22}^*)^{-1} D_{22} D_{12}^* \ll \gamma^2 I_Z$, *and the CARE*

$$\begin{cases} K_Y^* S_Y K_Y = A \mathcal{P}_{\mathrm{Y}} + \mathcal{P}_{\mathrm{Y}} A^* + B_2 B_2^*, \\ S_Y = \begin{bmatrix} D_{22} D_{22}^* & D_{22} D_{12}^* \\ D_{12} D_{22}^* & D_{12} D_{12}^* - \gamma^2 I \end{bmatrix}, \\ K_Y = -S_Y^{-1}\left(\begin{bmatrix} D_{22} \\ D_{12} \end{bmatrix} B_2^* + \begin{bmatrix} C_{2\mathrm{w}} \\ C_{1\mathrm{w}} \end{bmatrix} \mathcal{P}_{\mathrm{Y}} \right), \end{cases} \tag{12.14}$$

has a solution $(\mathcal{P}_{\mathrm{Y}}, S_Y, K_Y) \in \mathcal{B}(H) \times \mathcal{B}(Y \times Z) \times \mathcal{B}(H_1^*, Y \times Z)$ *s.t.* $\mathcal{P}_{\mathrm{Y}} \geq 0$, K_Y *is exponentially stabilizing for* $\left(\begin{array}{c|cc} A^* & C_2^* & C_1^* \end{array} \right)$, *and* $\lim_{s \to +\infty} \begin{bmatrix} C_2 \\ C_1 \end{bmatrix}_{\mathrm{w}} \mathcal{P}_{\mathrm{Y}} (s - A)^{-1} \begin{bmatrix} C_2^* & C_1^* \end{bmatrix} = 0$.

(3.) **(Coupling condition)** $\rho(\mathcal{P}_{\mathrm{X}} \mathcal{P}_{\mathrm{Y}}) < \gamma^2$.

If (1.)–(3.) are satisfied, then all suboptimal exponentially stabilizing DPF-controllers for Σ *are the ones parametrized in Theorem 12.1.8 (for* $\widetilde{\mathcal{A}} = \mathrm{MTIC}^{\mathrm{L}^1}_{\mathrm{exp}}$, *or for class (γ′) of Theorem 8.4.9 under the alternative assumption "(IV)" in (A1)); note that if* $D_{21} = 0$, *then all of them are well-posed.*

We remark from Theorem 11.1.4(iii)&(i) that (1.) implies that the FICP for Σ_{X} has a solution, and (2.) implies that the (dual) filter problem for Σ_{Y} has a solution.

A sufficient condition for (A1) is condition (A1)(I)1. of Theorem 12.1.4, by Corollary 9.5.12. See Theorem 12.1.11 for alternatives for (A1) and (A2).

Proof of Theorem 12.1.5: If there is a suboptimal exponentially stabilizing DPF-controller (possibly with internal loop), then (1.)–(3.) hold, by Lemma 12.5.22. (Thus, a fortiori, this is the case when there is a *well-posed* suboptimal exponentially stabilizing DPF-controller.)

Conversely, if (1.)–(3.) hold, then Hypothesis 12.5.1 and conditions (1.) and (4.) of Theorem 12.1.8 are satisfied, by Lemmas 12.5.21 and 12.5.20. Thus, then there are suboptimal exponentially stabilizing DPF-controllers for Σ, by Theorem 12.1.8 (whose assumptions are satisfied).

We remark from 3° of the proof of Lemma 12.5.20 that $S_Z = D_Z^* J D_Z$ when (1.)–(3.) hold. □

Remark 12.1.6 (Different forms of (1.)–(2.)) *We first recall from Theorem 9.8.12(b) that any solution of any (1.), (2.) or (4.) of this section is unique.*

We have used three forms of the $\mathcal{P}_{\mathrm{X}}$-CARE ("(1.)") and the $\mathcal{P}_{\mathrm{Y}}$-CARE ("(2.)"). The "weakest" forms (the standard forms) are given in Lemma 12.1.12 (they apply to any UR settings).

Those given in Theorem 12.1.5 are equivalent to those given in Lemma 12.1.12 combined with assumptions $S_X = D_X^* J_\gamma D_X$ and $S_Y = D_Y^* J_\gamma D_Y$; these assumptions are redundant when $\mathbb{N}_{\mathrm{u}}$ and $\widetilde{\mathbb{N}_{\mathrm{y}}}^{\mathrm{d}}$ satisfy Hypothesis 8.4.8, as noted in the proof of Lemma 12.5.21.

The strongest forms of (1.)–(2.) (the B_{w}^*-CARE forms) are given in Theorem 12.1.4; they are equivalent to either of the weaker ones whenever Hypothesis 9.2.1 is satisfied by $(\Sigma_{\mathrm{X}}, J_\gamma)$ and $(\Sigma_{\mathrm{Y}}, J_\gamma)$ (see (12.84)–(12.85)).

In all our results referring to any of these CAREs, one can always replace the CAREs by any weaker ones (of the three forms listed above).

Moreover, under the assumptions of Theorem 12.1.4, 12.1.5 or 12.1.11, we can use the middle form for the $\mathcal{P}_{\mathrm{Z}}$-CARE (4.) or (4.') too (i.e., require that $S_Z = D_Z^* J D_Z$). □

(This is straightforward; use Theorem 9.2.9(iii)&(iv) for the B_{w}^*-CARE forms.)

Condition (A2) is virtually equivalent to the I-coercivity of $\mathbb{D}_{11}$ and $\mathbb{D}_{22}^{\mathrm{d}}$:

Remark 12.1.7 (Different forms of (A2)) *(a) Assume that (A1) of Theorem 12.1.4 holds (or that (A1) of Theorem 12.1.5 holds). Assume that (A, B_1) is optimizable and (A, C_2) is estimatable (this is a necessary condition, by Theorem 7.3.12(a)) or that $D_{11}^* D_{11} \gg 0$ and $D_{22} D_{22}^* \gg 0$.*

Then condition (A2) of Theorem 12.1.4 holds iff $\mathbb{D}_{11}$ and $\mathbb{D}_{22}^{\mathrm{d}}$ are I-coercive over $\mathcal{U}_{\mathrm{exp}}$; in fact, then any of (i)–(iii) of Proposition 10.3.2 or Remark 10.3.3 (for $\Sigma_{\mathbb{D}_{11}}$ and $\Sigma_{\mathbb{D}_{22}^{\mathrm{d}}}$) are equivalent. One more equivalent condition is that Hypothesis 12.5.1 holds.

(b) If Σ is exponentially stable, then $\mathbb{D}_{11}$ and $\mathbb{D}_{22}^{\mathrm{d}}$ are I-coercive over $\mathcal{U}_{\mathrm{exp}}$ iff $\mathbb{D}_{11}^* \mathbb{D}_{11} \gg 0$ and $\mathbb{D}_{22} \mathbb{D}_{22}^* \gg 0$ (equivalently, iff Hypothesis 12.5.1 holds).

(c) The I-coercivity of $\mathbb{D}_{11}$ and $\mathbb{D}_{22}^{\mathrm{d}}$ over $\mathcal{U}_{\mathrm{exp}}$ is equivalent to (12.78) if the rest of Hypothesis 12.5.1 holds and (A, B) is optimizable.

In this chapter, the I-coercivity of $\mathbb{D}_{11}$ (resp. of $\mathbb{D}_{22}$) refers to the realization $\Sigma_{\mathbb{D}_{11}} := \left(\begin{array}{c|c} A & B_1 \\ \hline C_1 & D_{11} \end{array}\right)$ (resp. $\Sigma_{\mathbb{D}_{22}^{\mathrm{d}}} := \left(\begin{array}{c|c} A^* & C_2^* \\ \hline B_2^* & D_{22}^* \end{array}\right)$).

Proof: (a) By Proposition 10.3.2(e2) (use (e1) and (g2) for (V)), conditions (i)–(iii) are equivalent (note that this include I-coercivity and (A2)). The rest is given in Lemma 12.5.4 (use (b) for (V)) (since any of (I)–(IV) of Theorem 12.1.4(A1) implies (A1) of Theorem 12.1.5, by 2° of the proof of Theorem 12.1.4).

(b) This follows from (c) (take $\left[\begin{array}{c|c} \mathbb{K}_{\mathrm{u}} & \mathbb{F}_{\mathrm{u}} \end{array}\right] = 0 = \left[\begin{array}{c} \mathbb{H}_{\mathrm{y}} \\ \hline \mathbb{G}_{\mathrm{y}} \end{array}\right]$).

(c) By Lemma 12.5.2(i)&(vi), $\left[\begin{array}{c|c} \mathbb{K}_{\mathrm{u}} & \mathbb{F}_{\mathrm{u}} \end{array}\right]$ is exponentially stabilizing for Σ, hence $\left[\begin{array}{c|c} \mathbb{K}_{u1} & \mathbb{F}_{u11} \\ 0 & 0 \end{array}\right]$ is exponentially stabilizing for $\Sigma_{\mathbb{D}_{11}}$ (since the two closed-loop semigroups are equal).

Therefore $\mathbb{D}_{11}$ is I-coercive over $\mathcal{U}_{\exp}$ iff $\mathbb{N}_{u11}$ is (positively) I-coercive over $\mathcal{U}_{\text{out}}$, by Theorem 8.4.5(d). By Lemma 8.4.11(a2), this is the case iff $\mathbb{N}_{u11}^*\mathbb{N}_{u11} \gg 0$. By dual arguments we obtain that $\mathbb{D}_{22}^{\text{d}}$ is I-coercive over $\mathcal{U}_{\exp}$ iff $\widetilde{\mathbb{N}}_{y22}\widetilde{\mathbb{N}}_{y22}^* \gg 0$. □

As in, e.g., [IOW], the $\mathcal{P}_{\text{Z}}$-CARE is not determined by the exponentially stabilizing solution of the $\mathcal{P}_{\text{X}}$-CARE but by a(ny) exponentially stabilizing solution of the $\mathcal{P}_{\text{X}}$-IARE having $X_{11}, X_{22} \in \mathcal{GB}$ and $X_{21} = 0$ (see (9.114)). Analogously, for the parametrization of all controllers we use a modified solution of the $\mathcal{P}_{\text{Z}}$-IARE(see (12.94)):

Theorem 12.1.8 (All solutions to the 4BP) *Assume that Hypothesis 12.5.1 is satisfied with $\mathbb{N}_{\text{u}}, \mathbb{M}_{\text{u}} \in \widetilde{\mathcal{A}}$. Then there is a suboptimal exponentially stabilizing DPF-controller for Σ iff conditions (1.) and (4.) below hold:*

(1.) **($\mathcal{P}_{\text{X}}$-CARE)** *The CARE*

$$\begin{cases} K_X^* S_X K_X = A^*\mathcal{P}_{\text{X}} + \mathcal{P}_{\text{X}} A + C_1^* C_1, \\ \quad S_X = \begin{bmatrix} D_{11}^* D_{11} & D_{11}^* D_{12} \\ D_{12}^* D_{11} & D_{12}^* D_{12} - \gamma^2 I \end{bmatrix} + \lim_{s\to+\infty} B_{\text{w}}^* \mathcal{P}_{\text{X}} (s-A)^{-1} B, \\ \quad K_X = -S_X^{-1}\left(\begin{bmatrix} D_{11}^* \\ D_{12}^* \end{bmatrix} C_1 + B_{\text{w}}^* \mathcal{P}_{\text{X}} \right), \end{cases} \tag{12.15}$$

has a solution $(\mathcal{P}_{\text{X}}, S_X, K_X) \in \mathcal{B}(H) \times \mathcal{B}(U\times W) \times \mathcal{B}(H_1, U\times W)$ s.t. $\mathcal{P}_{\text{X}} \geq 0$, $S_{X11} \gg 0$, $S_{X22} - S_{X21} S_{X11}^{-1} S_{X12} \ll 0$ and K_X is exponentially stabilizing for $\left(\begin{array}{c|c} A & B \end{array} \right)$.

(4.) **($\mathcal{P}_{\text{Z}}$-CARE)** *The CARE*

$$\begin{cases} K_Z^* S_Z K_Z = A_Z^* \mathcal{P}_{\text{Z}} + \mathcal{P}_{\text{Z}} A_Z + B_2 X_{22}^{-1} X_{22}^{-*} B_2^*, \\ \quad S_Z = D_Z^* J D_Z + \lim_{s\to+\infty} (B_Z^*)_{\text{w}} \mathcal{P}_{\text{Z}} (s-A_Z)^{-1} B_Z, \\ \quad K_Z = -S_Z^{-1}\left(\begin{bmatrix} D_{22} X_{22}^{-1} \\ X_{12} X_{22}^{-1} \end{bmatrix} X_{22}^{-*} B_2^* + (B_Z^*)_{\text{w}} \mathcal{P}_{\text{Z}} \right), \end{cases} \tag{12.16}$$

has a solution $(\mathcal{P}_{\text{Z}}, S_Z, K_Z) \in \mathcal{B}(H) \times \mathcal{B}(Y\times Z) \times \mathcal{B}(\text{Dom}(A_Z), Y\times U)$ s.t. $\mathcal{P}_{\text{Z}} \geq 0$, $S_{Z11} \gg 0$, $S_{Z22} - S_{Z21} S_{Z11}^{-1} S_{Z12} \ll 0$ and K_Z is exponentially stabilizing for $\left(\begin{array}{c|c} A_Z & B_Z \end{array} \right)$.

Given the solution $(\mathcal{P}_{\text{X}}, S_X, K_X)$ of (1.), choose any $X \in \mathcal{GB}(U\times W)$ s.t. $X^* J_1 X = S_X$ and $X_{21} = 0$. Then by the operators A_Z, B_Z, C_Z, D_Z appearing in the $\mathcal{P}_{\text{Z}}$-CARE we mean the following (here $K_X = \begin{bmatrix} K_{X1} \\ K_{X2} \end{bmatrix} \in \mathcal{B}(H_1, U\times W)$):

$$\left[\begin{array}{c|c} A_Z & B_Z \\ \hline C_Z & D_Z \end{array} \right] = \left[\begin{array}{c|cc} A^* + K_{X2}^* (B_2^*)_{\text{w}} & C_2^* + K_{X2}^* D_{22}^* & -K_{X1}^* \\ \hline X_{22}^{-*} (B_2^*)_{\text{w}} & X_{22}^{-*} D_{22}^* & X_{22}^{-*} X_{12}^* \\ 0 & 0 & I \end{array} \right] \tag{12.17}$$

$$\in \mathcal{B}(\text{Dom}(A_Z) \times Y \times U, H\times W\times U); \tag{12.18}$$

$$\text{Dom}(A_Z) := \{x\in H \mid A_Z x \in H\} = \{x \in H_{C,K}^* \mid A_Z x\in H\}. \tag{12.19}$$

Moreover, any solutions of (1.) or (4.) are unique.

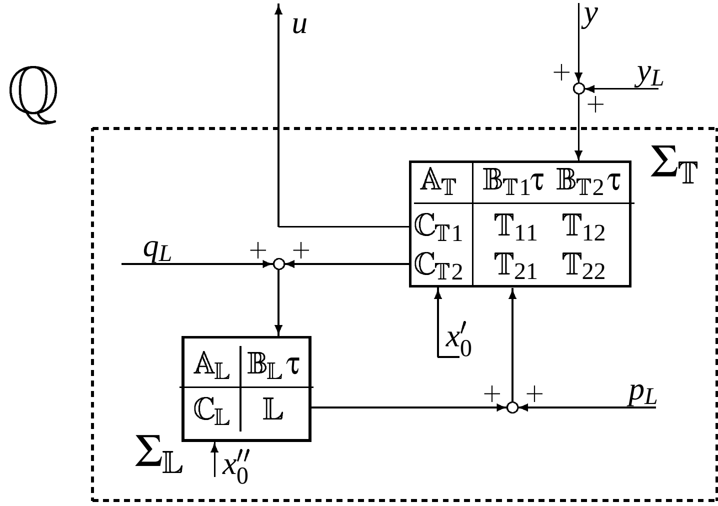

Figure 12.1: The suboptimal controller $\mathbb{Q} := \mathcal{F}_\ell(\mathbb{T},\mathbb{L}) : y \mapsto u$

Assume (1.) and (4.). Then we can construct all suboptimal exponentially stabilizing DPF-controllers for Σ as follows:

Choose $G \in \mathcal{GB}(Y \times U)$ s.t. $G^* J_1 G = S_Z$, $G_{21} = 0$ and $G_{11}, G_{22} \in \mathcal{GB}$, and then set $R := G \begin{bmatrix} I & 0 \\ 0 & M_{11}^* \end{bmatrix}$. If $D_{21} = 0$, then the assumptions of Proposition 12.5.19 are satisfied, hence then $\Sigma_{\mathbb{T}_{12}}$ is exponentially stabilizing for Σ, and all well-posed exponentially stabilizing suboptimal controllers for Σ are given by the connection "$\Sigma_{\mathbb{Q}}$" of $\Sigma_{\mathbb{T}}$ and $\Sigma_{\mathbb{L}}$ in Figure 12.1, where $\Sigma_{\mathbb{L}}$ is any exponentially stable realization of any $\mathbb{L} \in \mathrm{TIC}_{\exp}(Y,U)$ s.t. $\|\mathbb{L}\|_{\mathrm{TIC}} < 1$ (note that the I/O-map of controller $\Sigma_{\mathbb{Q}}$ is $\mathcal{F}_\ell(\mathbb{T},\mathbb{L})$; cf. (12.26)). Moreover, any exponentially stabilizing suboptimal controller with internal loop for Σ is equivalent to a well-posed one.

Here $\mathbb{T} \in \mathrm{TIC}_\infty(U \times Y)$ is a map that has a realization (denoted by $\Sigma_{\mathbb{T}} \in \mathrm{WPLS}(U \times Y, H, U \times Y)$) with the following generators:

$$A_{\mathbb{T}} := A + BK_I + B_{\mathbb{T}2} C_{\circlearrowleft 2} \tag{12.20}$$

$$B_{\mathbb{T}1} := -B_{\circlearrowleft 2} X_{22}^{-*}(D_{22}^* R_{11}^{-1} R_{12} + X_{12}^*) R_{22}^{-1} - B_{\circlearrowleft 1} R_{22}^{-1} \tag{12.21}$$

$$B_{\mathbb{T}2} := -B_{\circlearrowleft 2} X_{22}^{-*} D_{22}^* (R_{11}^* R_{11})^{-1} \tag{12.22}$$

$$C_{\mathbb{T}1} := -K_{I1} - R_{12}^* R_{11}^{-*} C_{\circlearrowleft 2} \tag{12.23}$$

$$C_{\mathbb{T}2} := R_{11}^{-*} C_{\circlearrowleft 2} \tag{12.24}$$

$$D_{\mathbb{T}} := \begin{bmatrix} R_{22}^* & -R_{12}^* R_{11}^{-*} \\ 0 & R_{11}^{-*} \end{bmatrix}, \tag{12.25}$$

where $B_{\circlearrowleft} := BX^{-1}$ and $C_{\circlearrowleft} := C + DK_X$.

For general $D_{21} \in \mathcal{B}(U,Y)$, the above formulae still parametrize all suboptimal exponentially stabilizing DPF-controllers for Σ modulo equivalence (see Definition 7.3.1) except that we have to add to $\mathbb{Q}$ the output feedback through $-D_{21} =: -E$, as in Figure 7.12 (and in Lemma 7.3.23), and that this internal loop need not be well-posed (but the combined system of Figure 7.12 is well-posed).

Note that the $\mathcal{P}_Z$-CARE is not uniquely defined (since X may vary slightly); this is not important, because condition (4.) (and $\mathcal{P}_Z$) is independent on the choice of X (under the above restrictions). An analogous formulation is given in [IOW] (see pp. 281 and 308–309 of [IOW]); the parametrizations on pp. 136–137 of [Keu] or on p. 297 of [GL] are simpler due to simplifying assumptions on the generators of Σ.

The condition $\mathbb{N}_u, \mathbb{M}_u \in \widetilde{\mathcal{A}}$ can be weakened significantly if we replace "iff" by "if" at the beginning of the theorem, as one observes from the proof (see Theorem 12.3.7(a) for the corresponding frequency-space claim).

We repeat that all exponentially stabilizing suboptimal controllers for $\mathbb{D}$ are given by

$$\mathbb{Q} = \mathcal{F}_\ell(\mathbb{T}, \mathbb{L}) \qquad (\mathbb{L} \in \mathrm{TIC}_{\exp}(Y,U) \text{ is s.t. } \|\mathbb{L}\|_{\mathrm{TIC}} < 1) \tag{12.26}$$

(when $D_{21} = 0$; cf. Lemma 7.3.23). As noted below Theorem 12.3.7, the controller $\mathbb{Q}$ in (12.47) is different for different parameters $\mathbb{L}$.

The maps $\mathbb{T}$ and $\mathbb{L}$ are the same as in Theorem 12.3.7(c) and Proposition 12.5.19, as one observes from the proof below, hence parts (a)–(g) of the theorem apply for Σ and $\Sigma_{\mathbb{T}}$.

In particular, $\mathbb{T} \in \mathrm{ULR}$, since $\mathbb{T}$ is given by (12.48) and $\widetilde{\mathbb{N}}_+, \widetilde{\mathbb{M}}_+ \in \widetilde{\mathcal{A}}$. Analogously, $\widetilde{\mathbb{Q}}_2, \widetilde{\mathbb{Q}}_1 \in \widetilde{\mathcal{A}}(Y,*)$ and $\widetilde{\mathbb{Q}}_2 \in \mathcal{G}\mathrm{TIC}(Y)$, where $\mathbb{Q} = \widetilde{\mathbb{Q}}_2^{-1}\widetilde{\mathbb{Q}}_1$ is the l.c.f. of $\mathbb{Q}$ parametrized by (12.47) (here $\widetilde{\mathbb{Q}}_*$, $\widetilde{\mathbb{N}}_+$, $\widetilde{\mathbb{M}}_+$ and $\mathbb{T}$ correspond to Σ with 0 in place of D_{21}; note that the actual controller has the additional loop through $-D_{21}$).

Remark 12.1.9 (Figure 12.1) *Figure 12.1 illustrates the map $\mathbb{Q} := \mathcal{F}(\mathbb{T}, \mathbb{L})$. The combined closed-loop connection of $\Sigma_{\mathbb{T}}$ and $\Sigma_{\mathbb{L}}$ is a system having $\begin{bmatrix} * & \mathbb{Q} \\ * & * \end{bmatrix}$ as its I/O map (see Lemma 7.3.2 with $\Sigma \mapsto \Sigma_{\mathbb{T}}$ and $\widetilde{\Sigma} \mapsto \Sigma_{\mathbb{L}}$). The signals p_L and q_L refer to external inputs and x_0' and x_0'' to the initial states.*

Thus, this combined system is not a realization of $\mathbb{Q}$ in the sense of Definition 6.1.6; however, there exists also a realization of $\mathbb{Q}$ in this strict sense, by Theorem 12.3.5. □

(Note that the parametrization of Figure 12.1 was given in (the Main) Theorem 5.4 of [Keu] too.)

Naturally, any $\mathbb{Q}$ given by (12.26) has infinitely many realizations (and so do $\mathbb{T}_{12}$ and $\mathbb{T}$), and not all of them are stabilizing for Σ. (Recall from Definition 7.3.1 that for a realization of $\mathbb{Q}$ to be stabilizing for Σ, this realization must be also internally stabilized by the connection.) The term "all controllers" has traditionally been used as above, meaning all (suboptimal [exponentially] stabilizing) $\mathbb{Q}$'s (with some realizations), not all realizations of such $\mathbb{Q}$'s.

Our choice to use pairs (J_γ, J_1) for the $\mathcal{P}_{\mathrm{X}}$-CARE (or (Factor1X) of Theorem 12.3.7) and (J_1, J_1) for $\mathcal{P}_{\mathrm{Z}}$-CARE (or (Factor1Z)) has slightly scaled the rows and columns of Σ_{Z} compared to those in [Keu] or [ZDG] (if $\gamma \neq 1$), but there is no essential difference (except that we have no simplifying assumptions such as "$D_{12} = 0 = D_{21}$"). This results in the formula $\mathcal{P}_{\mathrm{Z}} = \mathcal{P}_{\mathrm{Y}}(\gamma^2 I - \mathcal{P}_{\mathrm{X}}\mathcal{P}_{\mathrm{Y}})^{-1}$ instead of $\mathcal{P}_{\mathrm{Z}} = \mathcal{P}_{\mathrm{Y}}(I - \gamma^{-2}\mathcal{P}_{\mathrm{X}}\mathcal{P}_{\mathrm{Y}})^{-1}$ in Lemma 12.6.4.

Proposition 12.1.10 (Non-exponentially stabilizing H$^\infty$ 4BP) *Theorem 12.1.8 holds even iff we remove the word "exponentially" everywhere from the theo-*

rem and replace (1.) by (1.'), (4.) by (4.') and "$\mathrm{TIC}_{\mathrm{exp}}$*" by "*TIC*". Here we have referred to the following conditions:*

- *(1.')* **($\mathcal{P}_{\mathbf{X}}$-CARE)** *Condition (1.) holds except that on K_X we only require that K_X is P-stabilizing for Σ, and that $\mathbb{D}(I-\mathbb{F}_X)^{-1}$ and $(I-\mathbb{F}_X)^{-1}$ are r.c.*
- *(4.')* **($\mathcal{P}_{\mathbf{Z}}$-CARE)** *Condition (1.) holds except that on K_Z we only require that K_Z is P-stabilizing for $\Sigma_Z := \left(\begin{array}{c|c} A_Z & B_Z \\ \hline C_Z & D_Z \end{array}\right)$, and that $(I-\mathbb{F}_Z)(I-\widetilde{\mathbb{F}}) \in \mathcal{G}\mathrm{TIC}(Y\times U)$ and $\mathbb{K}_Z + (I-\mathbb{F}_Z)\widetilde{\mathbb{K}}$ is stable, where $\left[\ \mathbb{K}_Z \ \middle|\ \mathbb{F}_Z\ \right]$ is the pair generated by K_Z.*

Moreover, (1.) has a solution iff (1.') has a solution and Σ is exponentially stabilizable. If(f) Σ is exponentially stabilizable, then (1.) and (1.') have same solutions (if any), and so do (4.) and (4.'); thus, then any suboptimal stabilizing controller parametrized by the modified theorem is exponentially stabilizing iff $\Sigma_{\mathbb{L}}$ is exponentially stable.

Note that the assumptions of the theorem (by Lemma 12.5.9) and formulae (12.20)–(12.25) do not depend on D_{21}. However, for $\mathbb{Q}$ to be suboptimal and stabilizing for Σ, we must add the output feedback through $-D_{21}$ as in Figure 7.12 (with $E := D_{21}$); this combined connection is always well-posed (and stable). See the comments below Lemma 12.5.17 for why the r.c. condition in (1.') has a more complicated counterpart in (4.').

We conclude from Proposition 12.1.10 that if Hypothesis 12.5.1 is satisfied with $\mathbb{N}_{\mathrm{u}}, \mathbb{M}_{\mathrm{u}} \in \widetilde{\mathcal{A}}$, then the H^∞ 4BP can be solved as follows:

0. Choose some initial estimate $\gamma > 0$.
1. If (1.') does not have a solution, then there are no suboptimal controllers.
2. Otherwise, choose X as in (4.'). If (4.') does not have a solution, then there are no suboptimal controllers.
3. Iterate 1.–2. for different values of γ (by using, e.g., a binary search) so as to find a good approximate of the infimal (optimal) γ (but above it, so that (1.') and (4.') have solutions).
4. Choose G and construct $\mathbb{T}$ (and $\Sigma_{\mathbb{Q}}$) as in Theorem 12.1.8 to obtain all suboptimal controllers.

Usually we are looking for an exponentially stabilizing solution; if this is the case, then we must replace (1.') by (1.) and (4.') by (4.) above (see Theorem 12.1.8). If (A,B) is exponentially stabilizable, then the two pairs of conditions become equivalent (by, e.g., Lemma 12.5.2(ii)&(i) and Theorem 6.7.15(c1)).

Under the alternative, more easily verifiable assumptions (A1) and (A2) of Theorem 12.1.5 (or of Theorem 12.1.4, or under the assumptions of Theorem 12.1.11), we can use the following alternative procedure:

0. Choose some initial estimate $\gamma > 0$.
1. There are suboptimal solutions iff (1.)–(3.) of Theorem 12.1.5 hold.

2. Iterate 1. for different values of γ (by using, e.g., a binary search) so as to find a good approximate of the infimal (optimal) γ (but above it, so that (1.)–(3.) have solutions).
3. Set $\mathcal{P}_Z = \mathcal{P}_Y(\gamma^2 I - \mathcal{P}_X\mathcal{P}_Y)^{-1}$ to obtain a solution of (4.).
4. Choose G and construct $\mathbb{T}$ (and $\mathbb{Q}$) as in Theorem 12.1.8 to obtain all suboptimal controllers.

Thus, in this case it suffices to study the $\mathcal{P}_X$-CARE and the $\mathcal{P}_Y$-CARE to find an estimate of infimal γ, but we still need the $\mathcal{P}_Z$-CARE for finding a suboptimal controller (note that $\mathcal{P}_Z$-CARE depends on $\mathcal{P}_X$-CARE, whereas $\mathcal{P}_X$-CARE and $\mathcal{P}_Y$-CARE depend on the original system Σ and γ only and are therefore more suitable for computations). This latter procedure is the classical one.

For the nonexponentially stabilizable case covered by the former procedure, conditions analogous to (1.)–(3.) are necessary, by Lemma 12.1.12, but we do not know whether they are sufficient.

Proof of Theorem 12.1.8 and Proposition 12.1.10: (We prove here both Theorem 12.1.8 and Proposition 12.1.10 at once.)

(We note that if (1.) holds, then $(\mathcal{P}_X, J_1, \begin{pmatrix} XK_X & \big| & I-X \end{pmatrix})$ is an exponentially stabilizing solution of the $\mathcal{P}_X$-IARE; cf. Lemmas 11.1.7 and 12.5.12. Analogously, if also (4.) holds, then $(\mathcal{P}_Z, J_1, \begin{pmatrix} GK_Z & \big| & I-G \end{pmatrix})$ is an exponentially stabilizing solution of the $\mathcal{P}_Z$-IARE, as in Proposition 12.5.19, as one observes from II.2° below. Same "upper triangular" IARE (or "KPYS") solutions were used also on pp. 280–281 of [IOW] (in the corresponding finite-dimensional result).)

0° We first note that any P-stabilizing solution of a CARE is unique, by Theorem 9.8.12(b)&(s1).

Part I: There is a suboptimal stabilizing DPF-controller for Σ iff (1.') and (4.') hold:

I.1° *We only have to show that (4BP3) holds iff (1.') and (4.') hold:* By Lemma 12.5.3, Hypothesis 12.3.1 is satisfied. By Theorem 12.3.5(b) (and Theorem 6.6.28), (4BP1) is equivalent to the existence of a stabilizing DPF-controller *for* Σ. By Lemma 12.3.10, (4BP1)–(4BP3) of Theorem 12.3.7 are equivalent.

I.2° *(Factor1)⇔(1.'):* This follows from Lemma 12.5.12 (since a solution has necessarily $\mathbb{X},\mathbb{N},\mathbb{M} \in \widetilde{\mathcal{A}} \subset$ ULR $\subset$ UR, by Lemma 12.3.10(a)).

(A detailed verification that the condition on $\mathcal{P}_X$-CARE in Lemma 12.5.12 is equivalent to (1.') is given in 6° below. An analogous comment applies to (4.'), (1.) and (4.) too.)

I.3° *Consequences of (Factor1):* If (Factor1) holds, then Hypothesis 12.5.13 holds, by Lemma 12.5.12, and $\mathbb{X},\mathbb{N},\mathbb{M} \in \widetilde{\mathcal{A}} \subset$ ULR, as noted above; in particular, then $\mathbb{M}_{11} \in \mathcal{G}\mathrm{TIC}_\infty(U)$, by Proposition 6.3.1(c).

I.4° *(4BP3)⇒(1.')&(4.'):* Assume (4BP3). By 2°, (1.') holds. But (Factor2Z) has an ULR solution, by Lemma 12.3.10(a) (and Theorem 12.3.7(e1)), hence also (4.')=Lemma 12.5.17(ii) holds, by Lemma 12.5.17(c2).

I.5° *(1.')&(4.')⇒(4BP3):* If $D_{21} = 0$ and (1.') and (4.') hold, so that the facts listed in 3° hold, then we obtain (Factor2) from Lemma 12.5.17(a) (and Theorem 12.3.7(e1)), thus, then (4BP3) and hence also (4BP1) hold. But (4BP1) is independent of D_{21}, by Lemma 12.5.9, hence so is (4BP3); on the other hand, (1.') and (4.') are also independent of D_{21}, by Lemma 12.5.9. Thus, (1.') and (4.') imply (4BP3) regardless of D_{21}.

I.6° *The CAREs (1.') and (4.'):* Equation (12.17) follows from (12.94), Proposition 6.6.18(d4) and (6.145) (note that the $\mathcal{P}_Z$-CARE corresponds to $(\mathcal{P}, S_X, \begin{pmatrix} XK_X & \big| & I-X \end{pmatrix})$, where X is chosen as in (4.') (i.e., as in (4.)), by the last claim of Lemma 12.5.12; note also that we prove the equivalence regardless of the choice of X (within these restrictions)).

By Lemma 9.11.5(e), the "lim" (instead of w-lim) in (1.') (or (1.)) and (4.') (or (4.)) is equivalent to the solution being UR (since Σ_X and Σ_Z (if any) are necessarily UR). Actually, since any spectral factorization is necessarily in $\widetilde{\mathcal{A}}$, hence ULR, we could as well write w-lim.

I.7° *The set* $\mathrm{Dom}(A_Z)$*:* By Lemma A.4.6,

$$\mathrm{Dom}(A_Z) := \{x \in H \,\big|\, A_Z x \in H\}. \tag{12.27}$$

By Proposition 6.6.18(a1), the space "H_C" for $\Sigma_\circlearrowleft$ equals that for Σ_{ext} (of (6.132), which we denote by

$$H^*_{C,K} := (r - A^*)^{-1}[H + C^*[Z \times Y] + K^*[U \times W]] \tag{12.28}$$

$$= \{x_0 \in H \,\big|\, A^* x_0 + C^* \left[\begin{smallmatrix} z_0 \\ y_0 \end{smallmatrix}\right] + K^* \left[\begin{smallmatrix} u_0 \\ w_0 \end{smallmatrix}\right] \in H \text{ for some } (z_0, y_0, u_0, w_0) \in Z \times Y \times U \times W\} \tag{12.29}$$

(for any $r > \omega_A$); cf. Definition 6.1.17. Naturally, this is the space "H_B" for $\Sigma^{\mathrm{d}}_\circlearrowleft$, hence contained by the space "H_B" for $\Sigma_{\mathbb{E}^{\mathrm{d}}}$. By Proposition 6.6.18(a1), the space H_{B_Z} (for Σ_Z) equals "H_B" for $\Sigma_{\mathbb{E}^{\mathrm{d}}}$, and $\mathrm{Dom}(A_Z) \subset H_{B_Z}$, hence we can replace H in (12.27) by H_{B_Z}, hence by its superset $H^*_{C,K}$ (on which B^*_{w} is defined, by the regularity of Σ_{ext} and Proposition 6.2.8(a2), so that the formula for $\mathrm{Dom}(A_Z)$ below (12.17) is well defined).

By Proposition 6.6.18(d3), equation (12.17) actually holds on $H_{B_Z} \times U$, but the values of A_Z on $H_{B_Z} \setminus \mathrm{Dom}(A_Z)$ lie outside H.

Part II: All controllers: Within Part II we assume that (1.') and (4.') hold.

II.1° By I.3°, I.5° and Lemma 12.3.10(a), Hypothesis 12.5.13 holds and (Factor1X) and (Factor2Z) have solutions with $\mathbb{X}, \mathbb{N}, \mathbb{M}, \mathbb{E}, \mathbb{Z} \in \widetilde{\mathcal{A}} \subset$ ULR and $\mathbb{M}_{11} \in \mathcal{G}\mathrm{TIC}_\infty(U)$.

II.2° $R = Z^*$ *and* $\mathbb{Z}_{11} \in \mathcal{G}\mathrm{TIC}_\infty(Y)$ *when* $D_{21} = 0$*:* (Note that G can be obtained from (11.101).) By (a)2° and (d)1° of the proof of Lemma 12.5.17(a), the solution $\mathbb{Z}$ of (Factor2Z) obtained in I.5° satisfies $Z = \widetilde{X}^* G^*$ (where $G^* = \widetilde{Z}$). Since $\widetilde{X} = \begin{bmatrix} I & 0 \\ -D^*_{21} & M^*_{11} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & M^*_{11} \end{bmatrix}$, we have $R = Z^*$, where R is as in Theorem 12.1.8. Since $Z_{11} = G^*_{11} \in \mathcal{G}\mathcal{B}$, we have $\mathbb{Z}_{11} \in \mathcal{G}\mathrm{TIC}_\infty$, by Proposition 6.3.1(c).

II.3° *All controllers and their well-posedness when* $D_{21} = 0$*:* The assumptions of Proposition 12.5.19 are satisfied, by Part I and II.2°, hence we thus obtain the parametrization of all suboptimal stabilizing controllers.

To obtain (12.20)–(12.25), first write out the generators of $\Sigma_{\circlearrowleft}$, (12.102) (12.103) and (12.104) using (13.57) (see II.3° for $R = Z^*$), and then use these to obtain (12.20)–(12.25) (recall that $X^{-1}K = K_X$).

Since $D_{\mathbb{T}21} = 0$, by (12.25), the condition $I - \mathbb{L}\mathbb{T}_{21} \in \mathcal{G}\mathrm{TIC}_\infty(U)$ in (12.105) always holds (by Proposition 6.3.1(c))), hence all controllers parametrized by (12.26) are well posed.

II.4° *Case* $D_{21} \neq 0$*:* By Lemma 12.5.9, (1.') and (4.') are independent of D_{21}. Therefore, the above shows that all suboptimal stabilizing DPF-controllers for Σ' are given by (12.26), where Σ' is equal to Σ with 0 in place of D_{21}.

Thus, the claim on D_{21} at the end of Theorem 12.1.8 follows from Proposition 12.5.19(g) (alternatively, directly from Lemma 7.3.23).

Remarks on case $D_{21} \neq 0$*:* By Lemma 7.3.23, this output feedback connection of $\mathbb{Q}$ and $-D_{21}$ may be non-well-posed (and it corresponds to a controller with d.c. internal loop, as in the lemma), but when we add the connection with $\mathbb{D}$ (as in Figure 7.12), all signals in this final connection become well-posed. If $I + D_{21}\mathbb{Q} \in \mathcal{G}\mathrm{TIC}_\infty(Y)$, then the final controller becomes well-posed, an alternative formula for it is given by $\mathbb{Q}' = (I + \mathbb{Q}D_{21})^{-1}\mathbb{Q}$ $(= \mathbb{Q}(I + D_{21}\mathbb{Q})^{-1})$, as noted below Lemma 7.2.18.

The factorization (Factor1X) is independent of D_{21}, but the solution $\mathbb{Z}$ of (Factor2Z) used in II.1°–II.4° (for the application of Proposition 12.5.19) solves the condition (Factor2Z) corresponding to Σ', not that corresponding to Σ, i.e., we use the parametrization for the wrong (but equivalent) problem and correct it by the additional output feedback through $-D_{21}$ (since we do not know whether the solution of (Factor2Z) corresponding to Σ' has $\mathbb{Z}_{11}$ invertible, as required by Proposition 12.5.19; moreover, in this case we do not know whether $\mathbb{Q}$ is well-posed).

Part III: Conditions (1.) and (4.) compared to (1.') and (4.'):

If any of the equivalent conditions of Lemma 12.5.2 hold, e.g., Σ is exponentially stabilizable, then Parts I–II hold with the primes removed if we require the controller to be exponentially stabilizing (and use Theorem 12.3.5(e) instead of Theorem 12.3.5(b)). We observe from this and Part I that then (1.) and (1.') have same solutions, and so do (4.) and (4.'). Conversely, (1.) obviously implies that Σ (equivalently, (A, B)) is exponentially stabilizable.

Moreover, if Σ is exponentially stabilizable (i.e., $\begin{bmatrix} \mathbb{K}_u \mid \mathbb{F}_u \end{bmatrix}$ is exponentially stabilizing), then the above formulation of all suboptimal stabilizing controllers equals that of all suboptimal exponentially stabilizing controllers modulo the fact that $\Sigma_{\mathbb{L}}$ is required to be exponentially stable, as noted in Proposition 12.5.19(e). (Cf. the comments below Theorem 12.1.8.)

Note also that $\Sigma_{\mathbb{L}}$ must be optimizable and estimatable for the closed-loop system to be exponentially stable, by, e.g., Theorem 6.7.10(d)(viii), Lemma 6.7.11(c) and (twice) Lemma 6.7.18; by Theorem 6.7.10(d)(viii), this is the case iff $\Sigma_{\mathbb{L}}$ is exponentially stable. $\square$

We give here an alternative set of conditions under which the conditions (1.)–(3.) are necessary and sufficient:

Theorem 12.1.11 (MTIC$_{TZ}$: H$^\infty$ 4BP $\Leftrightarrow$ CAREs) *Assume that Hypothesis 12.5.1 is satisfied with $\mathbb{N}_u, \mathbb{M}_u, \widetilde{\mathbb{M}_y}, \widetilde{\mathbb{N}_y} \in \widetilde{\mathcal{A}}$ (e.g., that Σ is exponentially stable, $\mathbb{D} \in \widetilde{\mathcal{A}}$, $\mathbb{D}_{11}^*\mathbb{D}_{11} \gg 0$ and $\mathbb{D}_{22}\mathbb{D}_{22}^* \gg 0$) and that $\widetilde{\mathcal{A}}$ satisfies Hypothesis 8.4.8.*

Then there is a suboptimal exponentially stabilizing DPF-controller for Σ iff conditions (1.)–(3.) of Theorem 12.1.5 hold. If (1.)–(3.) hold, then all suboptimal exponentially stabilizing DPF-controllers for Σ are given by Theorem 12.1.8 (in particular, (1.) and (4.) of Theorem 12.1.8 hold).

Obviously, this is particularly useful for exponentially stable systems.

Proof: (If Σ is exponentially stable and $\mathbb{D} \in \widetilde{\mathcal{A}}$, then we can take $\begin{bmatrix} \mathbb{K}_u & \mathbb{F}_u \end{bmatrix} = 0$, $\begin{bmatrix} \mathbb{H}_y \\ \mathbb{G}_y \end{bmatrix} = 0$ in Hypothesis 12.5.1 to obtain that $\mathbb{N}_u = \mathbb{D} = \widetilde{\mathbb{N}_y} \in \widetilde{\mathcal{A}}$ and $\mathbb{M}_u = I = \widetilde{\mathbb{M}_y} \in \mathcal{B} \subset \widetilde{\mathcal{A}}$.)

By Lemma 12.5.20 and Theorem 12.1.8, conditions (1.)–(3.) are sufficient (and imply (1.)–(4.)); by Lemma 12.1.12, they are also necessary. □

The necessity of (1.)–(3.) can be shown under more general conditions:

Lemma 12.1.12 ($\widetilde{\mathcal{A}}$: H$^\infty$ 4BP $\Rightarrow$ CAREs) *Assume that Hypothesis 12.5.1 is satisfied with $\mathbb{N}_u, \mathbb{M}_u, \widetilde{\mathbb{M}_y}, \widetilde{\mathbb{N}_y} \in \widetilde{\mathcal{A}}$.*

If there is a suboptimal exponentially stabilizing DPF-controller for Σ, then (1.)–(3.) below hold.

If there is a suboptimal stabilizing DPF-controller for Σ, then (1.'), (2.') and (3.) hold.

(1.) **($\mathcal{P}_X$-CARE)** *Condition (1.) of Theorem 12.1.8 holds.*

(2.) **($\mathcal{P}_Y$-CARE)** *The dual of (1.) holds, i.e., the CARE*

$$\begin{cases} K_Y^* S_Y K_Y = A\mathcal{P}_Y + \mathcal{P}_Y A^* + B_2 B_2^*, \\ S_Y = \begin{bmatrix} D_{22}D_{22}^* & D_{22}D_{12}^* \\ D_{12}D_{22}^* & D_{12}D_{12}^* - \gamma^2 I \end{bmatrix} + \lim_{s\to+\infty} \begin{bmatrix} C_{2w} \\ C_{1w} \end{bmatrix} \mathcal{P}_Y (s-A)^{-1} \begin{bmatrix} C_2^* & C_1^* \end{bmatrix}, \\ K_Y = -S_Y^{-1}\left(\begin{bmatrix} D_{22} \\ D_{12} \end{bmatrix} B_2^* + \begin{bmatrix} C_{2w} \\ C_{1w} \end{bmatrix} \mathcal{P}_Y \right), \end{cases} \tag{12.30}$$

has a solution $(\mathcal{P}_Y, S_Y, K_Y) \in \mathcal{B}(H) \times \mathcal{B}(Y \times Z) \times \mathcal{B}(H_1^*, Y \times Z)$ s.t. $\mathcal{P}_Y \geq 0$, $S_{Y11} \gg 0$, $S_{Y22} - S_{Y21}S_{Y11}^{-1}S_{Y12} \ll 0$ and K_Y is exponentially stabilizing for $\begin{pmatrix} A^* & \big| & C_2^* & C_1^* \end{pmatrix}$.

(3.) **(Coupling condition)** $\rho(\mathcal{P}_X\mathcal{P}_Y) < \gamma^2$.

Here "(1.')" is the condition of Proposition 12.1.10 and "(2.')" is its dual condition (i.e., equal to (2.) with corresponding modifications).

However, we do not know whether the converse claims hold in general.

Proof: Assume that there is a suboptimal exponentially stabilizing DPF-controller for Σ (the proof below applies to the latter claim too, mutatis mutandis).

By Theorem 12.1.8, conditions (1.) and (4.) of Theorem 12.1.8 hold. Apply Lemma 12.5.6 to obtain that "(1.)" holds for Σ_{d} too (see p. 740), i.e., that (2.) holds.

It was noted in I.2°–I.3° the proof of Theorem 12.1.8 that (Factor1) is satisfied with $\mathbb{M}_{11} \in \mathcal{G}\mathrm{TIC}_\infty(U)$. We conclude from Lemma 12.5.18 and Lemma 12.6.4(a) that (3.) holds.

Remark — Why the converse is open: For the exponential claim, the problem is that since we have here given up the condition "$S = D^*JD$" of Hypothesis 8.4.8, we can no longer use part 3° of Lemma 12.5.20 to show that S_Z is as required in (4.), so that Theorem 12.1.8 would be applicable.

For the latter claim, the problem is that Lemma 12.6.4(b) does not say anything of the preservation of I/O-stabilization (from $\mathcal{P}_{\mathrm{Y}}$-DARE to $\mathcal{P}_{\mathrm{Z}}$-DARE), hence we would only obtain an internally P-stabilizing solution, which is not enough for derivation of (4.') (even if we would assume Hypothesis 8.4.8 to be able to establish the requirements on S_Z); in discrete-time, we face the same problem (although there the problem on S_Z disappears, by Lemma 12.6.4(c)). □

In fact, in the exponential case it suffices that Σ is somewhat smooth:

Lemma 12.1.13 (H^∞ 4BP ⇒ (1.)–(3.)) *Assume that* $(\Sigma_{\mathrm{X}}, J_\gamma), (\Sigma_{\mathrm{Y}}, J_\gamma) \in$ coerciveCARE *over* $\mathcal{U}_{\mathrm{exp}}$ *and that* $\mathbb{D}_{11}$ *and* $\mathbb{D}_{22}^{\mathrm{d}}$ *are I-coercive over* $\mathcal{U}_{\mathrm{exp}}$.

If there is an exponentially stabilizing DPF-controller for Σ, then (1.)–(3.) of Lemma 12.1.12 hold (with s-lim *in place of* lim*).*

See Remark 12.1.6 for different (equivalent) forms of (1.)–(3.), and Remark 12.1.7 for the above coercivity conditions. See the remark in the proof of Lemma 12.1.12 for why the converse is open.

Proof: (Note that if any of (1.)–(6.) of Remark 9.9.14 holds, then we need not replace lim by s-lim, by Lemma 9.11.5(e).)

We observe from Lemma 12.5.7 that the assumptions of Lemma 11.2.20 are satisfied for Σ_{X} (we need the coercivity assumption on $\mathbb{D}_{11}$ to satisfy Hypothesis 11.2.1). Consequently, (1.) is satisfied. By dual arguments (see Lemma 12.5.6), we obtain (2.).

Let $\mathbb{O}$ be an exponentially stabilizing DPF-controller for Σ. By discretization (see Theorem 13.4.4(e1)), we observe that $\Delta^{\mathrm{S}}\mathbb{O}$ is an exponentially stabilizing DPF-controller for $\Delta^{\mathrm{S}}\Sigma$, hence conditions (1.)–(3.) of Theorem 12.2.1 are satisfied (use Theorem 13.4.4(g) for its coercivity conditions (12.32) and (12.33)), even by same $\mathcal{P}_{\mathrm{X}}$ and $\mathcal{P}_{\mathrm{Y}}$, by Proposition 9.8.7(a) (and uniqueness, see Theorem 14.1.4(a)). Thus, (3.) holds. □

Notes

Our result, Theorem 12.1.4 (and Theorem 12.1.8), is of standard form and extends and generalizes at least most nonsingular state-space solutions to the H^∞ 4BP. In the finite-dimensional case, such earlier results include Theorem 1 of [GD88] Theorem 3 of [DGKF] Theorem 5.1 of [GGLD], Theorem 8.3.2 of [GL], Theorem 16.4 of [ZDG] and Theorem 10.3.1 of [IOW].

(Note that most of them interchange the subindices corresponding to u and w, as compared to our formulae. As explained on p. 317, we have used the "u comes before w" practice, which is more popular in the FICP literature, See also Lemma 12.6.2 and the rest of Section 12.6 for how the notation of [IOW] corresponds to that of ours.)

The early history of the problem is explained on p. 328 of [IOW], where it is said that the assumptions and formulae of [IOW] are more general than any earlier (nonsingular) ones; those assumptions and formulae are essentially the same as ours, except that they assume that $\gamma = 1$ and $D_{21} = 0$. In addition, we also treat non-well-posed controllers.

In some of the above results, one only speaks of $\mathbb{Q}$ stabilizing $\mathbb{D}$ (cf. Section 12.3), but in them it is assumed that $\mathbb{D}$ has an exponentially stabilizable and detectable realization and that such a realization is chosen for $\mathbb{Q}$ too, so that one ends up with our setting (see Theorem 7.3.11(c1)).

These results were extended to smooth Pritchard–Salamon systems in Theorem 5.4 of [Keu], by Bert van Keulen. Although our results allow for approximately twice as much unboundedness as the Pritchard–Salamon class does, the result in [Keu] is not exactly contained in ours: in [Keu], the Riccati equation "(2.)" is given on a space $\mathcal{W}$ embedded in $H(:= \mathcal{V})$. The two Riccati equations in [Keu] correspond to the "bounded B" case of the FICP (for Σ and for its "dual" Σ_{d}). (Note also that any exponentially stable (not necessarily smooth) Pritchard–Salamon system satisfies (A1)(V) of Theorem 12.1.4, by Theorem 6.9.6.)

All results mentioned above also make the coercivity assumptions (A2). The *singular* case, where (A2) is replaced by something weaker, is treated in, e.g., [Stoorvogel] for the finite-dimensional case; in that case the proofs and solutions become more complicated than in the standard setting. See Section 17.3 of [ZDG] or Section 5.4.2 of [Keu] for a discussion on how to circumvent (A2) by using "ε-perturbations" of the system (that satisfy (A2)) and then letting $\varepsilon \to 0+$.

The notes on pp. 446–447 of [ZDG] describe the historical development of solutions to the finite-dimensional H^∞ 4BP through several computationally difficult formulations to the simple "(1.)–(3.)" formulation of [GD88] and this section. Also the first few paragraphs of the notes on p. 628 are relevant to the H^∞ 4BP.

Outside this monograph, we do not know any research on nonexponentially stabilizing suboptimal controllers (cf. Proposition 12.1.10 and Lemma 12.1.12).

In Section 12.3, we shall solve the frequency-domain 4BP (the I/O map 4BP), see Theorems 12.3.6 and 12.3.7. (This is close to the Youla parameterization approach of [Doyle84] and [Francis87].) We obtain that the 4BP is equivalent to two nested spectral factorizations "(Factor1X)" and "(Factor2Z)" (equivalently, to two nested (J_1, J_1)-lossless coprime factorizations).

The CAREs (1.) and (4.) of Theorem 12.1.8 correspond to these two factorizations. The CARE (2.) is the dual of (1.) (and the 4BP is invariant under duality; see also Lemma 12.5.6).

In Lemma 12.5.18, we shall show that the $\mathcal{P}_{\mathrm{Z}}$-CARE (4.) is equivalent to (2.)&(3.), thus completing the proof that (1.)–(3.) are equivalent to the solvability of the 4BP. Since a direct continuous-time proof seems almost impossible (unless, e.g., the plant Σ has bounded generators), we have reduced the proof to the discrete time, where the standard computations can be extended to the infinite-dimensional case (see Lemma 12.6.4).

As in other chapters, here and in Section 12.5, we encounter the fact that the equivalence between the CAREs and the factorizations require certain regularity assumptions, and so does also the equivalence between the factorizations and the J-coercivity properties connected to the solvability of the 4BP (cf. Example 11.3.7); this is why most of our results have some kind of $\widetilde{\mathcal{A}}$ regularity assumption.

In discrete time, we have no such problems (since $\mathrm{tic}_{\mathrm{exp}} = \widetilde{\mathcal{A}}$ is a valid choice, by Theorem 14.3.2); see Theorem 12.2.1.

The parametrization (12.26) of all suboptimal controllers will be obtained in the frequency-domain theory (see Theorem 12.3.7), by reducing the problem to a frequency-domain FICP (see Theorem 12.3.7(c) and Proposition 12.5.19). To get this parametrization satisfactory, we must have a realization of the I/O map $\mathbb{T}$ (e.g., the one given by (12.20)–(12.25)); this will be done in Proposition 12.5.19, simply by following the steps guided by (12.49)–(12.50).

We also note that the solution in Theorem 12.1.8 is not symmetric; by replacing Σ by Σ_{d} (see p. 740) in the proofs, one would obtain (2.) in place of (1.) and a fourth Riccati equation ("(5.)") in place of (4.).

12.2 The discrete-time H^∞ problem (H^∞ 4bp)

If you only have a hammer, you tend to see every problem as a nail.
— Abraham Maslow (1908–1970)

As mentioned above, we assume that the discrete-time form of Standing Hypothesis 12.1.1 holds i.e., we consider the system

$$\begin{cases} x_{n+1} = Ax_n + B_1 u_n + B_2 w_n, \\ z_n = C_1 x_n + D_{11} u_n + D_{12} w_n, \\ y_n = C_2 x_n + D_{21} u_n + D_{22} w_n \end{cases} \qquad (n \in \mathbf{N}) \tag{12.31}$$

with initial state $x_0 \in H$, disturbance input $w \in \ell^2(\mathbf{N};W)$, control input $u \in \ell^2(\mathbf{N};U)$, objective output $z \in \ell^2(\mathbf{N};Z)$ and measurement output $y \in \ell^2(\mathbf{N};Y)$ (the controller input); here $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \in \mathcal{B}(H \times U \times W, H \times Z \times Y)$ are the generators of Σ (see Lemma 13.3.3). (As noted in Lemma 14.3.5, we can have, e.g., $\widetilde{\mathcal{A}} = \ell^1_+ *$ or $\widetilde{\mathcal{A}} = \mathrm{tic}_{\exp}$.)

We now present the discrete-time counterpart of the theory of Section 12.1, i.e., we try to find a controller $\mathbb{Q} : y \mapsto u$ (possibly with internal loop; as in the continuous-time case, internal loop is unnecessary at least when $D_{21} = 0$) s.t. it stabilizes the above system exponentially and makes the norm $\|w \mapsto z\|_{\mathcal{B}(\ell^2)}$ less than the given number $\gamma > 0$. (See the explanation on p. 36 for the H^∞ four-block problem.) We also record the discrete-time forms of all other results in this chapter (Theorem 12.2.2).

We first present the discrete-time counterpart of Theorem 12.1.4, and then we briefly list the other results of Section 12.1 that can be converted to discrete time (the most important of which is the parametrization of all suboptimal controllers, Theorem 12.1.8).

Theorem 12.2.1 (H^∞ 4BP $\Leftrightarrow$ DAREs) *Assume that there is $\varepsilon > 0$ s.t.*

$$(z - A)x_0 = B_1 u_0 \implies \|C_1 x_0 + D_{11} u_0\|_Z \geq \varepsilon(\|x_0\|_H + \|u_0\|_U) \quad \textit{and} \tag{12.32}$$

$$(z - A^*)x_0 = C_2^* y_0 \implies \|B_2^* x_0 + D_{22}^* y_0\|_W \geq \varepsilon(\|x_0\|_H + \|y_0\|_Y) \tag{12.33}$$

for all $x_0 \in H$, $u_0 \in U$, $y_0 \in Y$, $z \in \partial\mathbf{D}$. (Alternatively, we can assume that Hypothesis 12.5.1 is satisfied.)

Then there is a suboptimal exponentially stabilizing DPF-controller for Σ iff (1.)–(3.) hold:

(1.) **($\mathcal{P}_X$-DARE)** *the DARE*

$$\begin{cases} \mathcal{P}_X = A^* \mathcal{P}_X A + C_1^* C_1 - K_X^* S_X K_X, \\ S_X = \begin{bmatrix} D_{11}^* D_{11} & D_{11}^* D_{12} \\ D_{12}^* D_{11} & D_{12}^* D_{12} - \gamma^2 I \end{bmatrix} + B^* \mathcal{P}_X B, \\ K_X = -S_X^{-1}\left(\begin{bmatrix} D_{11}^* \\ D_{12}^* \end{bmatrix} C_1 + B^* \mathcal{P}_X A\right), \end{cases} \tag{12.34}$$

has a solution $(\mathcal{P}_X, S_X, K_X) \in \mathcal{B}(H) \times \mathcal{B}(U \times W) \times \mathcal{B}(H, U \times W)$ s.t. $\mathcal{P}_X \geq 0$, $S_{X11} \gg 0$, $S_{X22} - S_{X21} S_{X11}^{-1} S_{X12} \ll 0$ and $\rho(A + BK_X) < 1$;

(2.) **($\mathcal{P}_Y$-DARE)** *the DARE*

$$\begin{cases} \mathcal{P}_Y = A\mathcal{P}_Y A^* + B_2B_2^* - K_Y^* S_Y K_Y, \\ S_Y = \begin{bmatrix} D_{22}D_{22}^* & D_{22}D_{12}^* \\ D_{12}D_{22}^* & D_{22}D_{22}^* - \gamma^2 I \end{bmatrix} + \begin{bmatrix} C_2 \\ C_1 \end{bmatrix} \mathcal{P}_Y \begin{bmatrix} C_2^* & C_1^* \end{bmatrix}, \\ K_Y = -S_Y^{-1}(\begin{bmatrix} D_{22} \\ D_{12} \end{bmatrix} B_2^* + \begin{bmatrix} C_2 \\ C_1 \end{bmatrix} \mathcal{P}_Y A^*), \end{cases} \tag{12.35}$$

has a solution $(\mathcal{P}_Y, S_Y, K_Y) \in \mathcal{B}(H) \times \mathcal{B}(Y \times Z) \times \mathcal{B}(H, Y \times Z)$ *s.t.* $\mathcal{P}_Y \geq 0$, $S_{Y11} \gg 0$, $S_{Y22} - S_{Y21}S_{Y11}^{-1}S_{Y12} \ll 0$ *and* $\rho(A^* + \begin{bmatrix} C_2^* & C_1^* \end{bmatrix} K_Y) < 1$*;*

(3.) **(Coupling condition)** $\rho(\mathcal{P}_X\mathcal{P}_Y) < \gamma^2$.

If (1.)–(3.) hold, then $\Sigma_{\mathbb{T}_{12}}$ *is a suboptimal DPF-controller for* Σ *(the* central controller*), and all suboptimal DPF-controllers are parametrized in Theorem 12.1.8 (see Figure 12.1).*

Note that (12.34) is the DARE for Σ_X and J_γ (exactly as in Theorem 11.5.1), and (12.35) is the DARE for Σ_Y and J_γ. See also the remarks in Section 12.1.

Proof: (We shall again refer to continuous-time results, thereby meaning their discrete-time forms; cf. Theorem 13.3.13 and Theorem 12.2.2.)

Set $\widetilde{\mathcal{A}} :=$ tic$_{\text{exp}}$ (recall from Lemma 14.3.5 that tic$_{\text{exp}}$ satisfies Standing Hypothesis 12.0.1).

1° *Necessity of (1.)–(3.):* Necessity follows from, e.g., Lemma 12.1.13 (recall that "∈ coerciveCARE" is redundant in discrete time, as noted below Remark 9.9.14).

2° *Sufficiency under Hypothesis 12.5.1:* Even though we would assume no more than Standing Hypothesis 12.1.1, we would obtain from Lemma 12.6.4(b)&(c), that conditions (1.) and (4.) hold iff (1.)–(3.) hold.

Consequently, under (1.)–(3.) and Hypothesis 12.5.1, we obtain the existence of an exponentially stabilizing suboptimal DPF-controller from Theorem 12.1.8.

3° *Sufficiency under assumptions (12.32)–(12.33):* Assume (1.)–(3.) and (12.32)–(12.33). Then Hypothesis 12.5.1 is satisfied (even with "exponentially jointly" in place of "jointly") by Lemmas 12.6.7 and 12.6.6. Consequently, we can apply 2°.

4° *Remarks:* As noted in 2°, (1.) and (4.) hold iff (1.)–(3.) hold. However, the equivalence to the existence of an exponentially stabilizing DPF-controller requires further conditions (e.g., if $B = 0 = D$, then necessarily $S_X = 0$, so that (1.) cannot hold), such as the ones used in 2° or 3°.

Note from Proposition 15.2.2(c) that (12.32) and (12.33) say that $(\left(\begin{array}{c|c} A & B_1 \\ \hline C_1 & D_{11} \end{array}\right), I)$ and $(\left(\begin{array}{c|c} A^* & C_2^* \\ \hline B_2^* & D_{22}^* \end{array}\right), I)$ are I-coercive over $\mathcal{U}_{\text{exp}}$. as one can verify from the proof below, even weaker assumptions would suffice. □

Practically all our H^∞ 4BP results hold also in their discrete-time forms:

Theorem 12.2.2 (Discrete form of $\mathbf{H}^\infty$ 4BP results) *The following results hold also in their discrete-time forms (i.e., after the changes listed in Theorem 13.3.13):*

Lemmas 12.1.3 and 12.1.12, Theorems 12.1.11 and 12.1.8, and everything in Sections 12.3, 12.4 and 12.5.

Moreover, in the exponential case of Lemma 12.1.12 also the converse holds (this is Theorem 12.2.1 with the alternative assumption).

The $\mathcal{P}_Z$-DARE means the DARE for $\Sigma_Z =$(12.123) and J_1, hence the condition (4.) can now be written as

(4.) **($\mathcal{P}_Z$-DARE)** *The DARE (see (12.94))*

$$\begin{cases} K_Z^* S_Z K_Z = A_Z^* \mathcal{P}_Z A_Z - \mathcal{P}_Z + B_2 X_{22}^{-1} X_{22}^{-*} B_2^*, \\ \qquad S_Z = D_Z^* J D_Z + B_Z^* \mathcal{P}_Z B_Z, \\ \qquad K_Z = -S_Z^{-1} \left(\begin{bmatrix} D_{22} X_{22}^{-1} \\ X_{12} X_{22}^{-1} \end{bmatrix} X_{22}^{-*} B_2^* + B_Z^* \mathcal{P}_Z A_Z \right), \end{cases} \tag{12.36}$$

has a solution $(\mathcal{P}_Z, S_Z, K_Z) \in \mathcal{B}(H) \times \mathcal{B}(Y \times Z) \times \mathcal{B}(H, Y \times U)$ s.t. $\mathcal{P}_Z \geq 0$, $S_{Z11} \gg 0$, $S_{Z22} - S_{Z21} S_{Z11}^{-1} S_{Z12} \ll 0$ and $\rho(A_Z + B_Z K_Z) < 1$.

As noted around Example 14.2.9, we almost never have "$S = D^* J D$" in discrete time (thus, in practice we only meet the DARE equivalent of the "weakest" of the CAREs in Remark 12.1.6, and Theorem 12.1.11 becomes rather unnecessary).

See the remarks below Lemma 12.6.4 for the three DAREs; in particular, X must be chosen as in Lemma 12.6.1 (equivalently, as in Theorem 12.1.8).

Proof of Theorem 12.2.2: *Remark:* Recall that these changes include CARE$\mapsto$DARE, i.e., $A_*^* \mathcal{P}_* + \mathcal{P}_* A_* \mapsto A_*^* \mathcal{P}_* A_* - \mathcal{P}_*$ etc., as above.

The proof: This follows roughly by applying (13.63) also to the proofs (recall from Lemma 14.3.5 that $\widetilde{\mathcal{A}} := \mathrm{tic}_{\mathrm{exp}}$ satisfies Standing Hypothesis 12.0.1).

Alternatively, one could use Lemma 12.6.7, Lemma 12.6.6, Lemma 13.1.7, Lemma 6.6.11 and Lemma 13.3.12 to make the proof slightly shorter (and more "discrete-time self-contained"). □

Notes

For finite-dimensional systems, the discrete-time H^∞ 4BP is more complicated than the continuous-time H^∞ 4BP — the same holds for infinite-dimensional systems if we require the input and output operators to be bounded — but in general the continuous-time setting becomes very complicated. This is why part of the proof of our continuous-time results (in particular, the equivalence of (1.)–(3.) and (1.)&(4.)) has been reduced to discrete time (in the last two sections of this chapter).

Theorem 12.2.1 extends the classical nonsingular results to the infinite-dimensional case. Theorem 10.12.1 of [IOW] is possibly the most general of all the nonsingular finite-dimensional results; it is essentially Theorem 12.2.1 (and Theorem 12.1.8) with the assumptions that $\gamma = 1$ and $D_{21} = 0$. Section B.4.2 of [GL] contains a result close to (the discrete-time form of) Theorem 12.1.8. See also the notes on p. 706. The history of the solutions for the discrete-time H^∞ 4BP is explained on p. 501 of [GL].

The nonsingular finite-dimensional case (where (12.32)–(12.33) have been replaced by weaker assumptions) has been treated in [Stoorvogel].

12.3 The frequency-space (I/O) H^∞ 4BP

From a certain point onward there is no longer any turning back. That is the point that must be reached.

— Franz Kafka (1883–1924)

In this section, we solve the *frequency-domain (or I/O)* H^∞ *Four-Block Problem (I/O* H^∞ *4BP)*. This means that, given a plant $\mathbb{D} : \left[\begin{smallmatrix} u \\ w \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} z \\ y \end{smallmatrix}\right]$ and $\gamma > 0$, we determine whether there is a DPF-controller $\mathbb{Q} : y \mapsto u$ for $\mathbb{D}$ that makes the norm $\|w \mapsto z\| = \|\mathcal{F}_\ell(\mathbb{D}, \mathbb{Q})\|$ less than γ.

In Theorem 12.3.6, we extend to $\mathrm{MTIC}_{T\mathbf{Z}}$ (and beyond) the fact that this problem has a solution iff certain two nested lossless coprime factorizations exist (for rational maps this was established in [Green]), and we parameterize all solutions in terms of these factorizations. The exact conditions on the factorizations depend on whether we require $\mathbb{Q}$ to be well-posed (i.e., without internal loop) or not. In Theorem 12.3.7(a)&(d), the sufficiency part of the above equivalence is extended to general WPLSs (we also extend the necessity part under the assumption that certain maps admit spectral factorization). Theorem 12.3.5 connects these frequency domain solutions to the state-space problem.

First we list the standard I/O H^∞ 4BP assumptions and define the problem.

Standing Hypothesis 12.3.1 (I/O 4BP assumptions) *Throughout this section and Section 12.4, we assume that* $\mathbb{D} = \left[\begin{smallmatrix} \mathbb{D}_{11} & \mathbb{D}_{12} \\ \mathbb{D}_{21} & \mathbb{D}_{22} \end{smallmatrix}\right] \in \mathrm{TIC}_\infty(U \times W, Z \times Y)$, *that* $\gamma > 0$, *and that* $\mathbb{D}$ *has a d.c.f.* $\mathbb{D} = \mathbb{N}_\mathrm{u} \mathbb{M}_\mathrm{u}^{-1} = \widetilde{\mathbb{M}_\mathrm{y}}^{-1} \widetilde{\mathbb{N}_\mathrm{y}}$ *of the form*

$$\mathbb{D} = \begin{bmatrix} \mathbb{N}_{\mathrm{u}\,11} & \mathbb{N}_{\mathrm{u}\,12} \\ \mathbb{N}_{\mathrm{u}\,21} & \mathbb{N}_{\mathrm{u}\,22} \end{bmatrix} \begin{bmatrix} \mathbb{M}_{\mathrm{u}\,11} & \mathbb{M}_{\mathrm{u}\,12} \\ 0 & I \end{bmatrix}^{-1} = \begin{bmatrix} I & \widetilde{\mathbb{M}_\mathrm{y}}_{12} \\ 0 & \widetilde{\mathbb{M}_\mathrm{y}}_{22} \end{bmatrix}^{-1} \begin{bmatrix} \widetilde{\mathbb{N}_\mathrm{y}}_{11} & \widetilde{\mathbb{N}_\mathrm{y}}_{12} \\ \widetilde{\mathbb{N}_\mathrm{y}}_{21} & \widetilde{\mathbb{N}_\mathrm{y}}_{22} \end{bmatrix} \tag{12.37}$$

s.t. $\mathbb{N}_{\mathrm{u}\,21}$ *and* $\mathbb{M}_{\mathrm{u}\,11}$ *are r.c. and* $\widetilde{\mathbb{N}_\mathrm{y}}_{21}$ *and* $\widetilde{\mathbb{M}_\mathrm{y}}_{22}$ *are l.c. We also make the nonsingularity assumptions*

$$\mathbb{N}_{\mathrm{u}\,11}^* \mathbb{N}_{\mathrm{u}\,11} \gg 0, \quad \widetilde{\mathbb{N}_\mathrm{y}}_{22} \widetilde{\mathbb{N}_\mathrm{y}}_{22}^* \gg 0. \tag{12.38}$$

By Proposition 7.3.14 and Lemma 7.3.16, any d.c.f. of $\mathbb{D}$ of form (12.37) satisfies also the rest of the above hypothesis (when the hypothesis holds).

Obviously, the hypothesis is a generalization of the assumptions of Theorem 4.4 of [Green] and of those of Theorem 5.6 of [CG97]. Therefore, Theorem 12.3.6 generalizes those (frequency-domain) results. The d.c.f. assumption roughly says that $\mathbb{D}$ can be stabilized through y and u, and (12.38) is the standard nonsingularity assumption.

The results of this section will also be used in the proof of the results of Section 12.1; indeed, under assumptions (A1) and (A2) of Theorem 12.1.5 (or (A1) and (A2) of Theorem 12.1.4), the existence of an exponentially stabilizing controller for Σ (alternatively, conditions (1.)–(3.)) implies the above hypothesis, as noted in the proof of the theorem. Thus, the results of this section also apply to any classical (nonsingular) state-space H^∞ 4BPs.

The DPF-stabilizing controllers for $\mathbb{D}$ are Youla parametrized in Corollary 7.3.20, in which the map $\mathcal{F}_\ell(\mathbb{D},\mathbb{Q}) : w \mapsto z$ was defined; we repeat that definition here:

Lemma 12.3.2 ($\mathcal{F}_\ell(\mathbb{D},\mathbb{Q})$) *Let $\mathbb{Q}$ DPF-stabilize $\mathbb{D}$ (with internal loop). Then $\mathbb{Q}$ is a map with d.c. internal loop and $\mathbb{Q} = \mathbb{Q}_1\mathbb{Q}_2^{-1} = \widetilde{\mathbb{Q}}_2^{-1}\widetilde{\mathbb{Q}}_1$, where $\mathbb{Q}_1, \mathbb{Q}_2, \widetilde{\mathbb{Q}}_1, \widetilde{\mathbb{Q}}_2 \in$* TIC, *$\widetilde{\mathbb{M}_{\mathrm{y}}}_{22}\mathbb{Q}_2 - \widetilde{\mathbb{N}_{\mathrm{y}}}_{21}\mathbb{Q}_1 = I$ and $\widetilde{\mathbb{Q}}_2\mathbb{M}_{\mathrm{u}\,11} - \widetilde{\mathbb{Q}}_1\mathbb{N}_{\mathrm{u}\,21} = I$.*

The corresponding closed-loop $w \mapsto z$ map is given by

$$\mathcal{F}_\ell(\mathbb{D},\mathbb{Q}) := \mathbb{N}_{\mathrm{u}\,11}\mathbb{U} + \mathbb{N}_{\mathrm{u}\,12} = \widetilde{\mathbb{N}_{\mathrm{y}}}_{12} + \overline{\mathbb{U}}\widetilde{\mathbb{N}_{\mathrm{y}}}_{22}, \tag{12.39}$$

where $\mathbb{U} := \widetilde{\mathbb{Q}}_1\mathbb{N}_{\mathrm{u}\,22} - \widetilde{\mathbb{Q}}_2\mathbb{M}_{\mathrm{u}\,12}$, $\overline{\mathbb{U}} := \widetilde{\mathbb{N}}_{11}\mathbb{Q}_1 - \widetilde{\mathbb{M}}_{12}\mathbb{Q}_2$. □

For well-posed $\mathbb{Q}$, we have $\mathcal{F}_\ell(\mathbb{D},\mathbb{Q}) := \mathbb{D}_{12} + \mathbb{D}_{11}\mathbb{Q}(I - \mathbb{D}_{21}\mathbb{Q})\mathbb{D}_{22}$, by (7.65). (Note that, in the literature, the latter subindices are often interchanged, i.e., w comes before u.) As noted below Corollary 7.3.20, the map $\mathcal{F}_\ell(\mathbb{D},\mathbb{Q})$ depends on $\mathbb{D}$ and $\mathbb{Q}$ only.

We remind the reader that, in this chapter, we often drop "DPF-" and we allow the (DPF-)controllers to be non-well-posed (i.e., to have an internal loop).

We repeat here the definition of a solution of the I/O H^∞ 4BP:

Definition 12.3.3 (I/O H^∞ 4BP) *A map $\mathbb{Q}$ (with internal loop) is a* suboptimal stabilizing DPF-controller (for $\mathbb{D}$) *if $\mathbb{Q}$ DPF-stabilizes $\mathbb{D}$ and $\|\mathcal{F}_\ell(\mathbb{D},\mathbb{Q})\| < \gamma$.*

A solution of the I/O (H^∞) 4BP (for $\mathbb{D}$) *means a suboptimal stabilizing DPF-controller for $\mathbb{D}$.*

Recall from Section 7.2, that any well-posed (i.e., $\mathrm{TIC}_\infty(Y,U)$) map is a map with internal loop, hence the above definition covers all well-posed controllers too. In the results we shall also specify when there exists a well-posed solution.

The purpose of this suboptimal problem is that its solution can be used for a binary search over γ's to find an estimate of the optimal γ and an "almost optimal" $\mathbb{Q}$.

The solution of the 4BP is interplay between the original problem and its dual, hence we record the following (recall that $\mathbb{E}^{\mathrm{d}} := \boldsymbol{Я}\mathbb{E}^*\boldsymbol{Я}$):

Lemma 12.3.4 (Dual problem) *The map $\mathbb{D}_{\mathrm{d}} := \begin{bmatrix} \mathbb{D}_{22}^{\mathrm{d}} & \mathbb{D}_{12}^{\mathrm{d}} \\ \mathbb{D}_{21}^{\mathrm{d}} & \mathbb{D}_{11}^{\mathrm{d}} \end{bmatrix}$ also satisfies Standing Hypothesis 12.3.1. Moreover, $\mathbb{Q}$ is suboptimal for $\mathbb{D}$ iff $\mathbb{Q}^{\mathrm{d}}$ is suboptimal for $\mathbb{D}_{\mathrm{d}}$.*
□

(This follows from Proposition 7.3.4(d).

Next we note how a solution of the frequency-domain problem of this section leads to a solution of the corresponding state-space problem.

If Σ and $\widetilde{\Sigma}$ are realizations of $\mathbb{D}$ and $\mathbb{Q}$, respectively (if $\mathbb{Q}$ is considered as a map with internal loop, then $\widetilde{\Sigma}$ may be any realization of a representative of $\mathbb{Q}$), then, obviously, $\widetilde{\Sigma}$ I/O-DPF-stabilizes Σ iff $\mathbb{Q}$ DPF-stabilizes $\mathbb{D}$. If Σ is SOS-stabilizable, then $\widetilde{\Sigma}$ can be chosen to be SOS-DPF-stabilizing, i.e., such that the resulting closed-loop system is SOS-stable; similar claims hold also for stronger stabilizability properties of Σ:

Theorem 12.3.5 (I/O 4BP ⇒ 4BP) *Let $\mathbb{Q}$ be a stabilizing DPF-controller for $\mathbb{D}$ (with internal loop). Let Σ be a realization of $\mathbb{D}$. Given a realization $\widetilde{\Sigma}$ or $\mathbb{Q}$, the resulting closed-loop connection system will be denoted by Σ_I^o (see (7.60) and (6.125) for Σ_I^o; cf. Figure 7.11).*

Then Theorem 7.2.3 applies, in particular, the following holds:

- *(a) If Σ is SOS-stabilizable, then $\widetilde{\Sigma}$ can be chosen s.t. $\Sigma_I^o \in \mathrm{SOS}$.*
- *(b) If Σ is [strongly] r.c.-stabilizable, then $\widetilde{\Sigma}$ can be chosen s.t. Σ_I^o is [strongly] stable.*
- *(c) If Σ is stabilizable and [strongly] detectable, then $\widetilde{\Sigma}$ can be chosen s.t. Σ_I^o is [strongly] stable.*
- *(d) $\mathbb{D}$ and $\mathbb{Q}$ have such realizations that their closed-loop connection system Σ_I^o becomes strongly stable.*
- *(e) If Σ_{21} is exponentially jointly stabilizable and detectable, then $\mathbb{Q}$ DPF-stabilizes $\mathbb{D}$ exponentially with internal loop iff it has a realization that stabilizes Σ exponentially with an internal loop.*

Recall from Theorem 6.6.28 that $\mathbb{D}$ has a strongly jointly r.c.-stabilizable and l.c.-detectable realization, because it has a d.c.f., by Standing Hypothesis 12.3.1. Moreover, one can choose the realization so that it satisfies also Hypothesis 12.5.1, by Lemma 12.5.23.

Proof: (a)–(d) Because $\mathbb{Q}$ has a d.c.f., by Corollary 7.3.20, $\mathbb{Q}$ has a strongly jointly r.c.-stabilizable and l.c.-detectable realization $\widetilde{\Sigma}$, by Theorem 6.6.28 (if $\mathbb{Q}$ is a well-posed controller; in the general (non-well-posed) case we may take $\widetilde{\Sigma}$ to be a strongly stable realization (as in Definition 6.1.6) of a stable representative of $\mathbb{Q}$ (cf. Definition 7.2.11)).

By Theorem 7.2.3, $\widetilde{\Sigma}$ stabilizes Σ as in (a)–(d) (for (d) we use the fact that, by Theorem 6.6.28, Σ can be chosen to be as in (b)).

(e) This is contained in Lemma 7.3.6(b1) (even without any standing assumptions). □

Michael Green showed in [Green] (Theorem 4.4) that the frequency-domain H^∞ 4BP has a solution iff certain two nested spectral factorizations exist (in the rational finite-dimensional case). This result was extended to $\mathrm{MTIC}^{\mathrm{L}^1}_{\mathrm{exp}}$ with $\dim U \times W \times Y \times Z < \infty$ by Green and Ruth Curtain [CG97] (Theorem 5.6). The following (see (c)) is a direct generalization of these results:

Theorem 12.3.6 ($\widetilde{\mathcal{A}}$: H$^\infty$ 4BP (I/O)) *Assume that $\mathbb{N}_{\mathrm{u}}, \mathbb{M}_{\mathrm{u}} \in \widetilde{\mathcal{A}}$. Then conditions (4BP1$\widetilde{\mathcal{A}}$)–(4BP3$\widetilde{\mathcal{A}}$) are equivalent:*

- *(4BP1$\widetilde{\mathcal{A}}$) The I/O 4BP has a well-posed solution "in $\widetilde{\mathcal{A}}$"; i.e., there are $\widetilde{\mathbb{Q}}_2 \in \widetilde{\mathcal{A}}(U) \cap \mathcal{G}\mathrm{TIC}_\infty$ and $\widetilde{\mathbb{Q}}_1 \in \widetilde{\mathcal{A}}(Y,U)$ s.t. $\widetilde{\mathbb{Q}}_2^{-1}\widetilde{\mathbb{Q}}_1$ (DPF-)stabilizes $\mathbb{D}$ and makes $\mathcal{F}_\ell(\mathbb{D},\mathbb{Q}) < \gamma$.*
- *(4BP2$\widetilde{\mathcal{A}}$) (Factor1$\widetilde{\mathcal{A}}$) holds, and there is a solution $\widetilde{\mathbb{Q}}_1, \widetilde{\mathbb{Q}}_2 \in \widetilde{\mathcal{A}}$ to the* ASP

$$\begin{bmatrix} \widetilde{\mathbb{U}} & I \end{bmatrix} = \begin{bmatrix} \widetilde{\mathbb{Q}}_1 & \widetilde{\mathbb{Q}}_2 \end{bmatrix} \begin{bmatrix} \mathbb{N}_{22} & -\mathbb{N}_{21} \\ -\mathbb{M}_{12} & \mathbb{M}_{11} \end{bmatrix} \text{ for some } \widetilde{\mathbb{U}} \in \mathrm{TIC} \text{ with } \|\widetilde{\mathbb{U}}\| < 1 \tag{12.40}$$

s.t. $\widetilde{\mathbb{Q}}_2 \in \mathcal{G}\mathrm{TIC}_\infty(U)$.

(4BP3$\widetilde{\mathcal{A}}$) (Factor1$\widetilde{\mathcal{A}}$) and (Factor2$\widetilde{\mathcal{A}}$) hold.

Here we refer to the following:

(Factor1$\widetilde{\mathcal{A}}$) There is a r.c.f. $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$ *s.t.* $\widehat{\mathbb{M}}_{11}(+\infty), \widehat{\mathbb{M}}_{22}(+\infty) \in \mathcal{G}\mathcal{B}$.

$$\begin{bmatrix} \mathbb{N}_{11} & \mathbb{N}_{12} \\ \mathbb{M}_{21} & \mathbb{M}_{22} \end{bmatrix} \in \mathrm{TIC}(U \times W, Z \times W) \ \ \textit{is } (J_\gamma, J_1)\textit{-lossless.} \tag{12.41}$$

(Factor2$\widetilde{\mathcal{A}}$) The map $\mathbb{D}_+ := \left[\begin{smallmatrix} \mathbb{N}_{21} & \mathbb{N}_{22} \\ I & 0 \end{smallmatrix}\right] \left[\begin{smallmatrix} \mathbb{M}_{11} & \mathbb{M}_{12} \\ 0 & I \end{smallmatrix}\right]^{-1} \in \mathrm{TIC}_\infty(U \times W, Y \times U)$ *has a l.c.f.* $\mathbb{D}_+ = \widetilde{\mathbb{M}}_+^{-1}\widetilde{\mathbb{N}}_+$ *s.t.* $\mathbb{W}^{\mathrm{d}}$ *is* (J_1, J_1)*-lossless and* $\widehat{\widetilde{\mathbb{N}}}_{+21}(+\infty), \widehat{\widetilde{\mathbb{M}}}_{+22}(+\infty) \in \mathcal{G}\mathcal{B}(U)$*, where*

$$\mathbb{W} := \begin{bmatrix} \widetilde{\mathbb{N}}_{+12} & \widetilde{\mathbb{M}}_{+12} \\ \widetilde{\mathbb{N}}_{+22} & \widetilde{\mathbb{M}}_{+22} \end{bmatrix} \in \mathrm{TIC}(W \times U, Y \times U). \tag{12.42}$$

(a) Any r.c.f. of $\mathbb{D}$ *or l.c.f. of* $\mathbb{D}_+$ *is in* $\widetilde{\mathcal{A}}$*. If (Factor1$\widetilde{\mathcal{A}}$) and (Factor2$\widetilde{\mathcal{A}}$) hold, then* $\mathbb{N}, \mathbb{M}, \widetilde{\mathbb{N}}_+, \widetilde{\mathbb{M}}_+ \in \widetilde{\mathcal{A}}$*, and all well-posed suboptimal DPF-stabilizing controllers are given by*

$$\mathbb{Q} = \widetilde{\mathbb{Q}}_2^{-1}\widetilde{\mathbb{Q}}_1, \ \begin{bmatrix} \widetilde{\mathbb{Q}}_1 & \widetilde{\mathbb{Q}}_2 \end{bmatrix} := \begin{bmatrix} \mathbb{L} & I \end{bmatrix} \begin{bmatrix} \widetilde{\mathbb{M}}_{+11} & \widetilde{\mathbb{N}}_{+11} \\ \widetilde{\mathbb{M}}_{+21} & \widetilde{\mathbb{N}}_{+21} \end{bmatrix}, \ \mathbb{L} \in \mathrm{TIC}(Y,U), \ \|\mathbb{L}\| < 1 \tag{12.43}$$

with the additional condition that $\widetilde{\mathbb{Q}}_2 \in \mathcal{G}\mathrm{TIC}_\infty(U)$ *(e.g., take* $\mathbb{L} \in \widetilde{\mathcal{A}}$*,* $\widehat{\mathbb{L}}(+\infty) = 0$*).*

We have $\widetilde{\mathbb{Q}}_1, \widetilde{\mathbb{Q}}_2 \in \widetilde{\mathcal{A}}$ *iff* $\mathbb{L} \in \widetilde{\mathcal{A}}$*.*

(b) If $\widetilde{\mathbb{M}_{\mathrm{y}}}, \widetilde{\mathbb{N}_{\mathrm{y}}} \in \widetilde{\mathcal{A}}$*, then (4BP1$\widetilde{\mathcal{A}}$) holds iff there is a well-posed solution* $\mathbb{Q} = \mathbb{Q}_1\mathbb{Q}_2^{-1}$ *s.t.* $\mathbb{Q}_1, \mathbb{Q}_2 \in \widetilde{\mathcal{A}}$*.*

(c) Conditions "$\widehat{\mathbb{M}}_{22}(+\infty), \widehat{\widetilde{\mathbb{M}}}_{+22}(+\infty) \in \mathcal{G}\mathcal{B}(U)$*" are redundant if* $\dim U < \infty$.

(d) If $D_{21} = 0$*, then (4BP1$\widetilde{\mathcal{A}}$) is equivalent to (4BP1), i.e., if there is any suboptimal stabilizing DPF-controller for* $\mathbb{D}$*, then there is a well-posed suboptimal stabilizing DPF-controller for* $\mathbb{D}$ *"in* $\widetilde{\mathcal{A}}$*".* □

(This will be proved in Lemma 12.4.16.)

By $\widetilde{\mathcal{A}}(U)^{-1}\widetilde{\mathcal{A}}(Y,U)$ we mean maps of form $\widetilde{\mathbb{Q}}_2^{-1}\widetilde{\mathbb{Q}}_1$ s.t. $\widetilde{\mathbb{Q}}_1 \in \widetilde{\mathcal{A}}(Y,U)$, $\widetilde{\mathbb{Q}}_2 \in \widetilde{\mathcal{A}}(U)$ and $\widetilde{\mathbb{Q}}_2 \in \mathcal{G}\mathrm{TIC}_\infty$.

See Definition 6.4.4 for lossless factorizations. Note that we could replace "(J_1,J_1)-lossless" by "frequency-domain (J_1,J_1)-lossless" in (Factor1$\widetilde{\mathcal{A}}$) and (Factor2$\widetilde{\mathcal{A}}$), by Corollary 2.5.5.

The factorization $\mathbb{D}_+ := \left[\begin{smallmatrix} \mathbb{N}_{21} & \mathbb{N}_{22} \\ I & 0 \end{smallmatrix}\right] \left[\begin{smallmatrix} \mathbb{M}_{11} & \mathbb{M}_{12} \\ 0 & I \end{smallmatrix}\right]^{-1}$ is a r.c.f. when (Factor1$\widetilde{\mathcal{A}}$) holds, by Remark 12.4.5.

The above *ASP* (analytic system problem) formulations are due to [Green].

Above we assumed that $\mathbb{N}_{\mathrm{u}}, \mathbb{M}_{\mathrm{u}} \in \widetilde{\mathcal{A}}$. In the general case, we cannot guarantee the sufficient factorization conditions to be necessary, and the conditions become slightly more complicated unless we assume some regularity. This is stated below; we also give weaker sufficient conditions for the equivalence.

Theorem 12.3.7 (I/O $\mathbf{H}^\infty$ 4BP) *We consider the following conditions:*

(4BP1) The I/O 4BP has a solution, i.e., some $\mathbb{Q}$ stabilizes $\mathbb{D}$ and makes $\mathcal{F}_\ell(\mathbb{D},\mathbb{Q}) < \gamma$.

(4BP2) (Factor1) holds, and there is a solution $\widetilde{\mathbb{Q}}_1, \widetilde{\mathbb{Q}}_2 \in \mathrm{TIC}$ to the ASP

$$\begin{bmatrix} \widetilde{\mathbb{U}} & I \end{bmatrix} = \begin{bmatrix} \widetilde{\mathbb{Q}}_1 & \widetilde{\mathbb{Q}}_2 \end{bmatrix} \begin{bmatrix} \mathbb{N}_{22} & -\mathbb{N}_{21} \\ -\mathbb{M}_{12} & \mathbb{M}_{11} \end{bmatrix} \text{ for some } \widetilde{\mathbb{U}} \in \mathrm{TIC} \text{ with } \|\widetilde{\mathbb{U}}\| < 1. \tag{12.44}$$

(4BP3) (Factor1) and (Factor2) hold.

(Factor1) There is a r.c.f. $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$ s.t. $\mathbb{M}_{22} \in \mathcal{G}\mathrm{TIC}_\infty(W)$ and (12.41) holds.

(Factor1X) There is $\mathbb{X} \in \mathcal{G}\mathrm{TIC}(U \times W)$, s.t. $\mathbb{X}^* J_1 \mathbb{X} = \left[\begin{smallmatrix} \mathbb{N}_{u\,11} & \mathbb{N}_{u\,12} \\ 0 & I \end{smallmatrix}\right]^* J_\gamma \left[\begin{smallmatrix} \mathbb{N}_{u\,11} & \mathbb{N}_{u\,12} \\ 0 & I \end{smallmatrix}\right]$ and $\mathbb{X}_{11} \in \mathcal{G}\mathrm{TIC}(U)$.

(Factor2) The map $\mathbb{D}_+ := \left[\begin{smallmatrix} \mathbb{N}_{21} & \mathbb{N}_{22} \\ I & 0 \end{smallmatrix}\right] \left[\begin{smallmatrix} \mathbb{M}_{11} & \mathbb{M}_{12} \\ 0 & I \end{smallmatrix}\right]^{-1}$ with d.c. internal loop can be written as $\mathbb{D}_+ = \widetilde{\mathbb{M}}_+^{-1} \widetilde{\mathbb{N}}_+$ (cf. Remark 12.4.5) so that $\mathbb{W}^{\mathrm{d}}$ is (J_1, J_1)-lossless and $\mathbb{W}_{22} \in \mathcal{G}\mathrm{TIC}_\infty(U)$, where

$$\mathbb{W} := \begin{bmatrix} \widetilde{\mathbb{N}}_{+12} & \widetilde{\mathbb{M}}_{+12} \\ \widetilde{\mathbb{N}}_{+22} & \widetilde{\mathbb{M}}_{+22} \end{bmatrix} \in \mathrm{TIC}(W \times U, Y \times U). \tag{12.45}$$

(Factor2Z) There is $\mathbb{Z} \in \mathcal{G}\mathrm{TIC}(Y \times U)$ s.t. $\mathbb{E} J_1 \mathbb{E}^* = \mathbb{Z} J_1 \mathbb{Z}^*$, and $(\mathbb{Z}^{-1}\mathbb{E})_{22} \in \mathcal{G}\mathrm{TIC}$, where

$$\mathbb{E} := \begin{bmatrix} \mathbb{N}_{22} & -\mathbb{N}_{21} \\ -\mathbb{M}_{12} & \mathbb{M}_{11} \end{bmatrix} \in \mathrm{TIC}(W \times U, Y \times U). \tag{12.46}$$

(Note that we define (Factor2) only if (Factor1) holds.) The following holds:

(a) We have (4BP3)⇒(4BP2)⇒(4BP1). If $\left(\left[\begin{smallmatrix} \mathbb{N}_{u\,11} & \mathbb{N}_{u\,12} \\ 0 & I \end{smallmatrix}\right], J_\gamma\right) \in \mathrm{SpF}$, then (4BP2)⇔(4BP1). If (Factor1) holds and $(\mathbb{E}^{\mathrm{d}}, J_1) \in \mathrm{SpF}$, then (4BP3)⇔(4BP2)⇔(4BP1).

(b) Assume (Factor1). Then (4BP1)⇔(4BP2).

If $\widetilde{\mathbb{Q}}_1, \widetilde{\mathbb{Q}}_2$ satisfy (4BP2), then a solution of the I/O 4BP is given by $\mathbb{Q} = \widetilde{\mathbb{Q}}_2^{-1} \widetilde{\mathbb{Q}}_1$ (which is obviously a l.c.f.). Conversely, any solution $\mathbb{Q}$ of (4BP1) is of form $\mathbb{Q} = \widetilde{\mathbb{Q}}_2^{-1} \widetilde{\mathbb{Q}}_1$, where $(\widetilde{\mathbb{Q}}_1, \widetilde{\mathbb{Q}}_2)$ solves (4BP2).

(c) If (Factor1) and (Factor2) hold, then all suboptimal DPF-controllers (all solutions to the I/O 4BP) are given by the l.c.f.

$$\mathbb{Q} = \widetilde{\mathbb{Q}}_2^{-1} \widetilde{\mathbb{Q}}_1, \ \begin{bmatrix} \widetilde{\mathbb{Q}}_1 & \widetilde{\mathbb{Q}}_2 \end{bmatrix} := \begin{bmatrix} \mathbb{L} & I \end{bmatrix} \begin{bmatrix} \widetilde{\mathbb{M}}_{+11} & \widetilde{\mathbb{N}}_{+11} \\ \widetilde{\mathbb{M}}_{+21} & \widetilde{\mathbb{N}}_{+21} \end{bmatrix}, \ \mathbb{L} \in \mathrm{TIC}(Y, U), \ \|\mathbb{L}\| < 1 \tag{12.47}$$

(by $\|\mathbb{L}\| \le 1$ we get all $\mathbb{Q}$'s s.t. $\|\mathcal{F}_\ell(\mathbb{D},\mathbb{Q})\| \le \gamma$); the well-posed solutions are parametrized by (12.47) with the additional condition that $\widetilde{\mathbb{Q}}_2 \in \mathcal{G}\mathrm{TIC}_\infty(U)$.

If $\widetilde{\mathbb{N}}_{+21} \in \mathcal{G}\mathrm{TIC}_\infty(U)$ *(cf. (d)), then (12.47) can be written as* $\mathbb{Q} = \mathcal{F}_\ell(\mathbb{T},\mathbb{L}) := \mathbb{T}_{12} + \mathbb{T}_{11}\mathbb{L}(I - \mathbb{T}_{21}\mathbb{L})^{-1}\mathbb{T}_{22} \in \mathrm{TIC}_\infty(Y,U)$ *(for same* $\mathbb{L}$*'s s.t.* $\widetilde{\mathbb{Q}}_2 \in \mathcal{G}\mathrm{TIC}_\infty(U)$*; this parametrizes the well-posed suboptimal stabilizing controllers for* $\mathbb{D}$*), where*

$$\mathbb{T} := \begin{bmatrix} \widetilde{\mathbb{N}}_{+11} & I \\ \widetilde{\mathbb{N}}_{+21} & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 & \widetilde{\mathbb{M}}_{+11} \\ I & \widetilde{\mathbb{M}}_{+21} \end{bmatrix} \in \mathrm{TIC}_\infty(U \times Y). \tag{12.48}$$

(d) Assume that (Factor1) and (Factor2Z) are satisfied with $\mathbb{Z} \in \mathrm{ULR}$*. Then (Factor2) has a solution having* $\widetilde{\mathbb{N}}_{+21} \in \mathcal{G}\mathrm{TIC}_\infty(U)$ *iff there is a well-posed solution* $\mathbb{Q} = \widetilde{\mathbb{Q}}_2^{-1}\widetilde{\mathbb{Q}}_1$ *of the I/O 4BP s.t.* $\widetilde{\mathbb{Q}}_1, \widetilde{\mathbb{Q}}_2 \in \mathrm{TIC} \cap \mathrm{ULR}$.

(e1) We have (Factor1)⇔(Factor1X), and (Factor2)⇔(Factor2Z).

(e2) The solutions of (Factor1) and (Factor1X) correspond 1-1 to each other through formulae

$$\mathbb{N} = \mathbb{N}_{\mathrm{u}}\mathbb{X}^{-1}, \;\; \mathbb{M} = \mathbb{M}_{\mathrm{u}}\mathbb{X}^{-1}; \;\; \mathbb{X} = \mathbb{M}^{-1}\mathbb{M}_{\mathrm{u}}; \;\; \begin{bmatrix} \mathbb{N}_{\mathrm{u}11} & \mathbb{N}_{\mathrm{u}12} \\ 0 & I \end{bmatrix} \mathbb{X}^{-1} = \begin{bmatrix} \mathbb{N}_{11} & \mathbb{N}_{12} \\ \mathbb{M}_{21} & \mathbb{M}_{22} \end{bmatrix}. \tag{12.49}$$

The solutions of (Factor2) and (Factor2Z) correspond 1-1 to each other through formulae

$$\widetilde{\mathbb{N}}_+ = \mathbb{Z}^{-1}\begin{bmatrix} 0 & \mathbb{N}_{22} \\ I & -\mathbb{M}_{12} \end{bmatrix}, \;\; \widetilde{\mathbb{M}}_+ = \mathbb{Z}^{-1}\begin{bmatrix} I & -\mathbb{N}_{21} \\ 0 & \mathbb{M}_{11} \end{bmatrix}; \;\; \mathbb{Z}^{-1} = \begin{bmatrix} \widetilde{\mathbb{M}}_{+11} & \widetilde{\mathbb{N}}_{+11} \\ \widetilde{\mathbb{M}}_{+21} & \widetilde{\mathbb{N}}_{+21} \end{bmatrix} \tag{12.50}$$

(hence $\mathbb{W} := \begin{bmatrix} \widetilde{\mathbb{N}}_{+*2} & \widetilde{\mathbb{M}}_{+*2} \end{bmatrix} = \mathbb{Z}^{-1}\mathbb{E}$*).* □

(This will be proved in Lemma 12.4.14.)

The well-posedness and independence of the above conditions, as well as several additional facts are presented in the lemmas below.

Note that in $\left[\begin{smallmatrix} \mathbb{N}_{21} & \mathbb{N}_{22} \\ I & 0 \end{smallmatrix}\right] \in \mathrm{TIC}(U \times W, Y \times U)$ in (Factor2) does not have its identity operator on the diagonal. This choice (an analogous was done in [Green] and [CG97]) was done to avoid having to interchange the rows of $\mathbb{E}, \mathbb{Z}, \mathbb{W}$ etc., which would end up with $\mathbb{W}J_1\mathbb{W}^* = \left[\begin{smallmatrix} -I & 0 \\ 0 & I \end{smallmatrix}\right]$.

Obviously, the controller $\mathbb{Q}$ in (12.47) is different for different parameters $\mathbb{L}$. Note that this parametrizes all well-posed solutions one-to-one as well as all non-well-posed solutions one-to-one modulo equivalence (see Definition 7.2.11).

However, the solutions $\begin{bmatrix} \widetilde{\mathbb{Q}}_1 & \widetilde{\mathbb{Q}}_2 \end{bmatrix}$ of (12.44) are usually different from those of (12.47) (although both parametrize all solutions $\mathbb{Q}$ of the I/O 4BP, by Lemma 12.4.3(c)).

In [Green], the signature operator "$S = J_\gamma$" was used in (Factor1) and (Factor2) instead of this simplest choice "$S = J_1$" (this corresponds to the (J_γ, J_γ)-lossless r.c.f.'s used in [Green] and [CG97] instead of our (J_*, J_1)-lossless r.c.f.'s). Their choice would introduce several additional γ's in the proofs, and the second columns of certain maps would have to be multiplied by $\gamma^{\pm 1}$, but there is no essential difference. See also the corresponding remark below Theorem 12.1.8.

By Theorem 7.3.19, the standard assumptions (see Standing Hypothesis 12.3.1) imply that any stabilizing DPF-controller of the plant $\mathbb{D}$ has a d.c. internal loop (hence it has a d.c.f. if it is well-posed).

Though we use internal loop techniques to handle the temporary plant $\mathbb{D}_+$, the proofs could be written in the well-posed sense whenever $\mathbb{D}_+$ is well-posed, in particular, for the $\widetilde{\mathcal{A}}$ case treated in Theorem 12.3.6. We would mainly just have to refer to Sections 7.1 and 6.4 instead of Section 7.2 (the same change would be needed in Section 7.3 too).

Lemma 12.3.8 ((4BP1)–(4BP3) are independent on $\mathbb{N}_\mathrm{u}, \mathbb{M}_\mathrm{u}, \widetilde{\mathbb{N}_\mathrm{y}}, \widetilde{\mathbb{M}_\mathrm{y}}$) *Conditions (Factor1)–(Factor2Z) are independent of the preliminary factorizations $\mathbb{N}_\mathrm{u}\mathbb{M}_\mathrm{u}{}^{-1}$ and $\widetilde{\mathbb{M}_\mathrm{y}}{}^{-1}\widetilde{\mathbb{N}_\mathrm{y}}$ (of $\mathbb{D}$) satisfying Standing Hypothesis 12.3.1, as well as of factors $\mathbb{N}, \mathbb{M}, \mathbb{X}$ (if any) satisfying (Factor1) or (Factor1X) and of factors $\widetilde{\mathbb{N}}_+, \widetilde{\mathbb{M}}_+, \mathbb{Z}$ (if any) satisfying (Factor2[Z]) (i.e., they depend on $\mathbb{D}$ only).*

Moreover, the sets of allowable $(\mathbb{N}, \mathbb{M})$'s, $\mathbb{X}$'s, $\mathbb{Z}$'s and $\mathbb{Q}$'s in above conditions as well as the map $\mathbb{E}^* J_1 \mathbb{E}$ are independent of factors $\mathbb{N}_\mathrm{u}, \mathbb{M}_\mathrm{u}, \widetilde{\mathbb{N}_\mathrm{y}}, \widetilde{\mathbb{M}_\mathrm{y}}$. The solutions $\widetilde{\mathbb{Q}}_1, \widetilde{\mathbb{Q}}_2$ of (4BP2) are independent in the sense that they always define the same set of $\mathbb{Q}$'s (which is the set of $\mathbb{Q}$'s solving (4BP1)) through $\mathbb{Q} := \widetilde{\mathbb{Q}}_2^{-1}\widetilde{\mathbb{Q}}_1$.

Maps $\mathbb{D}_+$ and $\mathbb{E}$ and the sets of allowable $\widetilde{\mathbb{M}}_+$'s and $\widetilde{\mathbb{N}}_+$'s depend on $\mathbb{D}, \mathbb{M}, \mathbb{N}$ only. □

(This will be proved in Lemma 12.4.6.)

If $\dim U < \infty$, then the invertibility of $\mathbb{M}_{22}$, $\mathbb{X}_{11}$, $\widetilde{\mathbb{M}}_{+22}$ and $(\mathbb{Z}^{-1}\mathbb{E})_{22}$ becomes redundant:

Lemma 12.3.9 (Case $\dim U < \infty$) *If $\dim U < \infty$, then condition $\mathbb{M}_{22} \in \mathcal{G}\mathrm{TIC}_\infty(W)$ is redundant in (Factor1) and $\mathbb{W}_{22} \in \mathcal{G}\mathrm{TIC}_\infty(U)$ is redundant in (Factor2).* □

(This follows from Proposition 2.5.4(1).)

Lemma 12.3.10 (Case $\mathbb{N}_\mathrm{u}, \mathbb{M}_\mathrm{u} \in \widetilde{\mathcal{A}}$) *Assume that $\mathbb{N}_\mathrm{u}, \mathbb{M}_\mathrm{u} \in \widetilde{\mathcal{A}}$. Then (4BP1)–(4BP3) are equivalent, and claim (a) below holds. If (4BP1) holds, then we have the following:*

(a) All possible choices of $\mathbb{N}, \mathbb{M}, \mathbb{X}, \mathbb{E}, \mathbb{Z}, \mathbb{W}, \widetilde{\mathbb{M}}_+, \widetilde{\mathbb{N}}_+$ are in $\widetilde{\mathcal{A}}$.

(b) The solutions $\widetilde{\mathbb{Q}}_2^{-1}\widetilde{\mathbb{Q}}_1$ of the I/O 4BP are given by (12.47), and we have $\widetilde{\mathbb{Q}}_1, \widetilde{\mathbb{Q}}_2 \in \widetilde{\mathcal{A}} \Leftrightarrow \mathbb{L} \in \widetilde{\mathcal{A}}$.

(c) We can choose $\mathbb{X} \in \widetilde{\mathcal{A}}$ so that $\mathbb{M}_{11}, \mathbb{X}_{22}, \widetilde{\mathbb{M}}_+ \in \mathcal{G}\mathrm{TIC}_\infty$, $X = \begin{bmatrix} X_{11} & X_{12} \\ 0 & X_{22} \end{bmatrix}$, $M = \begin{bmatrix} M_{11} & M_{12} \\ 0 & X_{22}^{-1} \end{bmatrix}$, $X_{11}, X_{22}, M_{11} \in \mathcal{GB}$, and $\mathbb{D}_+$ becomes well-posed. The d.c.f. (12.61) of $\mathbb{D}_+$ is over $\widetilde{\mathcal{A}}$.

(d) There is a well-posed solution $\widetilde{\mathbb{Q}}_2^{-1}\widetilde{\mathbb{Q}}_1$ of the I/O 4BP with $\widetilde{\mathbb{Q}}_1, \widetilde{\mathbb{Q}}_2 \in \widetilde{\mathcal{A}}$ iff we can choose $\mathbb{Z}$ so that $(\mathbb{Z}^{-1})_{22} = \widetilde{\mathbb{N}}_{+21} \in \mathcal{G}\mathrm{TIC}_\infty$.

(e) We have $\widetilde{\mathbb{N}_\mathrm{y}}, \widetilde{\mathbb{M}_\mathrm{y}} \in \widetilde{\mathcal{A}}$ iff the d.c.f. $\mathbb{D} = \mathbb{N}_\mathrm{u}\mathbb{M}_\mathrm{u}{}^{-1} = \widetilde{\mathbb{M}_\mathrm{y}}{}^{-1}\widetilde{\mathbb{N}_\mathrm{y}}$ is over $\widetilde{\mathcal{A}}$. If this is the case, then also the d.c.f. $\mathbb{D}_{21} = \mathbb{N}_{\mathrm{u}\,21}\mathbb{M}_{\mathrm{u}\,11}^{-1} = \widetilde{\mathbb{M}_\mathrm{y}}{}_{22}^{-1}\widetilde{\mathbb{N}_\mathrm{y}}{}_{21}$ is over $\widetilde{\mathcal{A}}$.

(f) *We have the following implications: (4BP1$\widetilde{\mathcal{A}}$)⇒(4BP1), (4BP2$\widetilde{\mathcal{A}}$)⇒(4BP2), (4BP3$\widetilde{\mathcal{A}}$)⇒(4BP3), (Factor1$\widetilde{\mathcal{A}}$)⇒(Factor1), (Factor2$\widetilde{\mathcal{A}}$)⇒(Factor2) (and any solutions of former ones solve the latter ones too).*

(g) *The claims (a)–(e) also apply to any solutions of (Factor1$\widetilde{\mathcal{A}}$) and (Factor2$\widetilde{\mathcal{A}}$) (and for the corresponding $\mathbb{X}$ and $\mathbb{Z}$).* □

(This will be proved in Lemma 12.4.15.)

Lemma 12.3.11 ((Factor1&2))

(a) *If (Factor1) holds, then* $\mathbb{M}_{22}, \mathbb{X}_{11} \in \mathcal{G}\mathrm{TIC}$ *and* $\|\mathbb{X}_{21}\mathbb{X}_{11}^{-1}\| = \|(\mathbb{X}^{-1})_{22}^{-1}(\mathbb{X}^{-1})_{21}\| < 1$. *If, in addition, (Factor2) holds, then* $\mathbb{W}_{22} \in \mathcal{G}\mathrm{TIC}$ *and* $\|\mathbb{W}_{12}\mathbb{W}_{22}^{-1}\| < 1$.

(b) *If* $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$ *solves (Factor1) (resp.* $\mathbb{X}$ *solves (Factor1X)), then all solutions of (Factor1) (resp. of (Factor1X) for a fixed* $\mathbb{N}_{\mathrm{u}}$*) are given by*

$$\mathbb{D} = (\mathbb{N}E)(\mathbb{M}E)^{-1} \text{ (resp. } E^{-1}\mathbb{X}) \ , \ E^*J_1E = J_1, \ E \in \mathcal{G}\mathcal{B}(U \times W). \quad (12.51)$$

If $\mathbb{Z}$ *solves (Factor2Z), then all solutions are given by* $\mathbb{Z}' = \mathbb{Z}F^{-1}$, $F^*J_1F = J_1$, $F \in \mathcal{G}\mathcal{B}(Y \times U)$. *If* $\widetilde{\mathbb{M}}_+^{-1}\widetilde{\mathbb{N}}_+$ *solves (Factor2), then all solutions of (Factor2) for a fixed pair* $(\mathbb{N}, \mathbb{M})$ *are given by* $\mathbb{D}_+ = (F\widetilde{\mathbb{M}}_+)^{-1}(F\widetilde{\mathbb{N}}_+)$ *(hence* $\mathbb{W}' = F\mathbb{W}$*).*

(c) *If (Factor1) is satisfied, then every* $\mathbb{X} \in \mathcal{G}\mathrm{TIC}(U \times W)$ *s.t.* $\mathbb{X}^*J_1\mathbb{X} = \begin{bmatrix} \mathbb{N}_{\mathrm{u}\,11} & \mathbb{N}_{\mathrm{u}\,12} \\ 0 & I \end{bmatrix}^* J_\gamma \begin{bmatrix} \mathbb{N}_{\mathrm{u}\,11} & \mathbb{N}_{\mathrm{u}\,12} \\ 0 & I \end{bmatrix}$ *is a solution of (Factor1X). If (Factor2) is satisfied, then every* $\mathbb{Z} \in \mathcal{G}\mathrm{TIC}(Y \times U)$ *s.t.* $\mathbb{Z}J_1\mathbb{Z}^* = \mathbb{E}J_1\mathbb{E}^*$ *is a solution of (Factor2Z).*

(d) *If (Factor1X) holds and* $\mathbb{X} \in \mathrm{UR}$, *then we can have* $X_{21} = 0$, $X_{11}, X_{22} \in \mathcal{G}\mathcal{B}$. *If, in addition,* $\mathbb{N}_{\mathrm{u}}, \mathbb{M}_{\mathrm{u}} \in \mathrm{UR}$, *then it follows that* $\mathbb{D}, \mathbb{N}, \mathbb{M}, \mathbb{E} \in \mathrm{UR}$, $M_{21} = 0$, $M_{11}, M_{22} \in \mathcal{G}\mathcal{B}$.

(e) *Assume that* $D_{21} = 0$, *(Factor1X) holds,* $\mathbb{N}_{\mathrm{u}}, \mathbb{M}_{\mathrm{u}}, \mathbb{X} \in \mathrm{UR}$, *and X is chosen as in (d). Then* $E = \begin{bmatrix} N_{22} & 0 \\ -M_{12} & M_{11} \end{bmatrix}$, *and any UR solution of (Factor2) has* $Z_{11}, \widetilde{\mathbb{N}}_{+21} \in \mathcal{G}\mathcal{B}$. □

(This will be proved in Lemma 12.4.6.)

We finish this section by an intuitive explanation of the role of $\mathbb{D}_+$ of (Factor2$\widetilde{\mathcal{A}}$). This map obviously maps $\begin{bmatrix} u \\ w_\circlearrowleft \end{bmatrix} \mapsto \begin{bmatrix} y \\ u_\circlearrowleft \end{bmatrix}$ (because $\mathbb{M}$ maps $\begin{bmatrix} u_\circlearrowleft \\ w_\circlearrowleft \end{bmatrix} \mapsto \begin{bmatrix} u \\ w \end{bmatrix}$ and $\mathbb{N}$ maps $\begin{bmatrix} u_\circlearrowleft \\ w_\circlearrowleft \end{bmatrix} \mapsto \begin{bmatrix} z \\ y \end{bmatrix}$).

When (Factor1$\widetilde{\mathcal{A}}$) is satisfied, the map $\mathbb{D}$ equals the connection of $\begin{bmatrix} \mathbb{D}_{11}^{\curvearrowleft} & \mathbb{D}_{12}^{\curvearrowleft} \\ \underline{\mathbb{X}}_{21} & \underline{\mathbb{X}}_{22}^{-1} \end{bmatrix} : \begin{bmatrix} u_\circlearrowleft \\ w \end{bmatrix} \mapsto \begin{bmatrix} y \\ w_\circlearrowleft \end{bmatrix}$ and $\mathbb{D}_+$, as illustrated in Figure 12.2. Here $\mathbb{D}^{\curvearrowleft}$ is defined by (11.10) and $\underline{\mathbb{X}} := \begin{bmatrix} I & 0 \\ \mathbb{M}_{21} & \mathbb{M}_{22} \end{bmatrix}^{-1} = \begin{bmatrix} I & 0 \\ -\mathbb{M}_{22}^{-1}\mathbb{M}_{21} & \mathbb{M}_{22}^{-1} \end{bmatrix} : \begin{bmatrix} u_\circlearrowleft \\ w \end{bmatrix} \mapsto \begin{bmatrix} u_\circlearrowleft \\ w_\circlearrowleft \end{bmatrix}$.

Since (12.41):$\begin{bmatrix} u_\circlearrowleft \\ w_\circlearrowleft \end{bmatrix} \mapsto \begin{bmatrix} z \\ w \end{bmatrix}$ is (J_γ, J_1)-lossless, one can deduce that $\|w_\circlearrowleft \mapsto u_\circlearrowleft\| < 1$ iff $\|w \mapsto z\| < \gamma$. Thus, $\mathbb{Q}$ is suboptimal for $\mathbb{D}$ and γ iff $\mathbb{Q}$ is suboptimal for $\mathbb{D}_+$ and 1. It follows that we have reduced the solution to the case where $\mathbb{D}$ is of the special form (of $\mathbb{D}_+$) described in (Factor2$\widetilde{\mathcal{A}}$). These facts are often

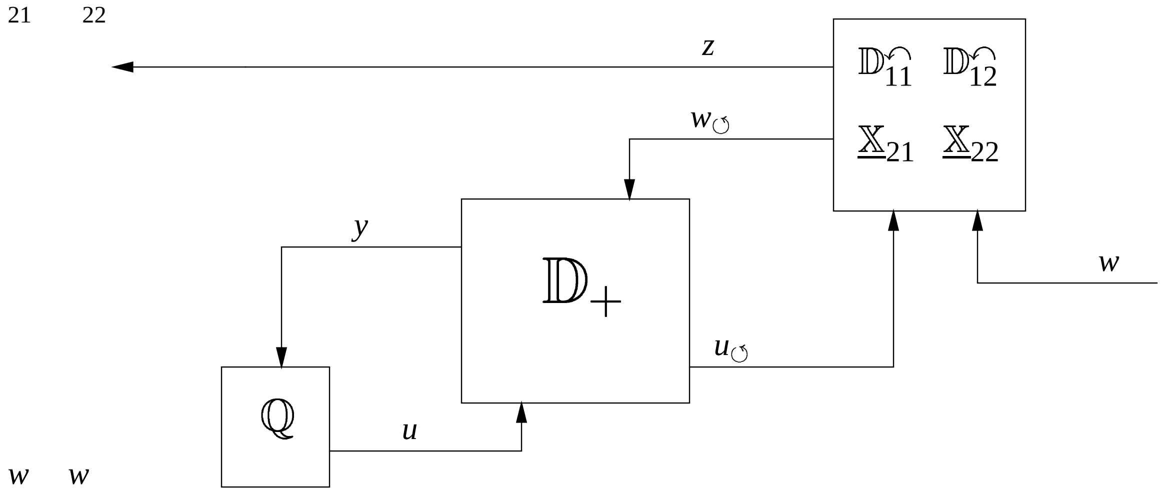

Figure 12.2: The map $\mathbb{D}_+$

used in the proof of the 4BP (see, e.g., pp. 421– of [ZDG]), but we use them only implicitly (in Section 12.4). In fact, in the more general setting of Theorem 12.3.7, the system $\mathbb{D}_+$ need not be well-posed.

Notes

Theorem 12.3.6 is a direct generalization of Theorem 4.4 of [Green] and of those of Theorem 5.6 of [CG97], as explained before the theorem.

We had essentially written this section during early 1998 (but at that stage, we needed improvements to our Riccati equation theory in order to solve the state-space H^∞ 4BP of Section 12.1). At the end of year 2000, Hans Zwart showed us the report [IZ00], which contains something like Theorems 12.3.6 and 12.3.7 with $\widetilde{\mathcal{A}} = \mathrm{MTIC}^{\mathrm{L}^1}$ and $\dim U \times W \times Z \times Y < \infty$; thus our assumptions are also more general than the ones in [IZ00]. Also [IZ00] uses pure frequency-domain methods, and its methods seem essentially finite-dimensional (see, e.g., Lemma 2.2.1(c1), Proposition 2.5.4, Lemma 7.1.4 and the remark in the proof of Lemma 2.2.2(c2)); due to the same reasons, the methods of [Green] and [CG97] do not apply to this general case.

Our formulae (and proofs) become mathematically more complete due to the fact that we allow for controllers with internal loop (hence we do not need any additional invertibility conditions); see Theorem 12.3.6 or Theorem 12.3.7(c)&(d) for well-posed solutions.

The frequency-domain results have their own merits, as explained in [CG97], but they can also be used to derive the solution to the corresponding state-space problem, and that is what we will do in the last two sections of this chapter.

Under sufficient regularity, one can formulate the two lossless factorizations as Riccati equations; this leads to Theorem 12.1.8 and Proposition 12.1.10, i.e., to CAREs “(1.)” and “(4.)”. Because “(4.)” is formulated in terms of a perturbed equation, one usually wishes to replace it with “(2.)” and “(3.)” of Theorem 12.1.5. This connection is established in Lemma 12.6.4 for the discrete-time setting and in Section 12.5 for the continuous-time setting (by discretization); the proofs require some extra regularity (as compared to Theorem 12.1.8 or to

Theorem 12.3.6) in the continuous-time case.

One might ask whether something similar could be done to the frequency-domain solution, i.e., whether (Factor1) and its dual condition with some coupling condition ("(3.)") would suffice, so that the perturbed factorization condition (Factor2) would not be needed. A positive answer to this question is given in Remark 12.5.25 (in the most popular setting where $\mathbb{D}$ has an exponential d.c.f.), but we still lack a simple formulation of the frequency-domain coupling condition.

Although the problem is symmetric, our solution is not. If we replace $\mathbb{D}$ by $\mathbb{D}_d$ in the proofs, we obtain Theorems 12.3.6 and 12.3.7 with the dual of (Factor1) (the factorization corresponding to "(2.)") in place of (Factor1) and, analogously, a fourth factorization condition in place of (Factor2) (the situation is the same with the state-space problem and the Riccati equations, as noted on p. 708).

12.4 Proofs for Section 12.3

When confronted by a difficult problem, you can solve it more easily by reducing it to the question, "How would the Lone Ranger have handled this?"

The proofs of the results of Section 12.3 are more complicated than in the rational case (cf. [Green]) or in the case of the Callier–Desoer class (cf. [CG97]), because we lack several algebraic properties satisfied by the rational transfer functions, and because the (possibly) infinite dimensions of input and output spaces cause additional problems.

We start by proving some results concerning (Factor1) (Lemma 12.4.2). Then we list additional conditions (Lemma 12.4.3) equivalent to (4BP1)–(4BP3) (under suitably assumptions). Thereafter, we go on to study (Factor2) (Lemma 12.4.4–) until we are ready to prove the implications and additional results of Section 12.3 (Lemma 12.4.7–).

The proofs will refer to the general WPLS case of Theorem 12.3.7, where we do not require $\mathbb{Q}$ to be well-posed (i.e., $\mathrm{TIC}_\infty(Y,U)$) but we allow $\mathbb{Q}$ to be any map with internal loop. However, the readers interested only in the cases illustrated in Theorem 12.3.6 and analogous results may consider well-posed $\mathbb{Q}$'s only, i.e., the case where $\mathbb{Q}_2 \in \mathcal{G}\mathrm{TIC}_\infty(Y)$ and $\widetilde{\mathbb{Q}}_2 \in \mathcal{G}\mathrm{TIC}_\infty(U)$.

We shall need the following notion:

Lemma 12.4.1 (4BP ⇒ FICP & FCP) *Let the 4BP for $\mathbb{D}$ have a solution. Then* $\begin{bmatrix} \mathbb{N}_{\mathrm{u}\,11} & \mathbb{N}_{\mathrm{u}\,12} \\ 0 & I \end{bmatrix}$ *and* $\begin{bmatrix} \widetilde{\mathbb{N}}_{\mathrm{y}\,22}^{\mathrm{d}} & \widetilde{\mathbb{N}}_{\mathrm{y}\,12}^{\mathrm{d}} \\ 0 & I \end{bmatrix}$ *are minimax J_γ-coercive.*

See Lemma 12.5.7 for the analogous and further state-space results under weaker assumptions.

Proof: From (12.39) we observe that there are solutions $\mathbb{U} \in \mathrm{TIC}$ of $\|\mathbb{N}_{\mathrm{u}\,11}\mathbb{U} + \mathbb{N}_{\mathrm{u}\,12}\| < \gamma$ and $\overline{\mathbb{U}} \in \mathrm{TIC}$ of $\|\widetilde{\mathbb{N}}_{\mathrm{y}\,22}^{\mathrm{d}}\overline{\mathbb{U}} + \widetilde{\mathbb{N}}_{\mathrm{y}\,12}^{\mathrm{d}}\| < \gamma$. It follows from Lemma 11.3.10 that $\begin{bmatrix} \mathbb{N}_{\mathrm{u}\,11} & \mathbb{N}_{\mathrm{u}\,12} \\ 0 & I \end{bmatrix}$ and $\begin{bmatrix} \widetilde{\mathbb{N}}_{\mathrm{y}\,22}^{\mathrm{d}} & \widetilde{\mathbb{N}}_{\mathrm{y}\,12}^{\mathrm{d}} \\ 0 & I \end{bmatrix}$ are minimax J_γ-coercive, hence J_γ-coercive, by Lemma 11.4.2. □

Lemma 12.4.2 *We give here partial proofs of Theorem 12.3.7 and Lemmas 12.3.8 and 12.3.11(a)–(c); these proofs will be completed in lemmas below.*

Proof: *I Theorem 12.3.7(e1)&(e2) (partially):*

1° *"(Factor1X)⇒(Factor1)":* Assume (Factor1X). Set $\mathbb{V} := \mathbb{X}^{-1}$, $\mathbb{Y} := \begin{bmatrix} \mathbb{N}_{\mathrm{u}\,11} & \mathbb{N}_{\mathrm{u}\,12} \\ 0 & I \end{bmatrix}\mathbb{V}$ to obtain $\mathbb{Y}^* J_\gamma \mathbb{Y} = J_1$. Set $\mathbb{N} := \mathbb{N}_{\mathrm{u}}\mathbb{V}$, $\mathbb{M} := \mathbb{M}_{\mathrm{u}}\mathbb{V}$ to obtain a r.c.f. (by Lemma 6.4.5(c)) with

$$\mathbb{N} = \begin{bmatrix} \mathbb{Y}_{11} & \mathbb{Y}_{12} \\ * & * \end{bmatrix}, \ \mathbb{M} = \begin{bmatrix} * & * \\ \mathbb{V}_{21} & \mathbb{V}_{22} \end{bmatrix} = \begin{bmatrix} * & * \\ \mathbb{Y}_{21} & \mathbb{Y}_{22} \end{bmatrix} \tag{12.52}$$

(because $\mathbb{M}_{\mathrm{u}} = \begin{bmatrix} * & * \\ 0 & I \end{bmatrix}$), satisfying (Factor1), because $\mathbb{Y}$ is (J_γ, J_1)-lossless, by Corollary 2.5.5 (since $\mathbb{Y}_{22} = \mathbb{M}_{22} \in \mathcal{G}\mathrm{TIC}_\infty$).

2° *"(Factor1)⇒(Factor1X)":* (This is the above proof backwards.) Assume (Factor1). By Lemma 6.4.5(c), we have $\mathbb{V} := \mathbb{M}_{\mathrm{u}}{}^{-1}\mathbb{M} \in \mathcal{G}\mathrm{TIC}(U \times W)$. Set $\mathbb{N} := \mathbb{N}_{\mathrm{u}}\mathbb{V}$, $\mathbb{M} := \mathbb{M}_{\mathrm{u}}\mathbb{V}$ to obtain that $\mathbb{Y} := \begin{bmatrix} \mathbb{N}_{\mathrm{u}11} & \mathbb{N}_{\mathrm{u}12} \\ 0 & I \end{bmatrix}\mathbb{V}$ satisfies $\mathbb{Y} = \begin{bmatrix} \mathbb{N}_{11} & \mathbb{N}_{12} \\ \mathbb{M}_{21} & \mathbb{M}_{22} \end{bmatrix}$. Thus, $\mathbb{Y}^* J_\gamma \mathbb{Y} = J_1$ implies that $\begin{bmatrix} \mathbb{N}_{\mathrm{u}11} & \mathbb{N}_{\mathrm{u}12} \\ 0 & I \end{bmatrix}^* J_\gamma \begin{bmatrix} \mathbb{N}_{\mathrm{u}11} & \mathbb{N}_{\mathrm{u}12} \\ 0 & I \end{bmatrix} = \mathbb{V}^{-*} J_1 \mathbb{V}^{-1}$. Therefore, $\mathbb{X} := \mathbb{V}^{-1}$ is as in (Factor1X) (use again Corollary 2.5.5).

3° *Equations (12.49) of (e2):* This follows from 1°–2°.

II Lemma 12.3.8 — The first claim on $\mathbb{N}_{\mathrm{u}}, \mathbb{M}_{\mathrm{u}}, \widetilde{\mathbb{M}_{\mathrm{y}}}, \widetilde{\mathbb{N}_{\mathrm{y}}}$ *(partially):* This is obvious for (4BP2), (Factor1) and (Factor2), hence for (4BP3) too; for (Factor1X) this follows from the above equivalence, for (Factor2Z) this will follow from the equivalence that will be established in Lemma 12.4.4.

III Lemma 12.3.11: (a), (c) and first half of (b): (a) For (Factor1X), this follows from Lemma 11.4.3(a)&(c) (with substitutions $\mathbb{D} \mapsto \begin{bmatrix} \mathbb{N}_{\mathrm{u}11} & \mathbb{N}_{\mathrm{u}12} \\ 0 & I \end{bmatrix}$, $J_\gamma \mapsto J_1$, $J \mapsto J_\gamma$; recall from Standing Hypothesis 12.3.1 that $\mathbb{N}_{\mathrm{u}}{}_{11}^{*}\mathbb{N}_{\mathrm{u}11} \gg 0$). The claims on (Factor2Z) follow analogously.

(c) This follows directly from Lemma 11.4.3(b).

(b) (the (Factor1[X]) part only) By ((c) and) Lemma 6.4.5(a), all factorizations of form (Factor1X) are given by $E^{-1}\mathbb{X}$, where $J_1 = E^* J_1 E$, $E \in \mathcal{G}\mathcal{B}(U \times W)$. (Recall that $\mathbb{N}_{\mathrm{u}}$ was taken fixed here.)

If $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$ is as in (Factor1), then the above proof of "(Factor1)⇒(Factor1X)" shows that $\mathbb{N} = \mathbb{N}_{\mathrm{u}}\mathbb{X}^{-1}$, where $\mathbb{X}$ is as above, hence such factorizations again differ by E only.

(Remark — $\mathbb{X}$ depends on $\mathbb{N}_{\mathrm{u}}$ as follows: If an r.c.f. $\mathbb{D} = \mathbb{N}_{\mathrm{u}}{}'(\mathbb{M}_{\mathrm{u}}{}')^{-1}$ is also of form $\mathbb{M}_{\mathrm{u}}{}' = \begin{bmatrix} * & * \\ 0 & I \end{bmatrix}$, then $\mathbb{N}_{\mathrm{u}}{}' = \mathbb{N}_{\mathrm{u}}\mathbb{U}$ and $\mathbb{M}_{\mathrm{u}}{}' = \mathbb{M}_{\mathrm{u}}\mathbb{U}$ for some $\mathbb{U} \in \mathcal{G}\mathrm{TIC}$ with $\mathbb{U} = \begin{bmatrix} * & * \\ 0 & I \end{bmatrix}$, by Lemma 7.3.16. It follows that $\begin{bmatrix} \mathbb{N}_{\mathrm{u}}{}'_{11} & \mathbb{N}_{\mathrm{u}}{}'_{12} \\ 0 & I \end{bmatrix} = \begin{bmatrix} \mathbb{N}_{\mathrm{u}11} & \mathbb{N}_{\mathrm{u}12} \\ 0 & I \end{bmatrix}\mathbb{U}$, hence $\mathbb{X}\mathbb{U}$ is a spectral factor of $\begin{bmatrix} \mathbb{N}_{\mathrm{u}}{}'_{11} & \mathbb{N}_{\mathrm{u}}{}'_{12} \\ 0 & I \end{bmatrix}$ iff $\mathbb{X}$ is a spectral factor of $\begin{bmatrix} \mathbb{N}_{\mathrm{u}11} & \mathbb{N}_{\mathrm{u}12} \\ 0 & I \end{bmatrix}$. Thus, for $\mathbb{N}_{\mathrm{u}}{}' = \mathbb{N}_{\mathrm{u}}\mathbb{U}$, $\mathbb{U} = \begin{bmatrix} * & * \\ 0 & I \end{bmatrix} \in \mathcal{G}\mathrm{TIC}$, (Factor1X) is satisfied by $\mathbb{X}' = E\mathbb{X}\mathbb{U}$, where E is as above.) □

To simplify the formulae in the proof, we show that the preliminary factorization $\mathbb{D} = \mathbb{N}_{\mathrm{u}}\mathbb{M}_{\mathrm{u}}{}^{-1}$ can be replaced by "a semioptimal preliminary factorization" $\mathbb{D} = \mathbb{N}_{\mathrm{u}}^{\frown}\mathbb{M}_{\mathrm{u}}^{\frown}{}^{-1}$. Then we establish further equivalent conditions for (4BP1):

Lemma 12.4.3 (Equivalent conditions (4BP1)–(4BP7)) *Assume (Factor1X) and set* $\mathbb{V} := \mathbb{X}^{-1}$, $\underline{\mathbb{V}} := \begin{bmatrix} I & 0 \\ \mathbb{V}_{21} & \mathbb{V}_{22} \end{bmatrix}$, $\underline{\mathbb{X}} := \underline{\mathbb{V}}^{-1}$, $\overline{\mathbb{X}} := \begin{bmatrix} \mathbb{X}_{11} & \mathbb{X}_{12} \\ 0 & I \end{bmatrix}$, $\overline{\mathbb{V}} := \overline{\mathbb{X}}^{-1}$, $\mathbb{M}_{\mathrm{u}}^{\frown} := \mathbb{M}_{\mathrm{u}}\overline{\mathbb{V}} = \begin{bmatrix} \mathbb{M}_{\mathrm{u}}^{\frown}{}_{11} & \mathbb{M}_{\mathrm{u}}^{\frown}{}_{12} \\ 0 & I \end{bmatrix}$, $\mathbb{N}_{\mathrm{u}}^{\frown} := \mathbb{N}_{\mathrm{u}}\overline{\mathbb{V}}$, $\mathbb{M} = \mathbb{M}_{\mathrm{u}}\mathbb{X}^{-1} = \mathbb{M}_{\mathrm{u}}^{\frown}\underline{\mathbb{V}}$ *and* $\mathbb{N} := \mathbb{N}_{\mathrm{u}}\mathbb{X}^{-1} = \mathbb{N}_{\mathrm{u}}^{\frown}\underline{\mathbb{V}}$. *Then* $\mathbb{X} = \underline{\mathbb{X}}\overline{\mathbb{X}}$, $\mathbb{V} = \overline{\mathbb{V}}\underline{\mathbb{V}}$, *and the following holds.*

(a) The r.c.f. $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$ *is as in (Factor1), and the r.c.f.* $\mathbb{D} = \mathbb{N}_{\mathrm{u}}^{\frown}\mathbb{M}_{\mathrm{u}}^{\frown}{}^{-1}$ *is as in Standing Hypothesis 12.3.1.*

(b) The conditions (4BP1), (4BP2) and (4BP4)–(4BP7) are equivalent and independent of $\mathbb{N}$, $\mathbb{M}$ *and* $\mathbb{X}$.

(c1) A map $\mathbb{Q}$ *with internal loop solves the 4BP iff it can be written as* $\mathbb{Q} = \widetilde{\mathbb{Q}}_2^{-1}\widetilde{\mathbb{Q}}_1$, *where* $\widetilde{\mathbb{Q}}_1, \widetilde{\mathbb{Q}}_2$ *satisfy (4BP2). Note that any maps* $\widetilde{\mathbb{Q}}_1$ *and* $\widetilde{\mathbb{Q}}_2$ *satisfying (4BP2) are necessarily l.c.*

The above claims hold also with (4BP4), (4BP5), (4BP6) and (4BP7) in place of (4BP2).

(c2) (For a fixed pair $(\widetilde{\mathbb{Q}}_1, \widetilde{\mathbb{Q}}_2)$ *we have "(4BP4)⇔(4BP5)" and "(4BP2)⇒(4BP6)⇔(4BP7)", whereas for* $\widetilde{\mathbb{Q}}_2^{-1}\widetilde{\mathbb{Q}}_1$ *(as a map with l.c. internal loop) all these are equivalent.)*

(c3) Assume that $\mathbb{N}_{\mathrm{u}}, \mathbb{M}_{\mathrm{u}} \in \mathrm{TIC}_{\mathrm{exp}}$. *If any of (4BP1), (4BP2) and (4BP4)–(4BP7) has a solution with* $\widetilde{\mathbb{Q}}_1, \widetilde{\mathbb{Q}}_2 \in \mathrm{TIC}_{\mathrm{exp}}$*, then so do all of them.*

(This also holds with $\mathcal{A}$ *in place of* $\mathrm{TIC}_{\mathrm{exp}}$ *if* $\mathcal{A} \underset{a}{\subset} \mathrm{TIC}$ *is closed under spectral factorization.)*

(d) If (4BP5) (and hence (4BP1)–(4BP7) except possibly (4BP3)) holds, then the l.c.f. $\widetilde{\mathbb{Q}}_2^{-1}\widetilde{\mathbb{Q}}_1 = \mathbb{Q}$ *given in (4BP5) is unique, and the following equations hold:*

$$\widetilde{\mathbb{U}}_o = (I - \mathbb{U}_o \mathbb{V}_{21})^{-1} \mathbb{U}_o \mathbb{V}_{22}, \tag{12.53}$$

$$\mathbb{U}_o = (I - \widetilde{\mathbb{U}}_o \underline{\mathbb{X}}_{21})^{-1} \widetilde{\mathbb{U}}_o \underline{\mathbb{X}}_{22} = \widetilde{\mathbb{U}}_o (\mathbb{V}_{21} \widetilde{\mathbb{U}}_o + \mathbb{V}_{22})^{-1}, \tag{12.54}$$

$$(I - \widetilde{\mathbb{U}}_o \underline{\mathbb{X}}_{21})^{-1} = I - \mathbb{U}_o \mathbb{V}_{21} \in \mathcal{G}\mathrm{TIC}, \tag{12.55}$$

$$(\mathbb{V}_{21} \widetilde{\mathbb{U}}_o + \mathbb{V}_{22})^{-1} = \underline{\mathbb{X}}_{22} + \underline{\mathbb{X}}_{21} \mathbb{U}_o \in \mathcal{G}\mathrm{TIC}. \tag{12.56}$$

(e) This lemma holds even if we replace "<" signs of the lemma by "≤" signs (this includes requiring that $\|\mathcal{F}_\ell(\mathbb{D}, \mathbb{Q})\| \le \gamma$ *instead of* $< \gamma$*).*

We have referred to the following conditions:

(4BP4) There are $\widetilde{\mathbb{Q}}_1, \widetilde{\mathbb{Q}}_2 \in \mathrm{TIC}$ *s.t.* $\widetilde{\mathbb{Q}}_2 \mathbb{M}_{\mathrm{u}}^{\curvearrowleft}{}_{11} - \widetilde{\mathbb{Q}}_1 \mathbb{N}_{\mathrm{u}}^{\curvearrowleft}{}_{21} = I$ *and* $\mathbb{U}_o := \widetilde{\mathbb{Q}}_1 \mathbb{N}_{\mathrm{u}}^{\curvearrowleft}{}_{22} - \widetilde{\mathbb{Q}}_2 \mathbb{M}_{\mathrm{u}}^{\curvearrowleft}{}_{12} \in \mathrm{TIC}$ *solves (the FICP)* $\|\mathbb{N}_{\mathrm{u}}^{\curvearrowleft}{}_{11} \mathbb{U}_o + \mathbb{N}_{\mathrm{u}}^{\curvearrowleft}{}_{12}\| < \gamma$*.*

(4BP5) There are $\widetilde{\mathbb{Q}}_1, \widetilde{\mathbb{Q}}_2 \in \mathrm{TIC}$ *s.t.* $-\widetilde{\mathbb{Q}}_1 \mathbb{N}_{\mathrm{u}}^{\curvearrowleft}{}_{21} + \widetilde{\mathbb{Q}}_2 \mathbb{M}_{\mathrm{u}}^{\curvearrowleft}{}_{11} = I$ *and* $\mathbb{U}_o := \widetilde{\mathbb{Q}}_1 \mathbb{N}_{\mathrm{u}}^{\curvearrowleft}{}_{22} - \widetilde{\mathbb{Q}}_2 \mathbb{M}_{\mathrm{u}}^{\curvearrowleft}{}_{12} \in \mathrm{TIC}$ *is of the form* $\mathbb{U}_o = \mathbb{U}_1 \mathbb{U}_2^{-1}$*, where* $\begin{bmatrix} \mathbb{U}_1 \\ \mathbb{U}_2 \end{bmatrix} = \underline{\mathbb{V}} \begin{bmatrix} \widetilde{\mathbb{U}}_o \\ I \end{bmatrix}$, $\widetilde{\mathbb{U}}_o \in \mathrm{TIC}$ *and* $\|\widetilde{\mathbb{U}}_o\| < 1$.

(4BP6) There are $\widetilde{\mathbb{Q}}_1, \widetilde{\mathbb{Q}}_2 \in \mathrm{TIC}$ *s.t.* $\widetilde{\mathbb{Q}}_2 \mathbb{M}_{11} - \widetilde{\mathbb{Q}}_1 \mathbb{N}_{21} \in \mathcal{G}\mathrm{TIC}(U)$*, and* $\|(\widetilde{\mathbb{Q}}_2 \mathbb{M}_{11} - \widetilde{\mathbb{Q}}_1 \mathbb{N}_{21})^{-1}(-\widetilde{\mathbb{Q}}_2 \mathbb{M}_{12} + \widetilde{\mathbb{Q}}_1 \mathbb{N}_{22})\|_{\mathrm{TIC}(W,U)} < 1$.

(4BP7) **(ASP)** *There are* $\widetilde{\mathbb{Q}}_1, \widetilde{\mathbb{Q}}_2 \in \mathrm{TIC}$ *s.t. the operators*

$$\begin{bmatrix} \widetilde{\mathbb{U}}_1 & \widetilde{\mathbb{U}}_2 \end{bmatrix} := \begin{bmatrix} \widetilde{\mathbb{Q}}_1 & \widetilde{\mathbb{Q}}_2 \end{bmatrix} \begin{bmatrix} \mathbb{N}_{22} & -\mathbb{N}_{21} \\ -\mathbb{M}_{12} & \mathbb{M}_{11} \end{bmatrix} \in \mathrm{TIC}(W \times U, U) \tag{12.57}$$

satisfy $\widetilde{\mathbb{U}}_2 \in \mathcal{G}\mathrm{TIC}$ *and* $\|\widetilde{\mathbb{U}}_2^{-1}\widetilde{\mathbb{U}}_1\| < 1$.

Thus, condition "(Factor1X) and (4BPn) hold" is equivalent to (4BP1) (and to (4BP2)) whenever $\left(\left[\begin{smallmatrix} \mathbb{N}_{\mathrm{u}\,11} & \mathbb{N}_{\mathrm{u}\,12} \\ 0 & I \end{smallmatrix}\right], J_\gamma\right) \in \mathrm{SpF}$, by Theorem 12.3.7(a).

Note that, for each $\mathbb{Q}$ solving the 4BP (cf. (c)), the pairs $(\widetilde{\mathbb{Q}}_1, \widetilde{\mathbb{Q}}_2)$ in (4BP2) and (4BP5) need not be the same (in fact this happens very seldom, when

$\widetilde{\mathbb{U}}\mathbb{V}_{22}^{-1}\mathbb{V}_{21} = 0$, as shown in the proof of (4BP2)⇒(4BP5)); we only know that they differ by an element of $\mathcal{G}$TIC (see Definition 7.2.11).

The equivalence "(4BP1)⇔(4BP4)" holds for the original r.c.f. $\mathbb{D} = \mathbb{N}_{\mathrm{u}}\mathbb{M}_{\mathrm{u}}{}^{-1}$ too, as one observes from the proof.

If one l.c.f. of $\mathbb{Q}$ satisfies the ASP given in (v) (and (iv) as well), then clearly every l.c.f. of $\mathbb{Q}$ does (cf. Lemma 6.4.5(d)).

Proof of Lemma 12.4.3: Note that we have chosen a "semicritical preliminary factorization" $\mathbb{D} = \mathbb{N}_{\mathrm{u}}^{\curvearrowleft}\mathbb{M}_{\mathrm{u}}^{\curvearrowleft}{}^{-1}$ to make the formulae in the proof (including those at the end of the lemma) simpler. By Lemma 12.3.8 (see Lemma 12.4.2), this causes no loss of generality.

The formula $\mathbb{V} = \overline{\mathbb{V}}\underline{\mathbb{V}}$ is from (A.9), and $\mathbb{X} = \underline{\mathbb{X}}\overline{\mathbb{X}}$ is its inverse.

(a) By Lemma 6.4.5(c), $\mathbb{N}_{\mathrm{u}}^{\curvearrowleft}\mathbb{M}_{\mathrm{u}}^{\curvearrowleft}{}^{-1}$ is a r.c.f. of (c). Obviously, $\mathbb{M}_{\mathrm{u}}^{\curvearrowleft} = \left[\begin{smallmatrix} * & * \\ 0 & I \end{smallmatrix}\right]$. The rest follows as in Corollary 7.3.17.

By 2° of the proof of Lemma 12.4.2, the r.c.f. $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$ is as in (Factor1).

I The equivalence: Let $\widetilde{\mathbb{Q}}_1 \in \mathrm{TIC}(Y,U)$ and $\widetilde{\mathbb{Q}}_2 \in \mathrm{TIC}(U)$ be l.c. (i.e., let $\mathbb{Q} := \widetilde{\mathbb{Q}}_2^{-1}\widetilde{\mathbb{Q}}_1$ be a map with l.c. internal loop; cf. Definition 7.2.11). We shall show below the equivalence of (4BP1)–(4BP7)$\setminus$(4BP3) for $\widetilde{\mathbb{Q}}_2^{-1}\widetilde{\mathbb{Q}}_1$ in the sense that given $n,m \in \{1,2,4,5,6,7\}$, there is $\mathbb{U}_n \in \mathcal{G}\mathrm{TIC}(U)$ s.t. $(\mathbb{U}_n\widetilde{\mathbb{Q}}_1, \mathbb{U}_n\widetilde{\mathbb{Q}}_n)$ satisfies (4BPn) iff there is $\mathbb{U}_m \in \mathcal{G}\mathrm{TIC}(U)$ s.t. $(\mathbb{U}_m\widetilde{\mathbb{Q}}_1, \mathbb{U}_m\widetilde{\mathbb{Q}}_n)$ satisfies (4BPm) (i.e., $\mathbb{Q}$ has a presentation satisfying (4BPn) iff $\mathbb{Q}$ has a presentation satisfying (4BPm)). Note that (4BP1) is independent on the presentation.

1° "(4BP1)⇔(4BP4)": This follows from Lemma 12.3.2.

2° "(4BP4)⇔(4BP5)": This follows directly from Theorem 11.3.6.

4° "(4BP6)⇔(4BP7)": Clearly (4BP7) is only a reformulation of (4BP6).

5° "(4BP7)⇔(4BP2)": If $(\widetilde{\mathbb{Q}}_1, \widetilde{\mathbb{Q}}_2)$ solves (4BP7), then $(\widetilde{\mathbb{U}}_2^{-1}\widetilde{\mathbb{Q}}_1, \widetilde{\mathbb{U}}_2^{-1}\widetilde{\mathbb{Q}}_2)$ solves (4BP2) (where $\widetilde{\mathbb{U}}_2$ is as in (4BP7)). Conversely, every solution of (4BP2) is a solution of (4BP7).

6° "(4BP5)⇒(d)&(4BP7)": We now assume (4BP5) (and hence (4BP1) and (4BP4) too) and will prove first the claims in part (d) and then (4BP7).

The uniqueness of the l.c.f. follows from Lemma 6.4.5(d). The first formula is equation $-\widetilde{\mathbb{Q}}_1\mathbb{N}_{\mathrm{u}}^{\curvearrowleft}{}_{21} + \widetilde{\mathbb{Q}}_2\mathbb{M}_{\mathrm{u}}^{\curvearrowleft}{}_{11} = I$ multiplied to the left by $\widetilde{\mathbb{Q}}_2^{-1}$.

Equations (12.53)–(12.56) follow from equations (11.89)–(11.92) of Theorem 11.3.6 (where we must use the spectral factor $\underline{\mathbb{X}} = \mathbb{M}^{-1}\mathbb{M}_{\mathrm{u}}^{\curvearrowleft}$ of $\begin{bmatrix} \mathbb{N}_{\mathrm{u}}^{\curvearrowleft}{}_{11} & \mathbb{N}_{\mathrm{u}}^{\curvearrowleft}{}_{12} \\ 0 & I \end{bmatrix}$) and the last three equations are derived as follows:

$$\widetilde{\mathbb{U}}_2 := \widetilde{\mathbb{Q}}_2\mathbb{M}_{11} - \widetilde{\mathbb{Q}}_1\mathbb{N}_{21} = \widetilde{\mathbb{Q}}_2\mathbb{M}_{\mathrm{u}}^{\curvearrowleft}{}_{11} + \widetilde{\mathbb{Q}}_2\mathbb{M}_{\mathrm{u}}^{\curvearrowleft}{}_{12}\mathbb{V}_{21} - \widetilde{\mathbb{Q}}_1\mathbb{N}_{\mathrm{u}}^{\curvearrowleft}{}_{21} - \widetilde{\mathbb{Q}}_1\mathbb{N}_{\mathrm{u}}^{\curvearrowleft}{}_{22}\mathbb{V}_{21} \tag{12.58}$$

$$= (\widetilde{\mathbb{Q}}_2\mathbb{M}_{\mathrm{u}}^{\curvearrowleft}{}_{11} - \widetilde{\mathbb{Q}}_1\mathbb{N}_{\mathrm{u}}^{\curvearrowleft}{}_{21}) - (\widetilde{\mathbb{Q}}_1\mathbb{N}_{\mathrm{u}}^{\curvearrowleft}{}_{22} - \widetilde{\mathbb{Q}}_2\mathbb{M}_{\mathrm{u}}^{\curvearrowleft}{}_{12})\mathbb{V}_{21} \tag{12.59}$$

$= I - \mathbb{U}_o\mathbb{V}_{21}$, by the equations in (4BP5). Now $\widetilde{\mathbb{U}}_2 = I - \mathbb{U}_o\mathbb{V}_{21} \in \mathcal{G}\mathrm{TIC}(U)$, by (12.55). Similarly,

$$\widetilde{\mathbb{U}}_1 = -\widetilde{\mathbb{Q}}_2\mathbb{M}_{12} + \widetilde{\mathbb{Q}}_1\mathbb{N}_{22} = -\widetilde{\mathbb{Q}}_2\mathbb{M}_{\mathrm{u}}^{\curvearrowleft}{}_{12}\mathbb{V}_{22} + \widetilde{\mathbb{Q}}_1\mathbb{N}_{\mathrm{u}}^{\curvearrowleft}{}_{22}\mathbb{V}_{22} = \mathbb{U}_o\mathbb{V}_{22}. \tag{12.60}$$

Therefore, $\widetilde{\mathbb{U}}_2^{-1}\widetilde{\mathbb{U}}_1 = (I - \mathbb{U}_o\mathbb{V}_{21})^{-1}\mathbb{U}_o\mathbb{V}_{22} = \widetilde{\mathbb{U}}_o$, whose norm is less than 1, hence (4BP5) implies (4BP7).

7° "(4BP2)⇒(4BP5)" (which completes the proof of the equivalence): Assume (4BP2). Then $\widetilde{\mathbb{G}} := \widetilde{\mathbb{Q}}_2 \mathbb{M}_{\mathrm{u}}^{\curvearrowleft}{}_{11} - \widetilde{\mathbb{Q}}_1 \mathbb{N}_{\mathrm{u}}^{\curvearrowleft}{}_{21} = \widetilde{\mathbb{Q}}_2 (\mathbb{M}_{11} - \mathbb{M}_{\mathrm{u}}^{\curvearrowleft}{}_{12}\mathbb{V}_{21}) - \widetilde{\mathbb{Q}}_1 (\mathbb{N}_{21} - \mathbb{N}_{\mathrm{u}}^{\curvearrowleft}{}_{22}\mathbb{V}_{21}) = \widetilde{\mathbb{Q}}_2 \mathbb{M}_{11} - \widetilde{\mathbb{Q}}_1 \mathbb{N}_{21} - \widetilde{\mathbb{Q}}_2 \mathbb{M}_{12} \mathbb{V}_{22}^{-1} \mathbb{V}_{21} + \widetilde{\mathbb{Q}}_1 \mathbb{N}_{22} \mathbb{V}_{22}^{-1} \mathbb{V}_{21} = I + \widetilde{\mathbb{U}} \mathbb{V}_{22}^{-1} \mathbb{V}_{21} \in \mathcal{G}\mathrm{TIC}(U)$, because $\|\mathbb{V}_{22}^{-1}\mathbb{V}_{21}\| < 1$, by (d) of the theorem.

Define $\widetilde{\mathbb{Q}}_2 := \widetilde{\mathbb{G}}^{-1}\widetilde{\mathbb{Q}}_2$, $\widetilde{\mathbb{Q}}_1 := \widetilde{\mathbb{G}}^{-1}\widetilde{\mathbb{Q}}_1$ to get a new l.c.f. of $\mathbb{Q}$, satisfying $\widetilde{\mathbb{Q}}_2 \mathbb{M}_{\mathrm{u}}^{\curvearrowleft}{}_{11} - \widetilde{\mathbb{Q}}_1 \mathbb{N}_{\mathrm{u}}^{\curvearrowleft}{}_{21} = I$, as required.

From the definitions of $\mathbb{N}$ and $\mathbb{M}$ (given in the lemma), we get $\widetilde{\mathbb{Q}}_1 \mathbb{N}_{\mathrm{u}}^{\curvearrowleft}{}_{22} - \widetilde{\mathbb{Q}}_2 \mathbb{M}_{\mathrm{u}}^{\curvearrowleft}{}_{12} = \widetilde{\mathbb{G}}^{-1}[\widetilde{\mathbb{Q}}_1 \mathbb{N}_{22} - \widetilde{\mathbb{Q}}_2 \mathbb{M}_{12}]\mathbb{V}_{22}^{-1} = (I + \widetilde{\mathbb{U}}\mathbb{V}_{22}^{-1}\mathbb{V}_{21})^{-1}[\widetilde{\mathbb{U}}]\mathbb{V}_{22}^{-1} = \widetilde{\mathbb{U}}(I + \mathbb{V}_{22}^{-1}\mathbb{V}_{21}\widetilde{\mathbb{U}})^{-1}\mathbb{V}_{22}^{-1} = \widetilde{\mathbb{U}}(\mathbb{V}_{22} + \mathbb{V}_{21}\widetilde{\mathbb{U}})^{-1} = \mathbb{U}_o$, if we set $\mathbb{U}_o := \mathbb{U}_1\mathbb{U}_2^{-1}$ and $\begin{bmatrix} \mathbb{U}_1 \\ \mathbb{U}_2 \end{bmatrix} := \underline{\mathbb{V}} \begin{bmatrix} \widetilde{\mathbb{U}} \\ I \end{bmatrix}$. Thus, (4BP3) is satisfied and hence the proof of the equivalence has been completed.

II — (b), (c), (d), (e):

(b) In I, we showed the equivalence in (b). Since $\mathbb{X}$ was arbitrary (and hence $\mathbb{N}$ and $\mathbb{M}$ too, by (12.49), which was established in Lemma 12.4.2), and (4BP1) is independent of $\mathbb{X}$, $\mathbb{N}$ and $\mathbb{M}$, hence so are (4BP2) and (4BP4)–(4BP7).

(c1) It follows directly from the equations in each (4BP$*$) that $\widetilde{\mathbb{Q}}_1, \widetilde{\mathbb{Q}}_2$ are l.c., as claimed.

As one observes from the proof, conditions (4BP2) and (4BP4)–(4BP7) have same solutions $(\widetilde{\mathbb{Q}}_1, \widetilde{\mathbb{Q}}_2)$ modulo the multiplication to the left by an element of $\mathcal{G}\mathrm{TIC}(U)$. By Definition 7.2.11, this means that the maps $\mathbb{Q} := \widetilde{\mathbb{Q}}_2^{-1}\widetilde{\mathbb{Q}}_1$ corresponding to these solutions are equal (but there are usually more than one solution $\mathbb{Q}$); by 2° these maps are exactly the solutions of (4BP1) (i.e., the suboptimal DPF-stabilizing controllers). Thus, (c1) has been established.

(c2) This can be observed from part I.

(c3) 1° *Weaker assumptions:* In fact, it suffices to assume that $\mathbb{N}_{\mathrm{u}}, \mathbb{M}_{\mathrm{u}}, \mathbb{X} \in \mathcal{A}$, that $\mathcal{A} \underset{a}{\subset} \mathrm{TIC}$ (see Definition 6.2.4), and that $\mathcal{A}$ is inverse-closed (i.e., $\mathcal{A} \cap \mathcal{G}\mathrm{TIC} = \mathcal{G}\mathcal{A}$).

The proof of Lemma 8.4.10 shows that if $\mathcal{A}$ is closed under spectral factorization (in the sense that $\widetilde{\mathbb{X}} \in \mathcal{G}\mathcal{A}$ whenever $\widetilde{\mathbb{D}} \in \mathcal{A}(U \times W, *)$, $\widetilde{J} = \widetilde{J}^* \in \mathcal{B}$, $\widetilde{S} \in \mathcal{G}\mathcal{B}$, $\widetilde{\mathbb{X}} \in \mathcal{G}\mathrm{TIC}(U \times W)$ and $\widetilde{\mathbb{X}}^* \widetilde{S} \widetilde{\mathbb{X}} = \widetilde{\mathbb{D}}^* \widetilde{J} \widetilde{\mathbb{D}}$), then $\mathcal{A}$ is inverse-closed.

2° *Suitable $\mathcal{A}$'s:* On the other hand, Lemma 6.4.7(c) shows that $\mathrm{TIC}_{\mathrm{exp}}$ is closed under spectral factorization; obviously, so is $\widetilde{\mathcal{A}}$, so that we can take $\widetilde{\mathcal{A}} := \mathrm{TIC}_{\mathrm{exp}}$ or $\widetilde{\mathcal{A}} := \mathcal{A}$.

3° *The proof using* 1°*:* By the assumptions, we have $\mathbb{V}, \underline{\mathbb{X}}, \underline{\mathbb{V}}, \overline{\mathbb{X}}, \overline{\mathbb{V}} \in \mathcal{G}\mathcal{A}$. Therefore, $\mathbb{N}, \mathbb{M}, \mathbb{M}_{\mathrm{u}}^{\curvearrowleft}, \mathbb{N}_{\mathrm{u}}^{\curvearrowleft} \in \mathcal{A}$. Consequently, it is easy to verify from part I that (c3) holds, since different $\widetilde{\mathbb{Q}}_*$'s are obtained from each other and the above $\mathcal{A}$ maps by using only algebraic operations in TIC (here we again need the inverse-closedness of $\mathcal{A}$).

(d) This was established in 6°.

(e) Obviously, the proof below applies also with the changes listed in (e). □

Lemma 12.4.4 ($\mathbb{D}_+$ & (Factor2)) *The map* $\mathbb{D}_+ := \begin{bmatrix} \mathbb{N}_{21} & \mathbb{N}_{22} \\ I & 0 \end{bmatrix} \begin{bmatrix} \mathbb{M}_{11} & \mathbb{M}_{12} \\ 0 & I \end{bmatrix}^{-1}$ *has a d.c. internal loop: we have the doubly coprime product*

$$\begin{bmatrix} \begin{bmatrix} \mathbb{M}_{11} & \mathbb{M}_{12} \\ 0 & I \end{bmatrix} & \begin{bmatrix} 0 & -I \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} \mathbb{N}_{21} & \mathbb{N}_{22} \\ I & 0 \end{bmatrix} & \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \end{bmatrix}^{-1} = \begin{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} & -\begin{bmatrix} 0 & -I \\ 0 & 0 \end{bmatrix} \\ -\begin{bmatrix} 0 & \mathbb{N}_{22} \\ I & -\mathbb{M}_{12} \end{bmatrix} & \begin{bmatrix} I & -\mathbb{N}_{21} \\ 0 & \mathbb{M}_{11} \end{bmatrix} \end{bmatrix} \tag{12.61}$$

$\in \mathcal{G}\mathrm{TIC}((U \times W) \times (Y \times U), (U \times W) \times (U \times Y))$. *Moreover, Theorem 12.3.7(e1)&(e2) hold; in particular, we have (Factor2)⇔(Factor2Z).*

Proof: We start with the last equivalence, based on the proof of Theorem 3.8 of [Green].

Obviously, (12.61) is true, hence $\mathbb{D}_+$ has a d.c. internal loop (see Definition 7.2.11) and can be written as $\mathbb{D}_+ = \widetilde{\mathbb{X}}_+^{-1}\widetilde{\mathbb{Y}}_+$, where

$$\widetilde{\mathbb{X}}_+ := \begin{bmatrix} I & -\mathbb{N}_{21} \\ 0 & \mathbb{M}_{11} \end{bmatrix} \in \mathrm{TIC}(Y \times U), \quad \widetilde{\mathbb{Y}}_+ := \begin{bmatrix} 0 & \mathbb{N}_{22} \\ I & -\mathbb{M}_{12} \end{bmatrix} \in \mathrm{TIC}(U \times W, Y \times U). \tag{12.62}$$

1° "(Factor2)⇒(Factor2Z)": Assume (Factor2). Then $\widetilde{\mathbb{M}}_+ = \mathbb{Z}^{-1}\widetilde{\mathbb{X}}_+$, $\widetilde{\mathbb{N}}_+ = \mathbb{Z}^{-1}\widetilde{\mathbb{Y}}_+$ for some $\mathbb{Z} \in \mathcal{G}\mathrm{TIC}(Y \times U)$, by Definition 7.2.11. Consequently,

$$\mathbb{Z}^{-1} = \begin{bmatrix} \widetilde{\mathbb{M}}_{+11} & \widetilde{\mathbb{N}}_{+11} \\ \widetilde{\mathbb{M}}_{+21} & \widetilde{\mathbb{N}}_{+21} \end{bmatrix} \text{ and } \mathbb{Z}^{-1}\mathbb{E} = \mathbb{Z}^{-1}\begin{bmatrix} \widetilde{\mathbb{Y}}_{+*2} & \widetilde{\mathbb{X}}_{+*2} \end{bmatrix} = \begin{bmatrix} \widetilde{\mathbb{N}}_{+*2} & \widetilde{\mathbb{M}}_{+*2} \end{bmatrix} = \mathbb{W}. \tag{12.63}$$

Therefore, (Factor2Z) holds.

2° "(Factor2Z)⇒(Factor2)": Assume (Factor2Z). Define $\widetilde{\mathbb{X}}_+, \widetilde{\mathbb{Y}}_+$ as above and set $\widetilde{\mathbb{M}}_+ = \mathbb{Z}^{-1}\widetilde{\mathbb{X}}_+$, $\widetilde{\mathbb{N}}_+ = \mathbb{Z}^{-1}\widetilde{\mathbb{Y}}_+$ to obtain $\widetilde{\mathbb{M}}_+^{-1}\widetilde{\mathbb{N}}_+ = \widetilde{\mathbb{X}}_+^{-1}\widetilde{\mathbb{Y}}_+$. Set $\mathbb{W} := \mathbb{Z}^{-1}\mathbb{E}$ to obtain that $\mathbb{W}J_1\mathbb{W}^* = J_1$, and that (12.63) holds. Then (Factor2) holds.

3° *Theorem 12.3.7(e1)&(e2):* Part II of the proof of Lemma 12.4.3 contains partial proofs of Theorem 12.3.7(e1)&(e2). The rest of (e1) and (e2) is obtained above. □

The factorization $\mathbb{D}_+ := \begin{bmatrix} \mathbb{N}_{21} & \mathbb{N}_{22} \\ I & 0 \end{bmatrix} \begin{bmatrix} \mathbb{M}_{11} & \mathbb{M}_{12} \\ 0 & I \end{bmatrix}^{-1}$ is a r.c.f. when (Factor1$\widetilde{\mathcal{A}}$) holds:

Remark 12.4.5 ($\mathbb{D}_+$) *If (Factor1) holds, then* $\begin{bmatrix} \mathbb{N}_{21} & \mathbb{N}_{22} \\ I & 0 \end{bmatrix} \in \mathrm{TIC}(U \times W, Y \times U)$ *and* $\begin{bmatrix} \mathbb{M}_{11} & \mathbb{M}_{12} \\ 0 & I \end{bmatrix} \in \mathrm{TIC}(U \times W)$ *are r.c. (even d.c.), by (12.61).* □

Lemma 12.4.6 *Lemmas 12.3.11 and 12.3.8 hold.*

Proof: 1.1° *Lemma 12.3.11(a)–(c):* Part II of the proof of Lemma 12.4.3 contains the proof of Lemma 12.3.11(a) and a partial proof of (b). The "(Factor2)" part of (b) is obtained as its (Factor1) part (note that the first columns of $\widetilde{\mathbb{M}}_+$ and $\widetilde{\mathbb{N}}_+$ are contained in $\mathbb{Z}^{-1}$, hence these depend on $\mathbb{D}$ only, where as their second columns are part of $\mathbb{W}$ and hence depend on $\mathbb{N}_{\mathrm{u}}, \mathbb{M}_{\mathrm{u}}$ in the same way as $\mathbb{E}$ does).

Part (c) was established in Lemma 12.4.3.

Part (d): 1.2° By (Factor1X), the assumptions of Proposition 11.3.4 are satisfied and (FI3s) holds. By Lemma 11.3.11, the solution of (FI3s) for $\begin{bmatrix} \mathbb{N}_{u11} & \mathbb{N}_{u12} \\ 0 & I \end{bmatrix}$ (hence of (Factor1X)) can be chosen so that X is as in (d) (since $X \in \mathcal{GB}(U \times W)$, by Proposition 6.3.1(b1)).

1.3° Assume that $\mathbb{X}$ is as above and $\mathbb{N}_u, \mathbb{M}_u \in \mathrm{UR}$. Then $\mathbb{D} = \mathbb{N}_u \mathbb{M}_u{}^{-1} \in \mathrm{UR}$ and $\mathbb{N}, \mathbb{M}, \mathbb{E} \in \mathrm{UR}$, by (12.49) and Proposition 6.3.1(b1). Consequently, $X^{-1} = \begin{bmatrix} X_{11}^{-1} & * \\ 0 & X_{22}^{-1} \end{bmatrix}$, $M = \begin{bmatrix} * & * \\ 0 & X_{22}^{-1} \end{bmatrix}$; by Lemma A.1.1(b), $M_{11} \in \mathcal{GB}(U)$. If $D_{21} = 0$, then $N = DM = \begin{bmatrix} * & * \\ 0 & D_{22}X_{22}^{-1} \end{bmatrix}$, hence then $E = \begin{bmatrix} N_{22} & -N_{21} \\ -M_{12} & M_{11} \end{bmatrix} = \begin{bmatrix} N_{22} & 0 \\ -M_{12} & M_{11} \end{bmatrix}$.

1.4° *Part (e):* (Note that $\mathbb{D}, \mathbb{N}, \mathbb{M}, \mathbb{E} \in \mathrm{UR}$, by (d). Thus, any UR solution $\widetilde{\mathbb{M}}_+, \widetilde{\mathbb{N}}_+$ of (Factor2) corresponds to a UR solution $\mathbb{Z}$ of (Factor2Z) and vice versa, by Theorem 12.3.7(e2).)

By Theorem 12.3.7(e2), we have $(Z^{-1})_{22} M_{11} = \widetilde{M}_{+22} = W_{22}$ and $\mathbb{W} \in \mathrm{UR}$. By (Factor2) and Proposition 6.3.1(b1), $W_{22} \in \mathcal{GB}(U)$, hence $(Z^{-1})_{22} \in \mathcal{GB}(U)$, hence $Z_{11} \in \mathcal{GB}(Y)$, by Lemma A.1.1(c1). But $\widetilde{N}_{+21} = (Z^{-1})_{22}$, hence (e) holds.

2° *Lemma 12.3.8:* This is given in the proof of Lemma 12.4.3 for $\mathbb{N}_u, \mathbb{M}_u, \widetilde{\mathbb{N}_y}, \widetilde{\mathbb{M}_y}$. For (4BP1) and (Factor1[X]) independence on $\mathbb{N}, \mathbb{M}, \mathbb{X}, \widetilde{\mathbb{N}}_+, \widetilde{\mathbb{M}}_+, \mathbb{Z}$ is obvious; for (4BP2) it follows from Lemma 12.4.3(b).

Fix a pair $(\mathbb{N}, \mathbb{M})$ and a corresponding $\mathbb{E}$. Let $\mathbb{E}'$ correspond to some E given in Lemma 12.3.11(b). By Lemma 6.4.8(c), $EJ_1E^* = J_1$. By setting $R := \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$ we get $R^*\mathbb{E}'R = \begin{bmatrix} \mathbb{M}_{1*} \\ \mathbb{N}_{2*} \end{bmatrix} E$ and $R^*J_1R = -J_1$, hence

$$\mathbb{E}'J_1\mathbb{E}'^* = -\begin{bmatrix} \mathbb{N}_{2*} \\ -\mathbb{M}_{1*} \end{bmatrix} J_1 \begin{bmatrix} \mathbb{N}_{2*} \\ -\mathbb{M}_{1*} \end{bmatrix}^*, \tag{12.64}$$

which is independent of E, i.e., depends on $\mathbb{D}$ only, hence so do the solutions $\mathbb{Z}$ of (Factor2Z[']). From the equivalences given in Theorem 12.3.7(e1), we obtain that also the solvability of (Factor2[']) depends on $\mathbb{D}$ only (but $\mathbb{D}_+$ depends on $\mathbb{N}, \mathbb{M}$, hence so do $\widetilde{\mathbb{N}}_+, \widetilde{\mathbb{M}}_+$). The claims concerning $\mathbb{Q}$ follow from Lemma 12.4.3(c). □

Now we can show that (4BP3) implies (4BP1)–(4BP7):

Lemma 12.4.7 ((Factor)⇒(4BP)) *Assume (Factor1) and (Factor2). Then (4BP1)–(4BP7) hold.*

Proof: We use the notation of Theorem 12.3.7(e1)–(e2) (see Lemma 12.4.4). Set $\begin{bmatrix} \widetilde{\mathbb{Q}}_1 & \widetilde{\mathbb{Q}}_2 \end{bmatrix} := \begin{bmatrix} 0 & I \end{bmatrix} \mathbb{Z}^{-1}$ to obtain

$$\begin{bmatrix} \widetilde{\mathbb{U}}_1 & \widetilde{\mathbb{U}}_2 \end{bmatrix} := \begin{bmatrix} \widetilde{\mathbb{Q}}_1 & \widetilde{\mathbb{Q}}_2 \end{bmatrix} \mathbb{E} = \begin{bmatrix} 0 & I \end{bmatrix} \mathbb{W} = \begin{bmatrix} \mathbb{W}_{21} & \mathbb{W}_{22} \end{bmatrix}. \tag{12.65}$$

From Lemma 12.3.11(a) we obtain that $\|\mathbb{W}_{22}^{-1}\mathbb{W}_{21}\| < 1$, hence (4BP7) is satisfied. Thus, (4BP1)–(4BP7) hold, by Lemma 12.4.3(b). □

We shall use the following to reduce (Factor2Z) to a (frequency-domain) H$^\infty$ FICP:

Lemma 12.4.8 ($\exists\mathbb{P}$) *Assume (4BP2). Then there is $\mathbb{P} \in \mathcal{G}\mathrm{TIC}(Y \times U)$ s.t. $\mathbb{P}\mathbb{E} = \begin{bmatrix} \mathbb{S} & 0 \\ \widetilde{\mathbb{U}} & I \end{bmatrix} \in \mathrm{TIC}(W \times U, Y \times U)$, where $\mathbb{S}\mathbb{S}^* \gg 0$, $\|\widetilde{\mathbb{U}}\| < 1$.*

As the proof shows, $\widetilde{\mathbb{U}}$ is the one appearing in (4BP2).

Proof: Assume (4BP2). By Lemma 12.4.3(c1)&(b), $\widetilde{\mathbb{Q}}_2^{-1}\widetilde{\mathbb{Q}}_1$ (from (4BP2)) stabilizes $\mathbb{D}$ (because it satisfies (4BP1)), and conditions (4BP1)–(4BP7) except possibly (4BP3) hold. In particular, $\begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} \begin{bmatrix} \widetilde{\mathbb{M}}_{\mathrm{y}\,22} & \widetilde{\mathbb{N}}_{\mathrm{y}\,21} \\ \widetilde{\mathbb{Q}}_1 & \widetilde{\mathbb{Q}}_2 \end{bmatrix} \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} \in \mathcal{G}\mathrm{TIC}$, by Theorem 7.3.19(iv'), hence

$$\begin{bmatrix} \widetilde{\mathbb{M}}_{\mathrm{y}\,22} & \widetilde{\mathbb{N}}_{\mathrm{y}\,21} \\ \widetilde{\mathbb{Q}}_1 & \widetilde{\mathbb{Q}}_2 \end{bmatrix} \in \mathcal{G}\mathrm{TIC}(Y \times U). \tag{12.66}$$

From the $(2,2)$- and $(2,1)$-blocks of equation $\widetilde{\mathbb{M}}_{\mathrm{y}}\overset{\frown}{\mathbb{N}}_{\mathrm{u}} = \widetilde{\mathbb{N}}_{\mathrm{y}}\overset{\frown}{\mathbb{M}}_{\mathrm{u}}$ we get

$$\widetilde{\mathbb{M}}_{\mathrm{y}\,22}\overset{\frown}{\mathbb{N}}_{\mathrm{u}\,22} - \widetilde{\mathbb{N}}_{\mathrm{y}\,21}\overset{\frown}{\mathbb{M}}_{\mathrm{u}\,12} = \widetilde{\mathbb{N}}_{\mathrm{y}\,22}, \ \ \widetilde{\mathbb{M}}_{\mathrm{y}\,22}\overset{\frown}{\mathbb{N}}_{\mathrm{u}\,21} - \widetilde{\mathbb{N}}_{\mathrm{y}\,21}\overset{\frown}{\mathbb{M}}_{\mathrm{u}\,11} = \widetilde{\mathbb{N}}_{\mathrm{y}\,22}\overset{\frown}{\mathbb{M}}_{\mathrm{u}\,21} = 0. \tag{12.67}$$

On the other hand, $\mathbb{E} = \begin{bmatrix} \mathbb{N}_{22} & -\mathbb{N}_{21} \\ -\mathbb{M}_{12} & \mathbb{M}_{11} \end{bmatrix} = \begin{bmatrix} \overset{\frown}{\mathbb{N}}_{\mathrm{u}\,22} & -\overset{\frown}{\mathbb{N}}_{\mathrm{u}\,21} \\ -\overset{\frown}{\mathbb{M}}_{\mathrm{u}\,12} & \overset{\frown}{\mathbb{M}}_{\mathrm{u}\,11} \end{bmatrix} \begin{bmatrix} \mathbb{V}_{22} & -\mathbb{V}_{21} \\ 0 & I \end{bmatrix}$ (here we have set $\mathbb{X} := \mathbb{M}^{-1}\mathbb{M}_{\mathrm{u}}$, $\mathbb{V} := \mathbb{X}^{-1}$). By combining this and (12.67) we get

$$\begin{bmatrix} \widetilde{\mathbb{M}}_{\mathrm{y}\,22} & \widetilde{\mathbb{N}}_{\mathrm{y}\,21} \end{bmatrix} \mathbb{E} = \begin{bmatrix} \widetilde{\mathbb{N}}_{\mathrm{y}\,22} & 0 \end{bmatrix} \begin{bmatrix} \mathbb{V}_{22} & -\mathbb{V}_{21} \\ 0 & I \end{bmatrix} = \begin{bmatrix} \widetilde{\mathbb{N}}_{\mathrm{y}\,22}\mathbb{V}_{22} & -\widetilde{\mathbb{N}}_{\mathrm{y}\,22}\mathbb{V}_{21} \end{bmatrix}. \tag{12.68}$$

This combined with (4BP2) gives us $\begin{bmatrix} \widetilde{\mathbb{M}}_{\mathrm{y}\,22} & \widetilde{\mathbb{N}}_{\mathrm{y}\,21} \\ \widetilde{\mathbb{Q}}_1 & \widetilde{\mathbb{Q}}_2 \end{bmatrix} \mathbb{E} = \begin{bmatrix} \widetilde{\mathbb{N}}_{\mathrm{y}\,22}\mathbb{V}_{22} & -\widetilde{\mathbb{N}}_{\mathrm{y}\,22}\mathbb{V}_{21} \\ \widetilde{\mathbb{U}} & I \end{bmatrix}$, where $\|\widetilde{\mathbb{U}}\| < 1$. Thus, $\mathbb{P}\mathbb{E} = \begin{bmatrix} \mathbb{S} & 0 \\ \widetilde{\mathbb{U}} & I \end{bmatrix}$, where

$$\mathbb{P} := \begin{bmatrix} I & \widetilde{\mathbb{N}}_{\mathrm{y}\,22}\mathbb{V}_{21} \\ 0 & I \end{bmatrix} \begin{bmatrix} \widetilde{\mathbb{M}}_{\mathrm{y}\,22} & \widetilde{\mathbb{N}}_{\mathrm{y}\,21} \\ \widetilde{\mathbb{Q}}_1 & \widetilde{\mathbb{Q}}_2 \end{bmatrix} \in \mathrm{TIC}(Y \times U), \tag{12.69}$$

and $\mathbb{S} := \widetilde{\mathbb{N}}_{\mathrm{y}\,22}(\mathbb{V}_{22} + \mathbb{V}_{21}\widetilde{\mathbb{U}})$ is onto ($\mathbb{S}\mathbb{S}^* \gg 0$), because $\widetilde{\mathbb{N}}_{\mathrm{y}\,22}$ is onto, by Standing Hypothesis 12.3.1, and $\mathbb{V}_{22} + \mathbb{V}_{21}\widetilde{\mathbb{U}} \in \mathcal{G}\mathrm{TIC}(W)$ (because $\|\mathbb{V}_{22}^{-1}\mathbb{V}_{21}\widetilde{\mathbb{U}}\| < 1$, by Lemma 12.3.11(a)). By (12.66) and Lemma A.1.1(b1), $\mathbb{P} \in \mathcal{G}\mathrm{TIC}(Y \times U)$. $\square$

Lemma 12.4.9 *Theorem 12.3.7(c) holds.*

Proof: 1° *(12.47):* Assume (Factor1) and (Factor2), so that (4BP2), (Factor1X) and (Factor2Z) hold, by Lemma 12.4.7 and Theorem 12.3.7(e1)&(e2) (see Lemma 12.4.4). We set $\mathbb{Z}^{-\mathrm{d}} := (\mathbb{Z}^{\mathrm{d}})^{-1}$ etc.

Set $\widetilde{\mathbb{Z}} := \mathbb{P}\mathbb{Z}$ to obtain $\widetilde{\mathbb{Z}}J_1\widetilde{\mathbb{Z}}^* = (\mathbb{P}\mathbb{E})J_1(\mathbb{P}\mathbb{E})^*$, where $\mathbb{P}$ is as in Lemma 12.4.8. Now equations $\mathbb{W} = \mathbb{Z}^{-1}\mathbb{E} = \widetilde{\mathbb{Z}}^{-1}\mathbb{P}\mathbb{E}$ and $(\mathbb{P}\mathbb{E})^{\mathrm{d}} = \begin{bmatrix} * & * \\ 0 & I \end{bmatrix}$ imply that $\mathbb{W}^{\mathrm{d}} = \begin{bmatrix} * & * \\ (\widetilde{\mathbb{Z}}^{-\mathrm{d}})_{21} & (\widetilde{\mathbb{Z}}^{-\mathrm{d}})_{22} \end{bmatrix}$, hence $(\widetilde{\mathbb{Z}}^{-\mathrm{d}})_{22} = \mathbb{W}^{\mathrm{d}}_{22} \in \mathcal{G}\mathrm{TIC}(U)$, by (Factor2Z), hence $\widetilde{\mathbb{Z}}^{\mathrm{d}}_{11} \in \mathcal{G}\mathrm{TIC}(Y)$, by Lemma A.1.1(c1).

Now each pair $\widetilde{\mathbb{Q}}_1,\widetilde{\mathbb{Q}}_2 \in \mathrm{TIC}$ corresponds to a unique pair $\overline{\mathbb{Q}}_1,\overline{\mathbb{Q}}_2 \in \mathrm{TIC}$ defined by $\begin{bmatrix}\overline{\mathbb{Q}}_1^{\mathrm{d}}\\ \overline{\mathbb{Q}}_2^{\mathrm{d}}\end{bmatrix} := \mathbb{P}^{-\mathrm{d}}\begin{bmatrix}\widetilde{\mathbb{Q}}_1^{\mathrm{d}}\\ \widetilde{\mathbb{Q}}_2^{\mathrm{d}}\end{bmatrix}$. The pairs $(\widetilde{\mathbb{Q}}_1,\widetilde{\mathbb{Q}}_2)$ satisfying (4BP2) correspond to pairs $\overline{\mathbb{Q}}_1,\overline{\mathbb{Q}}_2$ satisfying

$$\begin{bmatrix}\widetilde{\mathbb{U}}^{\mathrm{d}}\\ I\end{bmatrix} = \mathbb{E}^{\mathrm{d}}\begin{bmatrix}\widetilde{\mathbb{Q}}_1^{\mathrm{d}}\\ \widetilde{\mathbb{Q}}_2^{\mathrm{d}}\end{bmatrix} = \mathbb{E}^{\mathrm{d}}\mathbb{P}^{\mathrm{d}}\begin{bmatrix}\overline{\mathbb{Q}}_1^{\mathrm{d}}\\ \overline{\mathbb{Q}}_2^{\mathrm{d}}\end{bmatrix}, \tag{12.70}$$

for some $\widetilde{\mathbb{U}}$ with $\|\widetilde{\mathbb{U}}\| < 1$ $[\leq 1$ for $\mathbb{Q}$'s s.t. $\|\mathcal{F}_\ell(\mathbb{D},\mathbb{Q})\| \leq \gamma$; see Lemma 12.4.3(e)].

From (12.70) and $\widetilde{\mathbb{D}} := \mathbb{E}^{\mathrm{d}}\mathbb{P}^{\mathrm{d}} = \left[\begin{smallmatrix}* & *\\ 0 & I\end{smallmatrix}\right]$ we obtain that $\overline{\mathbb{Q}}_2 = I$, hence $\overline{\mathbb{Q}}_1^{\mathrm{d}}$ is as above iff it satisfies (the FICP) $\|\widetilde{\mathbb{U}}^{\mathrm{d}}\| := \|\widetilde{\mathbb{D}}_{11}\overline{\mathbb{Q}}_1^{\mathrm{d}} + \widetilde{\mathbb{D}}_{12}\| < 1$ $[\leq 1]$.

Thus, we may apply Corollary 11.3.5 (with substitutions $\mathbb{D} \mapsto \widetilde{\mathbb{D}}$, $\gamma \mapsto 1$, $\mathbb{X} \mapsto \widetilde{\mathbb{Z}}^{\mathrm{d}}$; note that $\widetilde{\mathbb{D}}_{11}^*\widetilde{\mathbb{D}}_{11} = (\mathbb{S}^{\mathrm{d}})^*\mathbb{S}^{\mathrm{d}} = (\mathbb{S}\mathbb{S}^*)^{\mathrm{d}} \gg 0$, since $\mathbb{S}$ (and hence $(\mathbb{S}^{\mathrm{d}})^*$ is onto)) to obtain that all $\overline{\mathbb{Q}}_1^{\mathrm{d}}$'s of this form are given by $\overline{\mathbb{Q}}_1^{\mathrm{d}} = \mathbb{L}_1\mathbb{L}_2^{-1}$, where $\begin{bmatrix}\mathbb{L}_1\\ \mathbb{L}_2\end{bmatrix} := \widetilde{\mathbb{Z}}^{-1}\left[\begin{smallmatrix}\mathbb{L}\\ I\end{smallmatrix}\right]$ and $\|\mathbb{L}\| < 1$ $[\leq 1]$ (hence $\mathbb{L}_2 \in \mathcal{G}\mathrm{TIC}(U)$, by Theorem 11.3.6). Writing it out, we have

$$\begin{bmatrix}\overline{\mathbb{Q}}_1^{\mathrm{d}}\\ I\end{bmatrix} = \begin{bmatrix}\mathbb{L}_1\\ \mathbb{L}_2\end{bmatrix}\mathbb{L}_2^{-1} = \widetilde{\mathbb{Z}}^{-\mathrm{d}}\begin{bmatrix}\mathbb{L}\\ I\end{bmatrix}\mathbb{L}_2^{-1}. \tag{12.71}$$

Thus, $\begin{bmatrix}\widetilde{\mathbb{Q}}_1^{\mathrm{d}}\\ \widetilde{\mathbb{Q}}_2^{\mathrm{d}}\end{bmatrix} = \mathbb{P}^{-\mathrm{d}}\widetilde{\mathbb{Z}}^{-\mathrm{d}}\left[\begin{smallmatrix}\mathbb{L}\\ I\end{smallmatrix}\right]\mathbb{L}_2^{-1} = \mathbb{Z}^{-\mathrm{d}}\left[\begin{smallmatrix}\mathbb{L}\\ I\end{smallmatrix}\right]\mathbb{L}_2^{-1}$.

By postmultiplying this by $\mathbb{L}_2$, we get another representative of $\widetilde{\mathbb{Q}}_2^{-1}\widetilde{\mathbb{Q}}_1$ (since $\mathbb{L}_2 \in \mathcal{G}\mathrm{TIC}(U)$), given by

$$\begin{bmatrix}\widetilde{\mathbb{Q}}_1' & \widetilde{\mathbb{Q}}_2'\end{bmatrix} = \mathbb{L}_2^{\mathrm{d}}\begin{bmatrix}\widetilde{\mathbb{Q}}_1 & \widetilde{\mathbb{Q}}_2\end{bmatrix} = \begin{bmatrix}\mathbb{L}^{\mathrm{d}} & I\end{bmatrix}\mathbb{Z}^{-1}, \tag{12.72}$$

But this "all solutions formula" is equal to (12.47) (see (12.50) for $\mathbb{Z}^{-1}$).

2° *Formula* $\mathbb{Q} = \mathcal{F}_\ell(\mathbb{T},\mathbb{L})$*:* The formula $\mathbb{Q} = \mathcal{F}_\ell(\mathbb{T},\mathbb{L})$ can be shown equal to (12.47) just by writing the two formulae out and simplifying slightly (note that $I - \mathbb{T}_{21}\mathbb{L} = I + \widetilde{\mathbb{N}}_{+11}\widetilde{\mathbb{N}}_{+21}^{-1}\mathbb{L}$ is invertible iff $\widetilde{\mathbb{Q}}_2 = \mathbb{L}\widetilde{\mathbb{N}}_{+11} + \widetilde{\mathbb{N}}_{+21} = (I + \mathbb{L}\widetilde{\mathbb{N}}_{+11}\widetilde{\mathbb{N}}_{+21}^{-1})\widetilde{\mathbb{N}}_{+21}$ is invertible, by Lemma A.1.1(f6), i.e., iff $\mathbb{Q}$ is well-posed (by Lemma 7.2.12(b))).

3° *Remark:* We needed $\mathbb{P}$ only to get a spectral factor (namely $\widetilde{\mathbb{Z}}^{\mathrm{d}}$) with invertible $(1,1)$-block and to establish the condition "$\mathbb{D}_1^*J\mathbb{D}_1 \gg 0$". Otherwise we could have applied Theorem 11.3.6 directly for $\mathbb{E}^{\mathrm{d}}$ instead of $\widetilde{\mathbb{D}}$.) □

If the coprimeness requirements of Standing Hypothesis 12.3.1 are satisfied "exponentially", then the existence of a solution the 4BP is equivalent to the existence of an exponentially stabilizing solution of the 4BP (at least when $\mathbb{N}_{\mathrm{u}},\mathbb{M}_{\mathrm{u}} \in \widetilde{\mathcal{A}}$, so that (4BP3) becomes necessary):

Proposition 12.4.10 (Exponentially stabilizing solutions of the (I/O) 4BP) *Assume that we can write "d.c.f. over* $\mathrm{TIC}_{\exp}$*", "r.c. over* $\mathrm{TIC}_{\exp}$*" and "l.c. over* $\mathrm{TIC}_{\exp}$*" to Standing Hypothesis 12.3.1 in place of "d.c.f.", "r.c." and "l.c.",* *respectively (this is the case if Hypothesis 12.5.1 holds and Σ is optimizable).*

If (4BP3) holds, then $\mathbb{N},\mathbb{M},\mathbb{X},\mathbb{Z},\mathbb{E} \in \mathrm{TIC}_{\mathrm{exp}}$, and all exponentially DPF-stabilizing suboptimal controllers (with internal loop) for $\mathbb{D}$ are parametrized by (12.47) with the additional requirement that $\mathbb{L} \in \mathrm{TIC}_{\mathrm{exp}}$.

Proof: One observes from the proof of Lemma 12.5.3 that the assumptions of the lemma are satisfied if Hypothesis 12.5.1 holds and Σ is optimizable.

1° $\mathbb{Q}$ *is suboptimal and exponentially DPF-stabilizing iff* $\mathbb{Q} = \widetilde{\mathbb{Q}}_2^{-1}\widetilde{\mathbb{Q}}_1$ *for some* $\widetilde{\mathbb{Q}}_1,\widetilde{\mathbb{Q}}_2 \in \mathrm{TIC}_{\mathrm{exp}}$ *satisfying (4BP4):*

From Corollary 7.3.20(i') and Remark 7.3.24 we observe that $\mathbb{Q}$ DPF-stabilizes $\mathbb{D}$ exponentially with internal loop iff $\mathbb{Q} = \widetilde{\mathbb{Q}}_2^{-1}\widetilde{\mathbb{Q}}_1$ and $\widetilde{\mathbb{Q}}_2\mathbb{M}_{\mathrm{u}\,11} - \widetilde{\mathbb{Q}}_1\mathbb{N}_{\mathrm{u}\,21} = I$ for some $\widetilde{\mathbb{Q}}_1,\widetilde{\mathbb{Q}}_2 \in \mathrm{TIC}_{\mathrm{exp}}$ (cf. Lemma 12.3.2).

2° $\mathbb{X}^{\pm 1},\mathbb{N},\mathbb{M},\mathbb{Z}^{\pm 1},\mathbb{E} \in \mathrm{TIC}_{\mathrm{exp}}$*:* As in the proof of Lemma 12.4.3(c3), we observe that $\mathbb{X}^{\pm 1},\mathbb{N},\mathbb{M} \in \mathrm{TIC}_{\mathrm{exp}}$, hence $\mathbb{E} \in \mathrm{TIC}_{\mathrm{exp}}$. By Lemma 6.4.7(c), it follows that $\mathbb{Z} \in \mathcal{G}\mathrm{TIC}_{\mathrm{exp}}$. (by Lemma 12.3.11(b), this applies to all possible choices of $\mathbb{X},\mathbb{N},\mathbb{M},\mathbb{Z},\mathbb{E}$).

3° *Sufficiency:* Therefore, for each $\mathbb{L} \in \mathrm{TIC}_{\mathrm{exp}}$, we have $\widetilde{\mathbb{Q}}_1,\widetilde{\mathbb{Q}}_2 \in \mathrm{TIC}_{\mathrm{exp}}$ in (12.47) (see (12.50) for $\mathbb{Z}^{-1}$). By Lemma 12.4.3(c3), $\mathbb{Q} := \widetilde{\mathbb{Q}}_2^{-1}\widetilde{\mathbb{Q}}_1$ corresponds to a solution of (4BP4) belonging to $\mathrm{TIC}_{\mathrm{exp}}$, hence $\mathbb{Q}$ is as in 1°.

4° *Necessity:* If $\mathbb{Q} = \underline{\mathbb{Q}}_2^{-1}\underline{\mathbb{Q}}_1$ is a suboptimal exponentially DPF-stabilizing controller (with an internal loop), then

$$\begin{bmatrix}\underline{\mathbb{Q}}_1 & \underline{\mathbb{Q}}_2\end{bmatrix} = \mathbb{U}\begin{bmatrix}\widetilde{\mathbb{Q}}_1 & \widetilde{\mathbb{Q}}_2\end{bmatrix} := \begin{bmatrix}\mathbb{U}\mathbb{L} & \mathbb{U}\end{bmatrix}\mathbb{Z}^{-1} \tag{12.73}$$

for some $\mathbb{U} \in \mathcal{G}\mathrm{TIC}$, by Lemma 6.4.5(d) and (12.47), hence then $\begin{bmatrix}\mathbb{U}\mathbb{L} & \mathbb{U}\end{bmatrix} = \mathbb{Z}\begin{bmatrix}\underline{\mathbb{Q}}_1 & \underline{\mathbb{Q}}_2\end{bmatrix} \in \mathrm{TIC}_{\mathrm{exp}}$, hence $\mathbb{U} \in \mathrm{TIC}_{\mathrm{exp}}$ (recall from 2° that $\mathbb{Z} \in \mathcal{G}\mathrm{TIC}_{\mathrm{exp}}$), hence $\mathbb{U} \in \mathcal{G}\mathrm{TIC}_{\mathrm{exp}}$, by Lemma 2.2.7. Consequently, then $\mathbb{L} = \mathbb{U}^{-1}\mathbb{U}\mathbb{L} \in \mathrm{TIC}_{\mathrm{exp}}$. □

Next we want to make $\mathbb{M}_{11}$ and $\mathbb{Z}_{11}$ invertible; therefore we need the following result:

Lemma 12.4.11 (X_{22} invertible) *Assume that $X \in \mathcal{GB}(U \times W)$, and that $\begin{bmatrix}T & I\end{bmatrix} X \begin{bmatrix}0\\ I\end{bmatrix} \in \mathcal{GB}(W)$ (resp. $X_{11} \in \mathcal{GB}$ and $\begin{bmatrix}T & I\end{bmatrix} X \begin{bmatrix}I\\ 0\end{bmatrix} = 0$) for some $T \in \mathcal{B}(U,W)$ with $\|T\| < 1$.*

Then there is $E \in \mathcal{GB}(U \times W)$ s.t. $E^*J_1E = J_1$, and $\widetilde{X} := EX$ satisfies $\widetilde{X}_{22} \in \mathcal{GB}(W)$ (resp. $\widetilde{X}_{21} = 0$ and $\widetilde{X}_{11},\widetilde{X}_{22} \in \mathcal{GB}$).

Thus, $\widetilde{X}^*J_1\widetilde{X} = X^*J_1X$.

Proof: We use below Lemma A.3.1(d)&(e2). Set $U := (I - T^*T)^{-1/2} \gg 0$, $V := (I - TT^*)^{-1/2} \gg 0$. By Lemma A.1.1(f6)&(d1), $V^2T = TU^2$ and the inverse of

$$E := \begin{bmatrix}U & 0\\ 0 & V\end{bmatrix}\begin{bmatrix}I & T^*\\ T & I\end{bmatrix} = \begin{bmatrix}U & UT^*\\ VT & V\end{bmatrix} \quad \text{is} \quad E^{-1} = \begin{bmatrix}U & -T^*V\\ -TU & V\end{bmatrix}. \tag{12.74}$$

By a simple computation one verifies that $E^*J_1E = J_1$.

If $\begin{bmatrix}T & I\end{bmatrix} X \begin{bmatrix}0\\ I\end{bmatrix} \in \mathcal{GB}(W)$, then $\widetilde{X}_{22} = V\begin{bmatrix}T & I\end{bmatrix} X \begin{bmatrix}0\\ I\end{bmatrix} \in \mathcal{GB}(W)$. If, instead, $\begin{bmatrix}T & I\end{bmatrix} X \begin{bmatrix}I\\ 0\end{bmatrix} = 0$ and $X_{11} \in \mathcal{GB}$, then $\widetilde{X}_{21} = V\begin{bmatrix}T & I\end{bmatrix} X \begin{bmatrix}I\\ 0\end{bmatrix} = 0$ and hence

$X_{11} = U\widetilde{X}_{11}$; therefore, $\widetilde{X}_{11} = U^{-1}X_{11} \in \mathcal{GB}$, and, consequently, $\widetilde{X}_{22} \in \mathcal{GB}$, by Lemma A.1.1(b2). $\square$

Lemma 12.4.12 (Well-posed $\mathbb{D}_+$) *If (Factor1) holds and* $\mathbb{X}, \mathbb{M}_{\rm u} \in$ ULR, *then we can take* $\mathbb{X}_{22}, \mathbb{M}_{11} \in \mathcal{G}TIC_\infty$ *(and* $X = \begin{bmatrix} X_{11} & X_{12} \\ 0 & X_{22} \end{bmatrix}$, $M = \begin{bmatrix} M_{11} & M_{12} \\ 0 & X_{22}^{-1} \end{bmatrix}$ *where* $X_{11}, X_{22}, M_{11} \in \mathcal{GB}$*); in particular,* $\mathbb{D}_+$ *becomes well-posed.*

Proof: (We use here Theorem 12.3.7(e1)&(e2) and Lemma 12.3.11(a)&(c); these have already been proved, in Lemmas 12.4.2 and 12.4.4.)

Let $\mathbb{X}$ satisfy (Factor1X). Now $X := \mathbb{X}(+\infty) \in \mathcal{GB}$ and $X_{11} \in \mathcal{GB}$, by Proposition 6.3.1(c). Clearly $\|X_{21}X_{11}^{-1}\| \le \|\mathbb{X}_{21}\mathbb{X}_{11}^{-1}\| < 1$ (see Lemma 12.3.11(a)), hence with $T := -X_{21}X_{11}^{-1}$ we obtain from Lemma 12.4.11 an operator $E \in \mathcal{GB}$ s.t. $\widetilde{\mathbb{X}} := E\mathbb{X}$ satisfies $\widetilde{\mathbb{X}}^* J_1 \widetilde{\mathbb{X}} = \mathbb{X}^* J_1 \mathbb{X}$ and $\widetilde{X}_{11}, \widetilde{X}_{22} \in \mathcal{GB}$, $\widetilde{X}_{21} = 0$; hence $\widetilde{\mathbb{X}}_{22} \in \mathcal{G}TIC_\infty(W)$, by Proposition 6.3.1(c).

By Lemma 12.3.11(c), also $\widetilde{\mathbb{X}}$ satisfies (Factor1X). The claims for $\mathbb{M} = \mathbb{M}_{\rm u}\widetilde{\mathbb{X}}^{-1}$ follow from the above (the invertibility of M_{11} and hence that of $\mathbb{M}_{11}$ follows from Lemma A.1.1(b2)&(b1)). $\square$

Lemma 12.4.13 (Well-posed $\mathbb{Q}$) *Assume that (Factor1) and (Factor2Z) hold with* $\mathbb{Z} \in \widetilde{\mathcal{A}}$*, and let some well-posed* $\mathbb{Q} = \widetilde{\mathbb{Q}}_2^{-1}\widetilde{\mathbb{Q}}_1$ *with* $\widetilde{\mathbb{Q}}_1, \widetilde{\mathbb{Q}}_2 \in \widetilde{\mathcal{A}}$ *solve the 4BP. Then we can redefine* $\mathbb{Z} \in \widetilde{\mathcal{A}}$ *s.t.* $(\mathbb{Z}^{-1})_{22} \in \mathcal{G}TIC_\infty$ *(i.e.,* $\widetilde{\mathbb{N}}_{+21} \in \mathcal{G}TIC_\infty$*).*

This also holds with ULR *in place of* $\widetilde{\mathcal{A}}$.

Proof: Let $\mathcal{A} = \widetilde{\mathcal{A}}$ or $\mathcal{A} =$ ULR (in fact, the lemma and this proof holds for $\mathcal{A}$ in place of $\widetilde{\mathcal{A}}$ whenever $\mathcal{B} \underset{a}{\subset} \mathcal{A} \underset{a}{\subset}$ ULR).

Set $\mathbb{L} := \begin{bmatrix} \widetilde{\mathbb{Q}}_1 & \widetilde{\mathbb{Q}}_2 \end{bmatrix} \mathbb{Z} \begin{bmatrix} 0 \\ I \end{bmatrix} \in \mathcal{A}$ as in Theorem 12.3.7(c).

Because $\mathcal{A} \subset$ ULR, we can set $T := \mathbb{L}(+\infty)$, $X := \mathbb{Z}^{-1}(+\infty)$, to obtain $\mathcal{GB} \ni \widetilde{\mathbb{Q}}_2(+\infty) = \begin{bmatrix} T & I \end{bmatrix} X \begin{bmatrix} 0 \\ I \end{bmatrix}$. By Proposition 6.3.1(c), $X \in \mathcal{GB}$. By Lemma 12.4.11 we get $(EX)_{22}$ invertible.

Thus, by setting $\mathbb{Z}' := \mathbb{Z}E^{-1} \in \mathcal{A}$ we get $(\mathbb{Z}'^{-1})_{22}(+\infty) = (EX)_{22} \in \mathcal{GB}$, and we see that $\mathbb{Z}'J_1\mathbb{Z}'^* = \mathbb{Z}J_1\mathbb{Z}^*$ $(= \mathbb{E}J_1\mathbb{E}^*)$ and $\mathbb{W}' := \mathbb{Z}'^{-1}\mathbb{E} = E\mathbb{W}$ also satisfies $\mathbb{W}'J_1\mathbb{W}'^* = J_1$, by Lemma 6.4.8(c). $\square$

Lemma 12.4.14 *Theorem 12.3.7 holds.*

Proof: Parts (e1) and (e2) were shown in Lemma 12.4.4, and part (c) in Lemma 12.4.9. We prove (a), (b) and (d) below.

(a) 1° *"(4BP3)⇒(4BP2)⇒(4BP1)":* "(4BP3)⇒(4BP2)" is obtained from Lemma 12.4.7, and "(4BP2)⇒(4BP1)" from Lemma 12.4.3(b).

2° *"(4BP1)⇒(4BP2)" when* $\left(\begin{bmatrix} \mathbb{N}_{\rm u\,11} & \mathbb{N}_{\rm u\,12} \\ 0 & I \end{bmatrix}, J_\gamma\right) \in$ SpF*:* Assume (4BP1). Then $\mathbb{D}' := \begin{bmatrix} \mathbb{N}_{\rm u\,11} & \mathbb{N}_{\rm u\,12} \\ 0 & I \end{bmatrix}$ is minimax J_γ-coercive, by Lemma 12.4.1, hence J-coercive.

Thus, if $(\mathbb{D}', J_\gamma) \in \mathrm{SpF}$, then (Factor1X) (and hence (Factor1)) holds, by Lemma 11.4.3(b). Consequently, (4BP2) holds, by Lemma 12.4.3(b).

3° *"(4BP2)⇒(4BP3)" when* $(\mathbb{E}^{\mathrm{d}}, J_1) \in \mathrm{SpF}$*:* Assume that (4BP2) holds and $(\mathbb{E}^{\mathrm{d}}, J_1) \in \mathrm{SpF}$. As in p. 530 of [Green], we set $\begin{bmatrix} \overline{\mathbb{Q}}_1 & \overline{\mathbb{Q}}_2 \end{bmatrix} := \begin{bmatrix} \widetilde{\mathbb{Q}}_1 & \widetilde{\mathbb{Q}}_2 \end{bmatrix} \mathbb{P}^{-1}$ (here $\mathbb{P}$ is from Lemma 12.4.8 and $\widetilde{\mathbb{Q}}_1, \widetilde{\mathbb{Q}}_2$ from (4BP2)) to obtain

$$\begin{bmatrix} \widetilde{\mathbb{U}} & I \end{bmatrix} = \begin{bmatrix} \widetilde{\mathbb{Q}}_1 & \widetilde{\mathbb{Q}}_2 \end{bmatrix} \mathbb{E} = \begin{bmatrix} \overline{\mathbb{Q}}_1 & \overline{\mathbb{Q}}_2 \end{bmatrix} \mathbb{P}\mathbb{E} = \begin{bmatrix} \overline{\mathbb{Q}}_1 & \overline{\mathbb{Q}}_2 \end{bmatrix} \begin{bmatrix} \mathbb{S} & 0 \\ \widetilde{\mathbb{U}} & I \end{bmatrix} = \begin{bmatrix} \overline{\mathbb{Q}}_1 \mathbb{S} + \overline{\mathbb{Q}}_2 \widetilde{\mathbb{U}} \\ \overline{\mathbb{Q}}_2 \end{bmatrix}. \tag{12.75}$$

Thus, $\overline{\mathbb{Q}}_2 = I$ and $\|\overline{\mathbb{Q}}_1 \mathbb{S} + \widetilde{\mathbb{U}}\| = \|\widetilde{\mathbb{U}}\| < 1$, i.e., $\|\mathbb{S}^{\mathrm{d}} \overline{\mathbb{Q}}_1^{\mathrm{d}} + \widetilde{\mathbb{U}}^{\mathrm{d}}\| < 1$, hence $\widetilde{\mathbb{D}} := \mathbb{E}^{\mathrm{d}} \mathbb{P}^{\mathrm{d}}$ is minimax J_1-coercive, by Lemma 11.3.10 (recall from Lemma 12.4.8 that $\widetilde{\mathbb{D}}_{11}^* \widetilde{\mathbb{D}}_{11} = (\mathbb{S}^{\mathrm{d}})^* (\mathbb{S}^{\mathrm{d}}) \gg 0$, as required by Hypothesis 11.3.1)), hence J_1-coercive, by Lemma 11.4.2, i.e., $\pi_+ (\mathbb{E}^{\mathrm{d}} \mathbb{P}^{\mathrm{d}})^* J_1 \mathbb{E}^{\mathrm{d}} \mathbb{P}^{\mathrm{d}} \pi_+$ is invertible. Therefore, also $\pi_+ (\mathbb{E}^{\mathrm{d}})^* J_1 \mathbb{E}^{\mathrm{d}} \pi_+$ is invertible, by Lemma 2.2.2(b)&(a1).

Since $(\mathbb{E}^{\mathrm{d}}, J_1) \in \mathrm{SpF}$, we have $(\mathbb{E}^{\mathrm{d}})^* J_1 \mathbb{E}^{\mathrm{d}} = (\widetilde{\mathbb{Z}}^{\mathrm{d}})^* S \widetilde{\mathbb{Z}}^{\mathrm{d}}$ for some $\widetilde{\mathbb{Z}}^{\mathrm{d}} \in \mathcal{G}\mathrm{TIC}(Y \times U)$ and $S \in \mathcal{G}\mathrm{TIC}(U)$. Consequently, $\widetilde{\mathbb{D}}^* J_1 \widetilde{\mathbb{D}} = (\widetilde{\mathbb{Z}}^{\mathrm{d}} \mathbb{P}^{\mathrm{d}})^* S (\widetilde{\mathbb{Z}}^{\mathrm{d}} \mathbb{P}^{\mathrm{d}})$ and $\widetilde{\mathbb{Z}}^{\mathrm{d}} \mathbb{P}^{\mathrm{d}} \in \mathcal{G}\mathrm{TIC}$. By Lemma 11.4.3(a), $\widetilde{\mathbb{D}}^* J_1 \widetilde{\mathbb{D}} = \mathbb{R}^* J_1 \mathbb{R}$ for some $\mathbb{R} \in \mathcal{G}\mathrm{TIC}(Y \times U)$ s.t. $\mathbb{R}_{11} \in \mathcal{G}\mathrm{TIC}(Y)$.

It follows that $(\mathbb{R}^{-1})_{22} \in \mathcal{G}\mathrm{TIC}(U)$, hence $(\mathbb{W}^{\mathrm{d}})_{22} \in \mathcal{G}\mathrm{TIC}(U)$, where $\mathbb{W}^{\mathrm{d}} := \widetilde{\mathbb{D}} \mathbb{R}^{-1}$ (we have $(\mathbb{W}^{\mathrm{d}})_{22} = \mathbb{R}_{22}^{-1}$, because $\widetilde{\mathbb{D}} = \left[\begin{smallmatrix} * & * \\ 0 & I \end{smallmatrix}\right]$, by Lemma 12.4.8). (Note that $\mathbb{W}^{\mathrm{d}} := \widetilde{\mathbb{D}} \mathbb{R}^{-1}$ is (J_1, J_1)-lossless, by Corollary 2.5.5(iii)&(i).)

Set $\mathbb{Z} := \mathbb{R}^{\mathrm{d}} \mathbb{P}^{-1}$, so that $\mathbb{W} = \mathbb{Z}^{-1} \mathbb{E}$ and hence (Factor2Z) (hence also (Factor2) holds.

(b) This is contained in Lemma 12.4.3(c1).

(d) By Definition 7.2.11 (and Lemma 7.2.12(b)), $\mathbb{Q} = \widetilde{\mathbb{Q}}_2^{-1} \widetilde{\mathbb{Q}}_1$ is well-posed iff $\widetilde{\mathbb{Q}}_2 \in \mathcal{G}\mathrm{TIC}_\infty$.

1° *If:* This follows from Lemma 12.4.13.

2° *Only if:* By Theorem 12.3.7(e2), we have $(\mathbb{Z}^{-1})_{22} = \widetilde{\mathbb{N}}_{+21}$. If this map is invertible and $\mathbb{Z} \in \mathrm{ULR}$, then $\mathbb{Z}^{-1} \in \mathrm{ULR}$, by Proposition 6.3.1(c), and we can take $\mathbb{L} = 0$ to obtain $\begin{bmatrix} \widetilde{\mathbb{Q}}_1 & \widetilde{\mathbb{Q}}_2 \end{bmatrix} = \begin{bmatrix} (\mathbb{Z}^{-1})_{21} & (\mathbb{Z}^{-1})_{22} \end{bmatrix} \in \mathrm{ULR}$; in particular $\widetilde{\mathbb{Q}}_2 = (\mathbb{Z}^{-1})_{22} \in \mathcal{G}\mathrm{TIC}$. □

Lemma 12.4.15 *Lemma 12.3.10 holds.*

Proof: Assume that $\mathbb{N}_{\mathrm{u}}, \mathbb{M}_{\mathrm{u}} \in \widetilde{\mathcal{A}}$. Then $\left(\left[\begin{smallmatrix} \mathbb{N}_{\mathrm{u}11} & \mathbb{N}_{\mathrm{u}12} \\ 0 & I \end{smallmatrix}\right], J_\gamma\right) \in \mathrm{SpF}$, hence we have "(4BP3)⇒(4BP2)⇔(4BP1)", by Theorem 12.3.7(a). If (4BP2) holds, then we have $\mathbb{X} \in \widetilde{\mathcal{A}}$, hence $\mathbb{E} \in \widetilde{\mathcal{A}}$, by Theorem 12.3.7(e2), hence $(\mathbb{E}^{\mathrm{d}}, J_1) \in \mathrm{SpF}$, hence (4BP3) holds, by Theorem 12.3.7(a). Thus, (4BP1)–(4BP3) are equivalent.

(a) Since $\mathbb{E} \in \widetilde{\mathcal{A}}$, we have $\mathbb{E}^{\mathrm{d}} \in \widetilde{\mathcal{A}}$, hence $\mathbb{Z}^{\mathrm{d}} \in \widetilde{\mathcal{A}}$, hence $\mathbb{Z} \in \widetilde{\mathcal{A}}$. It follows from Theorem 12.3.7(c)&(e2) that (a) holds.

(b) The first claim follows from Theorem 12.3.7(c), the second from the fact that $\mathbb{Z} \in \widetilde{\mathcal{A}} = \mathcal{G}\widetilde{\mathcal{A}}$ (and from (12.50)).

(c) By Lemma 12.4.12, we can take $\mathbb{M}_{11}, \mathbb{X}_{22} \in \mathcal{G}\mathrm{TIC}_\infty$ etc., hence $\mathbb{D}_+$ becomes well-posed. Consequently, $\widetilde{\mathbb{M}}_+ \in \mathcal{G}\mathrm{TIC}_\infty$, by Lemma 7.2.12(b). The d.c.f. (12.61) of $\mathbb{D}_+$ is over $\widetilde{\mathcal{A}}$, i.e., all its terms belong to $\widetilde{\mathcal{A}}$.

(d) This follows from Lemma 12.4.13 (because the converse is true by (12.47) with $\mathbb{L} = 0$).

(e) Choose $\widetilde{\mathbb{Q}}_1, \widetilde{\mathbb{Q}}_2 \in \widetilde{\mathcal{A}}$ so that $\mathbb{Q}$ is DPF-stabilizing $\mathbb{D}$. By Theorem 7.3.19(iii), $\mathbb{Q}$ DF-stabilizes $\mathbb{D}_{21}$. Because $\widetilde{\mathbb{Q}}_1, \widetilde{\mathbb{Q}}_2, \mathbb{N}_\mathrm{u}, \mathbb{M}_\mathrm{u} \in \widetilde{\mathcal{A}}$, it follows from Lemma 7.2.16 that the d.c.f. $\mathbb{D}_{21} = \mathbb{N}_{\mathrm{u}\,21}\mathbb{M}_{\mathrm{u}\,11}^{-1} = \widetilde{\mathbb{M}}_{\mathrm{y}\,22}^{-1}\widetilde{\mathbb{N}}_{\mathrm{y}\,21}$ is over $\widetilde{\mathcal{A}}$ if(f) $\widetilde{\mathbb{N}}_{\mathrm{y}\,21}, \widetilde{\mathbb{M}}_{\mathrm{y}\,22} \in \widetilde{\mathcal{A}}$. By (7.79), the "iff" in (e) follows analogously (because its converse is trivial).

(f) This is obvious (recall from Proposition 6.3.1(c) that $\mathbb{R} \in \mathcal{G}\mathrm{TIC}_\infty \Leftrightarrow \widehat{\mathbb{R}}(+\infty) \in \mathcal{GB}$ for any $\mathbb{R} \in \mathrm{TIC}_\infty \cap \mathrm{ULR}$, hence for any $\mathbb{R} \in \widetilde{\mathcal{A}}$).

(g) One observes this from the proof of Lemma 12.4.16 below. □

Lemma 12.4.16 *Theorem 12.3.6 holds.*

Proof: By Lemma 12.3.10(a)&(c), (Factor1) is now equivalent to (Factor1$\widetilde{\mathcal{A}}$).

The equivalence of (4BP1)–(4BP3) follows from Lemma 12.3.10; by Lemma 12.3.10(c)&(d), we can maintain the equivalence while strengthening the three assumptions to (4BP1$\widetilde{\mathcal{A}}$)–(4BP3$\widetilde{\mathcal{A}}$).

(We could equivalently drop the condition $\mathbb{M}_{11}(+\infty) \in \mathcal{GB}$ from (Factor1$\widetilde{\mathcal{A}}$), but we prefer having $\mathbb{D}_+$ well-posed in the formulation of (Factor2$\widetilde{\mathcal{A}}$).)

(a) This follows from Theorem 12.3.7(c) and Lemma 12.3.10(b).

(b) (In fact, it would suffice to assume that $\widetilde{\mathbb{N}}_{\mathrm{y}\,21}, \widetilde{\mathbb{M}}_{\mathrm{y}\,22} \in \widetilde{\mathcal{A}}$; cf. the proof of Lemma 12.3.10(e) above.)

"If" is trivial, "only if" follows from Lemma 7.2.16(a) (it provides a joint d.c.f. of $\mathbb{D}$ and $\mathbb{Q}$ in $\widetilde{\mathcal{A}}$, hence $\mathbb{Q} = \widetilde{\mathbb{X}}^{-1}\widetilde{\mathbb{Y}} = \mathbb{Y}\mathbb{X}^{-1}$ for some $\widetilde{\mathbb{Y}}, \widetilde{\mathbb{X}}, \mathbb{X}, \mathbb{Y} \in \widetilde{\mathcal{A}}$; recall from Lemma 7.2.12(b) that $\mathbb{X}, \widetilde{\mathbb{X}} \in \mathcal{G}\mathrm{TIC}_\infty$).

(c) This follows from Lemma 12.3.9.

(d) By Lemma 12.3.11(e), the requirement $\widetilde{\mathbb{N}}_{+21} \in \mathcal{GB}(U)$ in (Factor2$\widetilde{\mathcal{A}}$) can always be satisfied, hence (Factor2$\widetilde{\mathcal{A}}$) is equivalent to (Factor2) when (Factor1$\widetilde{\mathcal{A}}$) (equivalently, (Factor1)) holds. Thus, (4BP3$\widetilde{\mathcal{A}}$) is equivalent to (4BP3), hence now (4BP1)–(4BP3) and (4BP1$\widetilde{\mathcal{A}}$)–(4BP3$\widetilde{\mathcal{A}}$) are all equivalent to each other. □

Notes

The methods of [Green] (or [CG97]) do not apply in the general case, as explained on p. 721. However, we have been able to use part of them by using suitable modifications.

Much of Lemma 12.4.3 was established for rational transfer functions in Section 3 of [Green]. Part of Lemma 12.4.11 is from Lemma 3.6 of [Green]. In the proof of Lemma 12.4.9, we have borrowed from [Green, p. 530] the idea to reduce the ASP (4BP2) to a H^∞ FICP. Green and [CG97] use the well-known matrix complementation properties of their respective transfer function classes. Since general H^∞ functions do not possess such properties, by Lemma 4.1.10, we have constructed a suitable reducing operator ($\mathbb{P}$) explicitly, in Lemma 12.4.8, by using (4BP2).

12.5 Proofs for Section 12.1 — 4BP $\mathcal{P}_X, \mathcal{P}_Y, \mathcal{P}_Z$

> *Conjecture 1. The proof of every result requires at least three auxiliary lemmas.*
>
> *Addendum to Conjecture 1. This applies also to the proofs of auxiliary lemmas.*
>
> — K.M.

(Recall Standing Hypothesis 12.1.1.) In this section, we shall give proofs for the results of Section 12.1. Observe that, in most results of this section, the letters $\mathbb{X}$ and $\mathbb{M}$ correspond to "(1.)" (the $\mathcal{P}_\mathrm{X}$-CARE) or to the corresponding IARE, not to (Factor1X) of Theorem 12.3.7; see each statement for details.

The classical assumption is that $\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B}_1 \end{array}\right]$ is exponentially stabilizable and $\left[\begin{array}{c} \mathbb{A} \\ \hline \mathbb{C}_2 \end{array}\right]$ is exponentially detectable. By Lemma 13.3.17, in the discrete-time case this holds iff the system $\Sigma_{21} := \left[\begin{array}{c|c} \mathbb{A} & \mathbb{B}_1 \\ \hline \mathbb{C}_2 & \mathbb{D}_{21} \end{array}\right]$ is exponentially jointly stabilizable and detectable; the same is true in the continuous-time if, e.g., B_1 and C_2 are bounded (cf. Corollary 9.2.13(b)).

In that case, the corresponding pairs "$\left[\begin{array}{c|cc} \mathbb{K}_{\mathrm{u}\,1} & \mathbb{F}_{\mathrm{u}\,11} & \mathbb{F}_{\mathrm{u}\,12} \end{array}\right]$" and "$\left[\begin{array}{c} \mathbb{H}_{\mathrm{y}\,2} \\ \hline \mathbb{G}_{\mathrm{y}\,12} \\ \mathbb{G}_{\mathrm{y}\,22} \end{array}\right]$" (and interaction operator "$\mathbb{E}^{\mathrm{uy}}_{12}$") stabilize $\mathbb{A}$ and hence the whole Σ exponentially, by Lemma 13.3.8. However, sometimes the following, weaker assumption is enough for us:

Hypothesis 12.5.1 (H$^\infty$ 4BP) *Assume that* $\Sigma = \left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D} \end{array}\right] \in \mathrm{WPLS}(U \times W, H, Z \times Y)$*, and there are jointly stabilizing and detecting pairs*

$$\left[\begin{array}{c|c} \mathbb{K}_\mathrm{u} & \mathbb{F}_\mathrm{u} \end{array}\right] = \left[\begin{array}{c|cc} \mathbb{K}_{\mathrm{u}\,1} & \mathbb{F}_{\mathrm{u}\,11} & \mathbb{F}_{\mathrm{u}\,12} \\ 0 & 0 & 0 \end{array}\right] \quad \textit{and} \tag{12.76}$$

$$\left[\begin{array}{c} \mathbb{H}_\mathrm{y} \\ \hline \mathbb{G}_\mathrm{y} \end{array}\right] = \left[\begin{array}{cc} 0 & \mathbb{H}_{\mathrm{y}\,2} \\ \hline 0 & \mathbb{G}_{\mathrm{y}\,12} \\ 0 & \mathbb{G}_{\mathrm{y}\,22} \end{array}\right] \tag{12.77}$$

for Σ *with some interaction operator* $\left[\begin{smallmatrix} 0 & \mathbb{E}^{\mathrm{uy}}_{12} \\ 0 & 0 \end{smallmatrix}\right]$*. Moreover, we make the nonsingularity assumptions*

$$\mathbb{N}_{\mathrm{u}\,11}^*\mathbb{N}_{\mathrm{u}\,11} \gg 0, \quad \widetilde{\mathbb{N}_\mathrm{y}}_{22}\widetilde{\mathbb{N}_\mathrm{y}}_{22}^* \gg 0, \tag{12.78}$$

where $\mathbb{N}_\mathrm{u} := \mathbb{D}\mathbb{M}_\mathrm{u}$, $\widetilde{\mathbb{N}_\mathrm{y}} := \widetilde{\mathbb{M}_\mathrm{y}}\mathbb{D}$, $\mathbb{M}_\mathrm{u} := (I - \mathbb{F}_{\mathrm{u}\,11})^{-1}$, $\widetilde{\mathbb{M}_\mathrm{y}} := (I - \mathbb{G}_{\mathrm{y}\,22})^{-1}$.

Note also that joint stabilization is always doubly coprime, by Theorem 6.6.28, and that the interaction operator is necessarily of the above form (for pairs of the above form).

We strongly recommend for most readers to assume that the above pairs are exponentially jointly stabilizing and detecting, so that some of the proofs and results below become essentially simpler (and much of them can be ignored). This covers the case of an exponentially stabilizing controller, and the (rest of the) general case is not that important. Indeed, this assumption is necessary when one is interested in finding a suboptimal exponentially stabilizing controller (recall Definition 12.1.2):

Lemma 12.5.2 (Nonexp. 4BP & opt. $\Rightarrow$ exp. 4BP) *Assume Hypothesis 12.5.1. The following are equivalent:*

(i) Σ is optimizable;

(ii) Σ is estimatable;

(iii) Σ is optimizable and estimatable

(iv) $\begin{bmatrix} \mathbb{K}_{\mathrm{u}} & \big| & \mathbb{F}_{\mathrm{u}} \end{bmatrix}$ is exponentially stabilizing;

(v) $\begin{bmatrix} \mathbb{H}_{\mathrm{y}} \\ \hline \mathbb{G}_{\mathrm{y}} \end{bmatrix}$ is exponentially stabilizing;

(vi) $\begin{bmatrix} \mathbb{K}_{\mathrm{u}} & \big| & \mathbb{F}_{\mathrm{u}} \end{bmatrix}$ and $\begin{bmatrix} \mathbb{H}_{\mathrm{y}} \\ \hline \mathbb{G}_{\mathrm{y}} \end{bmatrix}$ are exponentially jointly stabilizing;

(vii) $\Sigma_{21} := \left[\begin{array}{c|c} \mathbb{A} & \mathbb{B}_1 \\ \hline \mathbb{C}_2 & \mathbb{D}_{21} \end{array}\right]$ is exponentially jointly stabilizable and detectable;

(viii) Σ is exponentially DPF-stabilizable with internal loop;

(ix) Σ_{21} is exponentially DF-stabilizable with internal loop.

If (i) holds, then the systems $\Sigma_{\circlearrowleft}$, $\Sigma_{\mathbb{E}^{\mathrm{d}}}$ and $\Sigma_{\mathbb{T}}$ of Lemmas 12.5.15 and 12.5.16 and Proposition 12.5.19 are then exponentially stable (under the assumptions of those results).

If $\begin{bmatrix} \mathbb{K}_{\mathrm{u}} & \big| & \mathbb{F}_{\mathrm{u}} \end{bmatrix}$ is strongly stabilizing for Σ, then the systems $\Sigma_{\circlearrowleft}$, $\Sigma_{\mathbb{E}^{\mathrm{d}}}{}^{\mathrm{d}}$ and $\Sigma_{\mathbb{T}}$ are strongly stable (under corresponding assumptions, as above).

Thus, when we can show that Hypothesis 12.5.1 is satisfied and Σ is optimizable, then, for solving the exponential H^∞ 4BP, it suffices to solve the I/O H^∞ 4BP of Section 12.3, i.e., the corresponding frequency-domain (or I/O map) problem, since then we obtain a solution of the exponential problem from Proposition 12.5.19.

An analogous claim obviously holds for the more general "nonexponential H^∞ 4BP" (see the discussion below Definition 12.1.2) instead of the exponential H^∞ 4BP, and also for the "strong H^∞ 4BP" where we require strong instead of exponential stability.

Proof of Lemma 12.5.2: 1° *The equivalence of (i)–(vi):* If Σ is estimatable, then $\begin{bmatrix} \mathbb{K}_{\mathrm{u}} & \big| & \mathbb{F}_{\mathrm{u}} \end{bmatrix}$ is exponentially stabilizing, by Theorem 6.7.15(c2), hence Σ is then optimizable. If Σ is optimizable, then $\begin{bmatrix} \mathbb{H}_{\mathrm{y}} \\ \hline \mathbb{G}_{\mathrm{y}} \end{bmatrix}$ is exponentially stabilizing, by the above and duality.

Thus, if (i) or (ii) holds, then so do (i)–(vi) (note that $\mathbb{A}_\flat = \mathbb{A}_L$ and $\mathbb{A}_\sharp = \mathbb{A}_{\tilde{L}}$ in terms of (6.133), (6.170), (6.168) and (6.171), so that also $\mathbb{A}_L$ and $\mathbb{A}_{\tilde{L}}$ are then exponentially stable). Moreover, then $\left[\begin{array}{c|c} \mathbb{K}_{u1} & \mathbb{F}_{u11} \\ 0 & 0 \end{array}\right]$ and $\left[\begin{array}{cc} 0 & \mathbb{H}_{\mathrm{y}\,2} \\ \hline 0 & \mathbb{G}_{\mathrm{y}\,22} \end{array}\right]$ are obviously jointly admissible, hence exponentially jointly stabilizing for $\Sigma_{21} = \left[\begin{array}{c|c} \mathbb{A} & \mathbb{B}_1 \\ \hline \mathbb{C}_2 & \mathbb{D}_{21} \end{array}\right]$ (since $(\Sigma_{21})_L$ and $(\Sigma_{21})_{\tilde{L}}$ have the same semigroups as Σ_L and $\Sigma_{\tilde{L}}$), so that also (vii) holds.

Conversely, any of (iii)–(vii) obviously implies (i) or (ii) (hence (i)–(vii)).

Finally, (vi) implies (viii) and (ix), by Theorem 7.3.12(b1), "(viii)⇒(i)" follows from Theorem 7.3.12(a), and "(ix)⇒(i)" follows from Theorem 7.2.4(a) and Lemma 6.7.4.,

2° $\Sigma_{\circlearrowleft}$, $\Sigma_{\mathbb{E}^{\mathrm{d}}}$ *and* $\Sigma_{\mathbb{T}}$ *are exponentially stable:* By Lemma 12.5.15, $\Sigma_{\circlearrowleft}$ is exponentially stable, hence so is $\Sigma_{\mathbb{E}^{\mathrm{d}}}$ (since $\mathbb{A}^{\mathrm{d}}_{\circlearrowleft}$ is). For $\Sigma_{\mathbb{T}}$ this is contained in Proposition 12.5.19.

3° *Strongly stable case:* See 2°. □

The theory of Section 12.3 can be applied under Hypothesis 12.5.1:

Lemma 12.5.3 (Hyp. 12.5.1 ⇒ Hyp. 12.3.1) *Hypothesis 12.5.1 implies Hypothesis 12.3.1 (with same* $\mathbb{N}_{\mathrm{u}}, \mathbb{M}_{\mathrm{u}}, \widetilde{\mathbb{N}}_{\mathrm{y}}, \widetilde{\mathbb{M}}_{\mathrm{y}}$*). Indeed, define* Σ_{Total} *by (12.87). Then*

$$\mathbb{X}_{\mathrm{u}} := I - \mathbb{G}_{\mathrm{y}} + \mathbb{D}\mathbb{Y}_{\mathrm{u}} = \begin{bmatrix} I & * \\ 0 & * \end{bmatrix}, \qquad \mathbb{Y}_{\mathrm{u}} := -\mathbb{M}_{\mathrm{u}}\mathbb{E}_{\mathrm{uy}} = \begin{bmatrix} 0 & -\mathbb{M}_{\mathrm{u}\,11}\mathbb{E}^{\mathrm{uy}}_{12} \\ 0 & 0 \end{bmatrix}, \tag{12.79}$$

$$\widetilde{\mathbb{X}}_{\mathrm{y}} := I - \mathbb{F}_{\mathrm{u}} + \widetilde{\mathbb{Y}}_{\mathrm{y}}\mathbb{D} = \begin{bmatrix} * & * \\ 0 & I \end{bmatrix} \quad \textit{and} \quad \widetilde{\mathbb{Y}}_{\mathrm{y}} := -\mathbb{E}_{\mathrm{uy}}\widetilde{\mathbb{M}}_{\mathrm{y}} = \begin{bmatrix} 0 & -\mathbb{E}^{\mathrm{uy}}_{12}\widetilde{\mathbb{M}}_{\mathrm{y}\,22} \\ 0 & 0 \end{bmatrix} \tag{12.80}$$

complement $\mathbb{M}_{\mathrm{u}}, \mathbb{N}_{\mathrm{u}}, \widetilde{\mathbb{M}}_{\mathrm{y}}, \widetilde{\mathbb{N}}_{\mathrm{y}}$ *to a d.c.f.; this d.c.f. satisfies Hypothesis 12.3.1.*

We shall use this below without further mention.

Conversely, if $\mathbb{D} \in \mathrm{TIC}_\infty$ satisfies Hypothesis 12.3.1, then $\mathbb{D}$ has a realization satisfying Hypothesis 12.5.1, by Lemma 12.5.23.

Proof: Assume Hypothesis 12.5.1. Then $\mathbb{X}_{\mathrm{u}}, \mathbb{Y}_{\mathrm{u}}, \widetilde{\mathbb{X}}_{\mathrm{y}}, \widetilde{\mathbb{Y}}_{\mathrm{y}}$ in (12.79)–(12.80) are stable (being parts of (6.170) and (6.171)) and complement $\mathbb{M}_{\mathrm{u}}, \mathbb{N}_{\mathrm{u}}, \widetilde{\mathbb{M}}_{\mathrm{y}}, \widetilde{\mathbb{N}}_{\mathrm{y}}$ to a d.c.f.; this d.c.f. satisfies Hypothesis 12.3.1. Indeed, the $(1,1)$-block of the equation

$$\begin{bmatrix} \widetilde{\mathbb{X}} & -\widetilde{\mathbb{Y}} \\ -\widetilde{\mathbb{N}} & \widetilde{\mathbb{M}} \end{bmatrix} \begin{bmatrix} \mathbb{M} & \mathbb{Y} \\ \mathbb{N} & \mathbb{X} \end{bmatrix} = I \tag{12.81}$$

(from (6.172)) is $\widetilde{\mathbb{M}}_{\mathrm{y}\,11}\mathbb{M}_{\mathrm{u}\,11} + (\mathbb{E}_{\sharp})_{12}\mathbb{N}_{\mathrm{u}\,21} = I$, hence $\mathbb{N}_{\mathrm{u}\,21}$ and $\mathbb{M}_{\mathrm{u}\,11}$ are r.c. Analogously, $\widetilde{\mathbb{N}}_{\mathrm{y}\,21}$ and $\widetilde{\mathbb{M}}_{\mathrm{y}\,22}$ are l.c. □

Next we note that Hypothesis 12.5.1 is equivalent to the standard H^∞ 4BP assumptions (under the regularity assumption (A1)):

Lemma 12.5.4 ((A,B_1) & (A,C_2) ⇒ Hypothesis 12.5.1) *Assume (A1) of Theorem 12.1.5. Then Hypothesis 12.5.1 is satisfied with "exponentially jointly" in place of "jointly" iff* (A,B_1) *is optimizable,* (A,C_2) *is estimatable and (A2) of Theorem 12.1.5 holds.*

If Hypothesis 12.5.1 is satisfied, then necessarily $\mathbb{N}_{\mathrm{u}}, \mathbb{M}_{\mathrm{u}}, \widetilde{\mathbb{N}}_{\mathrm{y}}, \widetilde{\mathbb{M}}_{\mathrm{y}} \in \mathrm{MTIC}^{\mathrm{L}^1}_{\mathrm{exp}}$ *(or in* (γ') *under the alternative "(IV)" in (A1)), and we can choose them so that* $\widehat{\mathbb{M}_{\mathrm{u}}}(+\infty) = I$*,* $\widehat{\widetilde{\mathbb{M}}_{\mathrm{y}}}(+\infty) = I$*.*

Proof: 1° (A,B_1) & (A,C_2) ⇐ *Hypothesis 12.5.1* This is contained in Lemma 12.5.2.

2° (A, B_1) & (A, C_2) $\Rightarrow$ *Hypothesis 12.5.1 except possibly (12.78):* By Corollary 9.2.13(b), there are $K \in \mathcal{B}(H, U)$ and $\mathtt{H} \in \mathcal{B}(Y, H)$ s.t. $A + B_1 K$ and $A + \mathtt{H} C_2$ are exponentially stable. Extend Σ by K and $\mathtt{H}$ (with $F = 0 = G = E$) to satisfy Hypothesis 12.5.1 with "exponentially jointly" in place of "jointly" except possibly (12.78).

By (the proof of) Corollary 9.2.13(c), $\mathbb{N}_{\mathrm{u}}, \mathbb{M}_{\mathrm{u}}, \widetilde{\mathbb{N}_{\mathrm{y}}}, \widetilde{\mathbb{M}_{\mathrm{y}}} \in \mathrm{MTIC}^{\mathrm{L}^1}_{\mathrm{exp}}$ (and they are d.c. over $\mathrm{MTIC}^{\mathrm{L}^1}_{\mathrm{exp}}$) and $\widehat{\mathbb{M}_{\mathrm{u}}}(+\infty) = I$, $\widehat{\widetilde{\mathbb{M}_{\mathrm{y}}}}(+\infty) = I$. The case of the alternative "(IV)" in (A1) follows similarly (use (c3) and (b) instead of (c1) of Lemma 6.8.4).

3° *Conditions (12.78)* $\Leftrightarrow$ *(A2):* The operator K is exponentially stabilizing for $\Sigma_{11} := \left(\begin{array}{c|c} A & B_1 \\ \hline C_1 & D_{11} \end{array}\right)$, with closed loop I/O map $\begin{bmatrix} \mathbb{N}_{\mathrm{u}\,11} \\ \mathbb{M}_{\mathrm{u}\,11} \end{bmatrix}$ (since $\mathbb{M}_{\mathrm{u}} := (I - \mathbb{F}_{\mathrm{u}})^{-1} =$ $\begin{bmatrix} (I-\mathbb{F}_{\mathrm{u}\,11})^{-1} & -(I-\mathbb{F}_{\mathrm{u}\,11})^{-1}\mathbb{F}_{\mathrm{u}\,12} \\ 0 & I \end{bmatrix}$).

By Lemma 8.4.11(a2), $\mathbb{N}_{\mathrm{u}\,11}^* \mathbb{N}_{\mathrm{u}\,11} \gg 0$ iff $\mathbb{N}_{\mathrm{u}\,11}$ is I-coercive; by Theorem 8.4.5(d), this is the case iff $\mathbb{D}_{11}$ is I-coercive over $\mathcal{U}_{\mathrm{exp}}$ (w.r.t. system Σ_{11}); by (e2)&(i)&(ii') of Proposition 10.3.2, this is the case iff $D_{11}^* D_{11} \gg 0$ and (12.11) holds; these are contained in the assumptions.

By dual arguments, we obtain that $\widetilde{\mathbb{N}_{\mathrm{y}}}_{22} \widetilde{\mathbb{N}_{\mathrm{y}}}_{22}^* \gg 0$ iff $D_{22} D_{22}^* \gg 0$ and (12.12) holds.

4° The final claims were observed in 2°. □

Lemma 12.5.5 ($\mathcal{P}_{\mathrm{X}}$ & $\mathcal{P}_{\mathrm{Y}}$ $\Rightarrow$ Hypothesis 12.5.1) *If conditions (A1), (A2), (1.) and (2.) of Theorem 12.1.5 are satisfied, then (A, B_1) is optimizable and (A, C_2) is estimatable.*

See Lemma 12.5.4 for more.

Proof: By Theorem 11.1.4(iii)&(i) (or Theorem 11.1.3(iii)&(i) in case (IV) of (A1)), the FICP for Σ_{X} has a solution; in particular, (A, B_1) is exponentially stabilizable. By duality, (A, C_2) is exponentially detectable. □

When solving $\mathcal{P}_{\mathrm{X}}$-part of the 4BP, we get the $\mathcal{P}_{\mathrm{Y}}$-part in the bargain due to duality:

Lemma 12.5.6 (Duality) *Hypotheses 12.1.1, 12.5.1, 12.3.1 and 12.0.1, and Definitions 12.1.2 and 12.3.3 are invariant under ("interchanged") duality.*

In particular, a map $\mathbb{Q} \in \mathrm{TIC}_\infty(Y, U)$ is a suboptimal [exponentially] stabilizing DPF-controller for Σ (resp. for $\mathbb{D}$) iff $\mathbb{Q}^{\mathrm{d}} \in \mathrm{TIC}_\infty(U, Y)$ is a suboptimal [exponentially] stabilizing DPF-controller for Σ_{d} (resp. for $\mathbb{D}_{\mathrm{d}}$). □

(Part of this follows from Proposition 7.3.4(d), the rest is obvious (note also that $\mathbb{O}$ is well-posed iff $\mathbb{O}^{\mathrm{d}}$ is). Part of the lemma is contained in Lemma 12.3.4.)

Here (and elsewhere), $\mathbb{D}_{\mathrm{d}} := \left[\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right] \mathbb{D}^{\mathrm{d}} \left[\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right]$, $\mathbb{C}_{\mathrm{d}} := \mathbb{C}^{\mathrm{d}} \left[\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right]$, $\mathbb{B}_{\mathrm{d}} := \left[\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right] \mathbb{B}^{\mathrm{d}}$ and $\mathbb{A}_{\mathrm{d}} := \mathbb{A}^{\mathrm{d}}$, $\Sigma_{\mathrm{d}} := \left[\begin{array}{c|c} \mathbb{A}_{\mathrm{d}} & \mathbb{B}_{\mathrm{d}} \\ \hline \mathbb{C}_{\mathrm{d}} & \mathbb{D}_{\mathrm{d}} \end{array}\right]$ etc.; i.e., we take causal adjoints of each map or system, and interchange the rows and columns corresponding to U and W and to Z and Y.

If the 4BP has a solution, then the "I/O FICP" and its dual problem have well-posed solutions:

Lemma 12.5.7 (H^∞ 4BP $\Rightarrow$ H^∞ FICP) *Assume that there is a suboptimal stabilizing DPF-controller for Σ (or for $\mathbb{D}$). Then there are $\mathbb{U} \in \mathrm{TIC}(W,U)$ and $\widetilde{\mathbb{U}} \in \mathrm{TIC}(Y,Z)$ s.t. $\|\mathbb{D}_{11}\mathbb{U}+\mathbb{D}_{12}\|_{\mathrm{TIC}} < \gamma$ and $\|\mathbb{D}_{12}+\widetilde{\mathbb{U}}\mathbb{D}_{22}\|_{\mathrm{TIC}} < \gamma$.*

Assume that there is a suboptimal exponentially stabilizing DPF-controller for Σ. Then (A,B_1) is optimizable, hence $\mathcal{U}_{\mathrm{exp}}^{\Sigma_{\mathrm{X}}}(x_0) \neq \emptyset$ for each $x_0 \in H$; $\gamma > \gamma_0$ for Σ_{X} (see Definition 11.1.2); and $\mathbb{U}$ can be chosen so that, in addition, $\left[\begin{smallmatrix} \mathbb{U} \\ I \end{smallmatrix}\right][\mathrm{L}^2(\mathbf{R}_+;W)] \subset \mathcal{U}_{\mathrm{exp}}^{\Sigma_{\mathrm{X}}}(0)$. Analogous claims hold for Σ_{Y}.

Proof: 1° The first claim follows from Proposition 7.3.4(e) (the stability of $\mathbb{R}^{-1}$ and $\mathbb{H}^{-1}$ (hence of $\mathbb{U} := (\mathbb{R}^{-1})_{12}$ and $\widetilde{\mathbb{U}} := (\mathbb{H}^{-1})_{12}$) follows from Proposition 7.3.4(b)).

2° Let $\widetilde{\Sigma}$ be an exponentially stabilizing DPF-controller for Σ. By Proposition 7.3.4(e), $w \mapsto u$ is given by $\mathbb{U} := (\mathbb{R}^{-1})_{12} \in \mathrm{TIC}(W,U)$, and $\mathcal{F}_\ell(\mathbb{D},\mathbb{O}) = \mathbb{D}_{11}\mathbb{U}+\mathbb{D}_{12}$.

However, for each $w \in \mathrm{L}^2(\mathbf{R}_+;W)$ (and $x_0 = 0$, $u_L = 0$, $y_L = 0$), all signals in the combined closed-loop system (see Figure 12.1) are in L^2 (since $\widetilde{\Sigma}$ is exponentially stabilizing, so that Σ_I^o is exponentially stable); in particular,

$$\mathrm{L}^2(\mathbf{R}_+;H) \ni x = \mathbb{B}\tau\left[\begin{smallmatrix} u \\ w \end{smallmatrix}\right] = \mathbb{B}\tau\left[\begin{smallmatrix} \mathbb{U} \\ I \end{smallmatrix}\right] w \tag{12.82}$$

(because in each form of feedback (cf. Summary 6.7.1), the state, output and input are unique solutions of (6.122)–(6.124), by Proposition 6.6.2).

We conclude that $\left[\begin{smallmatrix} \mathbb{U} \\ I \end{smallmatrix}\right] w \in \mathcal{U}_{\mathrm{exp}}(0) = \mathcal{U}_{\mathrm{exp}}^{\Sigma_{\mathrm{X}}}(0)$. Since w was arbitrary, we have $\gamma > \|\mathbb{D}_{11}\mathbb{U}+\mathbb{D}_{12}\| \geq \gamma_0$ (where $\gamma_0 := \gamma_{0,\Sigma_{\mathrm{X}}}$ is defined with Σ_{X} in place of Σ). By Theorem 7.3.12(a), (A,B_1) is optimizable, hence so is (A,B); in particular, $\mathcal{U}_{\mathrm{exp}}^{\Sigma_{\mathrm{X}}}(x_0) \neq \emptyset$ for all $x_0 \in H$. □

In finite-dimensional theory, one often looks for a map $\mathbb{Q} \in \mathrm{TIC}_\infty(Y,U)$ that "stabilizes Σ", a system. However, it seems that in such theory one always assumes an optimizable and estimatable realization of Σ, and the definition of "$\mathbb{Q}$ stabilizes Σ" refers to this realization explicitly or implicitly. Nevertheless, we shall show below that such a concept has a meaningful definition for general $\Sigma \in \mathrm{WPLS}$, and that for "L^1-smooth systems" (and any discrete-time systems), this definition guarantees the existence of a realization of $\mathbb{Q}$ that stabilizes Σ exponentially (in the ordinary sense):

Indeed, instead of requiring the existence of a DPF-stabilizing system with internal loop, it suffices to require the existence of a DPF-stabilizing map with internal loop (cf. the difference between Figures 7.11 and 12.3); thus, also this formally weaker condition is equivalent to (1.)–(3.):

Remark 12.5.8 ("Suboptimal exponentially stabilizing $\mathbb{Q}$ (not $\widetilde{\Sigma}$) for Σ") *Instead of finding a suboptimal system that stabilizes Σ exponentially, we may require the existence of a suboptimal map (a (well-posed) map $\mathbb{Q} \in \mathrm{TIC}_\infty(Y,U)$ or a map $\mathbb{Q}$ with internal loop) that stabilizes Σ exponentially.*

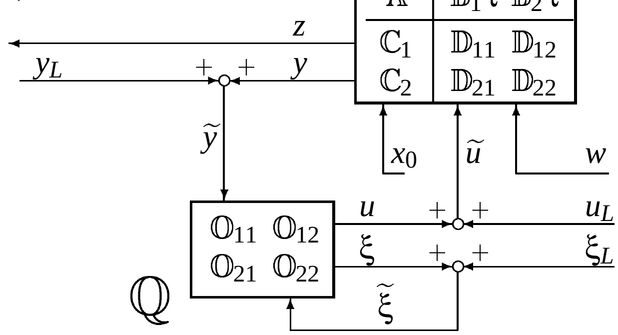

Figure 12.3: “DPF-controller $\mathbb{Q}$ with internal loop” for $\Sigma \in \mathrm{WPLS}(U \times W, H, Z \times Y)$

By this we mean the weakened form of Definition 7.3.1, where the second row and second column of Σ_I^o need not be exponentially stable (for an arbitrary realization of $\widetilde{\Sigma}$ of $\mathbb{Q}$; note that the realization affects only the second row and column of Σ_I^o). It follows that Σ_{21} is optimizable and estimatable.

Assume that (A1) of Theorem 12.1.5 holds.

Then any map $\mathbb{Q}$ that stabilizes Σ exponentially has a realization $\widetilde{\Sigma} \in$ $\mathrm{WPLS}(U \times W \times \Xi, \widetilde{H}, Z \times Y \times \Xi)$ that stabilizes Σ exponentially (if $\mathbb{Q}$ is well-posed, then $\widetilde{\Sigma}$ can be chosen to be well-posed). In particular, the existence of a suboptimal exponentially stabilizing $\mathbb{Q}$ is equivalent to the existence of a suboptimal exponentially stabilizing controller $\widetilde{\Sigma}$ (because the converse is trivial; note that “(1.)–(3.)” is a third equivalent condition if also (A2) holds, and that then the controller can be taken well-posed whenever $D_{21} = 0$, by Theorem 12.1.8, even if $\mathbb{Q}$ were not well-posed).

Note that in discrete time (A1) becomes redundant, and that any of (I)–(IV) of Theorem 12.1.4(A1) implies (A1) of Theorem 12.1.5.

Proof: (We only sketch the proof. Note that analogous definitions could be made with other attributes than “exponentially”, as well as for DF-stabilization (instead of DPF-stabilization studied in this chapter), but we have no use for such.)

1° *If $\mathbb{Q} \in \mathrm{TIC}_\infty(Y \times \Xi, U \times \Xi)$ DPF-stabilizes $\Sigma \in \mathrm{WPLS}(U \times W, H, Z \times Y)$ exponentially, then Σ_{21} is optimizable and estimatable:* Let $\widetilde{\Sigma}$ be a realization of $\mathbb{Q}$, let Σ_I^o be the corresponding closed-loop system (as in Definition 7.3.1), and let $\Sigma_{I,H}^o$ denote Σ_I^o with the second row and the second column removed.

By writing Σ_I^o out, one can verify that $\Sigma_{I,H}^o$ does not contain elements of $\widetilde{\Sigma}$ (other than its I/O map $\mathbb{Q}$). Indeed, for $\mathbb{D}_I^o$ this is trivial, for $\mathbb{C}_I^o$ we observe this from the fact that the left column of (7.74) does not contain such elements (since $(\mathbb{C}_I^o)_{31}$ consists of the elements of $(I - \mathbb{D}^o)^{-1}$ and $\mathbb{C}_2$ only), the formulae for $\mathbb{B}_I^o$ are analogous, and

$$\mathbb{A}_{I,H}^o := (\mathbb{A}_I^o)_{11} = \mathbb{A} + \mathbb{B}_1 \tau (\mathbb{C}_I^o)_{11}, \tag{12.83}$$

by (7.75); thus, also $\mathbb{A}_{I,H}^o$ is independent of $\widetilde{\Sigma}$. Therefore, the concept “$\mathbb{Q}$ stabilizes Σ exponentially” is independent of the realization $\widetilde{\Sigma}$ of $\mathbb{Q}$.

Assume that $\mathbb{Q}$ stabilizes Σ exponentially, i.e., that the four parts of $\Sigma_{I,H}^o$

satisfy 1.–4. of Definition 6.1.1 for some $\omega < 0$ (note that "$\mathbb{A}^o_{I,H}$" is not a semigroup in general). Then $(\mathbb{C}^o_I)_{11}$ is (exponentially) stable and $\mathbb{A}^o_{I,H}x_0 \in \mathrm{L}^2(\mathbf{R}_+;H)$ for all $x_0 \in H$. By (12.83), it follows that $\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B}_1 \end{array}\right]$ is optimizable. By dual arguments, $\left[\begin{array}{c|c} \mathbb{A} & \mathbb{C}_2 \end{array}\right]$ is estimatable.

2° *An optimizable and estimatable, hence exponentially stabilizing realization $\widetilde{\Sigma}$ of $\mathbb{Q}$ exists (under (A1)):* Assume that (A1) holds and that $\mathbb{Q}$ stabilizes Σ exponentially.

By 1°, Σ_{21} is optimizable and estimatable, hence the above claim holds, by Lemma 7.3.6(b1) (see also Theorem 7.3.11(c1)).

(This 2° applies both to (well-posed) $\mathbb{Q} \in \mathrm{TIC}_\infty(Y,U)$ (giving (well-posed) $\widetilde{\Sigma} \in \mathrm{WPLS}(U,\widetilde{H},Y)$) and to $\mathbb{Q} \in \mathrm{TIC}_\infty(Y\times\Xi,U\times\Xi)$ (giving $\widetilde{\Sigma} \in \mathrm{WPLS}(U\times\Xi,\widetilde{H},Y\times\Xi)$).

3° Assume (A1). If $\widetilde{\Sigma}$ stabilizes Σ exponentially, then $\mathbb{Q}$ stabilizes Σ exponentially (if Σ^o_I of Definition 7.3.1 is exponentially stable, then so is the corresponding sub"system" "$\Sigma^o_{I,H}$"); the converse was given in 2°. Thus, we have the first equivalence.

Under (A2), we obtain the second equivalence from Theorem 12.1.5. If $D_{21} = 0$, then all suboptimal exponentially stabilizing controllers are equivalent to well-posed ones, by Theorem 12.1.8. □

We often wish to assume that $D_{21} = 0$. Usually, this is not a problem:

Lemma 12.5.9 (D_{21} is irrelevant) *The validity of Hypothesis 12.5.1 remains unchanged if one alters D_{21} (ceteris paribus). The same holds for Hypotheses 12.5.13 and 12.3.1, and for the assumptions of Theorem 12.1.11.*

Conditions (1.) and (4.) of Theorem 12.1.8 are independent of D_{21}; the same holds for (1.') and (4.') of Proposition 12.1.10. If Hypothesis 12.3.1 is satisfied, then (4BP1) is independent of D_{21}.

However, D_{21} does affect the well-posedness and the exact form of the suboptimal controllers; cf., e.g., Proposition 12.5.19(g).

Proof: 1° *Hypothesis 12.5.1:* Obviously, the same pairs will do (but $\mathbb{N}_\mathrm{u}$ and $\widetilde{\mathbb{N}}_\mathrm{y}$ are affected).

2° *Hypothesis 12.5.13:* Observe that Σ_X is independent of D_{21}.

3° *Hypothesis 12.3.1:* This follows from Lemma 7.3.23 and Proposition 7.3.14(i)&(iii) (or from equation (7.103).

4° *Theorem 12.1.11:* When $\mathbb{D}$ is replaced by $\mathbb{D}+R$, where $R = \begin{bmatrix} 0 & 0 \\ R_{21} & 0 \end{bmatrix} \in \mathcal{B}(U\times W, Z\times Y)$, the map $\mathbb{M}_\mathrm{u}$ of Hypothesis 12.5.1 is unaffected but $\mathbb{N}_\mathrm{u}$ is replaced by $\mathbb{N}_\mathrm{u} + R\mathbb{M}_\mathrm{u} \in \widetilde{\mathcal{A}}$. An analogous remark applies to $\widetilde{\mathbb{M}}_\mathrm{y}$ and $\widetilde{\mathbb{N}}_\mathrm{y}$.

4° *(1.), (4.), (1.') and (4.'):* This is obvious since corresponding operators are independent on D_{21} except for the r.c. condition in (1.'), which can be handled with Lemma 6.5.7(b).

5° *(4BP1):* Assume Hypothesis 12.3.1 (it is independent of D_{21}, by the last claim of Lemma 7.3.23). Let $\mathbb{D}'$ be equal to $\mathbb{D}$ with 0 in place of D_{21}.

We observe from Lemma 7.3.23 that there is a suboptimal stabilizing DPF-controller for $\mathbb{D}$ (necessarily with d.c. internal loop, by the hypothesis and

Theorem 7.3.19(i)&(i')) iff there is a suboptimal stabilizing DPF-controller for $\mathbb{D}'$. □

The factorizations of Theorem 12.3.7 correspond to system Σ_X and to system Σ_Z of (12.94), hence, by Theorem 9.9.10, to their Riccati equations, that is, to "(1.)" and "(4.)" of Theorem 12.1.8 (the system Σ_Y provides the dual (H^∞ filter problem) Riccati equation "(2.)" of Theorems 12.1.4 and 12.1.5):

Definition 12.5.10 (ARE systems) *We define the following systems and map:*

$$\Sigma_X := \left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline \mathbb{C}_X & \mathbb{D}_X \end{array}\right] := \left[\begin{array}{c|cc} \mathbb{A} & \mathbb{B}_1 & \mathbb{B}_2 \\ \hline \mathbb{C}_1 & \mathbb{D}_{11} & \mathbb{D}_{12} \\ 0 & 0 & I \end{array}\right], \tag{12.84}$$

$$\Sigma_Y := \left[\begin{array}{c|c} \mathbb{A}_Y & \mathbb{B}_Y \\ \hline \mathbb{C}_Y & \mathbb{D}_Y \end{array}\right] := \left[\begin{array}{c|cc} \mathbb{A}^d & \mathbb{C}_2^d & \mathbb{C}_1^d \\ \hline \mathbb{B}_2^d & \mathbb{D}_{22}^d & \mathbb{D}_{12}^d \\ 0 & 0 & I \end{array}\right], \tag{12.85}$$

$$\mathbb{D}_d := \left[\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right]\mathbb{D}^d\left[\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right] = \begin{bmatrix} \mathbb{D}_{22}^d & \mathbb{D}_{12}^d \\ \mathbb{D}_{21}^d & \mathbb{D}_{11}^d \end{bmatrix}. \tag{12.86}$$

See Lemma 12.5.16 for Σ_Z and $\Sigma_{\mathbb{E}^d}$.

Here we write out a few formulae:

Lemma 12.5.11 (Σ_{Total}) *Assume Hypothesis 12.5.1. Then the pairs and $\mathbb{E}_{uy}$ of the hypothesis extend Σ to $\Sigma_{\mathrm{Total}} \in \mathrm{WPLS}(Z\times Y\times U\times W, H, Z\times Y\times U\times W)$, where*

$$\Sigma_{\mathrm{Total}} := \left[\begin{array}{c|cccc} \mathbb{A} & 0 & \mathbb{H}_{y\,2} & \mathbb{B}_1 & \mathbb{B}_2 \\ \hline \mathbb{C}_1 & 0 & \mathbb{G}_{y\,12} & \mathbb{D}_{11} & \mathbb{D}_{12} \\ \mathbb{C}_2 & 0 & \mathbb{G}_{y\,22} & \mathbb{D}_{21} & \mathbb{D}_{22} \\ \mathbb{K}_{u\,1} & 0 & \mathbb{E}_{12}^{uy} & \mathbb{F}_{u\,11} & \mathbb{F}_{u\,12} \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]. \tag{12.87}$$

Moreover, $\left[\begin{array}{c|c} \mathbb{K}_u & \mathbb{F}_u \end{array}\right]$ *stabilizes Σ_X to*

$$\Sigma_{X\flat} := \left[\begin{array}{c|cc} \mathbb{A}+\mathbb{B}_1\tau\mathbb{M}_{u\,11}\mathbb{K}_{u\,1} & \mathbb{B}_1\mathbb{M}_{u\,11} & \mathbb{B}_1\mathbb{M}_{u\,12}+\mathbb{B}_2 \\ \hline \mathbb{C}_1+\mathbb{N}_{u\,11}\mathbb{K}_{u\,1} & \mathbb{N}_{u\,11} & \mathbb{N}_{u\,11} \\ 0 & 0 & I \\ \mathbb{M}_{u\,11}\mathbb{K}_{u\,1} & \mathbb{M}_{u\,11}-I & \mathbb{M}_{u\,12} \\ 0 & 0 & 0 \end{array}\right] \tag{12.88}$$

here $\mathbb{M}_u := (I-\mathbb{F}_u)^{-1} = \left[\begin{smallmatrix} \mathbb{M}_{u\,11} & \mathbb{M}_{u\,12} \\ 0 & I \end{smallmatrix}\right]$.

Analogously, the pair $\left[\begin{array}{c|cc} \mathbb{H}_{y\,2}^{\ d} & \mathbb{G}_{y\,22}^{\ d} & \mathbb{G}_{y\,12}^{\ d} \\ 0 & 0 & 0 \end{array}\right]$ *stabilizes Σ_Y to*

$$\Sigma_{Y\flat} := \left[\begin{array}{c|cc} \mathbb{A}^d+\mathbb{C}_2^d\tau\widetilde{\mathbb{M}}_{y\,22}^{\ d}\mathbb{H}_{y\,2}^{\ d} & \mathbb{C}_2^d\widetilde{\mathbb{M}}_{y\,22}^{\ d} & \mathbb{C}_2^d\widetilde{\mathbb{M}}_{y\,21}^{\ d}+\mathbb{C}_1^d \\ \hline \mathbb{B}_2^d+\widetilde{\mathbb{N}}_{y\,22}^{\ d}\mathbb{H}_{y\,2}^{\ d} & \widetilde{\mathbb{N}}_{y\,22}^{\ d} & \widetilde{\mathbb{N}}_{y\,12}^{\ d} \\ 0 & 0 & I \\ \widetilde{\mathbb{M}}_{y\,22}^{\ d}\mathbb{H}_{y\,2}^{\ d} & \widetilde{\mathbb{M}}_{y\,22}^{\ d}-I & \widetilde{\mathbb{M}}_{y\,12}^{\ d} \\ 0 & 0 & 0 \end{array}\right] \tag{12.89}$$

(all elements above are from (6.171)); here $\widetilde{\mathbb{M}}_{\mathrm{yd}} := (I - \mathbb{G}_{\mathrm{y}}{}^{\mathrm{d}})^{-1} = \begin{bmatrix} (\widetilde{\mathbb{M}}_{\mathrm{yd}})_{11} & (\widetilde{\mathbb{M}}_{\mathrm{yd}})_{12} \\ 0 & I \end{bmatrix} = \left[\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right] \widetilde{\mathbb{M}_{\mathrm{y}}}^{\mathrm{d}} \left[\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right] = \begin{bmatrix} \widetilde{\mathbb{M}}_{\mathrm{y}}{}^{\mathrm{d}}_{22} & \widetilde{\mathbb{M}}_{\mathrm{y}}{}^{\mathrm{d}}_{12} \\ 0 & I \end{bmatrix}$.

Proof: The first claim is from Definition 6.6.21. One easily verifies the formulae. Since all elements of (12.88) are from (6.170) and all elements of (12.89) are from (6.171), it is obvious that these systems are stable. □

Now we are ready for the main part of the proof. We first show that (Factor1) is equivalent to the "$\mathcal{P}_{\mathrm{X}}$-CARE":

Lemma 12.5.12 ((Factor1) ⇔ $\mathcal{P}_{\mathrm{X}}$-CARE) *Assume that Hypothesis 12.5.1 holds and that* $\mathbb{N}_{\mathrm{u}}, \mathbb{M}_{\mathrm{u}} \in \mathrm{UR}$.

Then (Factor1) has a UR solution $\mathbb{N}, \mathbb{M}$ *iff the CARE for* Σ_{X} *and* J_γ *has a UR P-stabilizing solution* $(\mathcal{P}_{\mathrm{X}}, S_X, K_I)$ *s.t.* $\mathbb{D}(I - \mathbb{F})^{-1}$ *and* $(I - \mathbb{F})^{-1}$ *are [q.]r.c.,* $\mathcal{P}_{\mathrm{X}} \geq 0$, $S_{X11} \gg 0$ *and* $S_{X22} - S_{X21} S_{X11}^{-1} S_{X12} \ll 0$.

If $\left[\begin{array}{c|c} \mathbb{K}_{\mathrm{u}} & \mathbb{F}_{\mathrm{u}} \end{array}\right]$ *is exponentially stabilizing, then (Factor1) has a UR solution* $\mathbb{N}, \mathbb{M}$ *iff the CARE for* Σ_{X} *and* J_γ *has a UR exponentially stabilizing solution* $(\mathcal{P}_{\mathrm{X}}, S_X, K_I)$ *s.t.* $\mathcal{P}_{\mathrm{X}} \geq 0$, $S_{X11} \gg 0$ *and* $S_{X22} - S_{X21} S_{X11}^{-1} S_{X12} \ll 0$.

In either case, the IARE for Σ_{X} *and* J_γ *has a UR P-stabilizing solution* $(\mathcal{P}_{\mathrm{X}}, J_1, \left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right])$ *s.t.* $\mathbb{M} := (I - \mathbb{F})^{-1}$, $\mathbb{N} := \mathbb{D}\mathbb{M}$ *and* $\mathbb{X} := \mathbb{M}^{-1}\mathbb{M}_{\mathrm{u}}$ *satisfy (Factor1) and (Factor1X), and Hypothesis 12.5.13 is satisfied.*

Conversely, if Hypothesis 12.5.13 is satisfied, then the IARE for Σ_{X} *and* J_γ *has a unique UR P-stabilizing solution* $(\mathcal{P}_{\mathrm{X}}, J_1, \left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right])$ *s.t.* $(I - \mathbb{F})^{-1} = \mathbb{M}$, $\mathbb{D}\mathbb{M} = \mathbb{N}$.

Proof: Now $\mathbb{D} = \mathbb{N}_{\mathrm{u}} \mathbb{M}_{\mathrm{u}}{}^{-1}$ is UR, by Proposition 6.3.1(b1). By Theorem 12.3.7(e1)&(e2), (Factor1) and (Factor1X) are equivalent, and $\mathbb{X}$ is UR iff $\mathbb{N}$ and $\mathbb{M}$ are UR

1° *(Factor1X) iff CARE for* $\Sigma_{\mathrm{X}\flat}$*:* By Proposition 11.3.4(f), (Factor1X) (or (FI3s)) is equivalent to (FI4s) for $\Sigma_{\mathrm{X}\flat}$. If we add the requirement that $\mathbb{X}$ (or $\mathbb{F}$) is UR, then, by Proposition 11.3.4(a), (FI4s) is equivalent to (FI5s), i.e., to the condition that the CARE for Σ_{X} and J_γ has a UR stable, P-SOS-stabilizing solution $(\mathcal{P}_{\mathrm{X}}, S_X, K_I)$ s.t. $\mathcal{P}_{\mathrm{X}} \geq 0$, $S_{X11} \gg 0$ and $S_{X22} - S_{X21} S_{X11}^{-1} S_{X12} \ll 0$.

2° *(Factor1X) iff CARE for* Σ_{X} *when* $\left[\begin{array}{c|c} \mathbb{K}_{\mathrm{u}} & \mathbb{F}_{\mathrm{u}} \end{array}\right]$ *is exponentially stabilizing:* By Proposition 9.12.4 and Lemma 9.12.3(b), the exponentially stabilizing solutions of the CAREs for $(\Sigma_{\mathrm{X}\flat}, J_\gamma)$ and $(\Sigma_{\mathrm{X}}, J_\gamma)$ correspond to each other one-to-one (they have same $\mathbb{A}_{\circlearrowleft}$'s, by (b), hence one is exponentially stabilizing iff both are); by Theorem 9.8.12, they are equal. Since $\mathbb{F}_{\mathrm{u}}$ is UR, one is UR iff the other one is, by Proposition 9.12.4.

3° *(Factor1X) iff CARE for* Σ_{X}*:* Work as in 2°, let $\mathbb{F}_X$ and $\mathbb{F}_{X\flat}$ correspond to the solutions of the two CAREs, and set $\mathbb{X}_{X\flat} := I - \mathbb{F}_{X\flat}$ and $\mathbb{M}_X := (I - \mathbb{F}_X)^{-1}$.

By Lemma 9.12.3(b), condition (P) is preserved. The map $\mathbb{X}_{X\flat} := \mathbb{M}_X^{-1}\mathbb{M}_{\mathrm{u}}$ is in $\mathcal{G}$TIC iff $\mathbb{D}\mathbb{M}_X$ and $\mathbb{M}_X$ are r.c., by Lemma 6.4.5(c) (since $\mathbb{N}_{\mathrm{u}} = \mathbb{D}\mathbb{M}_{\mathrm{u}}$ and $\mathbb{M}_{\mathrm{u}}$ are r.c.). By Lemma 6.6.17(b)&(c), the solution for the CARE for $(\Sigma_{\mathrm{X}\flat}, J_\gamma)$ is stable and [SOS-]stabilizing iff $\left[\begin{array}{c|c} \mathbb{K}_{X\flat} & \mathbb{F}_{X\flat} \end{array}\right]$ is r.c.-stabilizing; by Lemma 6.7.11(a2), this is the case iff $\left[\begin{array}{c|c} \mathbb{K}_X & \mathbb{F}_X \end{array}\right]$ is q.r.c.-stabilizing for Σ (equivalently, r.c.-stabilizing, by Lemma 6.4.5(c)).

(Note that $(\mathcal{P}_{\rm X}, S_X, K_I)$ are the same in 2° and 3°.)

4° *X and M:* In 1°, we can choose X so that $X_{21} = 0$ and $X_{11}, X_{22} \in \mathcal{GB}$, by Proposition 11.3.4(e) (or by Lemma 11.3.13(i)&(iii') and Theorem 9.8.12), if we replace the CARE by the IARE and S_X by J_1. This is automatic if we are originally given $\mathbb{N}, \mathbb{M}$ satisfying Hypothesis 12.5.13 (the uniqueness claim follows from Theorem 9.8.12(b)(&(s1))).

The rest follows as above (note from 4° that $\mathbb{M}_X = \mathbb{M}_{\rm u}\mathbb{X}_{X\flat}^{-1}$, hence now $\mathbb{M}_X$ becomes equal to $\mathbb{M}$, by (12.49)). $\square$

We shall often assume the following:

Hypothesis 12.5.13 ((Factor1) with $X = \left[\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right]$) *Hypothesis 12.5.1 holds, $\mathbb{N}_{\rm u}, \mathbb{M}_{\rm u} \in$ UR, and $\mathbb{N}, \mathbb{M}$ is an UR solution of (Factor1) s.t. $X_{21} = 0$, where $\mathbb{X} := \mathbb{M}^{-1}\mathbb{M}_{\rm u}$.*

(When this hypothesis holds, we denote by $(\mathcal{P}_{\rm X}, J_1, \left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right])$ the solution of the $\mathcal{P}_{\rm X}$-IARE mentioned at the end of Lemma 12.5.12, and by $\Sigma_{\circlearrowleft}$ the corresponding (stable) closed-loop system.)

(The regularity assumptions might be somewhat (resp. completely) reduced, but this would require somewhat more complicated formulations of the results based on the above hypothesis (resp. and use of IAREs instead of CAREs as in (FI4s) instead of (FI5s) in Theorem 11.3.3).)

By (12.49), "$\mathbb{N}, \mathbb{M} \in$ UR" could be replaced by "$\mathbb{X} \in$ UR".

Lemma 12.5.14 *Make Hypothesis 12.5.13. Then $\mathbb{X}$ satisfies (Factor1X), $M = \begin{bmatrix} M_{11} & M_{12} \\ 0 & X_{22}^{-1} \end{bmatrix}$, and $X_{11}, X_{22}, M_{11} \in \mathcal{GB}$. Moreover, then $\mathbb{D}, \mathbb{X}, \mathbb{E} \in$* UR.

Proof: By Theorem 12.3.7(e2), $\mathbb{X}$ satisfies (Factor1X), hence $\mathbb{X}, \mathbb{X}_{11} \in \mathcal{G}TIC_\infty$. By Lemma A.1.1(b), $\mathbb{M}_{{\rm u}\,11} \in \mathcal{G}TIC_\infty(U)$. By Proposition 6.3.1(b1), it follows that $X, X_{11}, M_{u11} \in \mathcal{GB}$, hence $X_{22}, (X^{-1})_{11}, (X^{-1})_{22} \in \mathcal{GB}(W)$ and $(X^{-1})_{21} = 0$, by Lemma A.1.1(b). By (12.49), $M = \left[\begin{smallmatrix} * & * \\ 0 & I \end{smallmatrix}\right] X^{-1}$, hence also M and M_{11} are as above. By Proposition 6.3.1(b1), $\mathbb{D}, \mathbb{X}, \mathbb{E} \in$ UR. $\square$

We need to define explicitly the closed-loop system corresponding to $\mathcal{P}_{\rm X}$, so that we can define the "$\mathcal{P}_{\rm Z}$-CARE" further below:

Lemma 12.5.15 ($\boldsymbol{\Sigma_{\circlearrowleft}}$) *Make Hypothesis 12.5.13. Assume that $\left[\begin{array}{c|c} \mathbb{K}_{\rm u} & \mathbb{F}_{\rm u} \end{array}\right]$ is [[exponentially] strongly] stabilizing for Σ.*

Then there is a unique [[exponentially] strongly] P-stabilizing solution $(\mathcal{P}_{\rm X}, S_X, \left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right])$ of the IARE for $\Sigma_{\rm X}$ and J_1, and $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ is determined uniquely by requiring that $\mathbb{F} = I - \mathbb{M}^{-1}$. The pair $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ is [[exponentially] strongly] stabilizing for Σ too; we denote the corresponding closed-loop system by

$$\Sigma_{\circlearrowleft} := \left[\begin{array}{c|c} \mathbb{A}_{\circlearrowleft} & \mathbb{B}_{\circlearrowleft} \\ \hline \mathbb{C}_{\circlearrowleft} & \mathbb{D}_{\circlearrowleft} \\ \mathbb{K}_{\circlearrowleft} & \mathbb{F}_{\circlearrowleft} \end{array}\right], \tag{12.90}$$

so that $\mathbb{F}_{\circlearrowleft} = \mathbb{M} - I$, $\mathbb{D}_{\circlearrowleft} = \mathbb{N} = \mathbb{D}\mathbb{M}$.

It is important to note that $\Sigma_{\circlearrowleft}$ is a state-feedback perturbation of Σ, not of Σ_X, although K is determined by the IARE for Σ_X.

Proof: (Recall from Hypothesis 12.5.1 and Theorem 6.6.28 that $\left[\begin{array}{c|c} \mathbb{K}_u & \mathbb{F}_u \end{array}\right]$ is r.c.-stabilizing for Σ even without the extra assumption of the lemma.)

By the assumption, $\left[\begin{array}{c|c} \mathbb{K}_u & \mathbb{F}_u \end{array}\right]$ is [[exponentially] strongly] stabilizing for Σ, hence also for Σ_X. As in the proof of Lemma 12.5.12, we obtain the solution of the IARE mentioned above; being P-stabilizing, $\mathcal{P}_X$ is unique (see Theorem 9.8.12(b)&(s1); note that S_X and $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ is not unique in general before we fix $\mathbb{F}$). By Lemma 6.6.17(d)&(c), $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ is [[exponentially] strongly] stabilizing for Σ too. □

The third Riccati equation traditionally assosiated to the 4BP corresponds to the realization Σ_Z of $\begin{bmatrix} \mathbb{D}^d_{+12} & \mathbb{D}^d_{+22} \\ 0 & I \end{bmatrix} \in \mathrm{TIC}(Y \times U, W \times U)$ defined below:

Lemma 12.5.16 (Σ_Z and $\mathcal{P}_Z$-CARE) *Assume that Hypothesis 12.5.13 holds and that* $\mathbb{M}_{11} \in \mathcal{G}\mathrm{TIC}_\infty(U)$. *Then (see the hypothesis for $\Sigma_{\circlearrowleft}$)*

$$\left[\begin{array}{c|c} \widetilde{\mathbb{K}} & \widetilde{\mathbb{F}} \end{array}\right] := \left[\begin{array}{c|cc} 0 & 0 & 0 \\ \mathbb{B}^d_{\circlearrowleft,1} & \mathbb{N}^d_{21} & I - \mathbb{M}^d_{11} \end{array}\right] \tag{12.91}$$

is a stable admissible state feedback pair for

$$\Sigma_{\mathbb{E}^d} := \left[\begin{array}{c|c} \mathbb{A}_{\mathbb{E}^d} & \mathbb{B}_{\mathbb{E}^d} \\ \hline \mathbb{C}_{\mathbb{E}^d} & \mathbb{D}_{\mathbb{E}^d} \end{array}\right] := \left[\begin{array}{c|cc} \mathbb{A}^d_{\circlearrowleft} & \mathbb{C}^d_{\circlearrowleft,2} & -\mathbb{K}^d_{\circlearrowleft,1} \\ \hline \mathbb{B}^d_{\circlearrowleft,2} & \mathbb{N}^d_{22} & -\mathbb{M}^d_{12} \\ -\mathbb{B}^d_{\circlearrowleft,1} & -\mathbb{N}^d_{21} & \mathbb{M}^d_{11} \end{array}\right] \tag{12.92}$$

$$:= \left[\begin{array}{c|cc} \mathbb{A}^d + \mathbb{K}^d\mathbb{M}^d\tau\mathbb{B}^d & \mathbb{C}^d_2 + (\mathbb{N}\mathbb{K})^d_2 & -(\mathbb{M}\mathbb{K})^d_1 \\ \hline (\mathbb{B}\mathbb{M})^d_2 & (\mathbb{D}\mathbb{M})^d_{22} & -\mathbb{M}^d_{12} \\ -(\mathbb{B}\mathbb{M})^d_1 & -(\mathbb{D}\mathbb{M})^d_{21} & \mathbb{M}^d_{11} \end{array}\right]. \tag{12.93}$$

The top three rows (“the $\left[\begin{array}{c|c} \mathbb{A}_{\circlearrowleft} & \mathbb{B}_{\circlearrowleft} \\ \hline \mathbb{C}_{\circlearrowleft} & \mathbb{D}_{\circlearrowleft} \end{array}\right]$ part“) of the corresponding closed-loop system are given by

$$\Sigma_Z := \left[\begin{array}{c|c} \mathbb{A}_Z & \mathbb{B}_Z \\ \hline \mathbb{C}_Z & \mathbb{D}_Z \end{array}\right] := \left[\begin{array}{c|cc} \mathbb{A}^d + \mathbb{K}^d_2\tau\mathbb{X}^{-d}_{22}\mathbb{B}^d_2 & \mathbb{C}^d_2 + \mathbb{K}^d_2\mathbb{X}^{-d}_{22}\mathbb{D}^d_{22} & -\mathbb{K}^d_1 + \mathbb{K}^d_2\mathbb{X}^{-d}_{22}\mathbb{X}^d_{12} \\ \hline \mathbb{X}^{-d}_{22}\mathbb{B}^d_2 & \mathbb{X}^{-d}_{22}\mathbb{D}^d_{22} & \mathbb{X}^{-d}_{22}\mathbb{X}^d_{12} \\ 0 & 0 & I \end{array}\right] \tag{12.94}$$

$\in \mathrm{WPLS}(Y \times U, H, W \times U)$ *(see (12.17) for corresponding generators). In particular,*

$$\mathbb{D}_Z = \begin{bmatrix} \mathbb{X}^{-d}_{22}\mathbb{D}^d_{22} & \mathbb{X}^{-d}_{22}\mathbb{X}^d_{12} \\ 0 & I \end{bmatrix} = \begin{bmatrix} \mathbb{D}^d_{+12} & \mathbb{D}^d_{+22} \\ 0 & I \end{bmatrix} \in \mathrm{TIC}(Y \times U, W \times U). \tag{12.95}$$

The system $\Sigma_{\mathbb{E}^d}$ is stable; it is exponentially stable iff Σ is optimizable (by Lemma 12.5.2). □

(This is obvious. Note from Lemma 12.5.14 and Proposition 6.3.1(c) that the assumption on $\mathbb{M}_{11}$ is redundant if $\mathbb{M} \in \mathrm{ULR}$.)

The CARE (resp. IARE) for Σ_Z and J_1 is called the *$\mathcal{P}_Z$-CARE* (resp. *$\mathcal{P}_Z$-IARE*).

Lemma 12.5.17 ($\mathcal{P}_Z$-IARE $\Leftrightarrow$ (Factor2)) *Assume that Hypothesis 12.5.13 holds and $\mathbb{M}_{11} \in \mathcal{G}\mathrm{TIC}_\infty(U)$. Consider the following conditions:*

(i) (Factor2Z) has a UR solution $\mathbb{Z} \in \mathcal{G}\mathrm{TIC}(Y \times U)$ s.t. $Z_{12} = 0$ and $Z_{11}, Z_{22} \in \mathcal{G}\mathcal{B}$.

(ii) the CARE for Σ_Z and J_1 has a UR P-stabilizing solution $(\mathcal{P}_Z, S_Z, K_{Z,I})$ s.t. $\mathcal{P}_Z \geq 0$, $S_{Z11} \gg 0$, $S_{Z22} - S_{Z21}S_{Z11}^{-1}S_{Z12} \ll 0$, $(I - \mathbb{F}_{Z,I})(I - \widetilde{\mathbb{F}}) \in \mathcal{G}\mathrm{TIC}(Y \times U)$ and $\mathbb{K}_{Z,I} + (I - \mathbb{F}_{Z,I})\widetilde{\mathbb{K}}$ is stable.

(iii) the CARE for $\Sigma_{\mathbb{E}^{\mathrm{d}}}$ and J_1 has a UR stable, P-stabilizing solution $(\mathcal{P}_Z, S_E, K_E)$ s.t. $\mathcal{P}_Z \geq 0$, $S_{E11} \gg 0$ and $S_{E22} - S_{E21}S_{E11}^{-1}S_{E12} \ll 0$.

(ii') the CARE for Σ_Z and J_1 has a UR exponentially stabilizing solution $(\mathcal{P}_Z, S_Z, K_{Z,I})$ s.t. $\mathcal{P}_Z \geq 0$, $S_{Z11} \gg 0$ and $S_{Z22} - S_{Z21}S_{Z11}^{-1}S_{Z12} \ll 0$.

(iii') the CARE for $\Sigma_{\mathbb{E}^{\mathrm{d}}}$ and J_1 has a UR exponentially stabilizing solution $(\mathcal{P}_Z, S_E, K_E)$ s.t. $\mathcal{P}_Z \geq 0$, $S_{E11} \gg 0$ and $S_{E22} - S_{E21}S_{E11}^{-1}S_{E12} \ll 0$.

Then (c1)–(d) below hold; if $D_{21} = 0$, then also (a) and (b) hold:

(a) We have (i)$\Leftrightarrow$(ii)$\Leftrightarrow$(iii).

(b) If $\begin{bmatrix} \mathbb{K}_{\mathrm{u}} & \mathbb{F}_{\mathrm{u}} \end{bmatrix}$ is exponentially stabilizing, then (i)–(iii') are equivalent.

(c1) If (Factor2Z) has a solution, then (ii)–(iii) (and (ii') and (iii') if $\begin{bmatrix} \mathbb{K}_{\mathrm{u}} & \mathbb{F}_{\mathrm{u}} \end{bmatrix}$ is exponentially stabilizing) hold if we drop "UR" and the conditions on S_E and S_Z, and we replace "CARE" by "IARE".

(c2) If (Factor2Z) has a UR solution, then (ii) holds (and (ii') if $\begin{bmatrix} \mathbb{K}_{\mathrm{u}} & \mathbb{F}_{\mathrm{u}} \end{bmatrix}$ is exponentially stabilizing).

(d) If (ii) has a solution (or (ii') has and $\begin{bmatrix} \mathbb{K}_{\mathrm{u}} & \mathbb{F}_{\mathrm{u}} \end{bmatrix}$ is exponentially stabilizing), then $\mathbb{E}J_1\mathbb{E}^* = \mathbb{Z}J_1\mathbb{Z}^*$ for some $\mathbb{Z} \in \mathcal{G}\mathrm{TIC}(Y \times U)$ s.t. $\mathbb{E}^{\mathrm{d}}\mathbb{Z}^{-\mathrm{d}}$ is (J_1, J_1)-lossless and $Z_{11} \in \mathcal{G}\mathcal{B}(Y)$.

If, in addition, $\dim U < \infty$, then $\mathbb{Z}$ satisfies (Factor2Z) (and (Factor2$\widetilde{\mathcal{A}}$) if $\mathbb{N}_{\mathrm{u}}, \mathbb{M}_{\mathrm{u}} \in \widetilde{\mathcal{A}}$).

Recall from Theorem 12.3.7(a) that if $(\mathbb{E}^{\mathrm{d}}, J_1) \in \mathrm{SpF}$, then one more equivalent condition is that the 4BP has a solution. We remind that most readers should ignore the nonexponential 4BP (and hence (ii) and (iii) and most of the proof below); it combines a less important problem with more complicated proofs and solutions.

Note that we could drop the regularity assumption in (i) and formulate (ii[']) and (iii[']) analogously to (FI4) (see Theorem 11.3.3).

In the proof of Theorem 12.1.8 and Proposition 12.1.10, we strengthen the equivalence of (Factor2) and (ii[']) (by establishing a converse to (c2) without assuming that $D_{21} = 0$).

The condition in (ii) may seem complicated (it corresponds to $\mathbb{Z}^{\mathrm{d}}$ being a spectral factor of $(\mathbb{E}^{\mathrm{d}})^* J\mathbb{E}^{\mathrm{d}}$) compared to the simple r.c. condition of Lemma 12.5.12 (corresponding to $\mathbb{X}$ solving (Factor1)). The explanation is that the word "stable" in (iii) is, indeed, a r.c. condition (by Corollary 9.9.11), but since (12.91) need not be (q.)r.c.-stabilizing in general, the corresponding condition in (ii) is not a r.c. condition w.r.t. Σ_Z. This explains the corresponding difference in (1.') and (4.') of Proposition 12.1.10.

Proof of Lemma 12.5.17: (Much of this follows the lines of the proof of Lemma 12.5.12, we just need certain more complicated details. For each claim, we first give the proof where $\left[\begin{array}{c|c} \mathbb{K}_u & \mathbb{F}_u \end{array}\right]$ is assumed to be exponentially stabilizing; most readers will skip the more complicated and less important part (with a mere stabilization assumption).)

Remarks and notation: Since $\mathbb{N},\mathbb{M} \in \mathrm{UR}$, we have $\mathbb{D},\mathbb{M}^{-1},\mathbb{D}_+,\mathbb{D}_Z \in \mathrm{UR}$, As shown in the proof, the operators in (ii)–(iii') having same symbols are equal. The solutions of (ii)–(iii') (if any) are unique, by Theorem 9.8.12(b)&(s1).

We use the standard notation where $\mathbb{X} := I - \mathbb{F}$, $\mathbb{M} := \mathbb{X}^{-1}$, and $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ corresponds to K (with any subscripts). We also use the notation of Lemma 12.5.16.

The proof: We give the proofs in the order (c1), (c2), (d), (a), (b).

(c1) This follows from 2° and 4° (and 1° and 3° if $\left[\begin{array}{c|c} \mathbb{K}_u & \mathbb{F}_u \end{array}\right]$ is exponentially stabilizing).

1° *(Factor2Z) $\Rightarrow$ weakened (ii') (when $\left[\begin{array}{c|c} \mathbb{K}_u & \mathbb{F}_u \end{array}\right]$ is exponentially stabilizing):* Assume (Factor2Z). By Corollary 9.9.11 (and Theorem 9.9.10), the IARE for $\Sigma_{\mathbb{E}^d}$ and J_1 has an exponentially stabilizing solution $(\mathcal{P}_Z, J_1, \left[\begin{array}{c|c} \mathbb{K}_E & \mathbb{F}_E \end{array}\right])$, where $\mathbb{F}_E = I - \mathbb{Z}^d$ (note from Lemma 12.5.16 that $\Sigma_{\mathbb{E}^d}$ is exponentially stable).

From 3° we obtain an exponentially stabilizing solution $(\mathcal{P}_Z, J_1, \left[\begin{array}{c|c} \mathbb{K}_Z & \mathbb{F}_Z \end{array}\right])$ of the $\mathcal{P}_Z$-IARE. It only remains to be shown that $\mathcal{P}_Z \geq 0$.

By (9.224), we have $\mathbb{X}_Z = \mathbb{X}_E \widetilde{\mathbb{M}}$, hence $\mathbb{Z}^{-d} = \mathbb{M}_E = \widetilde{\mathbb{M}}\mathbb{M}_Z$ and

$$\mathbb{D}_{\circlearrowleft,Z} := \mathbb{D}_Z\mathbb{M}_Z = (\mathbb{E}^d\widetilde{\mathbb{M}})\mathbb{M}_Z = \mathbb{E}^d\mathbb{Z}^{-d} =: \mathbb{W}^d \in \mathrm{TIC}(Y \times U, W \times U), \quad (12.96)$$

where $\mathbb{M}_Z := (I - \mathbb{F}_Z)^{-1}$, $\mathbb{M}_E = (I - \mathbb{F}_E)^{-1}$. By (12.96) and Lemma 12.3.11(a), $(\mathbb{D}_{\circlearrowleft,Z})_{22} = \mathbb{W}^d_{22} \in \mathcal{G}\mathrm{TIC}(U)$. But $(\mathbb{M}_Z)_{22} = (\mathbb{D}_{\circlearrowleft,Z})_{22}$ (since $\mathbb{D}_Z = \left[\begin{smallmatrix} * & * \\ 0 & I \end{smallmatrix}\right]$, by (12.95)), and $(J_1)_{22} = -I \ll 0$, hence it follows from Lemma 11.2.18 (see the remark in its proof) that $\mathcal{P}_Z \geq 0$ (and that there is a suboptimal H^∞-FI-pair for Σ_Z and $\gamma = 1$ and that Hypothesis 11.2.1 is satisfied for Σ_Z over $\mathcal{U}_{\mathrm{exp}}$).

2° *(Factor2Z) $\Rightarrow$ weakened (ii):* Assume (Factor2Z). (We start almost as in 1°, but the proof of "$\mathcal{P}_Z \geq 0$" must be written more explicitly.)

By Corollary 9.9.11 (and Theorem 9.9.10), the IARE for $\Sigma_{\mathbb{E}^d}$ and J_1 has a stable, P-stabilizing solution $(\mathcal{P}_Z, J_1, \left[\begin{array}{c|c} \mathbb{K}_E & \mathbb{F}_E \end{array}\right])$, where $\mathbb{F}_E = I - \mathbb{Z}^d$ (note from Lemma 12.5.16 that $\Sigma_{\mathbb{E}^d}$ is stable).

From 4° we obtain a P-stabilizing solution $(\mathcal{P}_Z, J_1, \left[\begin{array}{c|c} \mathbb{K}_Z & \mathbb{F}_Z \end{array}\right])$ of the $\mathcal{P}_Z$-IARE (by $\Sigma_\circlearrowleft$ we shall denote the corresponding (stable) closed-loop system) s.t. $(I - \mathbb{F}_{Z,I})(I - \widetilde{\mathbb{F}}) \in \mathcal{G}\mathrm{TIC}(Y \times U)$ and $\mathbb{K}_{Z,I} + (I - \mathbb{F}_{Z,I})\widetilde{\mathbb{K}}$ is stable. It only remains to be shown that $\mathcal{P}_Z \geq 0$.

By (9.224), we have $\mathbb{X}_Z = \mathbb{X}_E\widetilde{\mathbb{M}}$, hence $\mathbb{Z}^{-\mathrm{d}} = \mathbb{M}_E = \widetilde{\mathbb{M}}\mathbb{M}_Z$ and

$$\mathbb{D}_{\circlearrowleft,Z} := \mathbb{D}_Z\mathbb{M}_Z = (\mathbb{E}^{\mathrm{d}}\widetilde{\mathbb{M}})\mathbb{M}_Z = \mathbb{E}^{\mathrm{d}}\mathbb{Z}^{-\mathrm{d}} =: \mathbb{W}^{\mathrm{d}} \in \mathrm{TIC}(Y\times U, W\times U), \quad (12.97)$$

where $\mathbb{M}_Z := (I-\mathbb{F}_Z)^{-1}$, $\mathbb{M}_E = (I-\mathbb{F}_E)^{-1}$. By (12.96) and Lemma 12.3.11(a), $(\mathbb{D}_{\circlearrowleft,Z})_{22} = \mathbb{W}^{\mathrm{d}}_{22} \in \mathcal{G}\mathrm{TIC}(U)$. But $(\mathbb{M}_Z)_{22} = (\mathbb{D}_{\circlearrowleft,Z})_{22}$ (since $\mathbb{D}_Z = \left[\begin{smallmatrix} * & * \\ 0 & I \end{smallmatrix}\right]$, by (12.95)), and $(J_1)_{22} = -I \ll 0$,

In 1°, we obtained $\mathcal{P}_Z \ge 0$ at this point from Lemma 11.2.18. Since this time $\mathcal{P}_Z$ need not be exponentially stabilizing, we shall apply Lemma 11.2.18 for $\Sigma^{\curvearrowleft}$ (see below) instead of Σ_Z in this case.

The pair $\left[\begin{array}{c|c} \overline{\mathbb{K}} & \overline{\mathbb{F}} \end{array}\right] := \left[\begin{array}{c|cc} \mathbb{K}_{Z1} & \mathbb{F}_{Z11} & \mathbb{F}_{Z12} \\ 0 & 0 & 0 \end{array}\right]$ is admissible for Σ_Z, because $\mathbb{F}_{Z11} \in \mathcal{G}\mathrm{TIC}_\infty(Y)$ (because $\mathbb{M}_{Z22} = (\mathbb{D}_{\circlearrowleft,Z})_{22} \in \mathcal{G}\mathrm{TIC}(U)$, as shown above); let $\Sigma^{\curvearrowleft}$ be the corresponding closed-loop system. Note that

$$\mathbb{D}^{\curvearrowleft} = \mathbb{D}_Z\overline{\mathbb{M}} = \begin{bmatrix} * & * \\ 0 & I \end{bmatrix}, \text{ where } \overline{\mathbb{M}} := (I-\overline{\mathbb{F}})^{-1} = \begin{bmatrix} * & * \\ 0 & I \end{bmatrix}. \quad (12.98)$$

By Lemma 9.12.3 $(\mathcal{P}, J_1, \left[\begin{array}{c|c} \mathbb{K}_\natural & \mathbb{F}_\natural \end{array}\right])$ is a solution of the IARE for $\Sigma^{\curvearrowleft}{}_0 := \left[\begin{array}{c|c} \mathbb{A}^{\curvearrowleft} & \mathbb{B}^{\curvearrowleft} \\ \hline \mathbb{C}^{\curvearrowleft} & \mathbb{D}^{\curvearrowleft} \end{array}\right]$ and J_1, where

$$\begin{aligned}\left[\begin{array}{c|c} \mathbb{K}_\natural & \mathbb{F}_\natural \end{array}\right] &:= \left[\begin{array}{c|c} \mathbb{K}-\mathbb{X}\mathbb{K}^{\curvearrowleft} & I-\mathbb{X}(I-\left[\begin{smallmatrix} \mathbb{F}_{Z11} & \mathbb{F}_{Z12} \\ 0 & 0 \end{smallmatrix}\right])^{-1} \end{array}\right] \\ &= \left[\begin{array}{c|cc} 0 & 0 & 0 \\ \hline \mathbb{M}_{Z22}^{-1}\mathbb{K}_{\circlearrowleft 2} & \mathbb{M}_{Z22}^{-1}\mathbb{M}_{Z21} & I-\mathbb{M}_{Z22}^{-1} \end{array}\right]\end{aligned} \quad (12.99)$$

(the last equality follows by direct computation). But $\left[\begin{array}{c|c} \mathbb{K}_\natural & \mathbb{F}_\natural \end{array}\right]$ is stable (because $\Sigma_{\circlearrowleft}$ is stable and $\mathbb{M}_{Z22} \in \mathcal{G}\mathrm{TIC}(U)$), and so is $(I-\mathbb{F}_\natural)^{-1} = \left[\begin{smallmatrix} I & 0 \\ \mathbb{M}_{Z21} & \mathbb{M}_{Z22} \end{smallmatrix}\right]$, hence $\Sigma^{\curvearrowleft}$ is stable, by Corollary 6.6.9 (see Definition 6.6.10), hence $\left[\begin{array}{c|c} \mathbb{K}_\natural & \mathbb{F}_\natural \end{array}\right]$ is r.c.-stabilizing, by Lemma 6.6.17(d).

Consequently, we can apply Lemma 11.2.18 to $\Sigma^{\curvearrowleft}$ to obtain that $\mathcal{P}_Z \ge 0$ (we have not established Hypothesis 11.2.1 for $\Sigma^{\curvearrowleft}$, but that hypothesis is redundant, as noted in the latter remark of the proof of Lemma 11.2.18; we did use the fact that $\mathbb{D}^{\curvearrowleft} = \left[\begin{smallmatrix} * & * \\ 0 & I \end{smallmatrix}\right]$, i.e., that Hypothesis 11.1.1 is satisfied).

3° *(c1)-form of (ii')⇔(iii'):* Given an exponentially stabilizing solution $(\mathcal{P}_Z, S_Z, \left[\begin{array}{c|c} \mathbb{K}_E & \mathbb{F}_E \end{array}\right])$ of the IARE for $\Sigma_{\mathbb{E}^{\mathrm{d}}}$ and J_1, an exponentially stabilizing solution $(\mathcal{P}_Z, S_Z, \left[\begin{array}{c|c} \mathbb{K}_Z & \mathbb{F}_Z \end{array}\right])$ of the $\mathcal{P}_Z$-IARE (i.e., of the IARE for Σ_Z and J_1) is then determined by

$$\mathbb{X}_E = \mathbb{X}_Z\widetilde{\mathbb{X}} \quad\text{and}\quad \mathbb{K}_E = \mathbb{K}_Z + \mathbb{X}_E(\widetilde{\mathbb{M}}\widetilde{\mathbb{K}}) = \mathbb{K}_Z + \mathbb{X}_Z\widetilde{\mathbb{K}}, \quad (12.100)$$

by Lemma 9.12.3(b) (this has same closed-loop semigroup, hence also this is exponentially stabilizing). Thus, (ii') implies (iii') after the changes listed in (c). The converse follows analogously.

(For future use we note that both solutions are UR if one is, because $\widetilde{\mathbb{F}}$ is UR).

4° *(c1)-form of (ii)⇔(iii):* In the same way as in 3°, we observe from Lemma 9.12.3(b), that the P-admissible solutions $(\mathcal{P}_Z, S_Z, \left[\begin{array}{c|c} \mathbb{K}_Z & \mathbb{F}_Z \end{array}\right])$ and $(\mathcal{P}_Z, S_Z, \left[\begin{array}{c|c} \mathbb{K}_E & \mathbb{F}_E \end{array}\right])$ of the two IAREs correspond to each other one-to-one through (12.100).

By Lemma 6.6.17(c), $\left[\begin{array}{c|c} \mathbb{K}_E & \mathbb{F}_E \end{array}\right]$ is stable and stabilizing iff $\mathbb{K}_E$ is stable and $I-\mathbb{F}_E \in \mathcal{G}\mathrm{TIC}(Y\times U)$. By (12.100), this is the case iff $(I-\mathbb{F}_Z)(I-\widetilde{\mathbb{F}}) \in \mathcal{G}\mathrm{TIC}(Y\times U)$ and $\mathbb{K}_E = \mathbb{K}_Z + \mathbb{X}_Z\widetilde{\mathbb{K}}$ is stable. Thus, (ii) implies (iii).

Assume (iii), so that $\mathbb{M}_E$ and $\mathbb{K}_E$ are stable. Then also $\mathbb{M}_Z = \widetilde{\mathbb{X}}\mathbb{M}_E$ is stable. By the above, it only remains to show that $\mathcal{P}_{\mathrm{Z}}$ is stabilizing for $\Sigma_{\mathbb{E}^{\mathrm{d}}}$ and J_1. By Lemma 9.12.3(b), it suffices to show that $\mathbb{M}_Z\mathbb{K}_Z$ is stable (since $\mathbb{M}_Z = \widetilde{\mathbb{X}}\mathbb{M}_E$ is stable and the rest of the closed-loop system is contained in that for Σ_{Z}). By (12.100),

$$\mathbb{M}_Z\mathbb{K}_Z = \mathbb{M}_Z\mathbb{K}_E - \mathbb{M}_Z\mathbb{X}_Z\widetilde{\mathbb{K}} = \mathbb{M}_Z\mathbb{K}_E - \widetilde{\mathbb{K}}, \tag{12.101}$$

which is stable (since also $\mathbb{K}_E$ and $\widetilde{\mathbb{K}}$ are stable). Thus, $\mathcal{P}_{\mathrm{Z}}$ is stabilizing, hence (ii) holds.

(c2) Much of (c2) follows from (c1). The CARE and the condition on S_Z are obtained from 2° (and in 1° if $\begin{bmatrix} \mathbb{K}_{\mathrm{u}} & \big| & \mathbb{F}_{\mathrm{u}} \end{bmatrix}$ is exponentially stabilizing) below.

(Note that (d) or (a)&(b) contains the converse implication if $D_{21} = 0$ or $\dim U < \infty$.)

1° *(ii') holds when* $\begin{bmatrix} \mathbb{K}_{\mathrm{u}} & \big| & \mathbb{F}_{\mathrm{u}} \end{bmatrix}$ *is exponentially stabilizing:* It was noted at the end of (c1)1° that Hypothesis 11.2.1 is satisfied by Σ_{Z} (for $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{exp}}$) and that $1 > \gamma_{\mathrm{FI}}$ (for Σ_{Z}), hence $1 > \gamma_0$, by (11.12).

Since $\mathbb{Z} \in \mathrm{UR}$, we have $\mathbb{F}_E, \mathbb{M}_E \in \mathrm{UR}$, (because $\mathbb{F}_E = I - \mathbb{Z}^{\mathrm{d}}$, by (c1)1°) by Proposition 6.3.1(b1), hence $\mathbb{M}_Z = \widetilde{\mathbb{X}}\mathbb{M}_E \in \mathrm{UR}$. Consequently, the triple $(\mathcal{P}_{\mathrm{Z}}, J_1, \begin{bmatrix} \mathbb{K}_Z & \big| & \mathbb{F}_Z \end{bmatrix})$ of (c1)1°&(c1)3° corresponds to a (unique) exponentially stabilizing solution $(\mathcal{P}_{\mathrm{Z}}, S_Z, K_{Z,I})$ of the $\mathcal{P}_{\mathrm{Z}}$-CARE, by Corollary 9.9.8. By Proposition 11.2.19(d1), we have $S_{Z11} \gg 0$ and $S_{Z22} - S_{Z21}S_{Z11}^{-1}S_{Z12} \ll 0$.

2° *(ii) holds:* Almost as in 1° above, we obtain a unique UR r.c.-stabilizing solution $(\mathcal{P}, S_\natural, K_{\natural,I})$ of the CARE for $\Sigma^{\curvearrowleft}{}_0$ and J_1 (use (c1)2°&(c1)4° and Corollary 9.9.8), and $S_\natural = X_\natural^* J_1 X_\natural$, where $X_\natural = I - F_\natural$, and that $S_{\natural 11} \gg 0$ and $S_{\natural 22} - S_{\natural 21}S_{\natural 11}^{-1}S_{\natural 12} \ll 0$.

The solution $(\mathcal{P}, J_1, \begin{bmatrix} \mathbb{K}_\natural & \big| & \mathbb{F}_\natural \end{bmatrix})$ of the CARE for $\Sigma^{\curvearrowleft}{}_0$ and J_1 corresponds analogously to a UR solution $(\mathcal{P}, S_Z, K_{Z,I})$ of the CARE for Σ_{Z} and J_1 (which also satisfies the (c1)-form of (ii)), where $S_Z = X^*J_1X = \overline{X}^*S_\natural\overline{X}$, $\overline{X} := X_\natural^{-1}X = \left[\begin{smallmatrix} X_{11} & X_{12} \\ 0 & I \end{smallmatrix}\right]$. Consequently, also S_Z satisfies the required condition, by Lemma 11.3.13(i)&(ii'').

(d) We prove "(ii')$\Rightarrow$(i)" in 1° (and 2°) assuming that $\begin{bmatrix} \mathbb{K}_{\mathrm{u}} & \big| & \mathbb{F}_{\mathrm{u}} \end{bmatrix}$ is exponentially stabilizing; the other case follows analogously (use (c1)4° instead of (c1)3°).

(N.B. By dropping the assumption $D_{21} = 0$ we lost the condition on S_E in (iii') (compared to (b)), hence do not longer know whether Z can be chosen as in (i) or whether W_{22} is invertible (the latter is true if $\dim U < \infty$).)

1° Assume (ii'). By Lemma 11.3.13(i)&(iii'), there is $\widetilde{Z} \in \mathcal{GB}(Y\times U)$ s.t. $\widetilde{Z}J_1\widetilde{Z}^* = S_Z$. Apply Theorem 9.8.12(s1) to redefine the solution so that we obtain another UR nonnegative exponentially stabilizing solution $(\mathcal{P}_{\mathrm{Z}}, J_1, \begin{bmatrix} \widetilde{\mathbb{K}}_Z & \big| & \widetilde{\mathbb{F}}_Z \end{bmatrix})$ of the IARE for Σ_{Z} and J_1 s.t. $I - \widetilde{\mathbb{F}}_Z = \widetilde{Z}^*$.

Set $Z := \widetilde{X}^*\widetilde{Z}$, so that $ZS_ZZ^* = \widetilde{X}^*S_Z\widetilde{X} =: S_E$. By (c1)3°, we get an UR exponentially stabilizing solution $(\mathcal{P}_{\mathrm{Z}}, J_1, \begin{bmatrix} \widetilde{\mathbb{K}}_E & \big| & \widetilde{\mathbb{F}}_E \end{bmatrix})$ of the IARE for $\Sigma_{\mathbb{E}^{\mathrm{d}}}$

and J_1 s.t. $\widetilde{X}_E = \widetilde{X}_Z\widetilde{X} = \widetilde{Z}^*\widetilde{X} = Z^*$. By Corollary 9.9.11, $\mathbb{Z}^{\rm d} := I - \widetilde{\mathbb{F}}_E \in \mathcal{G}\mathrm{TIC}(Y \times U)$. By Lemma 9.8.14, $\mathbb{W}^{\rm d} := \mathbb{E}^{\rm d}\mathbb{Z}^{-\rm d}$ is (J_1,J_1)-lossless. But $\widetilde{X} = [\begin{smallmatrix} I & 0 \\ * & * \end{smallmatrix}]$, by (12.91), hence $Z_{11} = \widetilde{Z}_{11} \in \mathcal{GB}(Y)$.

2° *Case* $\dim U < \infty$*:* Since $\mathbb{W}^{\rm d} := \mathbb{E}^{\rm d}\mathbb{Z}^{-\rm d}$ is (J_1,J_1)-lossless, we have $\mathbb{W}_{22} \in \mathcal{G}\mathrm{TIC}(U)$, by Proposition 2.5.4(1), hence (Factor2Z) is satisfied (hence so is (Factor2), hence also (4BP3), hence (4BP1) and (4BP2), by Theorem 12.3.7(a)&(e1))

(If $\mathbb{N}_{\rm u},\mathbb{M}_{\rm u} \in \widetilde{\mathcal{A}}$, then $\mathbb{N},\mathbb{M},\widetilde{\mathbb{M}}_+,\widetilde{\mathbb{N}}_+ \in \widetilde{\mathcal{A}}$, by Lemma 12.3.10(a). Thus, then (Factor2$\widetilde{\mathcal{A}}$) is satisfied, since $\widetilde{\mathbb{M}}_{+22} = \mathbb{W}_{22} \in \mathcal{G}\mathrm{TIC}(U)$ and $\widetilde{N}_{+21} = (Z^{-1})_{22} \in \mathcal{GB}(U)$ (since $Z_{11} \in \mathcal{GB}(Y)$), by (12.50).)

(a) By (c2), (i) implies (ii). The converse follows from 2°, and "(ii)⇔(iii)" from 1°.

1° *(ii)⇔(iii):* Let S_Z be as in (ii) (and $\left[\begin{array}{c|c} \mathbb{K}_Z & \mathbb{F}_Z \end{array}\right]$ corresponds to a solution of the CARE, i.e., $F_Z = 0$), then $X_E = X_Z\widetilde{X} = \widetilde{X}$, where $(\mathcal{P}_{\rm Z},S_Z,\left[\begin{array}{c|c} \mathbb{K}_E & \mathbb{F}_E \end{array}\right])$ is as in (c1)4°.

By Corollary 9.9.8, the corresponding solution $(\mathcal{P},S_E,K_E)$ of the CARE for $\Sigma_{\mathbb{E}^{\rm d}}$ and J_1 has the signature operator $S_E := \widetilde{X}^*S_Z\widetilde{X}$. By Lemma 11.3.13(ii"), also S_E is as in (iii) (since $\widetilde{X} = \left[\begin{smallmatrix} I & 0 \\ 0 & M_{11} \end{smallmatrix}\right]$ due to the assumption that $D_{21} = 0$). Thus, (ii) implies (iii). The converse is analogous.

2° *(ii)⇒(i):* This follows from (d), since now $Z := \widetilde{X}^*\widetilde{Z}$, by (d)1°, where $\widetilde{X} = \left[\begin{smallmatrix} I & 0 \\ 0 & M_{11}^* \end{smallmatrix}\right]$ and $\widetilde{Z}$ is as in (i).

(b) The proof of (a) will do with slight changes. □

Lemma 12.5.18 ($\mathcal{P}_{\rm Z} \Leftrightarrow \mathcal{P}_{\rm Y}$ & $\rho(\mathcal{P}_{\rm X}\mathcal{P}_{\rm Y}) < \gamma^2$) *Assume that Hypothesis 12.5.13 holds and* $\mathbb{M}_{11} \in \mathcal{G}\mathrm{TIC}_\infty(U)$*. Let* $\mathcal{P}_{\rm X}$ *be the corresponding P-stabilizing solution of the* $\mathcal{P}_{\rm X}$*-CARE (as in Lemma 12.5.12). Assume that the* $\mathcal{P}_{\rm Z}$*-eIARE and the* $\mathcal{P}_{\rm Y}$*-eIARE have internally P-stabilizing solutions* $\mathcal{P}_{\rm Z}$ *and* $\mathcal{P}_{\rm Y}$*, respectively.*

Then $\mathcal{P}_{\rm Z} \geq 0$ *iff* $\mathcal{P}_{\rm Y} \geq 0$ *and* $\rho(\mathcal{P}_{\rm X}\mathcal{P}_{\rm Y}) < \gamma^2$*. If* $\mathcal{P}_{\rm Z} \geq 0$*, then (a)–(c) of Lemma 12.6.4 hold for the solutions of the eDAREs.*

Proof: 1° *The assumptions of Lemma 12.6.4 are satisfied:* By Lemma 12.5.12, the IARE for $\Sigma_{\rm X}$ and J_γ has a UR P-stabilizing solution $(\mathcal{P}_{\rm X},J_1,\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right])$ s.t. $(I-\mathbb{F})^{-1} = \mathbb{M}$, $\mathbb{D}\mathbb{M} = \mathbb{N}$. Since $\mathbb{M}_{22} = (\mathbb{X}^{-1})_{22}$ (by (12.49)), we have $\mathbb{M}_{22} \in \mathcal{G}\mathrm{TIC}_\infty(W)$ (because $\mathbb{X}_{11} \in \mathcal{G}\mathrm{TIC}$), hence $\mathbb{X}'_{11} \in \mathcal{G}\mathrm{TIC}_\infty(U)$, where $\mathbb{X}' := \mathbb{M}^{-1}$, by Lemma A.1.1(c1).

By discretization (see Proposition 9.8.7(a)), we observe that $(\mathcal{P}_{\rm X},J_1,\Delta^{\rm S}\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right])$ a solution of the IARE for $\Delta^{\rm S}\Sigma_{\rm X}$ and J_γ; the corresponding solution $(\mathcal{P}_{\rm X},S',K')$ of the DARE ("$\mathcal{P}_{\rm X}$-DARE") satisfies the assumptions of Lemma 12.6.4, by Lemma 11.3.13(vi)&(i) (recall that $\mathbb{X}',\mathbb{X}'_{11} \in \mathcal{G}\mathrm{TIC}_\infty$ and that $\mathbb{X}'_{21}(\mathbb{X}'_{11})^{-1} = -\mathbb{M}_{22}^{-1}\mathbb{M}_{21} = \mathbb{X}_{21}(\mathbb{X}_{11})^{-1}$ (because $\mathbb{M}\mathbb{X}' = I$ and $\mathbb{M}_{2*} = (\mathbb{X}^{-1})_{2*}$, by (12.49)), hence $\|\mathbb{X}'_{21}(\mathbb{X}'_{11})^{-1}\| < 1$, by Lemma 12.3.11(a)).

2° *The equivalence:* By discretization (see again Proposition 9.8.7(a)), we obtain that $\mathcal{P}_{\rm X}$, $\mathcal{P}_{\rm Y}$, $\mathcal{P}_{\rm X}$ are P-stabilizing solutions of the $\mathcal{P}_{\rm X}$-eDARE, $\mathcal{P}_{\rm Y}$-eDARE and $\mathcal{P}_{\rm Z}$-eDARE (the ones corresponding to $(\Delta^{\rm S}\Sigma_{\rm X},J_\gamma)$, $(\Delta^{\rm S}\Sigma_{\rm Y},J_\gamma)$ and $(\Delta^{\rm S}\Sigma_{\rm Z},J_1)$), respectively. (Cf. Lemma 12.6.1.)

If $\mathcal{P}_Z \geq 0$, then $\mathcal{P}'_Y := \gamma^2\mathcal{P}_Z(I+\mathcal{P}_X\mathcal{P}_Z)^{-1}$ is an internally P-stabilizing solution of the $\mathcal{P}_Y$-eDARE and $\rho(\mathcal{P}_X\mathcal{P}'_Y) < \gamma$, by Lemma 12.6.4(b)&(a) (recall 1°), hence $\mathcal{P}'_Y = \mathcal{P}_Y$, by Theorem 14.1.4(b). The converse follows analogously from Lemma 12.6.4.

3° *(a)–(c):* If $\mathcal{P}_Z \geq 0$, then (a)–(c) of Lemma 12.6.4 hold for the DAREs, by 1° (but those S_Z and S_Y may differ from those corresponding to the CAREs, etc.). □

We have already shown that all suboptimal I/O maps for $\mathbb{D}$ are given by (12.48). Now we construct a stable realization for $\mathbb{T}$, so that we can present the formula for all solutions in the standard form (cf. Theorem 12.1.8, which is based on this):

Proposition 12.5.19 ((Factor1&2) ⇒ 4BP & all solutions) *Make Hypothesis 12.5.1. Assume that (Factor1X) and (Factor2Z) hold (let $\mathbb{X}$ and $\mathbb{Z}$ be their solutions) and that $\mathbb{Z}_{11} \in \mathcal{G}\mathrm{TIC}_\infty(Y)$ (as in Theorem 12.3.7(d)). Define*

$$\left[\begin{array}{c|c} \mathbb{K}_{\mathbb{E}^d} & \mathbb{F}_{\mathbb{E}^d} \end{array}\right] := \left[\begin{array}{c|c} -J_1^{-1}\pi_+(\mathbb{Z}^{-d})^*\mathbb{D}^*_{\mathbb{E}^d}J_1\mathbb{C}_{\mathbb{E}^d} & I-\mathbb{Z}^d \end{array}\right]. \tag{12.102}$$

(as in (9.140); this is generated by $\left[\begin{array}{c|c} Z^*K_E & I-Z^* \end{array}\right]$ if $(\mathcal{P}_Z, S_E, K_E)$ is a UR stable, P-stabilizing solution of the CARE for $\Sigma_{\mathbb{E}^d}$ and J_1). Then $\left[\begin{array}{c|c} \mathbb{K}_{\mathbb{E}^d} & \mathbb{F}_{\mathbb{E}^d} \end{array}\right]$ is an admissible state feedback pair for $\Sigma_{\mathbb{E}^d}$; in particular, $\left[\begin{array}{c|c} \mathbb{A}_{\mathbb{E}^d} & \mathbb{B}_{\mathbb{E}^d} \\ \hline \mathbb{K}_{\mathbb{E}^d} & \mathbb{F}_{\mathbb{E}^d} \end{array}\right] \in \mathrm{WPLS}(Y\times U, H, Y\times U)$. Set $\mathbb{R} := I-\mathbb{F}_{\mathbb{E}^d} = \mathbb{Z}^d$. Since $\mathbb{R}_{11} = \mathbb{Z}^d_{11} \in \mathcal{G}\mathrm{TIC}_\infty$, the output feedback operator $L := \left[\begin{smallmatrix} -I & 0 & I \\ 0 & 0 & 0 \end{smallmatrix}\right] \in \mathcal{B}(Y\times U\times Y, Y\times U)$ is admissible for

$$\Sigma_{\mathrm{alt}} := \left[\begin{array}{c|c} \mathbb{A}_{\mathrm{alt}} & \mathbb{B}_{\mathrm{alt}} \\ \hline \mathbb{C}_{\mathrm{alt}} & \mathbb{D}_{\mathrm{alt}} \end{array}\right] := \left[\begin{array}{c|cc} \mathbb{A}_{\mathbb{E}^d} & (\mathbb{B}_{\mathbb{E}^d})_1 & (\mathbb{B}_{\mathbb{E}^d})_2 \\ \hline -(\mathbb{K}_{\mathbb{E}^d})_1 & \mathbb{R}_{11} & \mathbb{R}_{12} \\ -(\mathbb{K}_{\mathbb{E}^d})_2 & \mathbb{R}_{21} & \mathbb{R}_{22} \\ 0 & I & 0 \end{array}\right], \tag{12.103}$$

and the corresponding closed-loop system is given by

$$\Sigma_{\mathrm{alt},L} := \left[\begin{array}{c|c} \mathbb{A}_{\mathrm{alt},L} & \mathbb{B}_{\mathrm{alt},L} \\ \hline \mathbb{C}_{\mathrm{alt},L} & \mathbb{D}_{\mathrm{alt},L} \end{array}\right] = \left[\begin{array}{c|cc} \mathbb{A}^d_{\mathbb{T}} & \mathbb{C}^d_{\mathbb{T}2} & \mathbb{C}^d_{\mathbb{T}1} \\ \hline 0 & I & 0 \\ \mathbb{B}^d_{\mathbb{T}1} & \mathbb{T}^d_{21} & \mathbb{T}^d_{11} \\ \mathbb{B}^d_{\mathbb{T}2} & \mathbb{T}^d_{22} & \mathbb{T}^d_{12} \end{array}\right] \tag{12.104}$$

$\in \mathrm{WPLS}(Y\times U, H, Y\times U\times Y)$. Then $\Sigma_{\mathbb{T}} := \left[\begin{array}{c|cc} \mathbb{A}_{\mathbb{T}} & \mathbb{B}_{\mathbb{T}1} & \mathbb{B}_{\mathbb{T}2} \\ \hline \mathbb{C}_{\mathbb{T}} & \mathbb{T}_{*1} & \mathbb{T}_{*2} \end{array}\right] \in \mathrm{WPLS}(U\times Y, H, U\times Y)$ is a realization of $\mathbb{T}$:=(12.48), and the following hold:

(a) *If we delete the middle column and bottom row of $\Sigma_{\mathbb{T}}$, we obtain a realization $\Sigma_{\mathbb{T}_{12}}$ of $\mathbb{T}_{12}$. The system $\Sigma_{\mathbb{T}_{12}}$ is a well-posed suboptimal controller (the "central controller") for Σ.*

(b) **(All well-posed $\Sigma_{\mathbb{Q}}$'s)** *All well-posed stabilizing suboptimal controllers $\Sigma_{\mathbb{Q}}$ for Σ are given by the connection of $\Sigma_{\mathbb{T}}$ and $\Sigma_{\mathbb{L}}$ in Figure 12.1 (cf. Remark 12.1.9), where the parameter $\mathbb{L}$ is as in (12.105) and $\Sigma_{\mathbb{L}}$ is any stable realization of $\mathbb{L}$.*

(c) **(All well-posed $\mathbb{Q}$'s)** *All well-posed stabilizing suboptimal controllers $\mathbb{Q} \in \mathrm{TIC}_\infty(Y,U)$ for $\mathbb{D}$ are given by*

$$\mathbb{Q} = \mathcal{F}_\ell(\mathbb{T},\mathbb{L}) \quad (\mathbb{L} \in \mathrm{TIC}(Y,U) \text{ is s.t. } \|\mathbb{L}\|_{\mathrm{TIC}} < 1 \text{ and } I - \mathbb{L}\mathbb{T}_{21} \in \mathcal{G}\mathrm{TIC}_\infty(U)). \tag{12.105}$$

(d) **(All (possibly ill-posed) solutions)** *By removing the condition $I - \mathbb{L}\mathbb{T}_{21} \in \mathcal{G}\mathrm{TIC}_\infty(U)$ we get all stabilizing suboptimal controllers (with internal loop) in any of (a)–(c).*

(e) **(All well-posed exponentially stabilizing solutions)** *If $\begin{bmatrix} \mathbb{K}_{\mathrm{u}} & \big| & \mathbb{F}_{\mathrm{u}} \end{bmatrix}$ is [exponentially] strongly stabilizing, then we can replace "stable" by "[exponentially] strongly stable" and "stabilizing" by "[exponentially] strongly stabilizing" everywhere in this proposition. [if we require in (12.105) that $\mathbb{L} \in \mathrm{TIC}_{\mathrm{exp}}(Y,U)$].*

(f) We can replace "$\|\mathbb{L}\|_{\mathrm{TIC}} < 1$" by "$\|\mathbb{L}\|_{\mathrm{TIC}} \le 1$" everywhere in this proposition if we replace "suboptimal" by "s.t. $\|\mathcal{F}_\ell(\mathbb{D},\mathbb{Q})\| \le \gamma$".

(g) If we add some element $E \in \mathcal{B}(U,Y)$ to D_{21}, then the parametrizations in (d) are is unchanged except that we have to add to $\mathbb{Q}$ the output feedback through $-E$, as in Lemma 7.3.23 and Figure 7.12.

(h) If $\mathbb{Z}$ is SR (resp. UR, SVR, UVR, SLR, ULR), then Σ_{alt}, $\Sigma_{\mathrm{alt},L}$ and $\Sigma_{\mathbb{T}}$ are SR (resp. UR, SVR, UVR, SLR, ULR).

See Theorem 12.1.8 for a more classic "all controllers" result. As noted below Theorem 12.1.8, "all suboptimal stabilizing controllers" refers to all maps $\mathbb{Q} \in \mathrm{TIC}_\infty$ (or, in (d), all maps $\mathbb{Q}$ with internal loop modulo equivalence) $\mathbb{Q} : y \mapsto u$ s.t. $\|\mathcal{F}_\ell(\mathbb{D},\mathbb{Q})\|_{\mathrm{TIC}(W,Z)} < \gamma$ (with some realization), not to all systems having such an I/O map. That is, in our "all solutions" formula, we do not distinguish between two solutions having the same "I/O map".[1]

Indeed, any, e.g., l.c.-detectable realization of a $\mathbb{Q}$ satisfying (12.105) will do — on the other hand, any solution must be a realization of some $\mathbb{Q}$ satisfying (12.105). In the exponential case mentioned in (e), all possible $\Sigma_{\mathbb{Q}}$'s are exactly all optimizable and estimatable realizations of $\mathbb{Q}$'s satisfying (12.105), by (c) and Theorem 7.3.11(c1).

We recall from Definition 6.1.6 that for any $\mathbb{L} \in \mathrm{TIC}$ (resp. $\mathbb{L} \in \mathrm{TIC}_{\mathrm{exp}}$), there exists a strongly (resp. exponentially) stable realization $\Sigma_{\mathbb{L}}$ of $\mathbb{L}$.

We leave it to the reader to write out $\mathbb{A}_{\mathbb{T}}$, $\mathbb{B}_{\mathbb{T}}$ and $\mathbb{C}_{\mathbb{T}}$ in terms of Σ, $\mathbb{K}_{\mathrm{u}}$, $\mathbb{F}_{\mathrm{u}}$, $\mathbb{X}$ and $\mathbb{Z}$; see Theorem 12.1.8 for their generators (under the regularity assumption $\mathbb{N}_{\mathrm{u}}, \mathbb{M}_{\mathrm{u}} \in \widetilde{\mathcal{A}}$).

Proof of Proposition 12.5.19: (We note that Lemma 12.5.15 and the part of Lemma 12.5.16 concerning $\Sigma_{\mathbb{E}^{\mathrm{d}}}$ are valid also under the assumptions of this proposition, with the same proofs, hence $\Sigma_{\mathbb{E}^{\mathrm{d}}}$ is well defined.)

1° *The proof of initial claims:* By Theorem 9.9.10(g1)&(a2)&(c1), $\begin{bmatrix} \mathbb{K}_{\mathbb{E}^{\mathrm{d}}} & \big| & \mathbb{F}_{\mathbb{E}^{\mathrm{d}}} \end{bmatrix}$ is stabilizing and J-critical over $\mathcal{U}_{\mathrm{out}}$ for $\Sigma_{\mathbb{E}^{\mathrm{d}}}$. If the CARE for $\Sigma_{\mathbb{E}^{\mathrm{d}}}$ and J_1 has a UR stable, P-stabilizing solution $(\mathcal{P}_{\mathrm{Z}}, S_E, K_E)$, then all J-critical pairs are generated by $\begin{bmatrix} QK_E & I-Q \end{bmatrix}$ ($Q \in \mathcal{G}\mathcal{B}$), by Theorem

[1] Here we use quotes because $\mathbb{Q}$ need not be well-posed in general.

9.9.10(d1) and Theorem 9.8.12(b)&(s1), hence only $\begin{bmatrix} Z^*K_E & I-Z^* \end{bmatrix}$ can generate $\left[\begin{array}{c|c} * & I-\mathbb{Z}^{\mathrm{d}} \end{array}\right]$.

By Lemma A.1.1(c1), we have $(\mathbb{Z}^{-1})_{22} \in \mathcal{G}\mathrm{TIC}_\infty(U)$; by (12.50), $(\mathbb{Z}^{-1})_{22} = \widetilde{\mathbb{N}}_{+21}$. Set $\mathbb{R} := \mathbb{Z}^d = I - \mathbb{F}_{\mathbb{E}^{\mathrm{d}}}$, $\mathbb{G} := \mathbb{R}^{-1} = \mathbb{Z}^{-\mathrm{d}}$. If we denote "$\begin{bmatrix} \mathbb{T}^{\mathrm{d}}_{21} & \mathbb{T}^{\mathrm{d}}_{11} \\ \mathbb{T}^{\mathrm{d}}_{22} & \mathbb{T}^{\mathrm{d}}_{12} \end{bmatrix}$" in (12.104) by $\mathbb{H}$, then, by (6.125),

$$\mathbb{H}\begin{bmatrix} 0 & I_Y \\ I_U & 0 \end{bmatrix} = \begin{bmatrix} \mathbb{R}_{21} & \mathbb{R}_{22} \\ I & 0 \end{bmatrix}\Big(I - \begin{bmatrix} I-\mathbb{R}_{11} & -\mathbb{R}_{12} \\ 0 & I \end{bmatrix}\Big)^{-1}\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \tag{12.106}$$

$$= \begin{bmatrix} \mathbb{R}_{21} & \mathbb{R}_{22} \\ I & 0 \end{bmatrix}\begin{bmatrix} 0 & I \\ \mathbb{R}_{11} & \mathbb{R}_{12} \end{bmatrix}^{-1} = \begin{bmatrix} 0 & I \\ \mathbb{G}_{11} & \mathbb{G}_{12} \end{bmatrix}\begin{bmatrix} \mathbb{G}_{21} & \mathbb{G}_{22} \\ I & 0 \end{bmatrix}^{-1} \tag{12.107}$$

$$= \begin{bmatrix} 0 & I \\ \widetilde{\mathbb{M}}^{\mathrm{d}}_{+11} & \widetilde{\mathbb{M}}^{\mathrm{d}}_{+21} \end{bmatrix}\begin{bmatrix} \widetilde{\mathbb{N}}^{\mathrm{d}}_{+11} & \widetilde{\mathbb{N}}^{\mathrm{d}}_{+21} \\ I & 0 \end{bmatrix}^{-1} = \mathbb{T}^d \tag{12.108}$$

(write the formulae out or multiply (11.88) by $\left[\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right]$ to the left and right to obtain the third equality; the fourth follows from (12.50) and the last one from (12.48)), as claimed, i.e., the closed-loop I/O map $\mathbb{D}_{\mathrm{alt},L}$ contains $\mathbb{T}^{\mathrm{d}}$ (permuted), so that we can pick corresponding rows, interchange the columns and take the causal adjoint of the system to obtain a realization of $\mathbb{T}$, as stated in the proposition.

(a) This follows from (b) by taking $\mathbb{L} = 0$.

(c) This follows from Theorem 12.3.7(c) (which is applicable, by Lemma 12.5.3 and the fact that $\widetilde{\mathbb{N}}_{+21} = (\mathbb{Z}^{-1})_{22} \in \mathcal{G}\mathrm{TIC}_\infty$, by 1° above).

(d)&(f) Also these follow from Theorem 12.3.7(c) for (c); the proofs of (a) and (b) show that the same holds for them.

(g) This follows from Lemma 7.3.23 (note that in this case the well-posedness of a solution need not be equivalent to $I - \mathbb{L}\mathbb{T}_{21} \in \mathcal{G}\mathrm{TIC}_\infty$).

(h) Use Proposition 6.3.1(b2) (and Lemma 6.2.5). (Note also that if $\mathbb{Z} \in \widetilde{\mathcal{A}}$ (e.g., $\mathbb{E} \in \widetilde{\mathcal{A}}$ or $\mathbb{N},\mathbb{M} \in \widetilde{\mathcal{A}}$), then $\mathbb{R} \in \widetilde{\mathcal{A}}$ and "$\mathbb{T} \in \widetilde{\mathcal{A}}^{-1}\widetilde{\mathcal{A}}$".)

(b)&(e) Assume that $\left[\begin{array}{c|c} \mathbb{K}_{\mathrm{u}} & \mathbb{F}_{\mathrm{u}} \end{array}\right]$ is [[exponentially] strongly] stabilizing for Σ. Let $\mathbb{L}$ be as in (c) or as in (d), and let $\Sigma_{\mathbb{L}}$ be an [[exponentially] strongly] stable realization of $\mathbb{L}$.

1° The system $\Sigma_{\mathbb{E}^{\mathrm{d}}}$ is [[exponentially] strongly*] stable, by Lemma 12.5.15, and Σ is [[exponentially] strongly] l.c.-detectable, by Theorem 6.6.28 [[shifted and Lemma 12.5.2(vi)]].

By (c), $\mathbb{Q}$ DPF-stabilizes $\mathbb{D}$ (possibly with an internal loop; cf. (d)), hence $\Sigma_{\mathbb{Q}}$ I/O-DPF-stabilizes Σ. We shall show in 2°–3° $\Sigma_{\mathbb{Q}}$ DPF-stabilizes Σ [[exponentially] strongly]; this establishes (b) and (e) [[this includes also (c) modified as in (e), becase then $\mathbb{Q}$ DPF-stabilizes $\mathbb{D}$ exponentially]].

2° $\Sigma_{\mathbb{T}}$ *is [[exponentially] strongly] l.c.-detectable:* Since $\left[\begin{array}{c|cc} 0 & 0 & 0 \end{array}\right]$ is a [[exponentially] strongly*] r.c.-stabilizing state feedback pair for Σ_{alt}, the pair (6.188) is admissible for $\Sigma_{\mathrm{alt},L}$ (use substitutions $\mathbb{K},\mathbb{F},\mathbb{K}_L,\mathbb{F}_L,\mathbb{K}_\flat,\mathbb{F}_\flat \mapsto 0$, $\Sigma \mapsto \Sigma_{\mathrm{alt}}$, $\Sigma_L \mapsto \Sigma_{\mathrm{alt},L}$, $\Sigma_\flat \mapsto \begin{bmatrix} \Sigma_{\mathrm{alt}} \\ [0\ 0] \end{bmatrix}$ in the proof of Lemma 6.7.11(c)), and the corresponding closed-loop system is given by

$$\left[\begin{array}{c|c} \mathbb{A}_\flat & \mathbb{B}_\flat \\ \hline \mathbb{C}_\flat & \mathbb{D}_\flat \\ \mathbb{K}_\flat & \mathbb{F}_\flat \end{array}\right] := \left[\begin{array}{c|cc} \mathbb{A}_{\text{alt}} & \mathbb{B}_{\text{alt}} & \\ \hline \mathbb{C}_{\text{alt}} & \mathbb{D}_{\text{alt}} & \\ -(\mathbb{K}_{\mathbb{E}^{\text{d}}})_1 & \mathbb{R}_{11}-I & \mathbb{R}_{12} \\ 0 & 0 & 0 \end{array}\right]. \tag{12.109}$$

Therefore, (6.188) is admissible for $\Sigma_{\mathbb{T}}^{\text{d}}\left[\begin{smallmatrix} I & 0 & 0 \\ 0 & 0 & I \\ 0 & I & 0 \end{smallmatrix}\right]$, and the corresponding closed-loop system is (12.109) with its second row (corresponding to $(\mathbb{C}_{\text{alt}})_1$) removed; in particular, (6.188) is [strongly*] stabilizing. Consequently, corresponding maps "$\mathbb{N} := \mathbb{D}_\flat$" and "$\mathbb{M} := \mathbb{F}_\flat + I$" (see Definition 6.6.10) are given by

$$\mathbb{N}' = \begin{bmatrix} \mathbb{R}_{21} & \mathbb{R}_{22} \\ I & 0 \end{bmatrix}, \quad \mathbb{M}' = \begin{bmatrix} \mathbb{R}_{11} & \mathbb{R}_{12} \\ 0 & I \end{bmatrix}; \tag{12.110}$$

these are [[exponentially]] r.c. (because $\left[\begin{smallmatrix} 0 & I \\ 0 & 0 \end{smallmatrix}\right]\mathbb{N}' + \left[\begin{smallmatrix} 0 & 0 \\ 0 & I \end{smallmatrix}\right]\mathbb{M}' = I$). Thus, $\Sigma_{\mathbb{T}}^{\text{d}}$ is [[exponentially] strongly*] r.c.-stabilizable (by Lemma 6.7.17, the permutation of columns does not matter), i.e., $\Sigma_{\mathbb{T}}$ is [[exponentially] strongly] l.c.-detectable.

3° *$\Sigma_{\mathbb{Q}}$ is [[exponentially] strongly] stabilizing:* Let Σ^o contain $\Sigma_{\mathbb{L}}$, $\Sigma_{\mathbb{T}}$ and Σ so that the static output feedback operator I corresponds to the connection $\mathcal{F}_\ell(\mathbb{D}, \mathcal{F}_\ell(\mathbb{T}, \mathbb{L}))$ (cf. (7.4) and Definition 7.3.1).

Since Σ (by 1°) and $\Sigma_{\mathbb{T}}$ (by 2°) and $\Sigma_{\mathbb{L}}$ (by assumption) are [[exponentially] strongly] l.c.-detectable, so is Σ^o, by Lemma 6.7.18 (applied twice). By Proposition 6.7.14(b)(2.), I is [[exponentially] strongly] stabilizing for Σ^o. Thus, all closed-loop maps in Figure 12.1 are [[exponentially] strongly] stable (because they are exactly the elements of Σ_I^o). As noted in 1°, this establishes (b) and (e).

(An alternative proof would apply Proposition 6.7.14(b)(1.)&(2.) for subsystems [[or the generators of Σ and $\Sigma_{\mathbb{T}}$ and optimizability and estimatability; cf. (12.20)]].) □

The $\mathcal{P}_{\text{X}}$-DARE, $\mathcal{P}_{\text{Y}}$-DARE and the coupling condition imply the existence of a suboptimal controller for Σ:

Lemma 12.5.20 (Sufficiency) *Assume that Hypothesis 12.5.1 is satisfied with $\mathbb{N}_{\text{u}}, \mathbb{M}_{\text{u}}, \widetilde{\mathbb{M}_{\text{y}}}, \widetilde{\mathbb{N}_{\text{y}}} \in \widetilde{\mathcal{A}}$, that $\widetilde{\mathcal{A}}$ satisfy Hypothesis 8.4.8, and that conditions (1.)–(3.) of Lemma 12.1.12 are satisfied.*

Then Hypothesis 12.5.13 and conditions (1.) and (4.) of Theorem 12.1.8 are satisfied; in particular, then there are suboptimal exponentially stabilizing DPF-controllers.

Proof: (We use Hypothesis 8.4.8 only in 3°.)

1° *Hypothesis 12.5.13 and (Factor1$\widetilde{\mathcal{A}}$):* By (1.) and Lemma 12.5.12, Hypothesis 12.5.13 is satisfied and hence (Factor1) has a solution with $X_{11}, X_{22}, M_{11}, M_{22} \in \mathcal{GB}$, by Lemma 12.5.14. By Lemma 12.3.10(a), $\mathbb{M}, \mathbb{N}, \mathbb{X} \in \widetilde{\mathcal{A}}$, hence (Factor1$\widetilde{\mathcal{A}}$) is satisfied.

2° *The* $\mathcal{P}_Z$*-CARE has an exponentially stabilizing solution* $(\mathcal{P}_Z, S_Z, K_Z)$*, where* $\mathcal{P}_Z = \mathcal{P}_Y(I - \mathcal{P}_X\mathcal{P}_Y)^{-1} \geq 0$*:* By 1° of the proof of Lemma 12.5.18, $\mathcal{P}_X$ is an exponentially stabilizing solution of the $\mathcal{P}_X$-DARE with S_X as in Lemma 12.6.4 (here we mean the DARE for $(\Delta^S\Sigma_X, J_\gamma)$). The situation with $\mathcal{P}_Y$ is analogous (by dual arguments).

By Lemma 12.6.4, the $\mathcal{P}_Z$-DARE (the DARE for $\Delta^S\Sigma_Z$ and J_1) has the exponentially stabilizing solution $\mathcal{P}_Z = \mathcal{P}_Y(I - \mathcal{P}_X\mathcal{P}_Y)^{-1} \geq 0$ with $S_{Z11} \gg 0$ and $S_{Z22} - S_{Z21}S_{Z11}^{-1}S_{Z12} \ll 0$ (recall "(3.)"). It follows from (the discretized, see Theorem 12.2.2) Lemma 12.5.17(ii')&(d) that $(\Delta^S\mathbb{E})J_1(\Delta^S\mathbb{E})^* = \mathbb{Z}_\Delta J_1 \mathbb{Z}_\Delta^*$ for some $\mathbb{Z}_\Delta \in \mathcal{G}\mathrm{tic}(Y_\Delta \times U_\Delta)$.

We deduce that $\pi_+\Delta^S(\mathbb{E}^d)^* J_1 \Delta^S(\mathbb{E}^d)\pi_+ \in \mathcal{GB}$, hence $\pi_+(\mathbb{E}^d)^* J_1 \mathbb{E}^d \pi_+ \in \mathcal{GB}(\mathrm{L}^2(\mathbf{R}_+; Y \times U))$, by Theorem 13.4.5(h2). Consequently, $(\mathbb{E}^d)^* J_1 \mathbb{E}^d = \mathbb{R}^* S_E \mathbb{R}$ for some $\mathbb{R} \in \mathcal{G}\widetilde{\mathcal{A}}^d(Y \times U)$ and $S_E \in \mathcal{GB}(Y \times U)$.

By Corollary 9.9.11, $\mathbb{R}$ corresponds to an exponentially stabilizing solution $\mathcal{P}_Z'$ of the $\mathcal{P}_Z$-IARE, hence of the $\mathcal{P}_Z$-DARE too. By uniqueness (Theorem 9.8.12(a)), $\mathcal{P}_Z' = \mathcal{P}_Z$. By Proposition 9.8.10 and Remark 9.8.2, $\mathcal{P}_Z$ is an exponentially stabilizing solution of the $\mathcal{P}_Z$-CARE (since $\mathbb{R} \in \mathcal{G}\widetilde{\mathcal{A}}^d \subset \mathcal{G}\mathrm{ULR}$ and hence $R \in \mathcal{GB}(Y \times U)$, by Proposition 6.3.1(b1)).

3° $S_{Z11} \gg 0$ *and* $S_{Z22} - S_{Z21}S_{Z11}^{-1}S_{Z12} \ll 0$ *(so that* $\mathcal{P}_Z$ *solves Lemma 12.5.17(ii')):* Set $A = B = C = 0$ (but keep the D of Σ), so that the $\mathcal{P}_X$-DARE and $\mathcal{P}_Y$-DARE have unique exponentially stabilizing solutions, given by $\mathcal{P}_X = 0 = \mathcal{P}_Y = K_X = K_Y$, and S_X and S_Y are of the standard form, by the signature-conditions in (1.) and (2.). By Lemma 12.6.4(a)–(c), $\mathcal{P}_Z = 0$ is the (unique) exponentially stabilizing solution of the $\mathcal{P}_Z$-DARE and $S_{Z11} \gg 0$ and $S_{Z22} - S_{Z21}S_{Z11}^{-1}S_{Z12} \ll 0$. But S_X, S_Y and S_Z are the same as in our problem (because $B = C = 0$), hence this proves our claim.

(Alternatively, one could write a somewhat shorter (but still not short) proof by going through the same computations as in the proof of Lemma 12.6.4(c) (including those on pp. 321-326 of [IOW]; one can, e.g., take $A = B = C = 0$ but the D of Σ must be kept).)

4° *(1.) and (4.) of Theorem 12.1.8 are satisfied:* Condition (1.) is contained in the assumptions, and condition (4.) was established in 2°–3° above.

5° *The existence of a solution:* By 4°, the assumptions of Theorem 12.1.8 are satisfied, hence there are suboptimal exponentially stabilizing DPF-controllers for Σ. □

The above also holds under different assumptions:

Lemma 12.5.21 *Suppose that the assumptions (A1) and (A2) and conditions (1.)–(3.) of Theorem 12.1.5 are satisfied. Then the assumptions of Lemma 12.5.20 are satisfied for* $\widetilde{\mathcal{A}} = \mathrm{MTIC}^{\mathrm{L}^1}_{\exp}$.

Proof: Recall from Theorem 8.4.9 that $\widetilde{\mathcal{A}} = \mathrm{MTIC}^{\mathrm{L}^1}_{\exp}$ satisfies Standing Hypothesis 12.0.1 and Hypothesis 8.4.8.

By Lemmas 12.5.5 and 12.5.4, Hypothesis 12.5.1 is satisfied (even with "exponentially jointly" in place of "jointly"), $\mathbb{N}_u, \mathbb{M}_u, \widetilde{\mathbb{N}_y}, \widetilde{\mathbb{M}_y} \in \widetilde{\mathcal{A}}$ (or $\in(\gamma')$

under "(IV)" of (A1)), and $\widehat{\mathbb{M}_{\mathrm{u}}}(+\infty)=I$, $\widehat{\widetilde{\mathbb{M}_{\mathrm{y}}}}(+\infty)=I$.

Note that conditions (1.)–(3.) of Theorem 12.1.5 are equal to those of Lemma 12.1.12 combined to conditions $S_X = D_X^* J_\gamma D_X$ and $S_Y = D_Y^* J_\gamma D_Y$. By Hypothesis 8.4.8 and Corollary 9.9.11, these additional conditions are now redundant. □

Lemma 12.5.22 (4BP: necessity) *Assume that there is a suboptimal exponentially stabilizing DPF-controller (possibly with internal loop), and that (A1) and (A2) of Theorem 12.1.5 hold. Then (1.)–(3.) of Theorem 12.1.5 hold.*

Proof: (The regularity condition (A1) on Σ is not superfluous, by, e.g., Example 11.3.7, but it may be weakened.)

Set $\widetilde{\mathcal{A}} := \mathrm{MTIC}^{\mathrm{L}^1}_{\mathrm{exp}}$ (or $\widetilde{\mathcal{A}} =(\gamma')$ in case "(IV)" of (A1)), so that Standing Hypothesis 12.0.1 is satisfied, by Theorem 8.4.9(c).

1° *Hypothesis 12.5.1:* By Theorem 7.3.12(a), (A, B_1) is optimizable and (A, C_2) is estimatable. By Lemma 12.5.4, it follows that Hypothesis 12.5.1 is satisfied (even with "exponentially jointly" in place of "jointly").

2° By Lemma 12.5.3 and 1°, Hypothesis 12.3.1 is satisfied with $\mathbb{N}_{\mathrm{u}}, \mathbb{M}_{\mathrm{u}}, \widetilde{\mathbb{N}_{\mathrm{y}}}, \widetilde{\mathbb{M}_{\mathrm{y}}} \in \widetilde{\mathcal{A}}$.

3° *(1.)–(3.) hold:* This follows from Theorem 12.1.11. □

Given just an I/O map, as in the frequency-domain problem of Section 12.3, we can choose a stabilizable realization as explained below (the advantage here is that we get the constructive formula (12.113)):

Lemma 12.5.23 *Assume Hypothesis 12.3.1. A strongly jointly stabilizable and detectable realization Σ for $\mathbb{D}$ can be chosen as follows.*

By the assumption, there are $\mathbb{Q}_1, \mathbb{Q}_2, \widetilde{\mathbb{Q}}_1, \widetilde{\mathbb{Q}}_2 \in \mathrm{TIC}$ *s.t.*

$$\begin{bmatrix} \mathbb{M}_{\mathrm{u}\,11} & \mathbb{Q}_1 \\ \mathbb{N}_{\mathrm{u}\,21} & \mathbb{Q}_2 \end{bmatrix}^{-1} = \begin{bmatrix} \widetilde{\mathbb{Q}}_2 & -\widetilde{\mathbb{Q}}_1 \\ -\widetilde{\mathbb{N}_{\mathrm{y}}}_{21} & \mathbb{M}_{\mathrm{u}\,22} \end{bmatrix} \in \mathcal{G}\mathrm{TIC}. \tag{12.111}$$

It follows that $\begin{bmatrix} \mathbb{M}_{\mathrm{u}} & \mathbb{Q}_{\mathrm{DF1}} \\ \mathbb{N}_{\mathrm{u}} & \mathbb{Q}_{\mathrm{DF2}} \end{bmatrix}, \begin{bmatrix} \widetilde{\mathbb{Q}}_{\mathrm{DF2}} & -\widetilde{\mathbb{Q}}_{\mathrm{DF1}} \\ -\widetilde{\mathbb{N}_{\mathrm{y}}} & \widetilde{\mathbb{M}_{\mathrm{y}}} \end{bmatrix} \in \mathcal{G}\mathrm{TIC}$, *where*

$$\mathbb{Q}_{\mathrm{DF2}} := \begin{bmatrix} I & 0 \\ 0 & \mathbb{Q}_2 \end{bmatrix}, \ \mathbb{Q}_{\mathrm{DF1}} := \begin{bmatrix} 0 & \mathbb{Q}_1 \\ 0 & 0 \end{bmatrix}, \ \widetilde{\mathbb{Q}}_{\mathrm{DF2}} := \begin{bmatrix} \widetilde{\mathbb{Q}}_2 & 0 \\ 0 & I \end{bmatrix}, \ \widetilde{\mathbb{Q}}_{\mathrm{DF1}} := \begin{bmatrix} 0 & \widetilde{\mathbb{Q}}_1 \\ 0 & 0 \end{bmatrix}. \tag{12.112}$$

Choose a strongly stable realization $\Sigma_\flat$ for $\begin{bmatrix} I-\mathbb{Q}_{\mathrm{DF2}} & \mathbb{N}_{\mathrm{u}} \\ \mathbb{Q}_{\mathrm{DF1}} & \mathbb{M}_{\mathrm{u}}-I \end{bmatrix}$. *Then* $\Sigma_{\mathrm{Total}} := (\Sigma_\flat)_{\left[\begin{smallmatrix} 0 & 0 \\ 0 & -I \end{smallmatrix}\right]}$ *satisfies Hypothesis 12.5.1 and its I/O map is given by (we denote the components of Σ_{Total} as in (12.87))*

$$\begin{bmatrix} \mathbb{G}_{\mathrm{y}} & \mathbb{D} \\ \mathbb{E}_{\mathrm{uy}} & \mathbb{F}_{\mathrm{u}} \end{bmatrix} := \begin{bmatrix} \begin{bmatrix} 0 & \mathbb{D}_{11}\mathbb{Q}_1 \\ 0 & I-\mathbb{Q}_2+\mathbb{D}_{21}\mathbb{Q}_1 \end{bmatrix} & \mathbb{D} \\ \begin{bmatrix} 0 & -\mathbb{M}_{\mathrm{u}\,11}^{-1}\mathbb{Q}_1 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} I-\mathbb{M}_{\mathrm{u}\,11}^{-1} & \mathbb{M}_{\mathrm{u}\,11}^{-1}\mathbb{M}_{\mathrm{u}\,12} \\ 0 & 0 \end{bmatrix} \end{bmatrix}. \tag{12.113}$$

The pairs $\begin{bmatrix} \mathbb{K}_{\mathrm{u}} & \big| & \mathbb{F}_{\mathrm{u}} \end{bmatrix}$ *and* $\begin{bmatrix} \mathbb{H}_{\mathrm{y}} \\ \hline \mathbb{G}_{\mathrm{y}} \end{bmatrix}$ *are actually strongly jointly stabilizing and detecting for Σ (with $\mathbb{E}_{\mathrm{uy}}$), these pairs are as in Hypothesis 12.5.1.*

If there is a suboptimal stabilizing controller for $\mathbb{D}$, then there is a suboptimal strongly stabilizing controller for the above system, by Theorem 12.3.5. Also an exponential version of the above lemma and claim hold (use Remark 6.1.9 and the "exponential" assumptions of Proposition 12.4.10).

Proof: (Note that $\mathbb{Q} := \mathbb{Q}_1\mathbb{Q}_2^{-1} = \widetilde{\mathbb{Q}}_1^{-1}\widetilde{\mathbb{Q}}_2$ DPF-stabilizes $\mathbb{D}$ with d.c. internal loop, by Corollary 7.3.20(iii), whereas $\mathbb{Q}_{\mathrm{DF1}}\mathbb{Q}_{\mathrm{DF2}}^{-1} = \widetilde{\mathbb{Q}}_{\mathrm{DF1}}^{-1}\widetilde{\mathbb{Q}}_{\mathrm{DF2}}$ DF-stabilizes $\mathbb{D}$ with d.c. internal loop, by Theorem 7.2.14(iii); cf. also Lemma 7.3.10.)

We obtain (12.111) from Lemma 6.5.8. But from $\begin{bmatrix} \mathbb{M}_{\mathrm{u}} & \mathbb{Q}_{\mathrm{DF1}} \\ \mathbb{N}_{\mathrm{u}} & \mathbb{Q}_{\mathrm{DF2}} \end{bmatrix}$ we obtain $\begin{bmatrix} I & 0 \\ 0 & \mathbb{T} \end{bmatrix}$ by four permutations, where $\mathbb{T} := \begin{bmatrix} \mathbb{M}_{\mathrm{u}\,11} & \mathbb{Q}_1 \\ \mathbb{N}_{\mathrm{u}\,21} & \mathbb{Q}_2 \end{bmatrix} \in \mathcal{G}\mathrm{TIC}$. Therefore, $\begin{bmatrix} \mathbb{M}_{\mathrm{u}} & \mathbb{Q}_{\mathrm{DF1}} \\ \mathbb{N}_{\mathrm{u}} & \mathbb{Q}_{\mathrm{DF2}} \end{bmatrix} \in \mathcal{G}\mathrm{TIC}$; analogously, $\begin{bmatrix} \widetilde{\mathbb{Q}}_{\mathrm{DF2}} & -\widetilde{\mathbb{Q}}_{\mathrm{DF1}} \\ -\widetilde{\mathbb{N}}_{\mathrm{y}} & \widetilde{\mathbb{M}}_{\mathrm{y}} \end{bmatrix} \in \mathcal{G}\mathrm{TIC}$.

The claims on strong joint stabilizability and detectability follow as in the proof of Theorem 6.6.28. □

We use the rest of this section to study the connection between the CAREs and the factorizations of Section 12.3. We mainly just sketch the proofs (and some of the statements), since we do not use these results elsewhere.

Lemma 12.5.24 ($\mathcal{P}_{\mathrm{X}}, \mathcal{P}_{\mathrm{Y}}$ directly) *Assume Hypothesis 12.5.1. The IAREs corresponding to (1.') and (2.') have solutions $\mathcal{P}_{\mathrm{X}}$ and $\mathcal{P}_{\mathrm{Y}}$, respectively, iff (Factor1X) and (Factor2Y) hold, where the latter is given by*

(Factor2Y) There is $\mathbb{Y} \in \mathcal{G}\mathrm{TIC}(Y \times Z)$, s.t. $\mathbb{Y}^* J_1 \mathbb{Y} = \begin{bmatrix} \widetilde{\mathbb{N}_{\mathrm{y}}}^{\mathrm{d}}_{22} & \widetilde{\mathbb{N}_{\mathrm{y}}}^{\mathrm{d}}_{12} \\ 0 & I \end{bmatrix}^* J_\gamma \begin{bmatrix} \widetilde{\mathbb{N}_{\mathrm{y}}}^{\mathrm{d}}_{22} & \widetilde{\mathbb{N}_{\mathrm{y}}}^{\mathrm{d}}_{12} \\ 0 & I \end{bmatrix}$ and $\mathbb{Y}_{11} \in \mathcal{G}\mathrm{TIC}(Y)$.

Assume (Factor1X) and (Factor2Y). Set $\mathbb{N} := \mathbb{N}_{\mathrm{u}}\mathbb{X}^{-1}$, $\widetilde{\mathbb{N}} := \mathbb{Y}_{\mathrm{d}}^{-1}\widetilde{\mathbb{N}_{\mathrm{y}}}$. Then

$$\mathcal{P}_{\mathrm{X}} = \mathbb{C}_{\flat\,1}^{\,*}\,(I - \mathbb{N}_{11}\pi_+\mathbb{N}_{11}^* + \mathbb{N}_{12}\pi_+\mathbb{N}_{12}^*)\,\mathbb{C}_{\flat\,1}, \tag{12.114}$$

$$\mathcal{P}_{\mathrm{Y}} = \mathbb{C}_{\mathrm{Y}\flat\,1}^{\,*}\left(I - \widetilde{\mathbb{N}}_{22}^{\mathrm{d}}\pi_+(\widetilde{\mathbb{N}}_{22}^{\mathrm{d}})^* + \widetilde{\mathbb{N}}_{12}^{\mathrm{d}}\pi_+(\widetilde{\mathbb{N}}_{12}^{\mathrm{d}})^*\right)\mathbb{C}_{\mathrm{Y}\flat\,1}, \tag{12.115}$$

where $\mathbb{C}_{\flat\,1} = \mathbb{C}_1 + \mathbb{N}_{\mathrm{u}\,11}\mathbb{K}_{\mathrm{u}\,1}$, $\mathbb{C}_{\mathrm{Y}\flat\,1} = \mathbb{B}_2^{\mathrm{d}} + \widetilde{\mathbb{N}_{\mathrm{y}}}^{\mathrm{d}}_{22}\mathbb{H}_{\mathrm{y}}{}_2^{\mathrm{d}}$.

See the remarks below Lemma 12.1.12 for (1.') and (2.'); by Lemma 12.5.12 (and Lemma 12.5.2 and duality), they are equivalent to (1.) and (2.) if Σ is exponentially stabilizable (but we need some regularity additions to get CAREs in place of IAREs).

Note that (Factor2Y) is not equivalent to (Factor2Z) but to the analogy of (Factor1X) for $\mathbb{D}_{\mathrm{d}}$.

Proof of Lemma 12.5.24: 1° *The equivalence:* By using Lemma 9.12.3, one can verify that the solutions of the IARE "(1.')" correspond 1-1 to the r.c.-stabilizing solutions of the IARE for $\Sigma_{\mathrm{X}\flat}$ and J_γ (see (12.88)); equivalently, to the solutions of (Factor1X) (of Theorem 12.3.7). By (9.141), we have

$$\mathcal{P}_{\mathrm{X}} = \begin{bmatrix} \mathbb{C}_{\flat\,1} \\ 0 \end{bmatrix}^* (J_\gamma - J_\gamma \mathbb{N}_X J_1^{-1}\pi_+\mathbb{N}_X^* J_\gamma) \begin{bmatrix} \mathbb{C}_{\flat\,1} \\ 0 \end{bmatrix}, \tag{12.116}$$

where $\mathbb{N}_X := \begin{bmatrix} (\mathbb{N}_u)_{11} & (\mathbb{N}_u)_{12} \\ 0 & I \end{bmatrix} \mathbb{X}^{-1}$.

The correspondence between (2.') and (Factor2Y) is analogous.

2° *(12.114) & (12.115):* We have $\mathbb{N}_X = \begin{bmatrix} \mathbb{N}_{11} & \mathbb{N}_{12} \\ * & * \end{bmatrix}$, where $\mathbb{N} := \mathbb{N}_u \mathbb{X}^{-1}$, hence

$$[\mathbb{N}_X J_1^{-1} \pi_+ \mathbb{N}_X]_{11} = \mathbb{N}_{11} \pi_+ \mathbb{N}_{11}^* - \mathbb{N}_{12} \pi_+ \mathbb{N}_{12}^*. \tag{12.117}$$

By (12.116) and (12.89), $\mathcal{P}_X = \mathbb{C}_{\flat 1}^* \left[I - \mathbb{N}_X J_1^{-1} \pi_+ \mathbb{N}_X^* \right]_{11} \mathbb{C}_{\flat 1}$. Combine this with (12.117) to obtain (12.114). Equation (12.115) is obtained analogously.

3° *A remark on* $\mathbb{P} := \mathbb{C}_{\flat 1} \mathbb{C}_{Y\flat}{}_1^* \boldsymbol{Я}$*:* Using Lemma 12.5.11 and the property "$\pi_+ \mathbb{D} \pi_- = \mathbb{C}\mathbb{B}$" of Definition 6.1.1 applied to whole Σ_{Total}, we get

$$\begin{aligned} \mathbb{P} &= (\mathbb{C}_1 + \mathbb{N}_{u\,11} \mathbb{K}_{u\,1})(\mathbb{B}_2 + \mathbb{H}_{y\,2} \widetilde{\mathbb{N}_y}_{22}) & (12.118) \\ &= \pi_+ \mathbb{D}_{12} \pi_- + \mathbb{N}_{u\,11} \pi_+ \mathbb{F}_{u\,12} \pi_- + \pi_+ \mathbb{G}_{y\,12} \pi_- \widetilde{\mathbb{N}_y}_{22} + \mathbb{N}_{u\,11} \pi_+ \mathbb{E}_{12} \pi_- \widetilde{\mathbb{N}_y}_{22} & (12.119) \\ &= \pi_+ \mathbb{D}_{12} \pi_- + \mathbb{N}_{u\,11} \pi_+ \mathbb{M}_{u\,11}^{-1} \mathbb{M}_{u\,12} \pi_- + \pi_+ \mathbb{N}_{u\,11} \pi_- \mathbb{M}_{u\,11}^{-1} \mathbb{Q}_1 \pi_- \widetilde{\mathbb{N}_y}_{22} & (12.120) \end{aligned}$$

where the last identity uses equations (12.113) and $\mathbb{D}_{11} = \mathbb{N}_{u\,11} \mathbb{M}_{u\,11}^{-1}$.

(The above components are in $\mathcal{B}(\mathrm{L}^2_\omega(\mathbf{R}_-;W), \mathrm{L}^2_\omega(\mathbf{R}_+;Y))$, where $\omega \in \mathbf{R}$ is s.t. $\Sigma_{\text{Total}} \in \text{WPLS}_\omega$; their sum is stable (since so is the left-hand-side $\mathbb{P} := \mathbb{C}_{\flat 1} \mathbb{C}_{Y\flat}{}_1^* \boldsymbol{Я} \in \mathcal{B}(\mathrm{L}^2(\mathbf{R}_-;W), \mathrm{L}^2(\mathbf{R}_+;Y)))$. Note this formula can be reduced to $\pi_+ \mathbb{D}_{12} \pi_-$ if Σ is stable.) □

In, e.g., Theorem 12.1.4, we given necessary and sufficient conditions for the standard H^∞ 4BP in terms of the original system "(1.)–(3.)", whereas Theorem 12.1.8 uses both the original system and a perturbed system (condition "(4.')").

In Theorem 12.3.6, we have given analogous necessary and sufficient conditions ("(Factor1)" and "(Factor3") in terms of the original and a perturbed system for the corresponding frequency-domain problem (I/O map problem). The following remark contains the analogy of "(1.)–(3.)" for this problem; note that we again get sufficiency only for the exponential problem (see Remark 12.6.9 for the simpler discrete-time counterpart of this remark):

Remark 12.5.25 ($\rho(XY) < \gamma^2$: I/O formulation) *Assume Hypothesis 12.3.1, so that the d.c.f. (12.111) exists (we need its map* $\mathbb{Q}_1 \in \text{TIC}(Y,U)$ *below for* $\mathbb{P}$*). Assume also that* $\mathbb{N}_u, \mathbb{M}_u, \widetilde{\mathbb{N}_y}, \widetilde{\mathbb{M}_y} \in \widetilde{\mathcal{A}}$*. Then (i) implies (iii) (see below). Here*

$$\widetilde{\mathbb{O}} := \mathbb{P}^* \left(I - \mathbb{N}_{11} \pi_+ \mathbb{N}_{11}^* + \mathbb{N}_{12} \pi_+ \mathbb{N}_{12}^* \right) \mathbb{P} \left(I - \widetilde{\mathbb{N}}_{22}^* \pi_- \widetilde{\mathbb{N}}_{22} + \widetilde{\mathbb{N}}_{12}^* \pi_- \widetilde{\mathbb{N}}_{12} \right), \tag{12.121}$$

$\mathbb{P}$ *is given by (12.120),* $\mathbb{N} := \mathbb{N}\mathbb{X}^{-1}$ *and* $\widetilde{\mathbb{N}} := \mathbb{Y}_d^{-1} \widetilde{\mathbb{N}_y}$.

Assume, in addition, that we have exponential coprimeness in Hypothesis 12.3.1 (see Proposition 12.4.10), still with $\mathbb{N}_u, \mathbb{M}_u, \widetilde{\mathbb{N}_y}, \widetilde{\mathbb{M}_y} \in \widetilde{\mathcal{A}}$*, and that* $\widetilde{\mathcal{A}}$ *satisfied Hypothesis 8.4.8; assume that (12.111) is chosen accordingly. Then the following are equivalent:*

(i) there is a suboptimal stabilizing controller for $\mathbb{D}$ *(i.e., (4BP1) holds);*

(ii) there is a suboptimal exponentially stabilizing controller for $\mathbb{D}$*;*

(iii) (Factor1X) and (Factor2Y) hold and $\rho(\widetilde{\mathbb{O}}) < \gamma$.

If (4BP1) holds and $D_{21} = 0$*, then each suboptimal stabilizing controller for* $\mathbb{D}$ *is equivalent to a well-posed one.*

We do not know whether (iii) implies (i) in the non-exponential case; the reasons for this are explained at the end of the proof of Lemma 12.1.12.

Proof: Note that now (4BP1)–(4BP3) are equivalent, by Lemma 12.3.10.

1° *"(i)⇒(iii)":* Assume (4BP1), so that (4BP3) and hence (Factor1X) holds. Then (4BP1) holds for $\mathbb{D}_{\mathrm{d}}$ too, by Proposition 7.3.4(d), hence so does (4BP3) for $\mathbb{D}_{\mathrm{d}}$, hence so does (Factor2Y) (since it equals "(Factor1X)" for $\mathbb{D}_{\mathrm{d}}$).

By Lemma 12.5.24, it follows that (1.') and (2.') have solutions (where Σ is chosen as in Lemma 12.5.23, so that Hypothesis 12.5.1 is satisfied). By Theorem 12.3.5(b), there is a suboptimal stabilizing DPF-controller for Σ, hence (3.) holds, by Lemma 12.1.12, i.e., $\rho(\mathcal{P}_{\mathrm{X}}\mathcal{P}_{\mathrm{Y}}) < \gamma$. By (12.114), (12.115), (12.118) and Lemma A.3.3(s2), $\rho(\widetilde{\mathbb{O}}) = \rho(\mathcal{P}_{\mathrm{X}}\mathcal{P}_{\mathrm{Y}})$.

2° *"(ii)⇒(i)":* This is trivial.

3° *"(iii)⇒(ii)":* Make now the additional assumptions in the remark. Since (12.111) was chosen to be in $\mathcal{G}\mathrm{TIC}_{\mathrm{exp}}$, we can use shifted Lemma 12.5.23 to satisfy Hypothesis 12.5.1 with $\begin{bmatrix} \mathbb{K}_{\mathrm{u}} & \big| & \mathbb{F}_{\mathrm{u}} \end{bmatrix}$ being exponentially stabilizing (cf. Lemma 12.5.2).

Assume (iii). By Lemma 12.5.24, it follows that (1.') and (2.') have solutions, hence so do (1.) and (2.). As in 1°, we observe that also (3.) holds, hence there is a suboptimal exponentially stabilizing controller for Σ by Theorem 12.1.11 (by Theorem 12.1.8, all such controllers are equivalent to well-posed controllers if $D_{21} = 0$). The I/O map of this controller is a suboptimal exponentially stabilizing controller for $\mathbb{D}$, hence (4BP1) holds.

Remark: In the discrete-time version of the remark, "still with $\mathbb{N}_{\mathrm{u}}, \mathbb{M}_{\mathrm{u}}, \widetilde{\mathbb{N}_{\mathrm{y}}}, \widetilde{\mathbb{M}_{\mathrm{y}}} \in \widetilde{\mathcal{A}}$, and that $\widetilde{\mathcal{A}}$ satisfied Hypothesis 8.4.8" becomes superfluous, since then Theorem 12.1.11 can be replaced by Theorem 12.2.1 and hence Hypothesis 8.4.8 is not required, so that $\widetilde{\mathcal{A}} = \mathrm{tic}_{\mathrm{exp}}$ becomes applicable (and we automatically have $\mathbb{N}_{\mathrm{u}}, \mathbb{M}_{\mathrm{u}}, \widetilde{\mathbb{N}_{\mathrm{y}}}, \widetilde{\mathbb{M}_{\mathrm{y}}} \in \mathrm{tic}_{\mathrm{exp}}$ under this exponential coprimeness assumption). □

(See the notes on p. 706.)

12.6 Proofs for Section 12.2 — 4bp $\mathcal{P}_X, \mathcal{P}_Y, \mathcal{P}_Z$

Labor omnia vicit improbus

— Vergil (70–19 B.C.)

Recall Standing Hypothesis 12.1.1. Recall also that when referring to continuous time theory (as above), we assume that substitutions (13.63) are made.

The main result of this section is Lemma 12.6.4, which generalizes the "(1.)–(3.) iff (1.) and (4.)" proof of [IOW] to our generality, and which is needed also for the continuous-time proofs for Section 12.1. We start with a few results that define required symbols and show the correspondence between them and the symbols of [IOW]. At the end of the section, there are some results that clarify certain important properties of the (discrete-time) H^∞ 4bp.

The DARE (12.34) will be called the $\mathcal{P}_X$*-DARE*, and the DARE (12.35) will be called the $\mathcal{P}_Y$*-DARE*. In this section, we shall establish the connection between these two DAREs and a third one, $\mathcal{P}_Z$*-DARE*, that will be defined below:

Lemma 12.6.1 ($\mathcal{P}_Z$-DARE) *Assume that the $\mathcal{P}_X$-DARE (12.34) has an internally P-stabilizing solution $(\mathcal{P}_X, S_X, K_I)$ s.t. $S_{X11} \gg 0$ and $S_{X22} - S_{X12}^* S_{X11} S_{X12} \ll 0$.*

Then the corresponding IARE has another internally P-stabilizing solution $(\mathcal{P}_X, J_1, \left[\begin{array}{c|c} \mathbb{K} & \mathbb{F}\end{array}\right])$ s.t. $X_{21} = 0$, where $\mathbb{X} := I - \mathbb{F} \in \mathrm{tic}_\infty(U \times W)$.

Fix such a solution and set $\mathbb{M} := \mathbb{X}^{-1}$, $\mathbb{N} := \mathbb{D}\mathbb{M}$. Then $X_{11}, X_{22} \in \mathcal{GB}$ and

$$K = XK_I = \begin{bmatrix} X_{11}K_{I1} + X_{12}K_{I2} \\ X_{22}K_{I2} \end{bmatrix}, \quad K_I = \begin{bmatrix} X_{11}^{-1}K_1 - X_{11}^{-1}X_{12}X_{22}^{-1}K_2 \\ X_{22}^{-1}K_2 \end{bmatrix}. \qquad (12.122)$$

Moreover, Lemma 12.5.16 applies (even under these weaker assumptions; if Hypothesis 12.5.13 is satisfied, then its triple $(\mathcal{P}_X, J_1, \left[\begin{array}{c|c} \mathbb{K} & \mathbb{F}\end{array}\right])$ has the above properties), and the generating operators of Σ_Z defined by are given by

$$\left[\begin{array}{c|c} A_Z & B_Z \\ \hline C_Z & D_Z \end{array}\right] := \left[\begin{array}{c|cc} A^* + K_{I2}^* B_2^* & C_2^* + K_{I2}^* D_{22}^* & -K_{I1}^* X_{11}^* \\ \hline X_{22}^{-*} B_2^* & X_{22}^{-*} D_{22}^* & X_{22}^{-*} X_{12}^* \\ 0 & 0 & I \end{array}\right]. \qquad (12.123)$$

*We define the $\mathcal{P}_Z$-*DARE *as the DARE for Σ_Z and J_1. We note that (the symbols on the left ocrrespond to the notation of [IOW], pp. 307–; this will be explained later)*

$$Q_\times := C_Z^* J_\gamma C_Z = B_2 X_{22}^{-1} X_{22}^{-*} B_2^*, \qquad (12.124)$$

$$\left[\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right] L_\times := D_Z^* J_\gamma C_Z = \begin{bmatrix} D_{22} \\ X_{12} \end{bmatrix} X_{22}^{-1} X_{22}^{-*} B_2^*, \qquad (12.125)$$

$$\left[\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right] R_\times \left[\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right] := D_Z^* J_\gamma D_Z = \begin{bmatrix} D_{22} \\ X_{12} \end{bmatrix} X_{22}^{-1} X_{22}^{-*} \begin{bmatrix} D_{22}^* & X_{12}^* \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & \gamma^2 I \end{bmatrix}. \qquad (12.126)$$

Note from Lemma 12.5.12 that here $\mathbb{X} = I - \mathbb{F} = \mathbb{M}^{-1}$ corresponds to $\mathcal{P}_X$ (i.e., to (Factor1)), not to (Factor1X). Note also that (12.123) corresponds to $\left[\begin{array}{c|c} K & I - X \end{array}\right]$ (more exactly, to the state feedback pair $\left[\begin{array}{c|cc} 0 & 0 & 0 \\ K_2 & -X_{21} & I - X_{22} \end{array}\right]$ for (extended) Σ), where $X := \widehat{\mathbb{X}}(+\infty)$, not to K_I, although we have written it in terms of K_I and X, not in terms of K and X.

Proof: By Lemma 11.3.13(i)&(iii), there is $X \in \mathcal{GB}(U \times W)$ s.t. $X^*J_1X = S_X$, $X_{21} = 0$. By Theorem 9.8.12(s1), $(\mathcal{P}_{\mathrm{X}}, J_1, \left(XK_I \mid I - X \right))$ is also a solution of the corresponding IARE with same stabilizability properties (note that $\left[\ \mathbb{K} \mid \mathbb{F} \ \right]$ is not unique but all possible choices are obtained parameterizes by $X \in \mathcal{B}(U \times W)$ s.t. $X_{21} = 0$ and $X^*J_1X = S_X$).

Conversely, any internally P-stabilizing solution of the IARE with $X_{21} = 0$ is as above, by Theorem 9.8.12(b)&(s1), hence $X_{11}, X_{22} \in \mathcal{GB}$, by Lemma 11.3.13(b3). Therefore, (12.122) holds. The rest is straightforward. □

Next we make another remark on the correspondence of our notation and that of [IOW]:

Lemma 12.6.2 ($\mathcal{P}_{\mathrm{X}}, \mathcal{P}_{\mathrm{Y}}$) *For $\mathcal{P}_{\mathrm{X}}$ and $\mathcal{P}_{\mathrm{Y}}$ Riccati equations, we note that Σ_X satisfies*

$$Q_c := C_X^* J_\gamma C_X = C_1^* C_1, \tag{12.127}$$

$$\left[\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right] L_c^* := D_X^* J_\gamma C_X = \begin{bmatrix} D_{11}^* C_1 \\ D_{12}^* C_1 \end{bmatrix}, \tag{12.128}$$

$$\left[\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right] R_c \left[\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right] := D_X^* J_\gamma D_X = \begin{bmatrix} D_{11}^* D_{11} & D_{11}^* D_{12} \\ D_{12}^* D_{11} & D_{12}^* D_{12} - \gamma^2 I \end{bmatrix} \tag{12.129}$$

(here the first terms on each line refer to symbols of [IOW], p. 307–, to which we will refer later), and Σ_Y^{d} satisfies

$$Q_o := C_{Y^{\mathrm{d}}}^* J_\gamma C_{Y^{\mathrm{d}}} = B_2 B_2^*, \tag{12.130}$$

$$\left[\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right] L_o^* := D_{Y^{\mathrm{d}}}^* J_\gamma C_{Y^{\mathrm{d}}} = \begin{bmatrix} D_{22} B_2^* \\ D_{12} B_2^* \end{bmatrix}, \tag{12.131}$$

$$\left[\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right] R_o \left[\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right] := D_{Y^{\mathrm{d}}}^* J_\gamma D_{Y^{\mathrm{d}}} = \begin{bmatrix} D_{22} D_{22}^* & D_{22} D_{12}^* \\ D_{12} D_{22}^* & D_{12} D_{12}^* - \gamma^2 I \end{bmatrix}. \tag{12.132}$$

Let X be as in Lemma 12.6.1, so that $X_{21} = 0$. Then, $S_X K_I = X^* J_\gamma X K_I = X^* J_\gamma K$, hence $K = J_\gamma^{-1} X^{-*}(S_X K_I)$, hence

$$K = -J_\gamma^{-1} X^{-*}(-S_X K_I) = \begin{bmatrix} -X_{11}^{-*} & 0 \\ -\gamma^{-2} X_{22}^{-*} X_{12}^* X_{11}^{-*} & \gamma^{-2} X_{22}^{-*} \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \end{bmatrix}, \tag{12.133}$$

$$\begin{bmatrix} T_1 \\ T_2 \end{bmatrix} = \begin{bmatrix} D_{11}^* C_1 + B_1^* \mathcal{P}_{\mathrm{X}} A \\ D_{12}^* C_1 + B_2^* \mathcal{P}_{\mathrm{X}} A \end{bmatrix} \tag{12.134}$$

$$K_1 = -X_{11}^{-*}(D_{11}^* C_1 + B_1^* \mathcal{P}_{\mathrm{X}} A), \quad K_2 = \gamma^{-2} X_{22}^{-*}(T_2 - X_{12}^* X_{11}^{-*} T_1) \tag{12.135}$$

□

(This is obvious. Note that we have $T_2^* = M_1$, $T_1^* = M_2$ in IOW-notation.)

We shall also need the following technical result:

Lemma 12.6.3 *Here (exceptionally) X, Y, Z refer to elements of $\mathcal{B}(H)$.*

(a) Let $\rho(XY) < 1$ and $X, Y \geq 0$. Then $I - XY \in \mathcal{GB}(H)$, $Z := Y(I - XY)^{-1} \geq 0$.

(b) Let $X, Z \geq 0$. Then $I + XZ \in \mathcal{GB}(H)$, $Y := Z(I + XZ)^{-1} \geq 0$, $\rho(XY) < 1$.

(c) *If the assumptions of (a) or (b) hold, then the assumptions of both (a) and (b) hold,* $\sigma(XY) \subset [0,1)$, $I + XZ = (I - XY)^{-1}$, $Y = Z(I + XZ)^{-1}$, *and* $Z = Y(I - XY)^{-1}$.

Proof: (a) Let $\rho(XY) < 1$. Then $1 \notin \sigma(XY)$, hence $I - XY \in \mathcal{GB}(H)$. Set $Z := Y(I - XY)^{-1}$, $W := I - Y^{1/2}XY^{1/2}$, so that $W = W^*$. Because $\rho(Y^{1/2}XY^{1/2}) = \rho(XY) < 1$, by Lemma A.3.3(s2), we have $\sigma(W) \subset 1 - (-1,1) = (0,2)$, in particular, $W \gg 0$. Therefore,

$$\langle Z(I - XY)x, (I - XY)x\rangle = \langle Yx, (I - XY)x\rangle = \langle Y^{1/2}x, WY^{1/2}x\rangle \geq 0 \quad \text{for all } x \in H. \tag{12.136}$$

Consequently, $Z \geq 0$. Moreover, $I + XZ = (I - XY + XY)(I - XY)^{-1} = (I - XY)^{-1}$, and $Z - ZXY = Y$ implies that $Y = Z(I + XZ)^{-1}$.

(b) By Lemma A.3.3(s2)&(s3), $\sigma(XZ) \cup \{0\} = \sigma(T) \cup \{0\}$, where $T := Z^{1/2}XZ^{1/2}$. But $T \geq 0$, hence $\sigma(XZ) \subset \mathbf{R}_+$, hence $\sigma(I + XZ) \subset [1,\infty)$; in particular, $I + XZ \in \mathcal{GB}(H)$. Moreover,

$$\langle Y(I + XZ)x, (I + XZ)x\rangle = \langle Zx, (I + XZ)x\rangle = \langle x, (Z + ZXZ)x\rangle \geq 0, \tag{12.137}$$

for all $x \in H$, i.e., for all $(I + XZ)x \in H$, hence $Y \geq 0$.

Furthermore, $\sigma(XY) = \sigma(I - (I + XZ)^{-1}) = 1 - 1/\sigma(I + XZ) \subset 1 - (0,1] = [0,1)$, hence $\rho(XY) < 1$. The final two equations are obtained from, e.g., $Y + YXZ = Z$.

(c) See the proofs of (b) and (a). □

Next we establish the connection between the $\mathcal{P}_{\mathrm{Y}}$-DARE and the $\mathcal{P}_{\mathrm{Z}}$-DARE, i.e., we show that "(1.) and (4.)" hold iff "(1.)–(3.)" hold (see Section 12.2):

Lemma 12.6.4 ($\mathcal{P}_{\mathrm{Z}} \Leftrightarrow \mathcal{P}_{\mathrm{Y}}$ & $\rho(\mathcal{P}_{\mathrm{X}}\mathcal{P}_{\mathrm{Y}}) < \gamma^2$) *Let the $\mathcal{P}_{\mathrm{X}}$-DARE have an internally P-stabilizing solution $(\mathcal{P}_{\mathrm{X}}, S_X, K_{X,I})$ s.t. $\mathcal{P}_{\mathrm{X}} \geq 0$, $S_{X11} \gg 0$ and $S_{X22} - S_{X21}S_{X11}^{-1}S_{X12} \ll 0$. Then the following are equivalent:*

(i) *the $\mathcal{P}_{\mathrm{Z}}$-eDARE has a solution $(\mathcal{P}_{\mathrm{Z}}, S_{\mathrm{Z}}, K_{\mathrm{Z}})$ s.t. $\mathcal{P}_{\mathrm{Z}} \geq 0$;*

(ii) *the $\mathcal{P}_{\mathrm{Y}}$-eDARE has a solution $(\mathcal{P}_{\mathrm{Y}}, S_Y, K_{\mathrm{Y}})$ s.t. $\mathcal{P}_{\mathrm{Y}} \geq 0$ and $\rho(\mathcal{P}_{\mathrm{X}}\mathcal{P}_{\mathrm{Y}}) < \gamma^2$.*

Assume that (i) or (ii) (hence both) holds. Let $(\mathcal{P}_{\mathrm{Z}}, S_Z, K_{\mathrm{Z}})$ and $(\mathcal{P}_{\mathrm{Y}}, S_Y, K_{\mathrm{Y}})$ be the corresponding solutions (i.e., one is given and the other is the one constructed in the proof). Then the following hold:

(a) *We have*

$$\mathcal{P}_{\mathrm{Z}} = \mathcal{P}_{\mathrm{Y}}(\gamma^2 I - \mathcal{P}_{\mathrm{X}}\mathcal{P}_{\mathrm{Y}})^{-1}, \qquad \mathcal{P}_{\mathrm{Y}} = \gamma^2\mathcal{P}_{\mathrm{Z}}(I + \mathcal{P}_{\mathrm{X}}\mathcal{P}_{\mathrm{Z}})^{-1}, \tag{12.138}$$

$$I + \mathcal{P}_{\mathrm{X}}\mathcal{P}_{\mathrm{Z}} = (I - \gamma^{-2}\mathcal{P}_{\mathrm{X}}\mathcal{P}_{\mathrm{Y}})^{-1}, \qquad A_{\circlearrowleft,\mathrm{Y}} = (I + \mathcal{P}_{\mathrm{X}}\mathcal{P}_{\mathrm{Z}})A_{\circlearrowleft,\mathrm{Z}}(I + \mathcal{P}_{\mathrm{X}}\mathcal{P}_{\mathrm{Z}})^{-1}, \tag{12.139}$$

where $A_{\circlearrowleft,\mathrm{Y}}$ and $A_{\circlearrowleft,\mathrm{Z}}$ are the corresponding closed-loop semigroup generators.

(b) *$\mathcal{P}_{\mathrm{Z}}$ is [strongly/exponentially] internally P-stabilizing iff $\mathcal{P}_{\mathrm{Y}}$ is.*

(c) *If $S_{Y11} \gg 0$, $S_{Y22} - S_{Y21}S_{Y11}^{-1}S_{Y12} \ll 0$, then $S_{Z11} \gg 0$ and $S_{Z22} - S_{Z21}S_{Z11}^{-1}S_{Z12} \ll 0$.*

Recall that an internally P-stabilizing solution is unique, and an internally exponentially stabilizing solution is exponentially stabilizing (see Theorem 14.1.4(b) and Lemma 13.3.8).

The $\mathcal{P}_X$-DARE, $\mathcal{P}_Y$-DARE and $\mathcal{P}_Z$-DARE, i.e., the DAREs for (Σ_X, J_γ), (Σ_Y, J_γ) and (Σ_Z, J_1) are given in (12.34), (12.35) and (12.36), respectively. See (12.84), (12.85) and (12.94) for corresponding systems and (12.123) for the generators A_Z, B_Z, C_Z and D_Z.

Proof of Lemma 12.6.4: (As before, the eDAREs refer to corresponding DAREs without the requirement $S_* \in \mathcal{GB}$. Note also that we have made no coercivity assumptions on Σ (e.g., (12.32)–(12.33) (or "(A2)") need not hold a priori).)

We shall follow the (finite-dimensional, exponentially stabilizing) proof of [IOW] and extend it to our generality (in some parts of the proof we are forced to develope different methods due to infinite dimensions); the page numbers and "(10.*nnn*)"s below refer to IOW.

Let $(\mathcal{P}_X, S_X, K_{X,I})$ be the internally P-stabilizing solution of the $\mathcal{P}_X$-DARE. Fix some X of the form of Lemma 12.6.1, so that $X_{21} = 0$ (and $S_X = X^* J_1 X$). Set $K := X K_{X,I}$, so that $(\mathcal{P}_X, J_1, \begin{pmatrix} K \mid I - X \end{pmatrix})$ is also internally P-stabilizing as in Lemma 12.6.1 (with the same $\mathbb{A}_\circlearrowleft$, $\mathbb{C}_\circlearrowleft$, $\mathbb{K}_\circlearrowleft$ as for $\begin{pmatrix} K_{X,I} \mid 0 \end{pmatrix}$).

We have the following correspondence of the [IOW]-notation (here on the left-hand-side) and ours

$$()^T = ()^*, \text{ "} > 0\text{" } = \text{"} \gg 0\text{"}, \ D = D\left[\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right], \ B = B\left[\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right], \ J = \left[\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right] J_1 \left[\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right] = \left[\begin{smallmatrix} -I & 0 \\ 0 & I \end{smallmatrix}\right], \tag{12.140}$$

$$V_c = \begin{bmatrix} V_{c11} & 0 \\ V_{c21} & V_{c22} \end{bmatrix} = \left[\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right] X \left[\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right], \ W_c = -\left[\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right] K, \ F = \left[\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right] K_{X,I} = \left[\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right] X^{-1} K, \tag{12.141}$$

$$R_\times = \left[\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right] D_{Z^{\mathrm{d}}}^* J_1 D_{Z^{\mathrm{d}}} \left[\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right], \ C_O = B_{Z^{\mathrm{d}}}^*, \ X = \mathcal{P}_X, \ Y = \mathcal{P}_Y, \ Z = \mathcal{P}_Z \tag{12.142}$$

(note that $\left[\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right]^{-1} = \left[\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right]^*$). Throughout the rest of this proof, symbols in quotation marks always refer to [IOW]-notation, in particular, the indices 1 and 2 corresponding to U and W are interchanged (as above). It follows that the $\mathcal{P}_X$-DARE is equal to (note that $-SK_{X,I} = -X^* J_1 X K_{X,I} = -X^* J_1 K = \left[\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right] V_c^* J W_c$) the equation system

$$W^* J W = W_{c2}^* W_{c2} - W_{c1}^* W_{c1} = A^* \mathcal{P}_X A - \mathcal{P}_X + C_1^* C_1 = \text{"} A^* X A - X + C_1^* C_1 \text{"}, \tag{12.143}$$

$$V_c^* J V_c = \begin{bmatrix} V_{c21}^* V_{c21} - V_{c11}^* V_{c11} & V_{c21}^* V_{c22} \\ V_{c22}^* V_{c21} & V_{c22}^* V_{c22} \end{bmatrix} = \left[\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right] S_X \left[\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right] = \text{"} \begin{bmatrix} D_{11} D_{11}^* - I + B_1^* X B_1 & D_{11}^* D_{12} + B_1^* X B_2 \\ D_{12}^* D_{11} + B_2^* X B_1 & D_{12}^* D_{12} + B_2^* X B_2 \end{bmatrix} \text{"}, \tag{12.144}$$

$$W_c^* J V_c = \begin{bmatrix} W_{c2}^* V_{c21} - W_{c1}^* V_{c11} \\ W_{c2}^* V_{22} \end{bmatrix} = \text{"} \begin{bmatrix} D_{11}^* C_1 \\ D_{12}^* C_1 \end{bmatrix} + \begin{bmatrix} B_1^* X A \\ B_2^* X A \end{bmatrix} \text{"}. \tag{12.145}$$

Part I: case $\gamma = 1$*:*

1° *Symplectic pencils:* Note that the conditions (CD1), (CD2) and (CD4) on p. 308 hold, respectively, iff the $\mathcal{P}_X$-DARE, $\mathcal{P}_Y$-DARE and $\mathcal{P}_Z$-DARE have exponentially stabilizing solutions with $S_{*11} \gg 0$ and $S_{*22} - S_{*21} S_{*11}^{-1} S_{*12} \ll 0$, by Lemma 11.1.7 (see also Lemma 12.6.2 and Lemma 12.6.1).

However, we start with only the assumption that the $\mathcal{P}_X$-DARE has an internally P-stabilizing solution, and we shall obtain analogous results as in [IOW], by following their proof.

Let $S = X^*J_1X$ and $K_{X,I}$ correspond to $X_I = I$ (i.e., to the solution of the DARE), and let K correspond to X (i.e., $K = XK_{X,I}$). Then $V_c = RXR$ and $W_c = -RK$, where $R = \left[\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right]$ in the notation of [IOW].

Let (M_Y, N_Y) and (M_Z, N_Z) be the symplectic pencils (see Lemma 14.2.5) corresponding to (CD2) and (CD4), respectively, with third and fourth rows and columns of M_Z and N_Z interchanged, in particular,

$$M_Y = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & -A & 0 & 0 \\ 0 & -C_1 & 0 & 0 \\ 0 & -C_2 & 0 & 0 \end{bmatrix} \text{ and } M_Z = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & -A - B_1X_{22}^{-1}K_2 & 0 & 0 \\ 0 & -X_{12}X_{22}^{-1}K_2 + K_1 & 0 & 0 \\ 0 & -C_2 - D_{22}X_{22}^{-1}K_2 & 0 & 0 \end{bmatrix} \tag{12.146}$$

satisfy $M_Y \in \mathcal{B}(H \times H \times Z \times Y)$ and $M_Z \in \mathcal{B}(H \times H \times Y \times U)$.

By repeating the routine but loooong computations (do not try this at home) of pp. 315–317 of [IOW], we see that the extensions $M_Y', N_Y' \in \mathcal{B}(H \times H \times Z \times Y \times W \times U)$ of M_Y and N_Y, and $M_Z', N_Z' \in \mathcal{B}(H \times H \times U \times Y \times W \times Z)$ of M_Z and N_Z, and the operators U_a and W_a given there satisfy

$$U_a M_Z' W_a = M_Y', \quad U_a N_Z' W_a = N_Y', \tag{12.147}$$

and U_a and W_a are invertible (use Lemma A.1.1(b1)&(b2) and elementary row and column operations for this invertibility).

(Note: we must have $()^*$ in place of $()^T$. Note also the following misprints: the top row of the biggest matrix in (10.161) should be $\begin{bmatrix} -S_c & S_cV_{c21}^*V_{c22}^* \end{bmatrix}$, the W_2^T in (10.162) should be W_{c2}^T (i.e., W_{c2}^*), and "(10.165)" on line 5 of p. 317 should be "(10.164)".)

2° "$\rho(\mathcal{P}_X\mathcal{P}_Y) < 1$" *is necessary:* Let the $\mathcal{P}_Z$-DARE have an internally P-stabilizing solution $\mathcal{P}_Z \geq 0$. Then $I + \mathcal{P}_X\mathcal{P}_Z \in \mathcal{GB}(H)$, $\mathcal{P}_Y := \mathcal{P}_Z(I + \mathcal{P}_X\mathcal{P}_Z)^{-1} \geq 0$ and $\rho(\mathcal{P}_X\mathcal{P}_Y) < 1$, by Lemma 12.6.3(a). Thus, we only have to show that $\mathcal{P}_Y$ is the (unique) internally P-stabilizing solution of $\mathcal{P}_Y$-eDARE.

Moreover, $N_ZV_Z = M_ZV_ZA_{\circlearrowleft,Z}$ and hence $N_Z'V_Z' = M_Z'V_Z'A_{\circlearrowleft,Z}$ (i.e., (10.166) and (10.169) hold), where

$$V_Z = \begin{bmatrix} I \\ \mathcal{P}_Z \\ K_Z \end{bmatrix}, \; V_Z' = \begin{bmatrix} V_Z \\ 0 \\ -\Theta\mathcal{P}_ZA_{\circlearrowleft,Z} \end{bmatrix}, \tag{12.148}$$

Θ is as on p. 316, K_Z is the corresponding state feedback operator and $A_{\circlearrowleft,Z}$ the corresponding closed-loop semigroup generator, by Lemma 14.2.5.

But (12.147) implies that $N_Y'W_a^{-1}V_Z' = N_Y'W_a^{-1}V_Z'A_{\circlearrowleft,Z}$ (this is (10.170)), and

$$W_a = \begin{bmatrix} I & -\mathcal{P}_X & 0 \\ 0 & I & 0 \\ * & * & * \end{bmatrix} \text{ implies that } W_a^{-1} = \begin{bmatrix} I & \mathcal{P}_X & 0 \\ 0 & I & 0 \\ * & * & * \end{bmatrix}, \tag{12.149}$$

It follows that $W_a^{-1}V_Z' = \left[\begin{smallmatrix} I+\mathcal{P}_X\mathcal{P}_Z \\ \mathcal{P}_Z \\ * \end{smallmatrix}\right]$. Consequently,

$$N_Y'V_Y' = M_Y'V_Y'A_{\circlearrowleft,Y}, \text{ where } V_Y' := W_a^{-1}V_Z'(I+\mathcal{P}_X\mathcal{P}_Z)^{-1} = \begin{bmatrix} I \\ \mathcal{P}_Y \\ * \end{bmatrix} \tag{12.150}$$

and $A_{\circlearrowleft,Y} = (I+\mathcal{P}_X\mathcal{P}_Z)A_{\circlearrowleft,Z}(I+\mathcal{P}_X\mathcal{P}_Z)^{-1}$. The four top rows of $N_Y'V_Y' = M_Y'V_Y'A_{\circlearrowleft,Y}$ are $N_YV_Y = M_YV_YA_{\circlearrowleft,Y}$, where V_Y consists of the four top rows of V_Y'. By Lemma 14.2.5, $\mathcal{P}_Y$ solves the $\mathcal{P}_Y$-eDARE, and $A_{\circlearrowleft,Y}$ is the corresponding closed-loop semigroup generator.

3° "$\rho(\mathcal{P}_X\mathcal{P}_Y) < 1$" *is sufficient:* (We go here 2° backwards.) Assume (ii). Then $\mathcal{P}_Z := \mathcal{P}_Y(I-\mathcal{P}_X\mathcal{P}_Y)^{-1} \geq 0$, by Lemma 12.6.3(a), and $N_YV_Y = M_YV_YA_{\circlearrowleft,Y}$ and hence $N_Y'V_Y' = M_Y'V_Y'A_{\circlearrowleft,Y}$, where

$$V_Y = \begin{bmatrix} I \\ \mathcal{P}_Y \\ K_Y \end{bmatrix}, \; V_Y' = \begin{bmatrix} V_Y \\ X_aV_YA_{\circlearrowleft,Y} \\ -X_bV_YA_{\circlearrowleft,Y} \end{bmatrix}, \tag{12.151}$$

where X_a and X_b are the operators "X_1" and "X_2" of (10.158). By (12.149), we have $V_Z' := W_aV_Y'(I-\mathcal{P}_X\mathcal{P}_Y)^{-1} = \left[\begin{smallmatrix} I \\ \mathcal{P}_Z \\ * \end{smallmatrix}\right]$. But $N_Y'W_a^{-1}V_Z' = M_Y'W_a^{-1}V_Z'A_{\circlearrowleft,Z}$, where $A_{\circlearrowleft,Z} := (I-\mathcal{P}_X\mathcal{P}_Y)A_{\circlearrowleft,Z}(I-\mathcal{P}_X\mathcal{P}_Y)^{-1}$, hence $N_Z'V_Z' = M_Z'V_Z'A_{\circlearrowleft,Z}$, hence $N_ZV_Z = M_ZV_ZA_{\circlearrowleft,Z}$, where $V_Z = \left[\begin{smallmatrix} I \\ \mathcal{P}_Z \\ * \end{smallmatrix}\right]$ are the four top rows of V_Z'. Consequently, $\mathcal{P}_Z$ is a solution of the $\mathcal{P}_Z$-eDARE.

(a) Given (i) or (ii), we obtain the connecting formulae (12.138) and (12.139) from (2°, 3° and) Lemma 12.6.3.

(b) By (a) and Lemma A.4.2(h1), $A_{\circlearrowleft,Y}$ is [strongly/exponentially] stable iff $A_{\circlearrowleft,Z}$ is. Moreover, if $\mathcal{P}_Z$ is internally P-stabilizing, so that $A_{\circlearrowleft,Z}$ and $A_{\circlearrowleft,Y}$ are bounded (by the above) and $\langle A_{\circlearrowleft,Z}^n x_0, \mathcal{P}_Z A_{\circlearrowleft,Z}^n x_0\rangle = \|\mathcal{P}_Z^{1/2}A_{\circlearrowleft,Z}^n x_0\|_H^2 \to 0$, as $n \to +\infty$, for all $x_0 \in H$, then $\mathcal{P}_Z A_{\circlearrowleft,Z}^n x_0 = \mathcal{P}_Z^{1/2}\mathcal{P}_Z^{1/2}A_{\circlearrowleft,Z}^n x_0 \to 0$, hence

$$\mathcal{P}_Y A_{\circlearrowleft,Y}^n x_0 = \mathcal{P}_Y(I+\mathcal{P}_X\mathcal{P}_Z)A_{\circlearrowleft,Z}^n(I+\mathcal{P}_X\mathcal{P}_Z)^{-1}x_0 = \mathcal{P}_Z A_{\circlearrowleft,Z}^n(I+\mathcal{P}_X\mathcal{P}_Z)^{-1}x_0 \to 0, \tag{12.152}$$

for all $x_0 \in H$, hence also $\mathcal{P}_Y$ is internally P-stabilizing (because $\{A_{\circlearrowleft,Y}^n\}$ is bounded).

(c) We follow here closely pp. 321–326 of IOW. Now

$$\text{“}R_\times + C_O Z C_O^*\text{''} = \left[\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right](D_{Z^d}{}^*J_1D_{Z^d} + B_{Z^d}{}^*\mathcal{P}_Z B_{Z^d})\left[\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right]S_Z\left[\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right], \tag{12.153}$$

where $C_O := \left[\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right]B_{Z^d}^* = \left[\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right]\left[\begin{smallmatrix} -X_{11}K_{I1} \\ C_2+D_{22}K_{I2} \end{smallmatrix}\right]$. We divide the proof in parts 1° and 2°.

1° $S_{Z11} \gg 0$*:* Make the definitions of p. 321. By p. 322, we have $\mathcal{V} \gg 0$ (use Lemma A.3.1(p2) for the final conclusion). Then

$$\text{''}\overline{D}_{22}\overline{D}_{22}^*\text{''} = D_{22}D_{22}^* + C_2\mathcal{P}_Y C_2^* = S_{Y11} \gg 0. \tag{12.154}$$

Consequently, $S_{Z11} \gg 0$, because $S_{Z11} = \text{''}\,\overline{D}_{22}\mathcal{V}\overline{D}_{22}^*\text{''}$, by the computations on p. 322.

2° $S_{Z22} - S_{Z21}S_{Z11}^{-1}S_{Z12} \ll 0$, *equivalently (10.198) holds:* Now "$\overline{D}_{12}^*\overline{D}_{12} =$

$V_{c22}^T V_{c22}$", i.e.,

$$\text{"}\overline{D}_{12}^*\overline{D}_{12}'' = D_{11}^* D_{11} + B_1^* \mathcal{P}_X B_1 = (S_X)_{11} = X_{11}^* X_{11} \gg 0 \tag{12.155}$$

Because $Q :='' \overline{D}_{12} X_{11}^{-1''} \in \mathcal{B}(U, Z \times H)$, so that $Q^* Q = I$, there is an unitary extension $U_c \in \mathcal{GB}(U \times V, Z \times H)$ of Q, by Lemma A.3.1(e3), where $V := \operatorname{Ran}(Q)^\perp$ is a closed subspace of $Z \times H$; thus, (10.200) holds. Similarly, $\overline{D}_{21}\overline{D}_{21}^* = S_{Y11} = Y_{11}^* Y_{11} \gg 0$, so that there is a unitary extension $U_o \in \mathcal{GB}(Y \times V', W \times H)$ of "$\overline{D}_{21}^T V_{o22}$"$= \overline{D}_{21}^* Y_{11}$, as in (10.201), where $V' = \operatorname{Ran}(\overline{D}_{21}^* Y_{11})^\perp \subset W \times H$. In (10.202) the partition is w.r.t. $\mathcal{B}(Y \times V', U \times V)$.

The rest is straightforward, hence (10.206)$\gg 0$ (i.e., $E \gg 0$ in (10.209), where $E \in \mathcal{GB}(Y \times V')$), hence (10.207)–(10.208) hold, by, e.g., Lemma A.3.1(d).

We arrive at the inequalities at the end of the proof. By Lemma A.3.3(s2), (10.208), and Lemma A.3.1(b1), the last inequality holds, hence (10.198)$= -S_{Z22} + S_{Z21} S_{Z11}^{-1} S_{Z12} \gg 0$.

Part II: The general case ($\gamma > 0$): Apply Part I to $\operatorname{diag}(I, \gamma^{-1} I, I)\Sigma$, and then apply Lemma 12.6.5 (e.g., we have $\rho(\gamma^{-2}\mathcal{P}_X\mathcal{P}_Y) < 1 \Leftrightarrow \rho(\mathcal{P}_X\mathcal{P}_Y) < \gamma^2$). □

For general $\gamma > 0$, we divide z by γ to reduce the 4bp for the case with $\gamma = 1$:

Lemma 12.6.5 ($\rho(\mathcal{P}_X\mathcal{P}_Y) < \gamma^2 \neq 1$) *The 4bp for Σ and $\gamma > 0$ corresponds to the 4bp for $\operatorname{diag}(I, \gamma^{-1} I, I)\Sigma$ and 1 (i.e., we multiply $\begin{bmatrix} \mathbb{C}_1 & | & \mathbb{D}_{11} & \mathbb{D}_{12} \end{bmatrix}$ by γ^{-1}). Moreover, the solutions of the DAREs for the latter problem correspond to those for the original problem as follows (both solutions exist iff either exists; in the claims on $\mathcal{P}_Z$ we assume that the assumptions of Lemma 12.6.1 hold):*

$$(\mathcal{P}_{X\gamma}, S_{X\gamma}, K_{X\gamma}) = (\gamma^{-2}\mathcal{P}_X, \gamma^{-2} S_X, K_X), \tag{12.156}$$

$$\Sigma_{X\gamma\circlearrowleft} = \operatorname{diag}(I; \gamma^{-1} I, I, I, I) \cdot \Sigma_{X\circlearrowleft}; \tag{12.157}$$

$$(\mathcal{P}_{Y\gamma}, S_{Y\gamma}, K_{Y\gamma}) = \left(\mathcal{P}_Y, \begin{bmatrix} I & 0 \\ 0 & \gamma^{-1} I \end{bmatrix} S_Y \begin{bmatrix} I & 0 \\ 0 & \gamma^{-1} I \end{bmatrix}, \begin{bmatrix} I & 0 \\ 0 & \gamma I \end{bmatrix} K_Y\right), \tag{12.158}$$

$$\Sigma_{Y\gamma\circlearrowleft} = \operatorname{diag}(I; I, \gamma I, I, \gamma I) \cdot \Sigma_{Y\circlearrowleft} \cdot \operatorname{diag}(I; I, \gamma^{-1} I); \tag{12.159}$$

$$(\mathcal{P}_{Y\gamma}, S_{Z\gamma}, K_{Z\gamma}) = \left(\gamma^2 \mathcal{P}_Z, \begin{bmatrix} \gamma I & 0 \\ 0 & I \end{bmatrix} S_Z \begin{bmatrix} \gamma I & 0 \\ 0 & I \end{bmatrix}, \begin{bmatrix} I & 0 \\ 0 & \gamma I \end{bmatrix} K_Z\right), \tag{12.160}$$

$$\Sigma_{Z\gamma\circlearrowleft} = \operatorname{diag}(I; \gamma I, \gamma I, I, \gamma I) \cdot \Sigma_{Z\circlearrowleft} \cdot \operatorname{diag}(I; I, \gamma^{-1} I); \tag{12.161}$$

in particular, $\rho(\mathcal{P}_X\mathcal{P}_Y) < \gamma^2 \Leftrightarrow \rho(\mathcal{P}_{X\gamma}\mathcal{P}_{Y\gamma}) < 1$, and the stabilizability of these solutions is invariant under this modification.

(This was used in the proof of Lemma 12.6.4.)

Proof: (In the lemma, $\Sigma_{X\gamma\circlearrowleft}$ is the closed-loop system corresponding to $\mathcal{P}_X$-DARE (hence to modified Σ_X and J_1); analogously for $\Sigma_{Y\gamma\circlearrowleft}$ and $\Sigma_{Z\gamma\circlearrowleft}$; in particular, the closed-loop semigroups are invariant.)

1° *$\mathcal{P}_X$-DARE and $\mathcal{P}_Y$-DARE:* (Note that here we have made no further assumptions than Standing Hypothesis 12.1.1.)

By writing the $\mathcal{P}_X$-DARE and the $\mathcal{P}_Y$-DARE out with substitutions $\Sigma \mapsto \operatorname{diag}(I, \gamma^{-1} I, I)\Sigma$ and $\gamma \mapsto 1$, one observes that the solutions of the modified and original $\mathcal{P}_X$-DARE and $\mathcal{P}_Y$-DARE correspond to each other through the above formulae.

2° $\mathcal{P}_Z$*-DARE:* The requirement that $X^* J_1 X = S_X$ (which is implicit in Lemma 12.6.1) results in $\gamma^{-1} X$ in place of the original X in (12.123), which affects (the extended) Σ_Z as in (12.161), hence the claims on $\mathcal{P}_Z$-DARE can be verified in the same way as those on the $\mathcal{P}_X$-DARE.

3° The claims at the end of the lemma follow from the equations. □

We end this section by recording discrete-time counterparts of three important lemmas and a remark of Section 12.5.

First we note that Hypothesis 12.5.1 is weaker than standard H^∞ 4BP assumptions:

Lemma 12.6.6 ((A,B_1) & (A,C_2) $\Rightarrow$ Hypothesis 12.5.1) *Assume that* (A,B_1) *is optimizable and* (A,C_2) *is estimatable.*

Then Hypothesis 12.5.1 is satisfied (even with "exponentially jointly" in place of "jointly") except possibly (12.78). Moreover, then condition (12.78) holds iff (12.32)–(12.33) are satisfied.

Proof: By Proposition 13.3.14 (and its proof), there are $K \in \mathcal{B}(H,U)$ and $\mathtt{H} \in \mathcal{B}(Y,H)$ s.t. $A + B_1 K$ and $A + \mathtt{H} C_2$ are exponentially stable. Extend Σ by K and $\mathtt{H}$ (with $F = 0 = G = E$) to satisfy Hypothesis 12.5.1 with "exponentially jointly" in place of "jointly" (cf. the proof of Lemma 13.3.17(a)&(b)) except possibly (12.78).

But K is exponentially stabilizing for $\Sigma_{11} := \left(\begin{array}{c|c} A & B_1 \\ \hline C_1 & D_{11} \end{array}\right)$, with closed loop I/O map $\begin{bmatrix} \mathbb{N}_{\mathrm{u}\,11} \\ \mathbb{M}_{\mathrm{u}\,11} \end{bmatrix}$ (since $\mathbb{M}_\mathrm{u} := (I - \mathbb{F}_\mathrm{u})^{-1} = \begin{bmatrix} (I-\mathbb{F}_{\mathrm{u}\,11})^{-1} & -(I-\mathbb{F}_{\mathrm{u}\,11})^{-1}\mathbb{F}_{\mathrm{u}\,12} \\ 0 & I \end{bmatrix}$).

By Lemma 8.4.11(a2), $\mathbb{N}_{\mathrm{u}\,11}^* \mathbb{N}_{\mathrm{u}\,11} \gg 0$ iff $\mathbb{N}_{\mathrm{u}\,11}$ is I-coercive; by Theorem 8.4.5(d), this is the case iff $\mathbb{D}_{11}$ is I-coercive over $\mathcal{U}_{\mathrm{exp}}$ (w.r.t. system Σ_{11}); by Proposition 15.2.2(f1)&(i)&(ii), this is the case iff (12.32) holds.

By dual arguments, we obtain that $\widetilde{\mathbb{N}}_{\mathrm{y}\,22} \widetilde{\mathbb{N}}_{\mathrm{y}\,22}^* \gg 0$ iff (12.33) holds. □

Lemma 12.6.7 ($\mathcal{P}_X$ & $\mathcal{P}_Y$ $\Rightarrow$ Hypothesis 12.5.1) *If conditions (1.) and (2.) of Theorem 12.2.1 are satisfied and (12.32)–(12.33) hold, then* (A,B_1) *is optimizable and* (A,C_2) *is estimatable.*

See Lemma 12.6.6 for more.

Proof: By Theorem 11.5.1(iii)&(i), (the ficp for Σ_X has a solution and) (A,B_1) is exponentially stabilizable. By dual arguments, (A,C_2) is exponentially detectable. □

Lemma 12.6.8 (Hypothesis 12.5.1 $\Rightarrow$ (12.32)–(12.33)) *Hypothesis 12.5.1 is satisfied with* $\begin{bmatrix} \mathbb{K}_\mathrm{u} & \big| & \mathbb{F}_\mathrm{u} \end{bmatrix}$ *and* $\begin{bmatrix} \mathbb{H}_\mathrm{y} \\ \hline \mathbb{G}_\mathrm{y} \end{bmatrix}$ *being exponentially [jointly] stabilizing iff* (A,B_1) *is optimizable,* (A,C_2) *is estimatable and (12.32)–(12.33) are satisfied.*

Proof: "If": This follows from Lemmas 12.6.7 and 12.6.6.

“Only if” Assume that Hypothesis 12.5.1 holds and that $\left[\begin{array}{c|c} \mathbb{K}_u & \mathbb{F}_u \end{array}\right]$ is exponentially stabilizing. Then (A,B_1) is optimizable and (A,C_2) is estimatable, by Lemma 12.5.2. From the end of the proof of Lemma 12.6.6, we observe that (12.32) and (12.33) hold. □

In discrete-time, the necessary and sufficient conditions “(Factor1X) and (Factor2Z)” for the solvability of the frequency-domain H^∞ 4BP can be formulated in terms of original data without any regularity assumptions:

Remark 12.6.9 ($\rho(XY) < \gamma^2$: I/O formulation) *Assume that Hypothesis 12.3.1 is satisfied with exponential coprimeness (as in Proposition 12.4.10). Then (i)–(iii) of Remark 12.5.25 are equivalent.* □

(This was remarked at the end of the proof of Remark 12.5.25.)

However, also in discrete time, our result in the non-exponential case is onedirectional, as in Remark 12.5.25.

(See the notes on p. 711.)

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Infinite-Dimensional Linear Systems, Optimal Control and Algebraic Riccati Equations

Volume 3/3 — Discrete-Time Theory & Appendices

Kalle Mikkola

$$x_{j+1} = Ax_j + Bu_j$$
$$y_j = Cx_j + Du_j$$

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Infinite-Dimensional Linear Systems, Optimal Control and Algebraic Riccati Equations

Volume 3/3 — Discrete-Time Theory & Appendices

Kalle Mikkola

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Helsinki University of Technology
Department of Engineering Physics and Mathematics
Institute of Mathematics

Kalle Mikkola: *Infinite-Dimensional Linear Systems, Optimal Control and Algebraic Riccati Equations*; Helsinki University of Technology Institute of Mathematics Research Reports A452 (2002).

Abstract: *In this monograph, we solve rather general linear, infinite-dimensional, time-invariant control problems, including the* H^∞ *and LQR problems, in terms of algebraic Riccati equations and of spectral or coprime factorizations. We work in the class of (weakly regular) well-posed linear systems (WPLSs) in the sense of G. Weiss and D. Salamon.*
Moreover, we develop the required theories, also of independent interest, on WPLSs, time-invariant operators, transfer and boundary functions, factorizations and Riccati equations. Finally, we present the corresponding theories and results also for discrete-time systems.

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