Chapter 11

H^∞ Full-Information Control Problem ($\|w \mapsto z\| < \gamma$)

Of all men's miseries, the bitterest is this: to know so much and have control over nothing.

— Herodotos

In this chapter, we shall solve the H^∞ Full-Information Control Problem (FICP), which is described on p. 33.

Our main results are presented in Section 11.1; applications to parabolic (analytic semigroup) systems are given in Corollary 9.5.11. Further results and proofs are given in Section 11.2 (including the extension of the (J_γ, J_1)-lossless factorization solution of [Green] and [CG97] to an MTIC setting, in Theorem 11.2.7), and the stable case in Section 11.3. Section 11.4 treats minimax J-coercivity, a property of the Popov operator, equivalent to the existence of a nonsingular solution to the H^∞ minimax problem.

The discrete-time H^∞ full-information control problem (ficp) is treated in Section 11.5, and corresponding proofs are given in Section 11.6. The reader might wish to read first these two sections in order to observe the basic characteristics of the H^∞ FICP in a simple setting before going into the technical details required by the unboundedness of the input and output operators in continuous time. A reader interested only in the main results should read only the introduction on p. 33 and Sections 11.5 and 11.1, in that order.

The necessity part of our proofs is based on the solution of the abstract H^∞ FICP (i.e., the FICP in the setting of Section 8.1), which is given in Section 11.7.

The methods used for the stable H^∞ FICP also apply to the (one-block) Nehari problem, where one wishes to estimate $d(\mathbb{D}, \mathrm{TIC}^*)$ or the Hankel norm $\|\pi_+ \mathbb{D} \pi_-\|$ of some $\mathbb{D} \in \mathrm{TIC}$. Therefore, we take a brief look at this problem in Section 11.8.

Standing Hypothesis 11.0.1 *Throughout this chapter (except in Section 11.7; see Hypothesis 11.7.1) we assume that H, U, W, Y, Z are Hilbert spaces, and that the space $\widetilde{\mathcal{A}}(U \times W)$ satisfies Hypothesis 8.4.7.*

(Cf. Theorem 8.4.9(a) and Definition 6.2.4.) Note also that Hypothesis 11.1.1 is assumed through Sections 11.1–11.3, 11.5 and 11.6, Hypothesis 11.8.1 is assumed through that Sections 11.8–11.9, and that Hypotheses 11.2.1, 11.3.1 11.6.1 and 11.7.1 are assumed through corresponding sections.

Thus, we can use the equivalence of Theorem 8.4.12 whenever we assume that $\mathbb{D} \in \widetilde{\mathcal{A}}(U \times W, Y)$, for any Hilbert space Y. (Note that this does not put any restrictions on U and W when one takes $\widetilde{\mathcal{A}}$ to be one of the classes in Theorem 8.4.9(1.). As noted in Lemma 14.3.5, classes $\ell^1_+(U \times W)*$ and $\mathrm{tic}_{\mathrm{exp}}(U \times W)$ satisfy the discrete-time version of Hypothesis 8.4.7.)

As in previous chapters, we denote by $\Sigma_{\circlearrowleft}$ the closed-loop system corresponding to the solution of the Riccati equation. By $\Sigma^{\curvearrowleft}$, we denote the corresponding "semi-closed-loop system" (where only the state feedback loop corresponding to the control u is closed but the second input, the disturbance w is unaffected); see Section 11.1 for details. The sub- or superscripts $\circlearrowleft$ and $\curvearrowleft$ are also used for the components and signals in those systems.

11.1 The H^∞ Full-Info Control Problem (FICP)

The goal of science is to build better mousetraps. The goal of nature is to build better mice.

We strongly recommend the reader to read the introduction to the H^∞ FICP problem (p. 33) and possibly also have a glance at the discrete-time results of Section 11.5 before going into the technical details of this section. The results of this section look like more complicated forms of Theorem 11.5.1, due to the possibly unbounded input and output operators.

In this section, we give necessary and sufficient conditions for the existence of a γ-suboptimal full-information or state-feedback controllers in terms of Riccati equations. For solutions in terms of lossless factorizations, see, e.g., Theorem 11.2.7 or Theorem 11.1.5(b).

In Theorems 11.1.3 and 11.1.4, we treate suboptimal exponentially stabilizing controllers (or "H^∞-FI-pairs") assuming "B_{w}^*-CARE" type regularity or a smoothing semigroup, respectively. In Theorem 11.1.6, we treate suboptimal output-stabilizing controllers assuming stronger nonsingularity (e.g., a copy of input contained in the output) and "B_{w}^*-CARE" type regularity. In Theorem 11.1.5, we treate suboptimal strongly stabilizing controllers assuming that the system is strongly q.r.c.-stabilizable with MTIC closed-loop system (the above three results do not assume any stabilizability). In all above results, we practically give sufficient and necessary conditions for a suboptimal state-feedback (and full-information) controllers to exist, in terms of a Riccati equation and the corresponding signature condition. In Remark 11.1.11 we treate the dual problem (the Full Control Problem).

As described on p. 33, in the H^∞ FICP, we have a system $\left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D}\end{smallmatrix}\right] \in \mathrm{WPLS}(U \times W, H, Z)$, and we wish to find, for each disturbance $w \in \mathrm{L}^2(\mathbf{R}_+; W)$, a "suboptimal" control $u \in \mathrm{L}^2(\mathbf{R}_+; U)$, i.e., one that is stabilizing and makes the (closed-loop) norm $\|w \mapsto z\|$ less than a given constant γ, where z is the output of the system under input $\left[\begin{smallmatrix} u \\ w \end{smallmatrix}\right]$ (and under initial state $x_0 = 0$). One often also requires that the control is given by some state feedback pair (with or without feedthrough).

We have $\|w \mapsto z\|_{\mathrm{L}^2 \to \mathrm{L}^2} < \gamma$ iff the (cost) function $\|z\|_2^2 - \gamma^2 \|w\|_2^2$ is uniformly negative w.r.t. w, i.e., iff $\|z\|_2^2 - \gamma^2\|w\|_2^2 \le -\varepsilon \|w\|_2^2$ for some $\varepsilon > 0$. Moreover, the J-critical control for this cost function and the corresponding Riccati equation lead to the solution of the H^∞ FICP, as will be shown in following sections.

Therefore, we usually augment Σ by the extra the row $\left[\begin{array}{c|cc} 0 & 0 & I \end{array}\right]$ to make the output equal to $y := \left[\begin{smallmatrix} z \\ w \end{smallmatrix}\right]$ (the input is $\left[\begin{smallmatrix} u \\ w \end{smallmatrix}\right] \in \mathrm{L}^2(\mathbf{R}_+; U \times W)$; see Figure 11.1 without the feedback row and loop), so that we can make the cost $\mathcal{J} := \langle y, Jy \rangle$ equal to $\|z\|_2^2 - \gamma^2 \|w\|_2^2$ by setting $J = J_\gamma := \left[\begin{smallmatrix} I & 0 \\ 0 & -\gamma^2 I \end{smallmatrix}\right]$:

Standing Hypothesis 11.1.1 (H^∞ Full-Information Control Problem (FICP)) *Throughout Sections 11.1–11.3, we make the following assumptions:*

$$\Sigma = \left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D} \end{array}\right] = \left[\begin{array}{c|cc} \mathbb{A} & \mathbb{B}_1 & \mathbb{B}_2 \\ \hline \mathbb{C}_1 & \mathbb{D}_{11} & \mathbb{D}_{12} \\ 0 & 0 & I \end{array}\right] \in \mathrm{WPLS}(U \times W, H, Z \times W). \tag{11.1}$$

The corresponding discrete-time assumptions (see (13.63)) are made in Sections 11.5 and 11.6.

If the generators of A, B, C and D of Σ are bounded, i.e., $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \in \mathcal{B}(H \times U \times W, H \times Z \times W)$, then this corresponds to the dynamics

$$\begin{cases} x' = Ax + B_1 u + B_2 w, \\ z = C_1 x + D_{11} u + D_{12} w \end{cases} \tag{11.2}$$

(and $w = Iw$) with initial state $x(0) = x_0 \in H$. In the case of a general weakly regular system Σ, equations (11.2) hold in the strong sense, see Theorem 6.2.13 for details. Note that $C = \left[\begin{smallmatrix} C_1 \\ 0 \end{smallmatrix}\right]$, $D = \left[\begin{smallmatrix} D_{11} & D_{22} \\ 0 & I \end{smallmatrix}\right]$.

As mentioned above, the suboptimal control is required to be "stabilizing". In the literature, this sometimes means that for any given initial state $x_0 \in H$ and disturbance $w \in \mathrm{L}^2(\mathbf{R}_+; W)$, the state feedback pair (or a more general control law $x_0, w \mapsto u$) must produce a control $u \in \mathrm{L}^2(\mathbf{R}_+; U)$ s.t. the output z (equivalently, $y := \left[\begin{smallmatrix} z \\ w \end{smallmatrix}\right]$) becomes stable, i.e., s.t. $\left[\begin{smallmatrix} u \\ w \end{smallmatrix}\right] \in \mathcal{U}_{\mathrm{out}}(x_0)$.

Often one also requires that $x \in \mathrm{L}^2(\mathbf{R}_+; H)$, i.e., that $\left[\begin{smallmatrix} u \\ w \end{smallmatrix}\right] \in \mathcal{U}_{\mathrm{exp}}(x_0)$ (see Lemma 8.3.3). In either case, we denote the set of corresponding controls by $\mathcal{U}_{\mathrm{u}}$ (in this section we set $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{exp}}$ or $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{out}}$, depending on the application):

Definition 11.1.2 (Suboptimal $\mathbf{H}^\infty$-FI-pair) *Throughout this section, we use the following notation:*

$$\mathcal{U}_{\mathrm{exp}}(x_0) = \{\left[\begin{smallmatrix} u \\ w \end{smallmatrix}\right] \in \mathrm{L}^2(\mathbf{R}_+; U \times W) \,\big|\, \mathbb{A}x_0 + \mathbb{B}\tau \left[\begin{smallmatrix} u \\ w \end{smallmatrix}\right] \in \mathrm{L}^2\}; \tag{11.3}$$

$$\mathcal{U}_{\mathrm{out}}(x_0) = \{\left[\begin{smallmatrix} u \\ w \end{smallmatrix}\right] \in \mathrm{L}^2(\mathbf{R}_+; U \times W) \,\big|\, \mathbb{C}x_0 + \mathbb{D} \left[\begin{smallmatrix} u \\ w \end{smallmatrix}\right] \in \mathrm{L}^2\}; \tag{11.4}$$

$$\mathcal{U}_{\mathrm{u}}(x_0, w) := \{u \in \mathrm{L}^2(\mathbf{R}_+; U) \,\big|\, \left[\begin{smallmatrix} u \\ w \end{smallmatrix}\right] \in \mathcal{U}_*^*(x_0)\} \quad (x_0 \in H,\ w \in \mathrm{L}^2(\mathbf{R}_+; W)), \tag{11.5}$$

$$\gamma_0 := \sup_{w \in \mathrm{L}^2(\mathbf{R}_+; W), \|w\|=1} \ \inf_{u \in \mathcal{U}_{\mathrm{u}}(0,w)} \|\mathbb{D}_{11} u + \mathbb{D}_{12} w\|_2. \tag{11.6}$$

An admissible state feedback pair of form $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right] = \left[\begin{array}{c|cc} \mathbb{K}_1 & \mathbb{F}_{11} & \mathbb{F}_{12} \\ 0 & 0 & 0 \end{array}\right]$ *(resp. admissible WR state feedback operator* $K = \left[\begin{smallmatrix} K_1 \\ 0 \end{smallmatrix}\right]$*) for* Σ *is called a* H$^\infty$-FI-pair *(resp.* H$^\infty$*-SF-operator) if*

$$\left[\begin{array}{c|c} \mathbb{K}_\circlearrowleft & \mathbb{F}_\circlearrowleft + I \end{array}\right] \begin{bmatrix} x_0 \\ 0 \\ w \end{bmatrix} \in \mathcal{U}_*^*(x_0) \quad \text{for all } x_0 \in H \text{ and } w \in \mathrm{L}^2(\mathbf{R}_+; W), \tag{11.7}$$

where $\Sigma_\circlearrowleft$ *is the corresponding closed-loop system (see Figure 11.1 and equation (11.8); we use prefices and suffices as in Definition 6.6.10).*

By γ_{FI} *(resp.* γ_{SF}*) we denote the infimum of the norm* $\|w \mapsto z\|_{\mathrm{L}^2(\mathbf{R}_+; W) \to \mathrm{L}^2(\mathbf{R}_+; Z)}$ *over all* H$^\infty$*-FI-pairs (resp. all* H$^\infty$*-SF-operators). Given* $\gamma > 0$*, a* H$^\infty$*-FI-pair or* H$^\infty$*-SF-operator is called* suboptimal *if* $\|\mathbb{D}_{\circlearrowleft 12}\|_{\mathrm{TIC}} < \gamma$.

By equation (11.8), we have $\|w \mapsto z\| = \|\mathbb{D}_{\circlearrowleft 12}\|_{\mathrm{TIC}}$. Therefore, the H$^\infty$ FICP means finding $\left[\begin{array}{c|c} \mathbb{K}_\circlearrowleft & \mathbb{F}_\circlearrowleft \end{array}\right]$ s.t. $\|\mathbb{D}_{\circlearrowleft 12}\|_{\mathrm{TIC}} < \gamma$. As one observes from Figure 11.1, the map $\left[\begin{array}{c|c} \mathbb{K}_\circlearrowleft & \mathbb{F}_\circlearrowleft \end{array}\right]$ maps $\left[\begin{smallmatrix} x_0 \\ u_\circlearrowleft \\ w \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} u - u_\circlearrowleft \\ 0 \end{smallmatrix}\right]$, i.e., from the initial state and

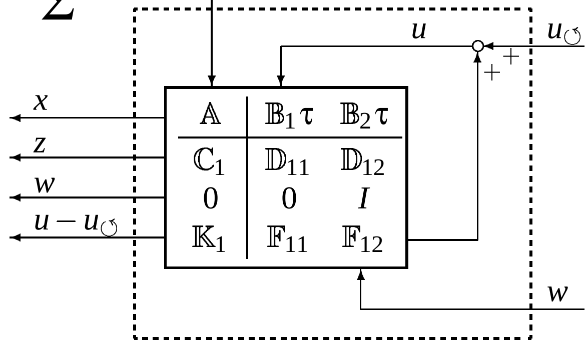

Figure 11.1: A WPLS controlled by a H$^\infty$-FI-pair

external inputs to the feedback signal, so that we must add $\left[\begin{smallmatrix} u_\circlearrowleft \\ w \end{smallmatrix}\right]$ to obtain the effective input $\left[\begin{smallmatrix} u \\ w \end{smallmatrix}\right]$ (the situation is the same as in Definition 6.6.10).

Note also that the controller (feedback) is only allowed to affect u. Therefore, the existence of a H$^\infty$-FI-pair for Σ can be described as "Σ is exponentially stabilizable through B_1" (cf. (11.11) and Figure 11.1). See Lemma 11.1.8 for details.

The standard "state-feedback" setting of Figure 6.3 has become the setting of Figure 11.1, because the feedback loop does not affect w (hence we have omitted the second, zero row of $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$) and because the second row of $\left[\begin{array}{c|c} \mathbb{C} & \mathbb{D} \end{array}\right]$ equals $\left[\begin{array}{c|cc} 0 & 0 & I \end{array}\right]$, so that the lower element of the output "$y = \left[\begin{smallmatrix} z \\ w \end{smallmatrix}\right]$" equals w. Recall from Definition 6.6.10, that our state feedback allows a feedforward term, hence it is actually a "full-information feedback". The corresponding closed-loop system $\Sigma^{\curvearrowleft} : \begin{bmatrix} x_0 \\ u_{\circlearrowleft} \\ w \end{bmatrix} \mapsto \begin{bmatrix} x \\ \left[\begin{smallmatrix} z \\ w \end{smallmatrix}\right] \\ \left[\begin{smallmatrix} u-u_{\circlearrowleft} \\ 0 \end{smallmatrix}\right] \end{bmatrix}$ is given by (cf. (6.134))

$$\Sigma^{\curvearrowleft} = \left[\begin{array}{c|cc} \mathbb{A}^{\curvearrowleft} & \mathbb{B}_1^{\curvearrowleft} & \mathbb{B}_2^{\curvearrowleft} \\ \hline \mathbb{C}_1^{\curvearrowleft} & \mathbb{D}_{11}^{\curvearrowleft} & \mathbb{D}_{12}^{\curvearrowleft} \\ 0 & 0 & I \\ \mathbb{K}_1^{\curvearrowleft} & \mathbb{F}_{11}^{\curvearrowleft} & \mathbb{F}_{12}^{\curvearrowleft} \\ 0 & 0 & 0 \end{array}\right] = \left[\begin{array}{c|cc} \mathbb{A} & \mathbb{B}_1 & \mathbb{B}_2 \\ \hline \mathbb{C}_1 & \mathbb{D}_{11} & \mathbb{D}_{12} \\ 0 & 0 & I \\ \mathbb{K}_1 & \mathbb{F}_{11} & \mathbb{F}_{12} \\ 0 & 0 & 0 \end{array}\right] \left[\begin{array}{c|cc} I & 0 & 0 \\ \hline \mathbb{X}_{11}^{-1}\mathbb{K}_1 & \mathbb{X}_{11}^{-1} & -\mathbb{X}_{11}^{-1}\mathbb{X}_{12} \\ 0 & 0 & I \end{array}\right] \tag{11.8}$$

$$= \left[\begin{array}{c|cc} \mathbb{A}+\mathbb{B}_1\tau\mathbb{X}_{11}^{-1}\mathbb{K}_1 & \mathbb{B}_1\mathbb{X}_{11}^{-1} & \mathbb{B}_2-\mathbb{B}_1\mathbb{X}_{11}^{-1}\mathbb{X}_{12} \\ \hline \mathbb{C}_1+\mathbb{D}_{11}\mathbb{X}_{11}^{-1}\mathbb{K}_1 & \mathbb{D}_{11}\mathbb{X}_{11}^{-1} & \mathbb{D}_{12}-\mathbb{D}_{11}\mathbb{X}_{11}^{-1}\mathbb{X}_{12} \\ 0 & 0 & I \\ \mathbb{X}_{11}^{-1}\mathbb{K}_1 & \mathbb{X}_{11}^{-1}-I & -\mathbb{X}_{11}^{-1}\mathbb{X}_{12} \\ 0 & 0 & 0 \end{array}\right] \tag{11.9}$$

$$= \left[\begin{array}{c|cc} \mathbb{A}_{\circlearrowleft}-\mathbb{B}_{\circlearrowleft 2}\tau\mathbb{M}_{22}^{-1}\mathbb{K}_{\circlearrowleft 2} & \mathbb{B}_{\circlearrowleft 1}-\mathbb{B}_{\circlearrowleft 2}\mathbb{M}_{22}^{-1}\mathbb{M}_{21} & \mathbb{B}_{\circlearrowleft 2}\mathbb{M}_{22}^{-1} \\ \hline \mathbb{C}_{\circlearrowleft 1}-\mathbb{N}_{12}\mathbb{M}_{22}^{-1}\mathbb{K}_{\circlearrowleft 2} & \mathbb{N}_{11}-\mathbb{N}_{12}\mathbb{M}_{22}^{-1}\mathbb{M}_{21} & \mathbb{N}_{12}\mathbb{M}_{22}^{-1} \\ 0 & 0 & I \\ \mathbb{K}_{\circlearrowleft 1}-\mathbb{M}_{12}\mathbb{M}_{22}^{-1}\mathbb{K}_{\circlearrowleft 2} & \mathbb{M}_{11}-\mathbb{M}_{12}\mathbb{M}_{22}^{-1}\mathbb{M}_{21}-I & \mathbb{M}_{12}\mathbb{M}_{22}^{-1} \\ 0 & 0 & I \end{array}\right] \tag{11.10}$$

$$= \left[\begin{array}{c|c} \mathbb{A}+\mathbb{B}\tau\mathbb{K}^{\curvearrowleft} & \mathbb{B}\overline{\mathbb{X}}^{-1} \\ \hline \mathbb{C}+\mathbb{D}\mathbb{K}^{\curvearrowleft} & \mathbb{D}\overline{\mathbb{X}}^{-1} \\ \mathbb{K}^{\curvearrowleft} & \overline{\mathbb{X}}^{-1}-I \end{array}\right], \tag{11.11}$$

and $\Sigma^{\curvearrowleft} \in \mathrm{WPLS}(U\times W, H, Z\times W\times U\times W)$ (here $\mathbb{X} := I-\mathbb{F}$, $\mathbb{M} := \mathbb{X}^{-1}$, $\mathbb{N} := \mathbb{D}\mathbb{M}$, $\overline{\mathbb{X}} := \left[\begin{smallmatrix} \mathbb{X}_{11} & \mathbb{X}_{12} \\ 0 & I \end{smallmatrix}\right] = (I+\mathbb{F}^{\curvearrowleft})^{-1}$; for (11.10) we used the fact that $\mathbb{M} = \overline{\mathbb{X}}^{-1}\underline{\mathbb{M}}$, where $\underline{\mathbb{M}} := \left[\begin{smallmatrix} I & 0 \\ \mathbb{M}_{21} & \mathbb{M}_{22} \end{smallmatrix}\right]$, by (A.9)).

If $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{exp}}$, then condition (11.7) holds iff $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ is exponentially stabilizing (see Remark 11.2.5 for more on (11.7)). Most existing literature for finite-dimensional control theory is written for $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{exp}}$, whereas also $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{out}}$ is popular in the infinite-dimensional case.

As obvious from the above definition, we are interested in the infimal value γ_0 of $\|w \mapsto z\|$ over all "stabilizing" controls (the norm of $w \mapsto \|\mathbb{D}_{11}u_{\min}(w) + \mathbb{D}_{12}w\|_2$), in the infimal value γ_{FI} of $\|w \mapsto z\|$ over all "stabilizing" full information controllers, and in the infimal value γ_{SF} of $\|w \mapsto z\|$ over all "stabilizing" pure state feedback controllers (thus, for γ_{SF}, we pose the additional condition that the feedthrough operator F of $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ exists and is equal to zero ($\mathbb{F} \in \mathrm{WR}$ and $F_{11} = 0 = F_{12}$); recall from Definition 6.6.10 that a state feedback pair is allowed to have any admissible feedthrough term). Since $\inf\emptyset = +\infty$, we obviously have

$$0 \le \gamma_0 \le \gamma_{\mathrm{FI}} \le \gamma_{\mathrm{SF}} \le +\infty. \tag{11.12}$$

Thus, $\gamma_0 < \gamma$ corresponds to the existence of a suboptimal control law (for $x_0 = 0$ only), $\gamma_{\mathrm{FI}} < \gamma$ to the existence of a causal control law and $\gamma_{\mathrm{SF}} < \gamma$ to the existence of a strictly causal control law (no feedforward term), with the additional restriction that γ_{FI} and γ_{SF} require the control law to be of state feedback form. If Σ is somewhat regular, then, under the standard assumptions that $\mathbb{D}_{11}$ is coercive and $D_{12} = 0$, we have $\gamma_0 = \gamma_{\mathrm{FI}} = \gamma_{\mathrm{SF}} < \infty$ for systems exponentially stabilizable through B_1, as illustrated in the following theorems. If $\mathbb{D} \in \mathrm{WR}$ and $\|D_{12}\| > \gamma_{\mathrm{FI}}$, then we necessarily have $\gamma_{\mathrm{SF}} > \gamma_{\mathrm{FI}}$ (if for $\mathbb{D} \in \mathrm{SR}$, since then

$$\|\widehat{\mathbb{D}}_{12}\|_{\mathrm{TIC}} = \|\widehat{\mathbb{D}}_{12}\|_{\mathrm{H}^\infty} \geq \|\widehat{D}_{12}\|_{\mathcal{B}(W,Z)} = \|D_{12}\|_{\mathcal{B}(W,Z)} > \gamma_{\mathrm{FI}}, \tag{11.13}$$

by Proposition 6.6.18(d4) (to be exact; this requires that we restrict ourselves to SR H^∞-SF-operators; when $\dim U < \infty$ or Σ is sufficiently regular, this is not a restriction, see, e.g., Theorem 11.1.3(a)). Example 11.1.9 illustrates this.

Now we are ready to present our results. See Proposition 10.3.2 and Theorem 9.2.3 for the two assumptions.

Theorem 11.1.3 ($\mathcal{U}_{\mathrm{exp}}$: H^∞ FICP $\Leftrightarrow$ B^*_{w}-CARE) *Assume that $\gamma > 0$ and that (1.) and (2.) hold.*

(1.) **(Nonsingularity)** *Assume $D_{11}^* D_{11} \gg 0$, and that there is $\varepsilon_1 > 0$ s.t.*

$$(ir - A)x_0 = Bu_0 \Rightarrow \|C_1 x_0 + D_{11} u_0\|_Z \geq \varepsilon_1 \|x_0\|_H \quad (x_0 \in H,\ u_0 \in U,\ r \in \mathbf{R}). \tag{11.14}$$

(2.) **(Regularity)** *Assume that Σ and J_γ satisfy Hypothesis 9.2.1 and that $\pi_{[0,1)} \mathbb{A} B \in \mathrm{L}^1([0,1); \mathcal{B}(U \times W, H))$ or $D^* J D \in \mathcal{G}\mathcal{B}(U \times W)$.*

Then (i)–(iii) are equivalent:

(i) $\gamma > \gamma_0$ and Σ is exponentially stabilizable through B_1;

(ii) $\gamma > \gamma_{\mathrm{FI}}$, i.e., there is a suboptimal H^∞-FI-pair for Σ;

(iii) $D_{12}^* D_{12} - D_{12}^* D_{11} (D_{11}^* D_{11})^{-1} D_{11}^* D_{12} \ll \gamma^2 I$, and (the B^*_{w}-CARE)

$$\left(B^*_{\mathrm{w}}\mathcal{P} + \begin{bmatrix} D_{11}^* \\ D_{12}^* \end{bmatrix} C_1\right)^* (D^* J_\gamma D)^{-1} \left(B^*_{\mathrm{w}}\mathcal{P} + \begin{bmatrix} D_{11}^* \\ D_{12}^* \end{bmatrix} C_1\right) = A^*\mathcal{P} + \mathcal{P}A + C_1^* C_1 \tag{11.15}$$

has a solution $\mathcal{P} \in \mathcal{B}(H, \mathrm{Dom}(B^*_{\mathrm{w}}))$ s.t. $\mathcal{P} \geq 0$ and $A + BK_{\mathrm{w}}$ generates an exponentially stable semigroup, where $K_{\mathrm{w}} := -(D^* J_\gamma D)^{-1}(B^*_{\mathrm{w}}\mathcal{P} + \begin{bmatrix} D_{11}^* \\ D_{12}^* \end{bmatrix} (C_{\mathrm{w}})_1)$.

Moreover, the following hold:

(a) Assume that $(\mathcal{P}, S, K)$ satisfies (iii) (here $S := D^* J_\gamma D$). Then

$$\left[\begin{array}{c|cc} -(D_{11}^* D_{11})^{-1}(D_{11}^* C_1 + (B_1^*)_{\mathrm{w}}\mathcal{P}) & 0 & -(D_{11}^* D_{11})^{-1} D_{11}^* D_{12} \\ 0 & 0 & 0 \end{array}\right] \tag{11.16}$$

generate a ULR (exponentially stabilizing) suboptimal H^∞-FI-pair.

There is a suboptimal H^∞-SF-operator iff $\|D_{12}\| < \gamma$; if this is the case, then

$$K_1 := \begin{bmatrix} I & 0 \end{bmatrix} K = -(S_{11} - S_{12} S_{22}^{-1} S_{21})^{-1} (D_{11}^* C_1 + (B_1^*)_{\mathrm{w}}\mathcal{P} - S_{12} S_{22}^{-1} (D_{12}^* C_1 + (B_2^*)_{\mathrm{w}}\mathcal{P})) \tag{11.17}$$

is a ULR (exponentially stabilizing) suboptimal H^∞-SF-operator, where $S := D^* J_\gamma D$.

(b) If (i)–(iii) hold, then the assumptions of Proposition 11.2.8 (including those of (a1)) are satisfied and (FI1)–(FI5) hold.

Applications to parabolic systems of this theorem and the ones to follow are given in Corollary 9.5.11. Recall that $J_\gamma := \begin{bmatrix} I & 0 \\ 0 & -\gamma^2 I \end{bmatrix}$, hence $D^* J_\gamma D = \begin{bmatrix} D_{11}^* D_{11} & D_{11}^* D_{12} \\ D_{12}^* D_{11} & D_{12}^* D_{12} - \gamma^2 I \end{bmatrix}$.

Under the normalizing conditions

$$D_{12} = 0, \quad D_{11}^* \begin{bmatrix} C_1 & D_{11} \end{bmatrix} = \begin{bmatrix} 0 & I \end{bmatrix}, \tag{11.18}$$

condition (iii) can be written as follows:

$$((B_1^*)_{\mathrm{w}} \mathcal{P})^* (B_1^*)_{\mathrm{w}} \mathcal{P} - \gamma^{-2} ((B_2^*)_{\mathrm{w}} \mathcal{P})^* (B_2^*)_{\mathrm{w}} \mathcal{P} = A^* \mathcal{P} + \mathcal{P} A + C_1^* C_1 \tag{11.19}$$

with the requirements that $\mathcal{P} \in \mathcal{B}(H, \mathrm{Dom}(B_{\mathrm{w}}^*))$, $\mathcal{P} \geq 0$, and $A + (\gamma^{-2} B_2 (B_2^*)_{\mathrm{w}} - B_1 (B_1^*)_{\mathrm{w}}) \mathcal{P}$ is exponentially stable. Now $S = J_\gamma := \left[\begin{smallmatrix} I & 0 \\ 0 & -\gamma^2 I \end{smallmatrix}\right]$ and $K = \begin{bmatrix} -(B_1^*)_{\mathrm{w}} \mathcal{P} \\ \gamma^{-2} (B_2^*)_{\mathrm{w}} \mathcal{P} \end{bmatrix} \in \mathcal{B}(H, U \times W))$, and if(f) (iii) holds, then $K_1 = -(B_1^*)_{\mathrm{w}} \mathcal{P} \in \mathcal{B}(H, U)$ is a suboptimal H^∞-SF-operator for Σ; this implies that the corresponding closed-loop state is controlled by the equation

$$x' = (A - B_1 (B_1^*)_{\mathrm{w}} \mathcal{P}) x + B_2 w \qquad \text{a.e.} \tag{11.20}$$

Note that equations (6.3.5) of [GL], (20.2.5) of [LR], (16.1) of [ZDG], and (4.11) of [Keu] are special cases of (11.19).

If B is bounded, then (11.19) takes the classical form

$$\mathcal{P}(B_1 B_1^* - \gamma^{-2} B_2 B_2^*) \mathcal{P} = A^* \mathcal{P} + \mathcal{P} A + C_1^* C_1, \tag{11.21}$$

hence $K = \begin{bmatrix} -(B_1^*)_{\mathrm{w}} \mathcal{P} \\ \gamma^{-2} (B_2^*)_{\mathrm{w}} \mathcal{P} \end{bmatrix} \in \mathcal{B}(H, U \times W) \in \mathcal{B}(H, U \times W)$, $K_1 = -B_1^* \mathcal{P} \in \mathcal{B}(H, U)$. Recall from Definition 9.8.1, that the CARE is given on $\mathcal{B}(\mathrm{Dom}(A), \mathrm{Dom}(A^*)^*) =: \mathcal{B}(H_1, H_{-1}^*)$; e.g., (11.19) holds iff

$$\begin{aligned} \langle (B_1^*)_{\mathrm{w}} \mathcal{P} x_0, (B_1^*)_{\mathrm{w}} \mathcal{P} x_1 \rangle - \gamma^{-2} \langle (B_2^*)_{\mathrm{w}} \mathcal{P} x_0, (B_2^*)_{\mathrm{w}} \mathcal{P} x_1 \rangle \\ = \langle A x_0, \mathcal{P} x_1 \rangle + \langle \mathcal{P} x_0, A x_1 \rangle + \langle C_1 x_0, C_1 x_1 \rangle \end{aligned} \tag{11.22}$$

for all $x_0, x_1 \in \mathrm{Dom}(A)$ (we can take $x_1 = x_0$ w.l.o.g., by Lemma A.3.5(a)).

All CAREs of this section and the next two sections equal the CARE for Σ and J_γ (under corresponding regularity assumptions, that is, some of the CAREs have been simplified).

Obviously, the K in (iii) is bounded (and hence $K_{\mathrm{w}} = K$) iff $\begin{bmatrix} D_{11}^* \\ D_{12}^* \end{bmatrix} C_1 \in \mathcal{B}(H, U)$; analogously, when $D_{11}^* C_1 \in \mathcal{B}(H, U)$, we may restrict us to bounded H^∞-FI-pairs (with generators $\left[\begin{array}{c|c} K & F \end{array}\right]$, $K = \left[\begin{smallmatrix} K_1 \\ 0 \end{smallmatrix}\right] \in \mathcal{B}(H, U)$, $F = \left[\begin{smallmatrix} F_{11} & F_{12} \\ 0 & 0 \end{smallmatrix}\right] \in \mathcal{B}(U \times W))$, i.e., if there is a suboptimal H^∞-FI-pair, then there is a bounded H^∞-FI-pair (by (a)).

Recall from Proposition 9.2.7 that any K of the form in (iii) (with $\mathcal{P} \in \mathcal{B}(H, \mathrm{Dom}(B_{\mathrm{w}}^*))$) is admissible (in particular, $A + BK_{\mathrm{w}}$ generates a C_0-semigroup). If $B \in \mathcal{B}(U, H)$, then (2.) is satisfied, the stabilizability assumption in (i) holds iff

(A, B_1) is exponentially stabilizable (or optimizable), and $\mathrm{Dom}(B_{\mathrm{w}}^*) = H$.

The equivalence (ii)⇔(iii) is (an extension of) the standard result that there is a suboptimal (exponentially stabilizing) state feedback controller through B_1 iff the Riccati equation has an exponentially stabilizing solution and the signature conditions on $D^* J_\gamma D$ are satisfied. Condition (i) means roughly that there is a suboptimal (exponentially stabilizing) control law $x_0, w \mapsto u$; thus "(i)⇔(ii)" says that such a law can always be realized by a state feedback controller.

If Σ is exponentially stable, then $\mathbb{A}_\circlearrowleft$, $\mathbb{C}_\circlearrowleft$, $\mathbb{K}_\circlearrowleft$ and $\mathcal{P}$ are given by (8.43)–(8.46), by (b) and Proposition 11.3.4(g), whenever (i)–(iii) hold.

One often has $\mathcal{J}(x_0, u, w) = \|x\|_2^2 + \varepsilon\|u\|_2^2 - \gamma^2\|w\|_2^2$ or something similar, so that (1.) is satisfied (see Proposition 10.3.2(e1) and 0.2°; the cost on x should be coercive at least for $x_0 = 0$ to satisfy (1.)); in fact, for this cost the nonsingularity assumptions of all results in this section and the following one are satisfied. The theorem does not hold without this assumption, that is, for "singular H^∞ problems", and such problems are rarely treated in the literature; see [Stoorvogel] for an exception (for finite-dimensional systems).

Proof of Theorem 11.1.3: We have tacitly assumed that $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{exp}}$ (also in (2.)). See Definition 9.2.6 for the B_{w}^*-CARE.

0.1° *Remark: Exponentially stabilizability through B_1:* By this we mean the existence of a H^∞-FI-pair (over $\mathcal{U}_{\mathrm{exp}}$), i.e., of an exponentially stabilizing state feedback pair that does not affect w (i.e., which is of form $\left[\begin{array}{c|c} \widetilde{\mathbb{K}} & \widetilde{\mathbb{F}} \end{array}\right] = \left[\begin{array}{c|cc} \widetilde{\mathbb{K}}_1 & \widetilde{\mathbb{F}}_{11} & \widetilde{\mathbb{F}}_{12} \\ 0 & 0 & 0 \end{array}\right]$, so that the feedback loop goes through B_1 only).

If $\pi_{[0,1)} \mathbb{A} B_1 u_0 \in \mathrm{L}^1([0,1); H)$ for all $u_0 \in U$ (this follows from the middle assumption in (2.)), then this assumption holds iff (A, B_1) is optimizable (equivalently, exponentially stabilizable), by Lemma 11.1.8 (see that lemma for further remarks).

0.2° *Remark: Assumption (1.):* Assumption (1.) implies that $\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B}_1 \\ \hline \mathbb{C}_1 & \mathbb{D}_{11} \end{array}\right]$ is positively I-coercive, by Proposition 10.3.2(g1)&(c). The converse holds if $D_{11}^* D_{11} \gg 0$ or $\mathbb{A}B \in \mathrm{L}^1_{\mathrm{loc}}$, by Proposition 10.3.2(e1)&(e2).

0.3° *Remark: $(D^* J_\gamma D)^{-1}$:* Set $S := D^* J_\gamma D$. Then $S_{11} = D_{11}^* D_{11} \gg 0$, by (1.), and the first condition in (iii) is equal to $S_{22} - S_{21} S_{11}^{-1} S_{12} \ll 0$. By Lemma 11.3.13(i)&(viii), these two together imply that $S \in \mathcal{GB}(U \times W)$, so that (11.15) is well defined.

1° *The equivalence of (i)–(iii):* Now (2.) or (3.) of Remark 9.9.14 is satisfied, hence we obtain the equivalence from Proposition 11.2.6, since (iii) is equivalent to (iii'), by which we denote "(iii)" of the proposition, as shown in 2° and 3° below. (Note that Hypothesis 11.2.1 is satisfied, by Lemma 11.2.2.)

2° *"(iii)⇒(iii')":* Assume (iii). Since $S := D^* J_\gamma D \in \mathcal{GB}(U \times W)$, by 0.3°, $\mathcal{P}$ is a solution of the B_{w}^*-CARE, hence an admissible and ULR solution of the CARE, by Proposition 9.2.7, and $\mathbb{A}_\circlearrowleft$ is generated by $A + BK_{\mathrm{w}}$, by (6.145), so that K is exponentially stabilizing.

Thus, (FI5) of Theorem 11.2.7 is satisfied. By Proposition 11.2.9, it follows that (FI2) holds, i.e., (ii) holds.

3° *"(iii')⇒(iii)":* This follows from Theorem 9.2.9(iii)&(iv) (as noted in 2°, the claim on $A + BK_{\mathrm{w}}$ holds iff K is exponentially stabilizing).

(b) This follows from Proposition 11.2.6 (whose assumptions are now satisfied, as noted in 1° above).

(a) This follows from Proposition 11.2.8(a1). □

We now present a CARE variant under assumptions that are weaker in certain sense:

Theorem 11.1.4 ($\mathcal{U}_{\mathrm{exp}}$: H$^\infty$ FICP ⇔ CARE: Case $\mathbb{A}B \in \mathrm{L}^1_{\mathrm{loc}}$) *Assume (1.) and (2.):*

(1.) **(Nonsingularity)** *Assume $D_{11}^*D_{11} \gg 0$, and that there is $\varepsilon_1 > 0$ s.t.*

$$(ir-A)x_0 = Bu_0 \Rightarrow \|C_1x_0 + D_{11}u_0\|_Z \geq \varepsilon_1\|x_0\|_H \quad (x_0 \in H,\ u_0 \in U,\ r \in \mathbf{R}). \tag{11.23}$$

(2.) **(Regularity)** *Assume that $\pi_{[0,1)}\mathbb{A}B \in \mathrm{L}^1([0,1);\mathcal{B}(U\times W,H))$, $\pi_{[0,1)}(C_1)_{\mathrm{w}}\mathbb{A} \in \mathrm{L}^1([0,1);\mathcal{B}(H,Z))$, and $\pi_{[0,1)}(C_1)_{\mathrm{w}}\mathbb{A}B \in \mathrm{L}^1([0,1);\mathcal{B}(U\times W,Z))$.*

Then (i)–(iii) are equivalent for any $\gamma > 0$:

(i) $\gamma > \gamma_0$, and (A,B_1) is exponentially stabilizable;

(ii) $\gamma > \gamma_{\mathrm{FI}}$, i.e., there is a suboptimal H$^\infty$-FI-pair for Σ;

(iii) $D_{12}^*D_{12} - D_{12}^*D_{11}(D_{11}^*D_{11})^{-1}D_{11}^*D_{12} \ll \gamma^2 I$, and the CARE

$$\left(B_{\mathrm{w}}^*\mathcal{P} + \begin{bmatrix} D_{11}^* \\ D_{12}^* \end{bmatrix} C_1\right)^* (D^*J_\gamma D)^{-1}\left(B_{\mathrm{w}}^*\mathcal{P} + \begin{bmatrix} D_{11}^* \\ D_{12}^* \end{bmatrix} C_1\right) = A^*\mathcal{P} + \mathcal{P}A + C_1^*C_1 \tag{11.24}$$

has an exponentially stabilizing solution $\mathcal{P} \in \mathcal{B}(H)$ s.t. $\mathcal{P} \geq 0$ and $\lim_{s\to+\infty} B_{\mathrm{w}}^*\mathcal{P}(s-A)^{-1}B = 0$.

In particular, $\gamma_{\mathrm{FI}} = \gamma_0$ if (A,B_1) is exponentially stabilizable (equivalently, optimizable). Moreover, Theorem 11.1.3(a)&(b) hold.

Most of the remarks made below Theorem 11.1.3 apply here too.

Proof: Condition (2.) implies that $\mathbb{D} \in \mathrm{MTIC}_\infty^{\mathrm{L}^1}(U\times W, Z\times W) \subset \mathrm{ULR}$, by Lemma 6.8.5.

0.1° *Remark: assumption (11.23) ⇔ $\mathbb{D}_{11}$ is I-coercive:* By Proposition 10.3.2(e2), (11.23) holds iff $\mathbb{D}_{11}$ is I-coercive (and also conditions (i)–(iii) of Proposition 10.3.2 (with $J \mapsto I$, $\Sigma \mapsto \left[\begin{array}{c|c} \mathbb{A} & \mathbb{B}_1 \\ \hline \mathbb{C}_1 & \mathbb{D}_{11} \end{array}\right]$) are equivalent to (11.23)).

0.2° *Remark: "(A,B_1) exponentially stabilizable":* By (2.) and Theorem 9.2.12, this holds iff (A,B_1) is optimizable, equivalently, iff Σ has an exponentially stabilizing (bounded) state feedback operator $\widetilde{K} = \begin{bmatrix} \widetilde{K}_1 \\ 0 \end{bmatrix} \in \mathcal{B}(H, U\times W)$.

See also 0.3° of the proof of Theorem 11.1.3.

1° *The equivalence of (i)–(iii):* Denote condition (iii) of Proposition 11.2.6 by "(iii-Prop)". Now Remark 9.9.14(5.) applies, hence we obtain the equivalence from Proposition 11.2.6, because condition (iii) is equivalent to (iii-Prop), as shown in 2° below. (Note that Hypothesis 11.2.1 is satisfied, by Lemma 11.2.2.)

2° *"(iii)⇔(iii-Prop)":* Condition (iii) says that (iii-Prop) holds, $\mathbb{F}$ is UR (by Lemma 9.11.5(e)) and $S = D^* J_\gamma D$. Conversely, if (iii-Prop) holds, then $S = D^* J_\gamma D$ and $\mathbb{F}$ is UR, by Remark 9.9.14(b)&(a).

(a)&(b) The original proofs of Theorem 11.1.3(a)&(b) apply mutatis mutandis. □

The case $\mathcal{U}_*^* = \mathcal{U}_{\text{out}}$ is much more complicated than the case $\mathcal{U}_*^* = \mathcal{U}_{\text{exp}}$ and hence very rarely treated in the literature. (We have not found any existing research of the H^∞ problems truely over $\mathcal{U}_{\text{out}}$ in the unstable case. Recall, however, that $\mathcal{U}_{\text{out}} = \mathcal{U}_{\text{exp}}$ for estimatable systems, hence many historical results are applicable for such systems, though none before for WPLSs.) We give a more thorough treatment of this case in Section 11.2, under weaker assumptions, but here we show that if Σ is strongly q.r.c.-stabilizable through B_1 with a rather regular closed-loop system, then we have the standard equivalence:

Theorem 11.1.5 ($\tilde{\mathcal{A}}$, $\mathcal{U}_{\text{out}} = \mathcal{U}_{\text{str}}$: H^∞ FICP ⇔ CARE) *Let $\mathcal{U}_*^* := \mathcal{U}_{\text{out}}$. Assume (1.) and (2.):*

(2.) **(Stabilizability)** *There is a strongly q.r.c.-stabilizing UR state feedback operator $\widetilde{K} = \begin{bmatrix} \widetilde{K}_1 \\ 0 \end{bmatrix}$ for Σ, and $\mathbb{D}_\flat \in \tilde{\mathcal{A}}$.*

(1.) **(Nonsingularity)** *There is $\varepsilon_1 > 0$ s.t. $\|\mathbb{D}_{\flat 11} u\|_2 \geq \varepsilon_1 \|u\|_2$ $(u \in \mathrm{L}^2(\mathbf{R}_+;U))$.*

Then (i)–(iii) are equivalent for each $\gamma > 0$:

(i) $\gamma > \gamma_0$;

(ii) $\gamma > \gamma_{\text{FI}}$*; i.e., there is a suboptimal H^∞-FI-pair for Σ;*

(iii) the CARE

$$\begin{cases} K^* S K = A^* \mathcal{P} + \mathcal{P} A + C_1^* C_1, \\ S = \begin{bmatrix} D_{11}^* D_{11} & D_{11}^* D_{12} \\ D_{12}^* D_{11} & D_{12}^* D_{12} - \gamma^2 I \end{bmatrix} + \lim_{s \to +\infty} B_{\text{w}}^* \mathcal{P} (s-A)^{-1} B, \\ K = -S^{-1} \left(B_{\text{w}}^* \mathcal{P} + \begin{bmatrix} D_{11}^* \\ D_{12}^* \end{bmatrix} C_1 \right). \end{cases} \tag{11.25}$$

has a q.r.c.-stabilizing solution $(\mathcal{P}, S, K)$, and $\mathcal{P} \geq 0$, $S_{11} \gg 0$ and $S_{22} - S_{21} S_{11}^{-1} S_{12} \ll 0$.

In particular, $\gamma_0 = \gamma_{\text{FI}}$. Moreover, the following hold:

(a) Assume that $(\mathcal{P}, S, K)$ satisfies (iii). Then K is UR and

$$\left[\begin{array}{c|cc} -S_{11}^{-1} (D_{11}^* C_1 + (B_1^*)_{\text{w}} \mathcal{P}) & 0 & -S_{11}^{-1} S_{12} \\ 0 & 0 & 0 \end{array} \right]. \tag{11.26}$$

generate a UR strongly q.r.c.-stabilizing suboptimal H^∞-FI-pair.

There is a suboptimal H^∞-SF-operator iff $S_{22} \ll 0$; if this is the case, then (11.17) is a UR strongly q.r.c.-stabilizing suboptimal H^∞-SF-operator.

(b) *Each of conditions (FI1)–(FI7) of Theorem 11.2.7 is equivalent to (i). If any of these holds, then the assumptions of Proposition 11.2.8 and Theorem 11.2.7 are satisfied.*

(c) *If (1.) of Theorem 11.1.6 holds, and the CARE has a UR solution with $\mathbb{A}_{\circlearrowleft}$, $\mathbb{C}_{\circlearrowleft}$ and $\mathbb{K}_{\circlearrowleft}$ strongly stable, then $\gamma \geq \gamma_{\mathrm{FI}} = \gamma_0$.*

See Theorem 11.2.7 and Corollary 11.2.11 for analogous results for $\mathcal{U}_{\mathrm{exp}}$, $\mathcal{U}_{\mathrm{sta}}$ and $\mathcal{U}_{\mathrm{out}}$ under slightly different assumptions.

Some of the remarks made below Theorem 11.1.3 apply here, too, with some minor modifications; in particular, also here we have $\gamma_0 = \gamma_{\mathrm{FI}}$ $(= \gamma_{\mathrm{SF}}$ if $D_{12} = 0)$.

An important special case of "(2.)" is the case where Σ is strongly stable and $\mathbb{D} \in \widetilde{\mathcal{A}}$ (take $\widetilde{K} = 0$). By Example 11.3.7(c), even for strongly stable Σ, condition "$\mathbb{D} \in \widetilde{\mathcal{A}}$" cannot be weakened to e.g., "$\mathbb{D} \in$ ULR" without making condition (iii) is strictly stronger than (i)–(ii) (indeed, the IARE corresponding to (iii) has a irregular (i.e., non-WR) q.r.c.-stabilizing solution leading to the minimax control and to a suboptimal irregular H^∞-FI-pair (there are also regular H^∞-FI-pairs, e.g., $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right] = \left[\begin{array}{c|c} 0 & 0 \end{array}\right]$), but the CARE does not have a stabilizing solution).

As noted below Theorem 11.1.6, the "almost equivalence" of (c) suffices for the binary search for γ_{FI}, hence we can use the simpler stabilization condition of (c) (instead of q.r.c.-stabilization) in this strictly nonsingular case.

Since $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{out}}$ means requiring the (effective) input ($\left[\begin{smallmatrix} u \\ w \end{smallmatrix}\right]$) and output ($y = \left[\begin{smallmatrix} z \\ w \end{smallmatrix}\right]$) to be stable ($\in \mathrm{L}^2$), we now must have $u, z \in \mathrm{L}^2$, for all $x_0 \in H$ and $w \in \mathrm{L}^2(\mathbf{R}_+; W)$. Because $\mathcal{U}_{\mathrm{out}} = \mathcal{U}_{\mathrm{str}}$ (by Lemma 8.3.3), it follows that then also the state becomes strongly stable ($\|x(t)\|_H \to 0$ as $t \to +\infty$).

Therefore, a H^∞-FI-pair is now an admissible state feedback pair of form $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right] = \left[\begin{array}{c|cc} \mathbb{K}_1 & \mathbb{F}_{11} & \mathbb{F}_{12} \\ 0 & 0 & 0 \end{array}\right]$ s.t. $u, z \in \mathrm{L}^2$ for all $x_0 \in H$ and $w \in \mathrm{L}^2(\mathbf{R}_+; W)$ in Figure 11.1 (see also (11.8)–(11.11) and Remark 11.2.5). By the above, it follows that $\|x(t)\|_H \to 0$, as $t \to +\infty$, for such x_0, w. Nevertheless, for $u_{\circlearrowleft} \neq 0$ the signals u and z (and x) are not required to be stable, hence the middle row of $\mathbb{D}^{\curvearrowleft}$ may be unstable, although we do have $u, z \in \mathrm{L}^2$ and x vanishing for any compactly supported $u_{\circlearrowleft} \in \mathrm{L}^2$, by (6.9).

Thus, for the solutions mentioned in (c), we do not know whether the H^∞-FI-pair defined by (11.26) is stabilizing. Therefore, if one has used (c) to find an estimate on $\gamma_{\mathrm{FI}} = \gamma_0$, one might wish to either 1. verify directly whether the corresponding (strongly internally and output-stabilizing) K is (strongly) stabilizing or 2. increase γ slightly to guarantee that $\gamma > \gamma_{\mathrm{FI}}$ and then find a internally stabilizing solution of the CARE (i.e., one with stable $\mathbb{A}_{\circlearrowleft}$), because then such a solution is necessarily the strongly q.r.c.-stabilizing one, by uniqueness (see Theorem 9.8.12(a)).

Proof of Theorem 11.1.5: (Note from the proof below that we can replace "UR" by "ULR" throughout this theorem.)

0.1° *Remark: Assumptions (1.)–(2.):* Here $\Sigma_\flat$ is the closed-loop system corresponding to $\widetilde{K}$ (cf. Definition 6.6.10). Recall from Theorem 6.6.28 that if $\widetilde{K}$ and some output injection pair are jointly strongly stabilizing and (I/O-)detecting for Σ, then $\widetilde{K}$ is strongly q.r.c.-stabilizing.

If Σ is strongly stable and $\mathbb{D}_\flat \in \widetilde{\mathcal{A}}$, then we can take $\widetilde{K} = 0$ in (2.).

0.2° *Remark: $\mathbb{D}_{11}$ is I-coercive* Assumption (1.) is equivalent to the (positive) I-coercivity of $\mathbb{D}_{\flat 11}$ (equivalently, of $\mathbb{D}_{11}$; in particular, it is independent of $\widetilde{K}$), by Lemma 8.4.11(b1).

1° *The equivalence of (i)–(iii), (a) and (b):* Except for (c), all claims in the lemma follow from Theorem 11.2.7 and Corollary 11.2.11 (use the fact that a solution of (11.25) is necessarily UR, by Lemma 9.11.5(e); note that Hypothesis 11.2.1 is satisfied, by 0.2°).

(c) By Theorem 9.8.5 and Theorem 9.9.1(c3), $\mathcal{P}$ is $\mathcal{U}_{\text{str}}$-stabilizing, hence $\mathcal{U}_{\text{out}}$-stabilizing. By Lemma 11.2.14(4.)&(b)&(a2) (with $s = +\infty$), it follows that (11.48) is a H^∞-FI-pair satisfying $\|\mathbb{D}_{12}^{\frown}\| \le \gamma$ (in fact, $\|\mathbb{D}_{12}^{\frown} u\|_2 < \gamma \|u\|_2$ for all $u \in \mathrm{L}^2(\mathbf{R}_+;U)$, by the remark in the proof of (a2)). Thus, then $\gamma \ge \gamma_{\text{FI}}$.

□

For $\mathcal{U}_{\text{exp}}$, one can obtain the equivalence "(ii)⇔(iii)" directly from Proposition 11.2.19 and Lemma 11.2.13 (or Lemma 11.2.14), provided that Σ is sufficiently regular (e.g., if Hypothesis 9.2.1 holds).

For any other $\mathcal{U}_*^*$ than $\mathcal{U}_{\text{exp}}$, there seems to be a gap between the proposition and the lemma: it seems that $\Sigma^{\frown}$ need not be stable enough. In Theorem 11.1.5, we used the q.r.c.-property to reduce the problem to the stable case, and the (stable) spectral factorization properties of $\widetilde{\mathcal{A}}$ to guarantee the existence of a strongly stabilizing solution in case that $\gamma > \gamma_0$.

In practice, the cost function is often of form $\mathcal{J}(x_0,u,w) := \|z_1\|_2^2 + \|Ru\|_2^2 - \gamma^2\|w\|_2^2$, where $R^*R \gg 0$. This is the case when $\left[\begin{array}{c|cc} \mathbb{C}_1 & \mathbb{D}_{11} & \mathbb{D}_{12} \end{array}\right] = \left[\begin{array}{c|cc} \mathbb{C}_{1a} & \mathbb{D}_{11a} & \mathbb{D}_{12a} \\ 0 & R & 0 \end{array}\right]$ and $\mathcal{B}(U) \ni R^*R \ge \varepsilon_+^2 I$ for some $\varepsilon_+ > 0$; obviously, condition (11.27) is then satisfied. Such a cost function forces u to be stable (for stable outputs, i.e., for a finite cost) and makes it possible to fill the gap mentioned above (as noted in (c) above):

Theorem 11.1.6 ($\boldsymbol{\mathcal{U}_{\text{out}}}$: $\mathbf{H^\infty}$ FICP ⇔ $\boldsymbol{B_{\text{w}}^*}$-CARE) *Assume that $\gamma > 0$ and $\mathcal{U}_*^* = \mathcal{U}_{\text{out}}$, and that (1.) and (2.) hold.*

(1.) **(Strict nonsingularity)** *We have $D_{11}^*D_{11} \gg 0$, and there is $\varepsilon_+ > 0$ s.t.*

$$\|\mathbb{C}_1 x_0 + \mathbb{D}_{11} u + \mathbb{D}_{12} w\|_2 \ge \varepsilon_+ \|u\|_2 \quad (u \in \mathrm{L}^2_{\varepsilon_+}(\mathbf{R}_+;U),\ w \in \mathrm{L}^2(\mathbf{R}_+;W),\ x_0 \in H). \tag{11.27}$$

(2.) **(Regularity)** *Assume that Σ and J_γ satisfy Hypothesis 9.2.1, and that $D^*J_\gamma D \in \mathcal{GB}(U \times W)$ or $B \in \mathcal{B}(U,H)$.*

If condition (iii) below holds, then there is a H^∞-FI-pair s.t. $\|w \mapsto z\| \le \gamma$. Conversely, if there is a H^∞-FI-pair s.t. $\|w \mapsto z\| < \gamma$, then (iii) holds.

(iii) $D_{12}^*D_{12} - D_{12}^*D_{11}(D_{11}^*D_{11})^{-1}D_{11}^*D_{12} \ll \gamma^2 I$, and (the B_{w}^*-CARE)

$$\left(B_{\text{w}}^*\mathcal{P} + \begin{bmatrix} D_{11}^* \\ D_{12}^* \end{bmatrix} C_1\right)^* (D^*J_\gamma D)^{-1} \left(B_{\text{w}}^*\mathcal{P} + \begin{bmatrix} D_{11}^* \\ D_{12}^* \end{bmatrix} C_1\right) = A^*\mathcal{P} + \mathcal{P}A + C_1^*C_1 \tag{11.28}$$

has a nonnegative $\mathcal{U}_{\text{out}}$-stabilizing solution $\mathcal{P} \in \mathcal{B}(H, \mathrm{Dom}(B_{\text{w}}^*))$.

Moreover, the following hold:

(a1) If $(\mathcal{P}, S, K)$ satisfies (iii), then (11.16) generate a ULR H^∞*-FI-pair s.t.* $\|w \mapsto z\| = \|\mathbb{D}_{12}^{\curvearrowright}\| \le \gamma$*.*

(a2) If there is a suboptimal H^∞*-SF-operator, then $\|D_{12}\| < \gamma$. Conversely, if $\|D_{12}\| < \gamma$ and (iii) has a solution $(\mathcal{P}, S, K)$, then K_1 =(11.17) is a ULR* H^∞*-SF-operator s.t.* $\|w \mapsto z\| = \|\mathbb{D}_{12}^{\curvearrowright}\| \le \gamma$*.*

The above "almost equivalence" is in practice as good as an equivalence: if we wish to find a state feedback controller s.t. $\|w \mapsto z\|$ is (approximately) minimized, then we can use a binary search over γ (solve (iii) above for different values of γ). See also the remarks below Theorem 11.1.3.

See the remarks below Theorem 11.1.5 for $\mathcal{U}_{\mathrm{out}}$ and for how stable is closed-loop system (in Figure 11.1) corresponding the H^∞-SF-operator defined by (11.17). See Definition 9.8.1 for "$\mathcal{U}_{\mathrm{out}}$-stabilizing" (which is equivalent to "one with stable $\mathbb{A}_{\circlearrowleft}$, $\mathbb{C}_{\circlearrowleft}$ and $\mathbb{K}_{\circlearrowleft}$" if Σ is strongly q.r.c.-stabilizable).

Note from Theorem 6.7.15(c2) that if Σ is estimatable, then (iii) is equivalent to (i)–(iii) of Theorem 11.1.3 (whose (a) and (b) then apply).

Proof of Theorem 11.1.6: (See Definition 9.2.6 for the B_{w}^*-CARE.)

0.1° *Remark on (2.):* Condition $D^* J_\gamma D \in \mathcal{GB}(U \times W)$ can be omitted if $B \in \mathcal{B}(U, H)$ (use Remark 9.9.14(1.)&(b) in 2°) or $D_{12}^* D_{12} - D_{12}^* D_{11} (D_{11}^* D_{11})^{-1} D_{11}^* D_{12} \ll \gamma^2 I$ (cf. 0.3° of the proof of Theorem 11.1.3).

0.2° *Remark: $\mathbb{D}_{11}$ is I-coercive* This follows from (1.). Thus, Hypothesis 11.2.1 is satisfied.

1° *(iii)⇒ "almost* H^∞*-FI-pair":* Since $S := D^* J_\gamma D \in \mathcal{GB}(U \times W)$, by 0.3° of the proof of Theorem 11.1.3, $\mathcal{P}$ is a solution of the B_{w}^*-CARE, hence an admissible and ULR solution of the CARE, by Proposition 9.2.7.

By Lemma 11.2.14(4.)&(b)&(a2) (with $s = +\infty$), it follows that (11.48) is a H^∞-FI-pair satisfying $\|\mathbb{D}_{12}^{\curvearrowright}\| \le \gamma$ (in fact, $\|\mathbb{D}_{12}^{\curvearrowright} u\|_2 < \gamma \|u\|_2$ for all $u \in \mathrm{L}^2(\mathbf{R}_+; U)$, by the remark in the proof of (a2)).

2° H^∞*-FI-pair⇒(iii):* (We give the proof for the case $D^* J D \in \mathcal{GB}(U \times W)$; use first Remark 9.9.14(1.)&(b) under the alternative assumptions $B \in \mathcal{B}(U, H)$.)

By Proposition 11.2.19(a1) and Theorem 9.2.9(i)&(iv)&(a2), the B_{w}^*-CARE (and hence the CARE and the IARE) has an ULR $\mathcal{U}_{\mathrm{out}}$-stabilizing solution $(\mathcal{P}, S, K)$ with $S = D^* J_\gamma D$. By Proposition 11.2.19(d1), $S_{22} - S_{21} S_{11}^{-1} S_{12} \ll 0$.

(a1) See 1°.

(a2) Drop Lemma 11.2.14(b) from 1° and replace (d1) by (d2) in 2°. □

In [IOW], the signature condition on S is formulated by using the following equivalence:

Lemma 11.1.7 *Instead of Standing Hypothesis 11.1.1, assume only that* $\left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right] \in \mathrm{WPLS}(U \times W, H, Y)$ *is WR, $J = J^* \in \mathcal{B}(Y)$ and $\gamma > 0$.*

(Even) then the CARE has a $\mathcal{U}_*^*$-stabilizing solution $(\mathcal{P}, S, K)$ s.t. $S_{11} \gg 0$ and $S_{22} - S_{21} S_{11}^{-1} S_{12} \ll 0$ iff the IARE has a WR $\mathcal{U}_*^*$-stabilizing solution

$(\mathcal{P}, J_\gamma, \left[\begin{array}{c|c} \widetilde{\mathbb{K}} & \widetilde{\mathbb{F}} \end{array}\right])$ s.t. $\widetilde{X}_{11}, \widetilde{X}_{22} \in \mathcal{GB}$, $\widetilde{X}_{21} = 0$, where $\widetilde{X} := I - \widetilde{F}$. All prefices and suffices apply (see Definition 9.8.1).

Indeed, this latter formulation is equivalent to a Kalman–Popov–Yakubovich system formulation in terms of [IOW]. Note that if U or W is finite-dimensional, then an equivalent condition is that the IARE has WR $\mathcal{U}_*^*$-stabilizing solution with $X_{21} = 0$ (i.e., with no feedforward from u to w).

Proof: (Here Y is an arbitrary Hilbert space.) Let $(\mathcal{P}, S, K)$ be as above. By Lemma 11.3.13(i)&(iii'), there is $\widetilde{X}$ as above s.t. $\widetilde{X}^* J_\gamma \widetilde{X} = S$. By Theorem 9.8.12(s1), the latter condition is satisfied. The converse is obtained analogously. The last claim can be observed from the $\Sigma_{\circlearrowleft E}$ of Theorem 9.8.12(s1). □

The following lemma clarifies our basic concepts:

Lemma 11.1.8 (FCC ⇔ optimizable) *Let $\mathcal{U}_*^* = \mathcal{U}_{\text{exp}}$. Then $\mathcal{U}_{\text{u}}(x_0, w) \neq \emptyset$ for all $x_0 \in H$ and all $w \in \mathrm{L}^2(\mathbf{R}_+; W)$ iff (A, B_1) is optimizable.*

If there is a H^∞-FI-pair, then (A, B_1) is exponentially stabilizable (hence optimizable). Conversely, if (A, B_1) is exponentially stabilizable (or optimizable) and $\pi_{[0,1)} \mathbb{A} B_1 u_0 \in \mathrm{L}^1([0,1); H)$ for all $u_0 \in U$ then there is a H^∞-FI-pair.

Thus, if there is a "stabilizing u" for each x_0 (and $w = 0$), then, actually, there is a "stabilizing u" for each x_0 and w_0:

Proof: 1° The equivalence follows from "(i)⇔(ii)" of Lemma 11.6.4, by discretization (note that "(i)⇐(ii)" is trivial).

2° If $\left[\begin{array}{c|cc} \mathbb{K} & \mathbb{F}_1 & \mathbb{F}_2 \end{array}\right]$ is a H^∞-FI-pair (equivalently, an exponentially stabilizing pair for Σ with second row equal to zero), then $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F}_1 \end{array}\right]$ is obviously an exponentially stabilizing pair for $\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B}_1 \end{array}\right]$.

3° Assume that (A, B_1) is exponentially stabilizable (or optimizable) and $\pi_{[0,1)} \mathbb{A} B_1 u_0 \in \mathrm{L}^1([0,1); H)$ for all $u_0 \in U$. Then there is an exponentially stabilizing $\widetilde{K}_1 \in \mathcal{B}(H, U)$ for $\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B}_1 \\ \hline \mathbb{C}_1 & \mathbb{D}_{11} \end{array}\right]$, hence $\begin{bmatrix} \widetilde{K}_1 \\ 0 \end{bmatrix}$ is a H^∞-SF-operator for Σ (by Lemma 6.6.11, it is admissible for Σ; obviously the two closed-loop semigroups $\mathbb{A}_\flat := \mathbb{A} + \mathbb{B}_1 (I - \widetilde{\mathbb{F}}_{11})^{-1} \widetilde{\mathbb{K}}_1$.) are equal, hence exponentially stable).

(This converse holds also under much weaker conditions: The exponential stabilizability of (A, B_1) means the existence of an admissible state feedback pair $\left[\begin{array}{c|c} \widetilde{\mathbb{K}}_1 & \widetilde{\mathbb{F}}_{11} \end{array}\right]$ for $\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B}_1 \end{array}\right]$; we just have to know that $\widetilde{K}_1$ and B_2 "fit" to the same WPLS, i.e., that (6.100) is satisfied under substitutions $C \mapsto \widetilde{K}_1$, $B \mapsto B_2$, since then we obtain an admissible state feedback pair with same (exponentially stable) closed-loop semigroup $\mathbb{A}_\flat$, as above.) □

Next we give an example, where the signature condition $S_{22} - S_{21} S_{11}^{-1} S_{12} \ll 0$ is satisfied but the stronger condition $S_{22} \ll 0$ is not, so that there is a suboptimal H^∞-FI-pair but no suboptimal H^∞-SF-operators (for $\gamma < 1$):

Example 11.1.9 **($\gamma_{\mathrm{SF}} > \gamma_{\mathrm{FI}}$)** Let $\mathbb{D} = D = \left[\begin{smallmatrix} I & I \\ 0 & I \end{smallmatrix}\right]$, $J = J_\gamma$, $B = 0 = C$, $A = -I$ and $U = W = Z$. Then $\mathbb{D}^* J_\gamma \mathbb{D} = \begin{bmatrix} I & I \\ I & (1-\gamma^2)I \end{bmatrix} = D^* J_\gamma D =: S \in \mathcal{GB}$, so that the CARE $-2\mathcal{P} = -K^* S K$, $K = 0$ has the unique solution $\mathcal{P} = 0$ (which is exponentially stabilizing).

By Theorem 11.1.3(iii)&(a), the pair (11.16)$= \left(\begin{smallmatrix} 0 & 0 & -I \\ 0 & 0 & 0 \end{smallmatrix}\right)$ is a suboptimal H$^\infty$-FI-pair for Σ (indeed, it leads to $\mathbb{D}_{12}^{\curvearrowleft} = 0$, by (11.8), hence to $\|w \mapsto z\| = \|\mathbb{D}_{12}^{\curvearrowleft}\| = 0 < \gamma$). Since this holds for any $\gamma > 0$, we have $\gamma_{\mathrm{FI}} = 0$.

However, each H$^\infty$-SF-operator $\left(\begin{smallmatrix} K_1 \\ 0 \end{smallmatrix}\right) \in \mathcal{B}(H_1, U \times W)$ leads to $u = 0$ (or $\mathbb{D}_{12}^{\curvearrowleft} = \mathbb{D}_{12} = I$, since $\mathbb{X} = I = \mathbb{M}$, because $B = 0$), hence to the cost

$$\|z\|_2 := \|\mathbb{D}_{11} u + \mathbb{D}_{12} w\|_2 = \|w\|_2, \tag{11.29}$$

so that any H$^\infty$-SF-operator is suboptimal iff $\gamma > 1$. Thus, $\gamma_{\mathrm{SF}} = 1$, whereas $\gamma_{\mathrm{FI}} = 0$ (in accordance to Theorem 11.1.3(iii)&(a), since $S_{22} - S_{21} S_{11}^{-1} S_{12} = -\gamma^2 \ll 0$ for all $\gamma > 0$ but $S_{22} = 1 - \gamma^2 \ll 0$ iff $\gamma > 1$). $\triangleleft$

The proofs of our H$^\infty$ FICP results are based on the "H$^\infty$ minimax game" (11.58) (not "maximin", since traditionally the cost $-\mathcal{J}$ has been used).

This game is often considered as a Stackelberg game where w, the *disturbance* (or evil player, nature, uncertainties, modeling errors, sensor noise, dark side of the force, ...) tries to maximize the cost, whereas u, the *control* (or good player, control engineer, our hero) bravely defends the stability of the system by trying to minimize the cost. Lucky for the good ones, the good player is allowed to act last (although not in a noncausal way, i.e., it has no knowledge on future disturbance, just past and present).

If $\gamma_0 < \gamma$, then, obviously, $\left[\begin{smallmatrix} u \\ w \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]$ is the unique solution of the game for $x_0 = 0$; if $\gamma > \gamma_0$, then there can be no solution (i.e., $\max_w \min_u \mathcal{J}(x_0, \left[\begin{smallmatrix} u \\ w \end{smallmatrix}\right]) = \infty$ for all x_0, since the quadratic term dominates the cost for a fixed x_0).

Naturally, $\mathcal{U}_*^*(x_0) \neq \emptyset$ is a necessary condition for each $x_0 \in H$; by Proposition 11.2.19(a), it is also sufficient for $\gamma_0 < \gamma$. Thus, this game is intimately connected to the H$^\infty$ FICP.

For any $x_0 \in H$, the solution $\left[\begin{smallmatrix} u \\ w \end{smallmatrix}\right]$, being a saddle point of the (J-coercive) cost function, constitute the unique J-critical control. This leads to the existence of a unique $\mathcal{U}_*^*$-stabilizing solution of the Riccati equation (11.15), with the saddle point ("minimax", worst disturbance and best control) input $\left[\begin{smallmatrix} u \\ w \end{smallmatrix}\right]$ being the corresponding state feedback for each initial state $x_0 \in H$, by the results of Section 9.9. See Proposition 11.2.19 for the proofs.

Conversely, given a $\mathcal{U}_*^*$-stabilizing solution $(\mathcal{P}, S, K)$ of the Riccati equation (11.15) satisfying the signature condition above the equation, we choose a modified $\mathcal{U}_*^*$-stabilizing solution $(\mathcal{P}, \widetilde{S}, \left[\begin{array}{c|c} \widetilde{\mathbb{K}} & \widetilde{\mathbb{F}} \end{array}\right])$ of the equation s.t. $\widetilde{S} = \begin{bmatrix} \widetilde{S}_{11} & 0 \\ 0 & \widetilde{S}_{22} \end{bmatrix}$, $\widetilde{S}_{11} \gg 0$, $\widetilde{S}_{22} \ll 0$. It follows that even when we drop the bottom row of $\left[\begin{array}{c|c} \widetilde{\mathbb{K}} & \widetilde{\mathbb{F}} \end{array}\right]$ (to obtain a H$^\infty$-FI-pair), the state feedback remains suboptimal, because if the disturbance ("the evil player") dares to deviate from the saddle point value, it is punished by the negative cost $\langle w_{\circlearrowleft}, \widetilde{S}_{22} w_{\circlearrowleft} \rangle < 0$ (as in (9.139)), where $w_{\circlearrowleft}$ is the deviation. This leads to the suboptimal H$^\infty$-FI-pair (11.16). See Lemma 11.2.14 for the proofs.

As mentioned above, these facts lead to the proofs of our results. In addition, this shows that for each $x_0 \in H$ and $w \in \mathrm{L}^2(\mathbf{R}_+;H)$, the cost $\|z\|_2^2 - \gamma^2\|w\|_2^2$ is at most (11.58). Thus, although the suboptimality requirement was posed for $x_0 = 0$ only, our suboptimal controller (11.16) (or (11.48) actually makes $\|z\|_2$ small also for $x_0 \neq 0$ (for the "worst w for a given $\|w\|_2$", the controller minimizes the cost, hence also $\|z\|_2$, and for the other w's with same norm, the value of $\|z\|_2$ becomes less though not necessarily minimal). For the same reasons, the same holds to the H^∞-SF-operator (11.17). We emphasize this observation:

Remark 11.1.10 *The controllers (11.16) and (11.17) are "worst-case-optimal" in certain sense also for $x_0 \neq 0$.* □

(By Theorem 9.9.1(h), the reference to (9.139) is allowed also for $\mathcal{U}_*^* \neq \mathcal{U}_{\exp}$ (for $x_0 \in H$, $u_\circlearrowleft = 0$, $w_\circlearrowleft \in \mathrm{L}^2(\mathbf{R}_+;W)$), because $\mathbb{C}_\circlearrowleft$ must be stable for a $\mathcal{U}_*^*$-stabilizing solution, and also $\begin{bmatrix} \mathbb{D}_{12}^{\circlearrowleft} \\ I \end{bmatrix}$ is stable, by Lemma 11.2.14(a).)

We finish this section by a remark on the dual of the FICP:

Remark 11.1.11 (Full Control Problem = FICP$^{\mathbf{d}}$) *The dual problem of the* H^∞ *FICP is the* H^∞ Full Control Problem, *where one looks for a (suitably) stabilizing output injection pair of form*

$$\begin{bmatrix} \mathbb{H} \\ \hline \mathbb{G} \end{bmatrix} = \begin{bmatrix} 0 & \mathbb{H}_2 \\ \hline 0 & \mathbb{G}_{12} \\ 0 & \mathbb{G}_{22} \end{bmatrix} \tag{11.30}$$

for some $\Sigma = \left[\begin{array}{c|cc} \mathbb{A} & 0 & \mathbb{B}_2 \\ \hline \mathbb{C}_1 & I & \mathbb{D}_{12} \\ \mathbb{C}_2 & 0 & \mathbb{D}_{22} \end{array}\right] \in \mathrm{WPLS}(Z \times W, H, Z \times Y)$ *(i.e., we may inject only the lower output, the "measurement"), s.t.* $\|\mathbb{D}_{\sharp\,12}\| < \gamma$ *(see Definition 6.6.21).*

By duality, we obtain a solution for this problem from any of the above solutions for H^∞ *FICP.*

For example, if $\mathcal{U}_*^* = \mathcal{U}_{\exp}$, $\mathbb{D}_{22}^{\mathrm{d}}$ *is I-coercive,* $\mathbb{A}B_2 \in \mathrm{L}^1_{\mathrm{loc}}$, $C_{\mathrm{w}}\mathbb{A} \in \mathrm{L}^1_{\mathrm{loc}}$, *and* $C_{\mathrm{w}}\mathbb{A}B_2 \in \mathrm{L}^1_{\mathrm{loc}}$, *then there is an exponentially stabilizing output injection pair of form (11.30) for* Σ *s.t.* $\|\mathbb{D}_{\sharp\,12}\| < \gamma$ *iff the following hold:*

$$D_{22}D_{22}^* \gg 0, \;\; D_{12}D_{12}^* - D_{12}D_{22}^*(D_{22}D_{22}^*)^{-1}D_{22}D_{12}^* \ll \gamma^2 I, \tag{11.31}$$

and there is a nonnegative exponentially stabilizing solution $\mathcal{P} \in \mathcal{B}(H)$ *of the Riccati equation*

$$\begin{cases} HSH^* = A\mathcal{P} + \mathcal{P}A^* + B_2B_2^*, \\ S = \begin{bmatrix} D_{22}^* & D_{12}^* \\ 0 & I \end{bmatrix}^* J_\gamma \begin{bmatrix} D_{22}^* & D_{12}^* \\ 0 & I \end{bmatrix}, \\ H^* = -S^{-1}\left(\begin{bmatrix} C_2 \\ C_1 \end{bmatrix}_{\mathrm{w}} \mathcal{P} + \begin{bmatrix} D_{22} \\ D_{12} \end{bmatrix} B_2^* \right) \end{cases} \tag{11.32}$$

s.t. $\lim_{s\to+\infty} \begin{bmatrix} C_2 \\ C_1 \end{bmatrix}_{\mathrm{w}} \mathcal{P}(s-A^*)^{-1} \begin{bmatrix} C_2^* & C_1^* \end{bmatrix} = 0.$

(Apply Theorem 11.1.4 to the system Σ_Y *:=(12.85) for the proof.)* □

The H^∞ FCP is also called the H^∞ filter problem, since it means that $\gamma > \|\mathbb{D}_{\sharp\,12}\|_{\mathrm{TIC}} = \|\mathbb{D}_{12} + \widetilde{\mathbb{M}}_{12}\mathbb{D}_{22}\|_{\mathrm{TIC}}$, where $\widetilde{\mathbb{M}} := (I - \mathbb{G})^{-1}$, i.e., that $-\widetilde{\mathbb{M}}_{12}y$ is an estimate of $z := \mathbb{D}_{12}w$, where $y := \mathbb{D}_{22}w$, $w \in \mathrm{L}^2(\mathbf{R}_+;W)$, with error of norm less than $\gamma\|w\|_2$.

Notes

The H^∞ problems were introduced by G. Zames [Zames]. The first solutions to the problem used frequency-domain methods; their history can be found in [Francis]. Our stable case solution (Section 11.3) is partially based on such methods. The state space solution of this section was given by J. Doyle et al. in [DGKF], for finite-dimensional systems under several simplifying assumptions, and that article also contains the early history of state-space methods. All these works provide solutions to the H^∞ 4BP, see the notes on p. 706 for more on that problem, whose special case the FICP is.

The formulation (11.1) of the H^∞ FICP has been used in several earlier results. The equivalence of (ii) and (iii) in Theorem 11.1.3 is an extension of [DGKF, p. 836], [ZDG, Section 16.4] and [GL, Section 6.3]. [ZDG] and [DGKF] also provide an "all suboptimal controllers" formula, whose extension is contained in Theorem 12.1.8.

The SF-variant of (ii)–(iii) (of, e.g., Theorem 11.1.3(a)) is an extension of [Keu, Theorem 4.4], [IOW, Theorem 10.9.1] and [LR, Theorem 20.2.1]. The results in [Keu] also contain the equivalence with (i).

Except for [Keu], which treats Pritchard–Salamon systems (and hence assumes that $B \in \mathcal{B}(U,H)$), all of the above results assume that U, W, H and Z are finite-dimensional, but otherwise [IOW] has as general assumptions as we do. (Use Proposition 10.3.2 to observe that the assumptions of the above results are stronger than those of ours. Note also that since all results mentioned above assume a bounded B, Hypothesis 9.2.1 is satisfied.)

In the general case (see, e.g., (11.25), (11.17) and (11.26)), the formulae become similar to their discrete-time counterparts (e.g., $S \neq D^*JD$), given in Section 11.5 and in, e.g., [GL, (B.2.31), p. 487] and in [GL, Remark B.2.1, p. 488].

For parabolic (analytic) systems, the equivalence of (i) and (iii) is given in [MT94a] (repeated in [LT00a]), for a setting that allows the input and output operators to be more unbounded than we do (in Theorem 9.5.11; they take $\gamma = 0$ but allow for any $\beta < 1$). The cost function in [MT94a] is rather specific (namely $C = \left[\begin{smallmatrix} R \\ 0 \end{smallmatrix}\right]$, $D = \left[\begin{smallmatrix} 0 \\ I \end{smallmatrix}\right]$, $R \in \mathcal{B}(H,Y)$, so that the signature condition becomes redundant; condition (iii) is also otherwise modified due to different regularity assumptions) and the system is assumed to be estimatable; moreover, condition (ii) is not treated.

Almost the same result is given in [MT94b], for all estimatable WPLSs that have C, D, R as above (in particular, C must be bounded, hence these systems are ULR) with the additional requirements that B_2 is bounded and Σ is exactly reachable in finite time (but $\mathbb{A}$ need not be analytic). In this result, the CARE is treated as in Section 9.7; in particular, non-well-posed solutions are allowed. Thus, Proposition 11.2.19 and Theorem 9.7.3 extend the necessity part of this result to arbitrary regular WPLSs. However, the converse is not true without suitable signature conditions, as illustrated in Example 11.2.17.

The nonsingularity assumptions of all above results are the same or stronger than those of ours. For singular finite-dimensional systems, the H^∞ problems have been solved in [Stoorvogel].

In Theorems 11.1.3 and Theorem 11.1.6, we have not stated that $\gamma_0 = \gamma_{\mathrm{FI}}$; however, this follows from the theorems if (2.) holds independently of γ, e.g., if Hypothesis 9.2.2 is satisfied.

The state-space results mentioned above treat the case $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{exp}}$. In the frequency-domain setting, one usually works with stable rational H^∞ transfer functions, in which case there is no difference between "$\mathcal{U}_{\mathrm{exp}}$ and $\mathcal{U}_{\mathrm{out}}$", but there are also more general solutions, involving only the I/O maps (or transfer functions), such as in, e.g., [FF].

See the notes on p. 652 for solutions of the H^∞ FICP in terms of spectral and "J-lossless" factorizations. Historical remarks on the stable H^∞ FICP are given on p. 669.

11.2 The H^∞ FICP: proofs

Thu chanted a song of wizardry,
of piercing, opening, of treachery,
revealing, uncovering, betraying.
Then sudden Felagund there swaying
sang in answer a song of staying,
resisting, battling against power,
of secrets kept, strength like a tower,
and trust unbroken, freedom, escape;
of changing and of shifting shape,
of snares eluded, broken traps,
the prison opening, the chain that snaps.
— J.R.R. Tolkien (1892–1973), "The Lay of Leithian"

In this section, we shall prove the results of the previous section and present some new, more technical ones. In Theorem 11.2.7 we solve the H^∞ FICP in terms of J-lossless factorizations, assuming q.r.c.-stabilizability with MTIC. The assumptions are then weakened in Proposition 11.2.8. Most other results of this section are rather technical generalizations, parts of proofs, or counter-examples against further reduction of assumptions.

In addition to Standing Hypotheses 11.0.1 and 11.1.1, we assume the following:

Standing Hypothesis 11.2.1 (H^∞ Full-Information Control Problem (FICP)) *Throughout this section, we make the following assumptions: Hypothesis 9.0.1 is satisfied (with* $U \mapsto U \times W$ *and* $Y \mapsto Z \times W$*),* $\gamma > 0$ *and there is* $\varepsilon_+ > 0$ *s.t.* $\|\mathbb{D}_{11} u\|_2 \geq \varepsilon_+ \|\left[\begin{smallmatrix} u \\ 0 \end{smallmatrix}\right]\|_{\mathcal{U}_*^*}$ *for all* $u \in \mathcal{U}_{\mathrm{u}}(0,0)$.

The first assumption says that $\mathcal{U}_*^*$ is something reasonable (and it is satisfied if, e.g., $\mathcal{U}_*^* \in \{\mathcal{U}_{\mathrm{out}}, \mathcal{U}_{\mathrm{sta}}, \mathcal{U}_{\mathrm{str}}, \mathcal{U}_{\mathrm{exp}}\}$). The third assumption is the standard nonsingularity assumption:

Lemma 11.2.2 ($\mathbb{D}_{11}$ I-coercive) *(Drop Standing Hypothesis 11.2.1 for the moment).*

If $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{exp}}$ *(resp.* $\mathcal{U}_{\mathrm{out}}$*,* $\mathcal{U}_{\mathrm{str}}$*,* $\mathcal{U}_{\mathrm{sta}}$*) and* $\gamma > 0$*, then Hypothesis 11.2.1 holds iff* $\mathbb{D}_{11}$ *is I-coercive over* $\mathcal{U}_{\mathrm{exp}}$ *(resp.* $\mathcal{U}_{\mathrm{out}}$*,* $\mathcal{U}_{\mathrm{str}}$*,* $\mathcal{U}_{\mathrm{sta}}$*).*

Thus, we obtain several equivalent assumptions from Proposition 10.3.2 (resp. Proposition 10.3.1, the two propositions and Lemma 8.3.3, -”-). Recall from Definition 8.4.1 that I-coercivity is equivalent to positive I-coercivity. It is up to the reader to choose $\mathcal{U}_*^*$, i.e., to decide which controls shall be allowed (cf. (11.6) and (11.7)).

Proof of Lemma 11.2.2: (By I-coercivity, we refer to realization $\Sigma_{11} := \left[\begin{array}{c|c} \mathbb{A} & \mathbb{B}_1 \\ \hline \mathbb{C}_1 & \mathbb{D}_{11} \end{array}\right]$ of $\mathbb{D}_{11}$.)

The lemma follows directly from the definitions. Indeed, we obviously have

$$\|\left[\begin{smallmatrix} u \\ 0 \end{smallmatrix}\right]\|_{\mathcal{U}_{\mathrm{exp}}^{\Sigma}} = \|u\|_{\mathcal{U}_{\mathrm{exp}}^{\Sigma_{11}}} \quad \text{for all} \quad u \in \mathcal{U}_{\mathrm{u}}(0,0) = \mathcal{U}_{\mathrm{exp}}^{\Sigma_{11}}(0); \tag{11.33}$$

the same holds with $\mathcal{U}_{\mathrm{out}}$, $\mathcal{U}_{\mathrm{str}}$ or $\mathcal{U}_{\mathrm{sta}}$ in place of $\mathcal{U}_{\mathrm{exp}}$. $\square$

In addition to Definition 11.1.2, we need some extra notation:

Definition 11.2.3 *Throughout this chapter, we use also the following notation:*

$$\mathbb{D}_1 := \begin{bmatrix} \mathbb{D}_{11} \\ 0 \end{bmatrix} \quad \mathbb{D}_2 := \begin{bmatrix} \mathbb{D}_{12} \\ I \end{bmatrix}, \quad Y := Z \times W, \quad J := J_\gamma := \begin{bmatrix} I & 0 \\ 0 & -\gamma^2 I \end{bmatrix} \tag{11.34}$$

The cost function is given by

$$\mathcal{J}(x_0,u,w) := \mathcal{J}(x_0,\left[\begin{smallmatrix} u \\ w \end{smallmatrix}\right]) := \langle y, J_\gamma y\rangle_{\mathrm{L}^2(\mathbf{R}_+;Y)}, \text{ where } y := \mathbb{C}x_0 + \mathbb{D}\left[\begin{smallmatrix} u \\ w \end{smallmatrix}\right] \tag{11.35}$$

(for $x_0 \in H$, $\left[\begin{smallmatrix} u \\ w \end{smallmatrix}\right] \in \mathcal{U}_*^*(x_0)$).

Thus, $\mathbb{D} = \begin{bmatrix} \mathbb{D}_1 & \mathbb{D}_2 \end{bmatrix} \in \mathcal{B}(U \times W, Y)$; this short-hand-notation makes many formulae simpler.

Lemma 11.2.4 *We have $\gamma > \gamma_0$ iff there is $\varepsilon > 0$ s.t. $\inf_{u \in \mathcal{U}_{\mathrm{u}}(0,w)} \mathcal{J}(0,u,w) \le -\varepsilon \|w\|_2^2$ for all $w \in \mathrm{L}^2(\mathbf{R}_+;W)$.*

A H^∞-FI-pair (or H^∞-SF-operator) is suboptimal for Σ iff $\mathbb{D}_2^{\curvearrowleft *} J_\gamma \mathbb{D}_2^{\curvearrowleft} \ll 0$.

Thus, a H^∞-FI-pair is suboptimal if it makes $\mathcal{J}$ uniformly negative w.r.t. w. H^∞-FI-pairs and H^∞-SF-operators and $\Sigma^{\curvearrowleft}$ are defined as in Definition 11.1.2.

Proof: 1° *Case $\gamma > \gamma_0$:* Given $\left[\begin{smallmatrix} u \\ w \end{smallmatrix}\right] \in \mathrm{L}^2(\mathbf{R}_+;U \times W)$, we have $\langle y, J_\gamma y\rangle = \|z\|_2^2 - \gamma^2\|w\|_2^2$, where $\left[\begin{smallmatrix} z \\ w \end{smallmatrix}\right] := y := \mathbb{D}\left[\begin{smallmatrix} u \\ w \end{smallmatrix}\right]$, hence the cost function $\mathcal{J}$ becomes uniformly negative w.r.t. w ($\mathcal{J}(0,u_w,w) \le -\varepsilon\|w\|_2^2$ for all w and some $\varepsilon > 0$) iff the control law $w \mapsto u_w$ makes the norm $\|w \mapsto z\|$ less than γ, i.e., iff $\|z\|_2^2 - \gamma^2\|w\|_2^2 \le \varepsilon\|w\|_2^2$ for some $\varepsilon > 0$.

2° *Suboptimality:* Since $\mathbb{D}_2^{\curvearrowleft} = \begin{bmatrix} \mathbb{D}_{12}^{\curvearrowleft} \\ I \end{bmatrix}$ (see (11.8)), we have $\mathbb{D}_2^{\curvearrowleft *} J_\gamma \mathbb{D}_2^{\curvearrowleft} = (\mathbb{D}_{12}^{\curvearrowleft})^* \mathbb{D}_{12}^{\curvearrowleft} - \gamma^2 I \ll 0$, iff $\|\mathbb{D}_{12}^{\curvearrowleft}\| < \gamma$, by Lemma A.3.1(e2). $\square$

A H^∞-FI-pair is a state feedback (through u only) pair for which the (controlled) input $\left[\begin{smallmatrix} u \\ w \end{smallmatrix}\right]$ is in $\mathcal{U}_*^*$ for all $x_0 \in H$ and $w \in \mathrm{L}^2(\mathbf{R}_+;W)$:

Remark 11.2.5 *Let $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right] = \left[\begin{array}{c|cc} \mathbb{K}_1 & \mathbb{F}_{11} & \mathbb{F}_{12} \\ 0 & 0 & 0 \end{array}\right]$ be an admissible state feedback pair for Σ. Then $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ is a H^∞-FI-pair iff the closed-loop control $u := \mathbb{K}_1^{\curvearrowleft} x_0 + \mathbb{F}_{12}^{\curvearrowleft} w$ produced by $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ is in $\mathcal{U}_{\mathrm{u}}(x_0,w)$ for each $x_0 \in H$ and $w \in \mathrm{L}^2(\mathbf{R}_+;W)$. In particular, $\mathbb{C}^{\curvearrowleft}$ and $\mathbb{D}_2^{\curvearrowleft}$ must be stable.*

Thus, if $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{exp}}$ (resp. $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{out}}$), then $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ is a H^∞-FI-pair iff the controller makes the control, state and output (resp. control and output) stable for all x_0 and w; equivalently, iff $\Sigma^{\curvearrowleft}$ is exponentially stable (resp. iff $\mathbb{C}_1^{\curvearrowleft}$, $\mathbb{K}_1^{\curvearrowleft}$, $\mathbb{D}_{12}^{\curvearrowleft}$ and $\mathbb{F}_{12}^{\curvearrowleft}$ are stable).

Proof: The first equivalence is trivial. Assume that $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{exp}}$ (the case $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{out}}$ is analogous and hence omitted). By definition, $\left[\begin{smallmatrix} u \\ w \end{smallmatrix}\right] \in \mathcal{U}_{\mathrm{exp}}(x_0)$ iff $\left[\begin{smallmatrix} u \\ w \end{smallmatrix}\right], x, y \in \mathrm{L}^2$. By Lemma A.4.5, the closed-loop system is exponentially stable iff $x := \mathbb{A}^{\curvearrowleft} x_0 \in \mathrm{L}^2$ for all $x_0 \in H$; conversely, if this is the case, then also the output $\begin{bmatrix} y \\ u \\ w \end{bmatrix}$ is stable, so that $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ is a H^∞-FI-pair. $\square$

The $\mathcal{U}_{\text{exp}}$-results of Section 11.1 are based on the following:

Proposition 11.2.6 ($\mathcal{U}_{\text{exp}}$: (i)–(iii)) *Suppose that $\mathcal{U}_*^* = \mathcal{U}_{\text{exp}}$ and that some of (1.)–(6.) of Remark 9.9.14 hold. Then (i)–(iii) are equivalent:*

(i) $\gamma > \gamma_0$, and there is an exponentially stabilizing H^∞-FI-pair for Σ;

(ii) $\gamma > \gamma_{\text{FI}}$, i.e., there is a suboptimal H^∞-FI-pair for Σ;

(iii) The CARE has a UR exponentially stabilizing solution $(\mathcal{P}, S, K)$ s.t. $\mathcal{P} \geq 0$, $S_{11} \gg 0$ and $S_{22} - S_{21}S_{11}^{-1}S_{11} \ll 0$.

Moreover, if (ii) holds, then the assumptions of Proposition 11.2.8 (also those of (a1) and (a2)) are satisfied, (FI1)–(FI5) hold, and the solution of (iii) is unique and ULR.

Note that "there is a H^∞-FI-pair for Σ" means that Σ is exponentially stabilizable through B_1. Cf. also Lemma 11.1.8.

Proof: 0° *Weakening the assumptions:* In fact, it suffices that $(\Sigma, J_\gamma) \in$ coerciveCARE over $\mathcal{U}_*^* = \mathcal{U}_{\text{exp}}$ except that we have to require an UR state feedback operator instead of a SR one (implications (ii)$\Rightarrow$(i)$\Rightarrow$(iii) are true even without this extra requirement); in particular, also (7.) and (8.) or Remark 9.9.14 with "UR" in place of "SR" are sufficient.

1° *(iii)$\Rightarrow$(ii):* This follows from Proposition 11.2.9 (implication (FI5)$\Rightarrow$(FI2)).

2° *(ii)$\Rightarrow$(i):* This follows from (11.12).

3° *(i)$\Rightarrow$(iii) and Proposition 11.2.8:* Assume (i). By Proposition 11.2.19(a1) and Remark 9.9.14(a), the CARE has an ULR $\mathcal{U}_{\text{exp}}$-stabilizing solution $(\mathcal{P}, S, K)$ (and $\mathbb{D} \in$ ULR). Since $\mathbb{D}_\circlearrowleft^* J_\gamma \mathbb{D}_\circlearrowleft = I^*SI$, by Theorem 9.9.1(g2), also the closed-loop I/O map $\mathbb{D}_\flat$ corresponding to any exponentially stabilizing H^∞-FI-pair has a spectral factorization, by Lemma 6.7.13. Therefore, we can apply Proposition 11.2.8(a1), to observe that $S_{11} \gg 0$ and $S_{22} - S_{21}S_{11}^{-1}S_{12} \ll 0$, so that (iii) and (FI1)–(FI5) hold. $\square$

If Σ is smoothly exponentially stabilizable through u, then we have the classical equivalence (for $\mathcal{U}_*^* = \mathcal{U}_{\text{exp}}$; we also give here a variant of this result for $\mathcal{U}_*^* = \mathcal{U}_{\text{out}}$):

Theorem 11.2.7 ($\widetilde{\mathcal{A}}$: FICP) *Assume that $\widetilde{K} = \begin{bmatrix} \widetilde{K}_1 \\ 0 \end{bmatrix}$ is a UR state feedback operator for Σ with closed-loop system $\Sigma_\flat$ s.t. $\mathbb{D}_\flat \in \widetilde{\mathcal{A}}$. Assume also that $\mathcal{U}_*^* = \mathcal{U}_{\text{exp}}$ and $\widetilde{K}$ is exponentially stabilizing (resp. $\mathcal{U}_*^* = \mathcal{U}_{\text{out}}$ and $\widetilde{K}$ is [q.]r.c.-SOS-stabilizing). Then (FI1)–(FI5) are equivalent:*

(FI1) $\gamma > \gamma_0$; i.e., $\inf_{u \in \mathcal{U}_{\text{u}}(0,\cdot)} \mathcal{J}(0, u, \cdot) \ll 0$;

(FI2) $\gamma > \gamma_{\text{FI}}$; i.e., there is a suboptimal H^∞-FI-pair for Σ;

(FI3) $\mathbb{D}_\flat^* J_\gamma \mathbb{D}_\flat = \mathbb{X}_\natural^* J_1 \mathbb{X}_\natural$, where $\mathbb{X}_\natural, \mathbb{X}_{\natural 11} \in \mathcal{G}\mathrm{TIC}$;

(FI4) *the IARE has an exponentially stabilizing (resp. P-[q.]r.c.-SOS-stabilizing) solution* $(\mathcal{P}, S, \begin{bmatrix} \mathbb{K} \mid \mathbb{F} \end{bmatrix})$, *and* $\mathcal{P} \geq 0$, *and* $\widetilde{S} := (\widehat{\mathbb{X}}^* S \widehat{\mathbb{X}})(s_0)$ *satisfies* $\widetilde{S}_{11} \gg 0$ *and* $\widetilde{S}_{22} - \widetilde{S}_{21}\widetilde{S}_{11}^{-1}\widetilde{S}_{12} \ll 0$ *for some (equivalently, all)* $s_0 \in \mathbf{C}$ *with* $\operatorname{Re} s > \max\{0, \omega_A\}$.

(FI5) *the CARE (11.36) has a UR exponentially stabilizing (resp. a UR P-[q.]r.c.-SOS-stabilizing) solution* $(\mathcal{P}, S, K)$, *and* $\mathcal{P} \geq 0$, $S_{11} \gg 0$ *and* $S_{22} - S_{21}S_{11}^{-1}S_{12} \ll 0$.

Moreover, the following hold:

(a) *There is a suboptimal* H^∞*-SF-operator iff (FI5) has a solution with* $S_{22} \ll 0$*; if this is the case, then* K_1 *=(11.40) is a UR exponentially (resp. [q.]r.c.-SOS-) stabilizing suboptimal* H^∞*-SF-operator.*

(b) *For any solutions of (FI3)–(FI5) (resp. (FI3)–(FI7)) we have* $\mathbb{X}_\natural \in \widetilde{\mathcal{A}}$, $\mathbb{X}_{\natural 11} \in \mathcal{G}\widetilde{\mathcal{A}}(U)$, $\mathbb{M}_{22} \in \mathcal{G}\widetilde{\mathcal{A}}(W)$, $\mathbb{D}, \mathbb{F}, \mathbb{D}_\circlearrowleft, \mathbb{F}_\circlearrowleft, \mathbb{N}, \mathbb{M}^{\pm 1}, \mathbb{X}_\natural^{\pm 1}, \mathbb{D}_\flat \in \mathrm{UR}$.

Moreover, if there is a suboptimal H^∞*-FI-pair, then (11.39) generate a UR suboptimal exponentially (resp. [q.]r.c.-SOS-) stabilizing* H^∞*-FI-pair.*

(c) *If any of (FI1)–(FI5) have solutions, then the assumptions of Proposition 11.2.8 are satisfied (also those of (a1)&(a2); in particular,* $(\mathcal{P}, S, K)$ *is unique).*

(d) *If* $\begin{bmatrix} \widetilde{\mathbb{K}} \mid \widetilde{\mathbb{F}} \end{bmatrix}$ *is q.r.c.-SOS-stabilizing, then (FI1)–(FI7) (and (FI8) if* $\widetilde{K}$ *is ULR; and (FI9) if* $\dim U < \infty$ *or* $\dim W < \infty$*) are equivalent:*

(FI6) $\mathbb{D}$ *has a* (J_γ, J_1)*-inner [q.]r.c.f.* $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$ *with* $\mathbb{M}_{22} \in \mathcal{G}\mathrm{TIC}(W)$*;*
(FI7) $\mathbb{D}$ *has a* (J_γ, J_1)*-lossless [q.]r.c.f.* $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$ *s.t.* $\mathbb{M}_{22} \in \mathcal{G}\mathrm{TIC}_\infty(W)$*;*
(FI8) $\mathbb{D}$ *has a* (J_γ, J_1)*-lossless [q.]r.c.f.* $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$ *s.t.* $M_{22} \in \mathcal{G}\mathcal{B}(W)$*;*
(FI9) $\mathbb{D}$ *has a* (J_γ, J_1)*-lossless [q.]r.c.f.* $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$.

(e) *This theorem also holds with "*$\mathrm{TIC} \cap \mathrm{ULR}$*" in place of "*$\widetilde{\mathcal{A}}$*" if any of Remark 9.9.14(1.)–(6.) holds. Moreover, this theorem always holds with "ULR" in place of "UR".*

Naturally, $\mathbb{X} := I - \mathbb{F}$ in (FI4). Note from (11.58) that the state feedback K of (FI5) (or $\begin{bmatrix} \mathbb{K} \mid \mathbb{F} \end{bmatrix}$ of (FI4)) produces the unique "minimax" control. In this theorem the CARE can also be written as (11.25), by Lemma 9.11.5(e).

Recall from Definition 6.4.4 that $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$ is a (J_γ, J_1)-inner [q.]r.c.f. iff $\mathbb{N}$ and $\mathbb{M}$ are [q.]r.c., $\mathbb{M} \in \mathcal{G}\mathrm{TIC}_\infty$, and $\mathbb{N}^* J_\gamma \mathbb{N} = J_1$ (and $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$).

By Example 11.2.16, condition (FI9) is not sufficient in general (by 11.2.15, 11.2.17 and 11.1.9, also the conditions on $\mathbb{X}_{11}$ and S are not superfluous).

Proof of Theorem 11.2.7: 1° $\mathbb{D}, \widetilde{\mathbb{X}}^{\pm 1} \in \mathrm{UR}$, $\mathbb{D}_\flat \in \mathrm{ULR}$: By the assumptions, $\mathbb{D}, \widetilde{\mathbb{X}} := I - \widetilde{\mathbb{F}} \in \mathrm{UR}$. By Proposition 6.3.1(b1), also $\widetilde{\mathbb{M}} := \widetilde{\mathbb{X}}^{-1}$ is UR. Since $\mathbb{D}_\flat \in \widetilde{\mathcal{A}} \subset \mathrm{ULR}$, it follows that $\mathbb{D} = \mathbb{D}_\flat \widetilde{\mathbb{X}}$ is UR.

2° *(FI1)–(FI7)⇒SpF:* By Proposition 11.2.9 and Lemma 11.2.10, any of (FI3)–(FI5) (resp. (FI3)–(FI7)) implies that (FI3) holds, in particular, that $\mathbb{D}_\flat^* J_\gamma \mathbb{D}_\flat$ has a spectral factorization $\mathbb{X}_\diamond^* S_\diamond \mathbb{X}_\diamond$ (with $S_\diamond = J_1$).

If (FI1) or (FI2) holds, then (FI1) holds, by (11.12), hence then $\mathbb{D}$ is J_γ-coercive, by Proposition 11.2.19, hence so is $\mathbb{D}_\flat$, by Theorem 8.4.5(d)&(g1), hence $\mathbb{D}_\flat^* J_\gamma \mathbb{D}_\flat$ has a spectral factorization $\mathbb{X}_\diamond^* S_\diamond \mathbb{X}_\diamond$ with $\mathbb{X}_\diamond \in \mathcal{G}\widetilde{\mathcal{A}}(U \times W)$ (since $\mathbb{D}_\flat \in \widetilde{\mathcal{A}}$).

3° *The equivalence of (FI1)–(FI7):* By 1°&2°, the assumptions of Proposition 11.2.8(a2) (and (a1)) are satisfied (since $\mathbb{D}_\flat, \mathbb{X}_\diamond \in \widetilde{\mathcal{A}} \subset$ ULR $\subset$ UR) whenever any of (FI1)–(FI5) (resp. (FI1)–(FI7)) is satisfied. Thus, these conditions are equivalent, by Proposition 11.2.8(d).

(c) This was established above.

(d) This was shown in 3° above for (FI6) and (FI7). The rest follows from Lemma 11.2.10, whose proof shows that $\mathbb{M}_{22} = (\mathbb{X}_\natural^{-1})_{22}$ (hence $\mathbb{M}_{22} \in$ ULR, because necessarily $\mathbb{X}_\natural^{-1} \in \widetilde{\mathcal{A}} \subset$ ULR).

(a) This follows from Proposition 11.2.8(a2) (see 3° above).

(b) b.1° *(FI3)&(FI6)–(FI9):* By 1°&3° above, $\mathbb{D}, \widetilde{\mathbb{X}}, \widetilde{\mathbb{M}} \in$ UR and $\mathbb{D}_\flat, \mathbb{X}_\natural, \mathbb{X}_\diamond \in \widetilde{\mathcal{A}} \subset$ ULR (from the proof of Lemma 11.2.10, we observe that this holds for solutions of (FI8) and (FI9), except that $\mathbb{X}_{\natural 11}$ and $\mathbb{M}_{22}$ might be noninvertible if the additional assumptions in (d) are not met).

Therefore, $\mathbb{X}_{\natural 11} \in \mathcal{G}\widetilde{\mathcal{A}}$, hence $\mathbb{M}_{22} = (\mathbb{X}_\natural^{-1})_{22} \in \mathcal{G}\widetilde{\mathcal{A}}$ (and $\mathbb{M}_{21} = (\mathbb{X}_\natural^{-1})_{21} \in \widetilde{\mathcal{A}}(U,W)$) if $\mathbb{M}$ is as in (FI6), by the proof of Proposition 11.2.8(d); note that then also $\mathbb{M} = \widetilde{\mathbb{M}}\mathbb{X}_\natural^{-1}$ is UR.

b.2° *(FI4)&(FI5):* (Here $\left[\ \mathbb{K} \ \middle|\ \mathbb{F}\ \right]$ refers to a pair solving (FI4) or (FI5), $\mathbb{X} := I - \mathbb{F}$, $\mathbb{M} := \mathbb{X}^{-1} = \mathbb{F}_\circlearrowleft + I$, $\mathbb{N} := \mathbb{D}_\circlearrowleft := \mathbb{D}\mathbb{M}$, as in Definition 9.8.1.)

By Proposition 11.2.8 (and b.1°), $\mathbb{F} = I - E'\mathbb{X}_\diamond\widetilde{\mathbb{X}} \in$ UR, hence $\mathbb{X}, \mathbb{M}, \mathbb{F}_\circlearrowleft, \mathbb{N}, \mathbb{D}_\circlearrowleft \in$ UR.

b.3° *Suboptimal* H^∞*-FI-pairs:* This follows from Proposition 11.2.8(a1) (its assumptions hold by (c), since now we have assumed (FI2)).

(e) In fact, this theorem also holds with "TIC$\cap$UR" in place of "$\widetilde{\mathcal{A}}$" if either $\mathbb{D}_\flat$ is not J-coercive or $\mathbb{D}_\flat$ has an UR spectral factorization (except that (FI8) might become strictly stronger than the other conditions; this is not the case when the factor is ULR), as shown below.

1° $\widetilde{\mathcal{A}}$*:* Indeed, the proof only uses from "$\mathbb{D}_\flat \in \widetilde{\mathcal{A}}$" the facts that $\mathbb{D}_\flat$ is UR and that if $\mathbb{D}_\flat$ is J_γ-coercive, then $\mathbb{D}_\flat$ has a UR spectral factorization (if we replace $\widetilde{\mathcal{A}}$ by TIC$\cap$UR in (a)). The only exception is that the necessity of (FI8) was shown above assuming that $\mathbb{X}_\natural \in$ ULR (so that even it is true in 2° below).

2° *ULR:* Indeed, if $\widetilde{K}$ is ULR, then so are all the other operators claimed to be UR in this theorem; cf. the proof of (b).

(Even "SR" would be otherwise acceptable but it might lead to problems with (FI5) unless we make some additional assumption in (FI5) (cf. Proposition 2.2.5) or we assume that $\dim W < \infty$.) □

In most classical results, one assumes that $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{exp}}$ (or exponential detectability, which implies that $\mathcal{U}_{\mathrm{out}} = \mathcal{U}_{\mathrm{exp}}$), that Σ is exponentially stabilizable and that B is bounded (this includes [Keu]). By Theorem 9.2.12, this implies that we can take a bounded exponentially stabilizing $\widetilde{K}$ and choose the $\widetilde{\mathcal{A}}$ of Theorem 8.4.9(γ), so that the assumptions of Theorem 11.2.7 are satisfied.

However, without sufficient regularity assumptions, the above equivalence does not hold, at least in case $\mathcal{U}_*^* = \mathcal{U}_{\text{out}}$: By Example 11.3.7(b), (FI1) does not imply any of (FI3)–(FI5) in general (not even for strongly stable uniformly half-plane-regular systems (so that one can take $\widetilde{K} = 0$); in this example, the $\mathcal{U}_*^*$-stabilizing solution the CARE is not I/O-stabilizing; in Example 11.3.7(a), there is no $\mathcal{U}_*^*$-stabilizing solution of the CARE, nor of the IARE). Although this counter-example only treats the cases $\mathcal{U}_{\text{out}}$, $\mathcal{U}_{\text{sta}}$ and $\mathcal{U}_{\text{str}}$, it is believed that an example similar to Example 11.3.7(a) could be constructed for $\mathcal{U}_{\text{exp}}$ too; see the comments below the example.

To avoid this problem, we made the $\widetilde{\mathcal{A}}$-assumption above, and in the general WPLS result below we have to make a weaker spectral factorization assumption (which is necessary for (FI3), hence for (FI4) and (FI5) too, by Proposition 11.2.9 and Lemma 11.2.10):

Proposition 11.2.8 (FICP) *Assume that* $\left[\begin{array}{c|c} \widetilde{\mathbb{K}} & \widetilde{\mathbb{F}} \end{array}\right] = \left[\begin{array}{c|cc} \widetilde{\mathbb{K}}_1 & \widetilde{\mathbb{F}}_{11} & \widetilde{\mathbb{F}}_{12} \\ 0 & 0 & 0 \end{array}\right]$ *is a state feedback pair for* Σ *with closed-loop system* $\Sigma_\flat$, *and that* $\mathbb{D}_\flat^* J_\gamma \mathbb{D}_\flat = \mathbb{X}_\diamond^* S_\diamond \mathbb{X}_\diamond$ *for some* $\mathbb{X}_\diamond \in \mathcal{G}\mathrm{TIC}(U \times W)$ *and* $S_\diamond \in \mathcal{G}\mathcal{B}(U \times W)$. *Assume also that* $\mathcal{U}_*^* = \mathcal{U}_{\text{exp}}$ *and* $\left[\begin{array}{c|c} \widetilde{\mathbb{K}} & \widetilde{\mathbb{F}} \end{array}\right]$ *is exponentially stabilizing (resp.* $\mathcal{U}_*^* = \mathcal{U}_{\text{out}}$ *and* $\left[\begin{array}{c|c} \widetilde{\mathbb{K}} & \widetilde{\mathbb{F}} \end{array}\right]$ *is [q.]r.c.-SOS-stabilizing).*

Then (FI1)–(FI4) of Theorem 11.2.7 are equivalent to each other and implied by (FI5). Also the following hold:

(a1) **(CARE)** *Assume that* $\mathbb{D}$ *is WR and the CARE*

$$\begin{cases} K^*SK = A^*\mathcal{P} + \mathcal{P}A + C_1^*C_1, \\ S = \begin{bmatrix} D_{11}^*D_{11} & D_{11}^*D_{12} \\ D_{12}^*D_{11} & D_{12}^*D_{12} - \gamma^2 I \end{bmatrix} + \underset{s\to+\infty}{\text{w-lim}}\, B_{\mathrm{w}}^*\mathcal{P}(s-A)^{-1}B, \\ K = -S^{-1}(B_{\mathrm{w}}^*\mathcal{P} + \begin{bmatrix} D_{11}^* \\ D_{12}^* \end{bmatrix} C_1). \end{cases} \tag{11.36}$$

has a UR exponentially (resp. P-[q.]r.c.-SOS-)stabilizing solution $(\mathcal{P}, S, K)$. *Then* $\mathcal{P}$, S *and* K *are unique.*

Moreover, (FI1) holds iff $S_{11} \gg 0$ *and* $S_{22} - S_{21}S_{11}^{-1}S_{12} \ll 0$; *if this is the case, then (11.39) generate a UR exponentially (resp. [q.]r.c.-SOS-) stabilizing suboptimal* H^∞*-FI-pair and (FI5) holds.*

There is a suboptimal H^∞*-SF-operator iff* $S_{11} \gg 0$ *and* $S_{22} \ll 0$; *if this is the case, then* $K_1 =$*(11.40) is a UR exponentially (resp. [q.]r.c.-SOS-) stabilizing suboptimal* H^∞*-SF-operator.*

(a2) Assume that $\mathbb{D}$ *is WR,* $\widetilde{\mathbb{F}}$ *and* $\mathbb{X}_\diamond$ *are UR and* $\widetilde{F} = 0$. *Then (FI1)–(FI5) are equivalent, and the CARE has a UR exponentially (resp. P-[q.]r.c.-SOS-)stabilizing solution* $(\mathcal{P}, S, K)$.

(b1) The condition on $\widetilde{S}$ *in (FI4) is independent on the choice of* S *and* $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ *(and* $s_0 \in \mathbf{C}^+$*), and* $\mathcal{P}$ *is unique. Condition* $\mathbb{X}_{\natural 11} \in \mathcal{G}\mathrm{TIC}(U)$ *in (FI3) is independent on* $\mathbb{X}_\natural$ *(by (c1)).*

(b2) An exponentially (resp. P-[q.]r.c.-SOS-)stabilizing solution of the CARE is unique.

(b3) A solution of (FI3) or (FI4) is unique modulo an invertible constant.

(c1) If (FI1) holds, $\mathbb{D}_\flat^* J_\gamma \mathbb{D}_\flat = \mathbb{X}_\natural^* J_1 \mathbb{X}_\natural$ and $\mathbb{X} \in \mathcal{G}\mathrm{TIC}$, then $\mathbb{X}_{11} \in \mathcal{G}\mathrm{TIC}(U)$ and $\|\mathbb{X}_{21}\mathbb{X}_{11}^{-1}\|_{\mathrm{TIC}} < 1$.

(c2) Any solution $\mathcal{P}$ of (FI5) is unique and a solution of (FI4).

(c3) If $\mathbb{X}_\natural$ and $\mathbb{F}$ are as in (FI3) and (FI4), respectively, then $\mathbb{X}_\natural := E\mathbb{X}_\diamond$ and $(I-\mathbb{F}) = E'\mathbb{X}_\diamond(I-\widetilde{\mathbb{F}})$ for some $E, E' \in \mathcal{G}\mathcal{B}(U \times W)$.

(d) If $\left[\begin{array}{c|c} \widetilde{\mathbb{K}} & \widetilde{\mathbb{F}} \end{array}\right]$ is q.r.c.-SOS-stabilizing, then (FI1)–(FI4) are equivalent to (FI6) and to (FI7) (and to (FI9) if $\dim U < \infty$ or $\dim W < \infty$).

(e) Any UR solution of (FI3) can be redefined s.t. $X = \begin{bmatrix} X_{11} & X_{12} \\ 0 & X_{22} \end{bmatrix}$, $X_{11}, X_{22} \in \mathcal{G}\mathcal{B}$.

(f) Even without the above spectral factorization assumption (that of $\mathbb{X}_\diamond$ and $S_\diamond$), we have (FI5)⇒(FI4)⇔(FI3)⇒(FI2)⇒(FI1), part (d) is true with (FI3) in place of (FI1), and part (a1) is still true (whenever $(\mathcal{P}, S, K)$ exists).

By Proposition 11.2.9, (FI5) and (FI4) are sufficient for (a1) and for (FI1)–(FI4) without any further assumptions. Unfortunately, (FI1) implies neither of (FI4) and (FI5) in general. By Example 11.3.7(c), condition (FI5) is strictly stronger than (FI1)–(FI4).

Proof: (Note from Lemma 6.7.13 that the existence of $\mathbb{X}_\diamond$ and $S_\diamond$ (if any) is independent on $\left[\begin{array}{c|c} \widetilde{\mathbb{K}} & \widetilde{\mathbb{F}} \end{array}\right]$.) We set $\widetilde{\mathbb{X}} := I - \widetilde{\mathbb{F}}$, $\widetilde{\mathbb{M}} := \widetilde{\mathbb{X}}^{-1}$. By (FI1s)–(FI5s), we refer to the conditions of Theorem 11.3.3 for $\left[\begin{array}{c|c} \mathbb{A}_\flat & \mathbb{B}_\flat \\ \hline \mathbb{C}_\flat & \mathbb{D}_\flat \end{array}\right]$.

1° *(FI1)⇔(FI2)⇔(FI3)⇔(FI1s)–(FI4s):* By Lemma 11.2.22, condition (FI1s) of Proposition 11.3.4 for $\Sigma_\flat^1 := \left[\begin{array}{c|c} \mathbb{A}_\flat & \mathbb{B}_\flat \\ \hline \mathbb{C}_\flat & \mathbb{D}_\flat \end{array}\right]$ is equivalent to (FI1), and (FI2s) for $\Sigma_\flat^1$ to (FI2). By Proposition 11.3.4, (FI1s)–(FI4s) are equivalent (for $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{exp}}$ this follows from the fact that $\mathcal{U}_{\mathrm{exp}}^{\Sigma_\flat} = \mathcal{U}_{\mathrm{out}}^{\Sigma_\flat}$, by Theorem 8.4.5(e)). Condition (FI3) is trivially (FI3s) for $\Sigma_\flat^1$.

2° *(FI4)⇔(FI4s):* By Lemma 9.12.3(d1)&(d2) (and uniqueness), the solutions of (FI4) and (FI4s) correspond to each other as in (9.224).

Indeed, then $\mathbb{X}_\natural = \mathbb{X}\widetilde{\mathbb{M}}$, by (9.224), hence $S' := \widehat{\mathbb{X}_\natural}^* S \widehat{\mathbb{X}_\natural} = \widehat{\widetilde{\mathbb{M}}}^* \widetilde{S} \widehat{\widetilde{\mathbb{M}}}$ where $\widetilde{S} := \widehat{\mathbb{X}}^* S \widehat{\mathbb{X}}$ (at some $s_0 \in \mathbf{C}^+$). Thus, the claim follows from Lemma 11.3.13(i)&(ii") (the "(hence all)" claim), because $\widetilde{\mathbb{M}} = \left[\begin{smallmatrix} * & * \\ 0 & I \end{smallmatrix}\right]$ (note that if $\omega > \omega_A$, then $\widehat{\mathbb{X}} = \left[\begin{smallmatrix} * & * \\ 0 & I \end{smallmatrix}\right] \in \mathrm{H}^\infty(\mathbf{C}_\omega^+; \mathcal{B}(U \times W))$, by Lemma 6.1.10 and Theorem 6.2.1).

3° *(FI5)⇒(FI1):* This is given in Proposition 11.2.9.

(a1) If (FI1) holds, then so does (FI2), by 1°. Since $\mathcal{P}$ is $\mathcal{U}_*^*$-stabilizing (see 3° above), the signature conditions $S_{11} \gg 0$ and $S_{22} - S_{21}S_{11}^{-1}S_{12} \ll 0$ are necessary, by Proposition 11.2.19(d1)&(d2). The rest follows from Lemma 11.2.13.

(a2) a2.1° $(\mathcal{P}, S, K)$ *exists:* Now $\widetilde{\mathbb{M}}$ is UR and $\widetilde{M} = \widetilde{X}^{-1} = I$, by Proposition 6.3.1(a3), hence $\mathbb{D}_\flat = \mathbb{D}\widetilde{\mathbb{M}}$ is WR. By Corollary 9.9.11, the IARE for $\Sigma_\flat$ has a UR exponentially (resp. stable and P-[q.]r.c.-SOS-)stabilizing solution with zero feedthrough (use Lemma 6.7.15(c2) in case $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{exp}}$), hence so does that for Σ, by Lemma 9.12.3(d1)&(d2) (and uniqueness).

a2.2° *(FI1)–(FI5) are equivalent:* We have already shown above that (FI1)–(FI4) are equivalent and that (FI5) implies (FI4). By a2.1° and (a1), a solution of (FI4) is necessarily a solution of (FI5).

Remark: In this case (FI1)–(FI5) are equivalent to (FI1s)–(FI5s) (see Proposition 11.3.4(a)). Now $\mathcal{P}$ and S are the same in (FI5) and (FI5s) (if either (hence both) holds), and $K = \widetilde{K} + K'$, where K corresponds to (FI5) and K' to (FI5s), by (9.226).

(b1)–(c3)&(e) Most of these follow easily from the above and Proposition 11.3.4; the rest follow as in its proof.

(d) This follows form Lemma 11.2.10.

Remark: We necessarily have $\mathbb{M}(I - \mathbb{F}) \in \mathcal{G}\mathrm{TIC}(U \times W)$ for solutions $\mathbb{M}$ of (FI6) and $\mathbb{F}$ of (FI4), by the proof of Lemma 11.2.10.

(f) 1° *(FI5)⇒(FI4)⇔(FI3)⇒(FI2)⇒(FI1):* If (FI3) holds, then so do (FI1)–(FI4) (since we can take $\mathbb{X}_\diamond := \mathbb{X}$), by this proposition. Implications (FI5)⇒(FI4)⇒(FI3) follow from Lemma 11.2.9.

2° *(a1)&(d):* The assumption that (FI5) holds is more than sufficient for the proof of (a1). The modified part (d) still follows from Lemma 11.2.10. □

As explained above, the equivalence of (FI1)–(FI4) does not hold under a mere stabilizability assumption. However, we still have the following sufficiency results of proposition and lemma below:

Proposition 11.2.9 (FICP: CARE ⇒ H$^\infty$-SF-operator) *Assume that $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{out}}$ or $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{exp}}$. If (FI4) or (FI5) holds, then (FI1)–(FI4) hold and the assumptions of Lemma 11.2.10 and Proposition 11.2.8 are satisfied.*

Thus, if (FI5) holds, then we can apply Proposition 11.2.8(a1).

Proof: 1° *(FI4)⇒(FI1)–(FI3):* Assume (FI4). Redefine S and $\begin{bmatrix} \mathbb{K} \mid \mathbb{F} \end{bmatrix}$ by (11.52), so that $S_{11} \gg 0 \gg S_{22}$. By Lemma 11.2.14(a), $\begin{bmatrix} \overline{\mathbb{K}} \mid \overline{\mathbb{F}} \end{bmatrix} :=$ is a suboptimal H$^\infty$-FI-pair and $\mathbb{M}_{22}^{-1} \in \mathrm{TIC}$.

If $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{exp}}$, then $\begin{bmatrix} \overline{\mathbb{K}} \mid \overline{\mathbb{F}} \end{bmatrix}$ is exponentially stabilizing, by Remark 11.2.5. If $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{out}}$, then we observe from (11.10) that $\begin{bmatrix} \overline{\mathbb{K}} \mid \overline{\mathbb{F}} \end{bmatrix}$ is stable and SOS-stabilizing, and from (11.11) that also $\overline{\mathbb{X}}^{-1} = \mathbb{M}\underline{\mathbb{M}}$ and $\mathbb{D}^{\curvearrowleft} = \mathbb{D}_{\circlearrowleft}\underline{\mathbb{M}}^{-1}$ are [q].r.c., where $\underline{\mathbb{M}} := \begin{bmatrix} I & 0 \\ \mathbb{M}_{21} & \mathbb{M}_{22} \end{bmatrix} \in \mathcal{G}\mathrm{TIC}$, by Lemma 6.4.5(c), hence also $\begin{bmatrix} \overline{\mathbb{K}} \mid \overline{\mathbb{F}} \end{bmatrix}$ is [q].r.c.-SOS-stabilizing,

In either case, we have shown that the assumptions of Lemma 11.2.10 are satisfied, hence (FI1)–(FI4) hold and the assumptions of Proposition 11.2.8 are satisfied.

2° *(FI5)⇒(FI1)–(FI4):* The proof is analogous to that of 1° (see Lemma 11.2.13 and 3° of the proof of Lemma 11.2.10). □

Lemma 11.2.10 (FICP: SpF/IARE $\Rightarrow$ H$^\infty$-FI-pair) *Assume that* $\left[\begin{array}{c|c} \widetilde{\mathbb{K}} & \widetilde{\mathbb{F}} \end{array}\right] = \left[\begin{array}{c|cc} \widetilde{\mathbb{K}}_1 & \widetilde{\mathbb{F}}_{11} & \widetilde{\mathbb{F}}_{12} \\ 0 & 0 & 0 \end{array}\right]$ *is a state feedback pair for* Σ *with closed-loop system* $\Sigma_\flat$. *Assume also that* $\mathcal{U}_*^* = \mathcal{U}_{\rm exp}$ *and* $\left[\begin{array}{c|c} \widetilde{\mathbb{K}} & \widetilde{\mathbb{F}} \end{array}\right]$ *is exponentially stabilizing (resp.* $\mathcal{U}_*^* = \mathcal{U}_{\rm out}$ *and* $\left[\begin{array}{c|c} \widetilde{\mathbb{K}} & \widetilde{\mathbb{F}} \end{array}\right]$ *is [q.]r.c.-SOS-stabilizing).*

Either of (FI3) and (FI4) implies that (FI1)–(FI4) and the assumptions of Proposition 11.2.8 are satisfied. If $\left[\begin{array}{c|c} \widetilde{\mathbb{K}} & \widetilde{\mathbb{F}} \end{array}\right]$ *is q.r.c.-SOS-stabilizing, then (FI3) is equivalent to (FI6) and to (FI7) (and to (FI8) if* $\mathbb{M}_{22} \in$ ULR *and to (FI9) if* $\dim U < \infty$ *or* $\dim W < \infty$*).*

Proof: 1° *(FI3):* For (FI3) this is obvious: take $\mathbb{X}_\diamond := \mathbb{X}_\natural$, $S_\diamond := J_1$.

2° *(FI6):* By Lemma 6.4.8(b), the solutions of (FI3) and (FI6) correspond to each other through $\mathbb{M} = \widetilde{\mathbb{M}}\mathbb{X}_\natural^{-1}$ (if we neglect $\mathbb{X}_{\natural 11}$ and $\mathbb{M}_{22}$), hence $\mathbb{M}_{22} = (\mathbb{X}_\natural^{-1})_{22}$ (because $\widetilde{\mathbb{M}} = [\begin{smallmatrix} * & * \\ 0 & I \end{smallmatrix}]$). But $(\mathbb{X}_\natural^{-1})_{22} \in \mathcal{G}\mathrm{TIC}(W)$ iff $\mathbb{X}_{\natural 11} \in \mathcal{G}\mathrm{TIC}(U)$, by Lemma A.1.1(c1), hence (FI3) and (FI6) are equivalent.

3° *(FI7), (FI8) and (FI9):* We obtain "(FI6)$\Leftrightarrow$(FI7)" from Corollary 2.5.5 (since $\mathbb{N} = \mathbb{D}\mathbb{M}$, so that $\mathbb{N}_{22} = \mathbb{M}_{22}$, because $\mathbb{D} = [\begin{smallmatrix} * & * \\ 0 & I \end{smallmatrix}]$), "(FI6)$\Leftrightarrow$(FI9)" from Proposition 2.5.4 (if $\dim U < \infty$ or $\dim W < \infty$; not in general!), and "(FI7)$\Leftrightarrow$(FI8)" from Proposition 6.3.1(c) (if $\mathbb{M}_{22} \in$ ULR).

4° *(FI4):* Assume (FI4). Let $\Sigma_\circlearrowleft$ be the closed-loop system corresponding to $(\mathcal{P}, S, \left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right])$. Then $\mathbb{D}_\circlearrowleft^* J_\gamma \mathbb{D}_\circlearrowleft = S = I^* S I$, by Theorem 9.9.10(a2)&(a1). Thus, also $\mathbb{D}_\flat^* J_\gamma \mathbb{D}_\flat$ has a spectral factorization, by Lemma 6.7.13 (in case of $\mathcal{U}_{\rm exp}$) or Lemma 6.4.5(c) (in case of $\mathcal{U}_{\rm out}$, since then $\mathbb{D}_\flat = \mathbb{D}_\circlearrowleft \mathbb{U}$, hence $\mathbb{D}_\flat^* J_\gamma \mathbb{D}_\flat = \mathbb{U}^* S \mathbb{U}$, for some $\mathbb{U} \in \mathcal{G}\mathrm{TIC}$). Therefore, the (preliminary) assumptions of Proposition 11.2.8 are satisfied, hence also (FI1)–(FI3) hold. □

The above results also hold for $\mathcal{U}_{\rm str}$ or $\mathcal{U}_{\rm sta}$ in place of $\mathcal{U}_{\rm out}$, mutatis mutandis:

Corollary 11.2.11 (FICP for $\mathcal{U}_{\rm str}$ and $\mathcal{U}_{\rm sta}$) *Assume that* $\left[\begin{array}{c|c} \widetilde{\mathbb{K}} & \widetilde{\mathbb{F}} \end{array}\right]$ *is [strongly] q.r.c.-stabilizing. Then the following hold for Theorem 11.2.7, Propositions 11.2.8 and 11.2.9 and Lemma 11.2.10:*

We have $\mathcal{U}_{\rm out} = \mathcal{U}_{\rm sta}[= \mathcal{U}_{\rm str}]$*, and "SOS-" can be omitted [or replaced by strongly; moreover, "P-SOS-" can be replaced by strongly].*

Proof: By Theorem 8.4.5(g2), $\mathcal{U}_{\rm out} = \mathcal{U}_{\rm sta}[= \mathcal{U}_{\rm str}]$. By Theorem 6.7.15(a1)[(a2)], prefices "q.r.c.-SOS-stabilizing" and "q.r.c.-stabilizing", [and "strongly q.r.c.-stabilizing"] are equivalent for admissible pairs for Σ or $\left[\begin{array}{c|c} \mathbb{A}_\flat & \mathbb{B}_\flat \\ \hline \mathbb{C}_\flat & \mathbb{D}_\flat \end{array}\right]$. [Obviously, that "strongly" implies "P-".] For $\left[\begin{array}{c|c} \mathbb{A}_\flat & \mathbb{B}_\flat \\ \hline \mathbb{C}_\flat & \mathbb{D}_\flat \end{array}\right]$, "q.r.c.-SOS-stabilizing" and "stable and SOS-stabilizing" are equivalent, by Lemma 6.6.17(b). □

In the setting of Proposition 11.2.8, all suboptimal (i.e., s.t. $\|w \mapsto z\| < \gamma$) stable causal time-invariant control laws $\mathbb{U} : w \mapsto u$ can be formulated as follows:

Lemma 11.2.12 (All suboptimal TIC controllers) *Assume that* $\begin{bmatrix} \widetilde{\mathbb{K}} & \widetilde{\mathbb{F}} \end{bmatrix} = \left[\begin{smallmatrix} \widetilde{\mathbb{K}}_1 & \widetilde{\mathbb{F}}_{11} & \widetilde{\mathbb{F}}_{12} \\ 0 & 0 & 0 \end{smallmatrix}\right]$ *is a q.r.c.-SOS-stabilizing state feedback pair for* Σ *with closed-loop system* $\Sigma_\flat$.

Then all $\mathbb{U} \in \mathrm{TIC}(W,U)$ *s.t.* $\|\mathbb{D}_{11}\mathbb{U} + \mathbb{D}_{12}\|_{\mathrm{TIC}} < \gamma$ *are given by*

$$\{\widetilde{\mathbb{M}}_{11}\mathbb{U}_{\mathrm{st}} + \widetilde{\mathbb{M}}_{12} \mid \mathbb{U}_{\mathrm{st}} \in \mathrm{TIC}(W,U) \textit{ is s.t. } \|\widetilde{\mathbb{N}}_{11}\mathbb{U}_{\mathrm{st}} + \widetilde{\mathbb{N}}_{12}\| < \gamma\}. \tag{11.37}$$

Given a solution for (FI3), we obtain the parametrization of all such (closed-loop suboptimal TIC control laws) $\mathbb{U}_{\mathrm{st}}$ from Theorem 11.3.6.

Recall from Theorem 6.7.15(c2), that if Σ is estimatable, then any exponentially stabilizing state feedback pair is exponentially q.r.c.-stabilizing (hence q.r.c.-SOS-stabilizing).

If we drop the q.r.c.-condition, then "$\mathbb{U}_{\mathrm{st}} \in \mathrm{TIC}$" must be replaced by "$\mathbb{U}_{\mathrm{st}} \in \mathrm{TIC}$ & $\widetilde{\mathbb{M}}_{11}\mathbb{U}_{\mathrm{st}} + \widetilde{\mathbb{M}}_{12} \in \mathrm{TIC}$".

The existence of a solution $\mathbb{U}$ to the I/O map problem (or frequency-domain problem) formulated above obviously implies that $\gamma > \gamma_0$. Recall that if the assumptions of, e.g., Theorem 11.3.6 hold, then $\gamma_0 = \gamma_{\mathrm{FI}}$, hence then also the converse holds (since this problem obviously lies between those corresponding to γ_0 and γ_{FI}), so that (FI3) is applicable for the above parametrization whenever a solution exists.

Proof of Lemma 11.2.12: (N.B. We observe from the proof that the theorem also holds with "$\le \gamma$" in place of "$< \gamma$".) As elsewhere, we have set

$$\widetilde{\mathbb{M}} := (I - \widetilde{\mathbb{F}})^{-1} = \left[\begin{smallmatrix} * & * \\ 0 & I \end{smallmatrix}\right] \in \mathrm{TIC} \quad \widetilde{\mathbb{N}} := \mathbb{D}\widetilde{\mathbb{M}} = \left[\begin{smallmatrix} * & * \\ 0 & I \end{smallmatrix}\right] \in \mathrm{TIC}. \tag{11.38}$$

1° $\mathbb{U}_{\mathrm{st}} \Longrightarrow \mathbb{U}$ *(sufficiency):* Let $\mathbb{U}_{\mathrm{st}} \in \mathrm{TIC}(W,U)$. Set $\mathbb{U} := \widetilde{\mathbb{M}}_{11}\mathbb{U}_{\mathrm{st}} + \widetilde{\mathbb{M}}_{12} \in \mathrm{TIC}(W,U)$, so that $\left[\begin{smallmatrix} \mathbb{U} \\ I \end{smallmatrix}\right] = \widetilde{\mathbb{M}}\left[\begin{smallmatrix} \mathbb{U}_{\mathrm{st}} \\ I \end{smallmatrix}\right]$. Then $\mathbb{N}\left[\begin{smallmatrix} \mathbb{U}_{\mathrm{st}} \\ I \end{smallmatrix}\right] = \mathbb{D}\mathbb{M}\left[\begin{smallmatrix} \mathbb{U}_{\mathrm{st}} \\ I \end{smallmatrix}\right] = \mathbb{D}\left[\begin{smallmatrix} \mathbb{U} \\ I \end{smallmatrix}\right]$, hence $\mathbb{N}_{11}\mathbb{U}_{\mathrm{st}} + \mathbb{N}_{12} = \mathbb{D}_{11}\mathbb{U} + \mathbb{D}_{12}$, so that $\mathbb{U}$ is suboptimal for $\mathbb{D}$ iff $\mathbb{U}_{\mathrm{st}}$ is suboptimal for $\mathbb{N}$.

2° $\mathbb{U} \Longrightarrow \mathbb{U}_{\mathrm{st}}$ *(necessity):* Let $\mathbb{U} \in \mathrm{TIC}(W,U)$ be s.t. $\|\mathbb{D}_{11}\mathbb{U} + \mathbb{D}_{12}\|_{\mathrm{TIC}} < \gamma$. Set $\mathbb{U}_{\mathrm{st}} := \begin{bmatrix} I & 0 \end{bmatrix}\mathbb{M}^{-1}\left[\begin{smallmatrix} \mathbb{U} \\ I \end{smallmatrix}\right]$. Since $\mathbb{D}\left[\begin{smallmatrix} \mathbb{U} \\ I \end{smallmatrix}\right] \in \mathrm{TIC}$ and $\left[\begin{smallmatrix} \mathbb{U} \\ I \end{smallmatrix}\right] \in \mathrm{TIC}$, we have $\mathbb{M}^{-1}\mathbb{M}\left[\begin{smallmatrix} \mathbb{U}_{\mathrm{st}} \\ I \end{smallmatrix}\right] \in \mathrm{TIC}$, by Lemma 6.5.6(b), i.e., $\mathbb{U}_{\mathrm{st}} \in \mathrm{TIC}$. Thus, we have the setting of 1°. □

In the next two lemmas we list some implications of the Riccati equation and formulate a sufficient condition for suboptimality (see Lemma 11.2.14(a)):

Lemma 11.2.13 (General $\mathcal{U}_*^*$: CARE ⇒ FICP) *Assume that CARE has a UR* $\mathcal{U}_*^*$*-stabilizing solution* $(\mathcal{P},S,K)$ *s.t.* $\mathcal{P} \ge 0$, $S_{11} \gg 0$ *and* $S_{22} - S_{21}S_{11}^{-1}S_{12} \ll 0$. *Then the assumptions of Lemma 11.2.14 are satisfied (including (4.)).*

In particular, if $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{exp}}$ *(or* $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{out}}$ *and* $(I - \mathbb{F})^{-1} \in \mathrm{TIC}$*), then (11.48) is a suboptimal* H^∞*-FI-pair for* Σ*, with generators*

$$\left[\begin{array}{c|cc} -S_{11}^{-1}(D_{11}^* C_1 + (B_1^*)_{\mathrm{w}}\mathcal{P}) & 0 & -S_{11}^{-1}S_{12} \\ 0 & 0 & 0 \end{array}\right]; \tag{11.39}$$

if, in addition, $S_{22} \ll 0$, *then*

$$K_1 := \begin{bmatrix} I & 0 \end{bmatrix} K = -(S_{11} - S_{12}S_{22}^{-1}S_{21})^{-1}(D_{11}^*C_1 + (B_1^*)_{\mathrm{w}}\mathcal{P} - S_{12}S_{22}^{-1}(D_{12}^*C_1 + (B_2^*)_{\mathrm{w}}\mathcal{P})) \tag{11.40}$$

is a UR suboptimal H^∞*-SF-operator for* Σ*.*

Proof: (By Theorem 9.8.12(s4)&(s3), $\mathcal{P}$, S and K are unique.)

1° *All claims except the formulae (11.39) and (11.40):* By Proposition 9.8.10, $(\mathcal{P}, S, \begin{bmatrix} \mathbb{K} & | & \mathbb{F} \end{bmatrix})$ is a UR $\mathcal{U}_*^*$-stabilizing solution of the IARE. Obviously, (4.) is satisfied for $s_0 = +\infty$. The claim on K_1 follows from (a) of Lemma 11.2.14, that on (11.48) from (b) and (a) (with $\widetilde{S} = S$).

Thus, it only remains to establish (11.39) and (11.40).

2° *The generators of (11.48) are given by (11.39):* If $(\mathcal{P}, S, K)$ is a UR solution of the CARE (i.e., $s_0 = +\infty$ and $\widetilde{S} = S$), then $F = 0$, hence $\overline{F} = \begin{bmatrix} 0 & -S_{11}^{-1}S_{12} \\ 0 & 0 \end{bmatrix}$, and

$$(\overline{K})_1 = K_1 + S_{11}^{-1}S_{12}K_2 = -\begin{bmatrix} I & S_{11}^{-1}S_{12} \end{bmatrix} S^{-1}(D^*J_\gamma C + B_{\mathrm{w}}^*\mathcal{P}) \tag{11.41}$$

$$= -\begin{bmatrix} I & S_{11}^{-1}S_{12} \end{bmatrix} \begin{bmatrix} I & -S_{11}^{-1}S_{12} \\ 0 & I \end{bmatrix} \begin{bmatrix} S_{11}^{-1} & 0 \\ 0 & (S_{22}')^{-1} \end{bmatrix} \begin{bmatrix} I & -S_{11}^{-1}S_{12} \\ 0 & I \end{bmatrix}^* (D^*J_\gamma C + B_{\mathrm{w}}^*\mathcal{P}) \tag{11.42}$$

$$= -\begin{bmatrix} S_{11}^{-1} & 0 \end{bmatrix} (D^*J_\gamma C + B_{\mathrm{w}}^*\mathcal{P}) = -S_{11}^{-1}(D_{11}^*C_1 + (B_1^*)_{\mathrm{w}}\mathcal{P}). \tag{11.43}$$

3° *We have* $K_1 =$*(11.40):* By Lemma A.1.1(c1) (substitute $A \mapsto S$),

$$\mathcal{GB}(U) \ni (S^{-1})_{11}^{-1} = S_{11} - S_{12}S_{22}^{-1}S_{21} =: S_{11}'. \tag{11.44}$$

Set $L := D^*J_\gamma C + B_{\mathrm{w}}^*\mathcal{P}$. Then

$$K_1 = -\begin{bmatrix} I & 0 \end{bmatrix} S^{-1}L = -\begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ -S_{22}^{-1}S_{21} & I \end{bmatrix} \begin{bmatrix} (S_{11}')^{-1} & 0 \\ 0 & S_{22}^{-1} \end{bmatrix} \begin{bmatrix} I & -S_{12}S_{22}^{-1} \\ 0 & I \end{bmatrix} L \tag{11.45}$$

$$= -(S_{11}')^{-1} \begin{bmatrix} I & -S_{12}S_{22}^{-1} \end{bmatrix} L = -(S_{11}')^{-1}(L_1 - S_{12}S_{22}^{-1}L_2) \tag{11.46}$$

=(11.17). □

Our sufficiency results are based in the following lemma (all three cases of (a2) are used in Section 11.1):

Lemma 11.2.14 (General $\mathcal{U}_*^*$: IARE ⇒ FICP) *Assume that IARE has a* $\mathcal{U}_*^*$*-stabilizing solution* $(\mathcal{P}, S, \begin{bmatrix} \mathbb{K} & | & \mathbb{F} \end{bmatrix})$ *s.t.* $\mathcal{P} \geq 0$*. Assume, in addition, that at least one of (1.)–(4.) holds:*

(1.) $\min(\dim U, \dim W) < \infty$*;*

(2.) $\mathbb{X}_{11} \in \mathcal{G}\mathrm{TIC}_\infty(U)$*;*

(3.) $\widehat{\mathbb{X}}_{11}(s_0) \in \mathcal{GB}(U)$ *for some* $s_0 \in \mathbf{C}_\alpha^+$*;*

(4.) $\widetilde{S} := \widehat{\mathbb{X}}(s_0)^*S\widehat{\mathbb{X}}(s_0)$ *safisfies* $\widetilde{S}_{11} \gg 0$ *and* $\widetilde{S}_{22} - \widetilde{S}_{21}\widetilde{S}_{11}^{-1}\widetilde{S}_{12} \ll 0$ *for some* $s_0 \in \mathbf{C}_\alpha^+$.

(In (3.) and (4.), we allow for any $\alpha \geq \max\{0,\vartheta\}$ *s.t.* $\mathbb{F} \in \mathrm{TIC}_\alpha$*; for* $\mathbb{F} \in \mathrm{UR}$ *we also allow for* $s_0 = +\infty$*. In (2.) and (3.), we can allow right-invertibility (being onto) instead of invertibility.)*

Then the following is true:

(a1) If $S_{22} \ll 0$*, then* $\begin{bmatrix} \mathbb{K} & | & \mathbb{F} \end{bmatrix}$ *satisfies* $\mathbb{X}_{11}, \mathbb{M}_{22} \in \mathcal{G}\mathrm{TIC}_\infty$, $\mathbb{D}_{\widehat{12}}, \mathbb{M}_{22}^{-1}, \mathbb{X}_{21}\mathbb{X}_{11}^{-1}, \mathbb{M}_{22}^{-1}\mathbb{M}_{21} \in \mathrm{TIC}$ *and* $\|\mathbb{D}_{\widehat{12}}\|_{\mathrm{TIC}} \leq \gamma$*.*

(a2) Assume that 1. $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{exp}}$*, or that 2.* $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{out}}$ *and* $\mathbb{M} \in \mathrm{TIC}$*, or that 3.* $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{out}}$ *and there is* $\varepsilon > 0$ *s.t.*

$$\|\mathbb{C}_1 x_0 + \mathbb{D}_{11} u + \mathbb{D}_{12} w\|_2 \geq \varepsilon \|u\|_2 \quad (u \in \mathrm{L}^2_\varepsilon(\mathbf{R}_+;U),\ w \in \mathrm{L}^2(\mathbf{R}_+;W),\ x_0 \in H). \tag{11.47}$$

If $S_{22} \ll 0$*, then* $\begin{bmatrix} \overline{\mathbb{K}} & | & \overline{\mathbb{F}} \end{bmatrix} := \left[\begin{smallmatrix} \mathbb{K}_1 \\ 0 \end{smallmatrix} \middle| \begin{smallmatrix} \mathbb{F}_{11} & \mathbb{F}_{12} \\ 0 & 0 \end{smallmatrix}\right]$ *is a* H^∞*-FI-pair; in cases 1. and 2. it is suboptimal (and* $\|\mathbb{D}_{\widehat{12}}\| \leq \gamma$ *also in case 3.).*

(b) If (4.) holds, then the $\mathcal{U}_*^*$*-stabilizing solution* $(\mathcal{P}, S', \begin{bmatrix} \mathbb{K}' & | & \mathbb{F}' \end{bmatrix})$ *defined by (11.52) satisfies the assumptions of (a) (i.e.,* $\mathcal{P} \geq 0$*,* $S'_{22} \ll 0$ *and (4.) holds for* $\widehat{\mathbb{X}}'(s_0)S'\widehat{\mathbb{X}}'(s_0)$*); the corresponding pair "*$\begin{bmatrix} \overline{\mathbb{K}} & | & \overline{\mathbb{F}} \end{bmatrix}$*" is given by*

$$\left[\begin{array}{c|cc} \mathbb{K}_1 + \widetilde{S}_{11}^{-1}\widetilde{S}_{12}\mathbb{K}_2 & \mathbb{F}_{11} + \widetilde{S}_{11}^{-1}\widetilde{S}_{12}\mathbb{F}_{21} & \mathbb{F}_{12} + \widetilde{S}_{11}^{-1}\widetilde{S}_{12}(\mathbb{F}_{22} - I) \\ 0 & 0 & 0 \end{array}\right] \tag{11.48}$$

In (a), we have used the standard notation $\mathbb{X} := I - \mathbb{F}$, $\mathbb{M} := \mathbb{X}^{-1}$, $\mathbb{N} := \mathbb{D}\mathbb{M}$, $\mathbb{D}_{\widehat{12}} := \mathbb{N}_{12}\mathbb{M}_{22}^{-1}$; cf. Definition 9.8.4 and (11.8).

Note that we can choose, e.g., $\alpha = \max\{0, \vartheta, \omega_A + 1\}$. Recall that for $\mathcal{U}_{\mathrm{out}}$ and $\mathcal{U}_{\mathrm{exp}}$ one requires that the control is stable, i.e., $\vartheta = 0$.

Proof of Lemma 11.2.14: *Remark 1:* This lemma also holds without the assumption on the existence of ε_+ (see Standing Hypothesis 11.2.1); indeed, this proof does not use it even implicitly. The same remark applies also to Lemma 11.2.13.

Remark 2: By Theorem 9.8.12(s4), $\widetilde{S}$ (and $\mathcal{P}$) is independent on the choice of a $\mathcal{U}_*^*$-stabilizing $(\mathcal{P}, S, \begin{bmatrix} \mathbb{K} & | & \mathbb{F} \end{bmatrix})$. However, conditions (2.) and (3.) may depend on $\begin{bmatrix} \mathbb{K} & | & \mathbb{F} \end{bmatrix}$: if $\mathbb{D}^* J_\gamma \mathbb{D} = J_1$, $S = -J_1$ and $X = \left[\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right]$, then $\widetilde{S} = J_1$ and (4.) (and (1.) if we choose so) hold although (2.) and (3.) are false for this solution (but not for all solutions, by (b)); this problem did not occur with the CARE (see Lemma 11.2.13).

(a1) By Theorem 9.9.1(a1)&(e1), $\Sigma_{\circlearrowleft}$ is a J_γ-critical control in state feedback form. By Theorem 8.3.9(a1'), $\mathbb{C}_{\circlearrowleft}$ is stable and $\mathbb{K}_{\circlearrowleft}$ is ϑ-stable. By Lemma 6.1.11 (and Remark 6.1.9), $\mathbb{M} = \mathbb{F}_{\circlearrowleft} + I \in \mathrm{TIC}_\omega(U \times W)$ for all $\omega > \vartheta$.

1° $\mathbb{M}_{22}$ *satisfies some of (1)–(5) of Proposition 2.2.5:* If any of (1.)–(3.) holds, then $\mathbb{X}_{11}$ satisfies some of Proposition 2.2.5(1)–(5), hence so does $\mathbb{M}_{22}$, by Lemma A.1.1(c1) (if $\dim U < \infty$, then $\mathbb{M}_{22}$ satisfies (4), which is observed by exchanging the columns and rows of $\mathbb{M}$).

If (4.) holds, then we can apply Lemma 11.3.13(b2) to $\widetilde{S} := \widehat{\mathbb{X}}(s_0)^* S \widehat{\mathbb{X}}(s_0)$ to observe that $\widehat{\mathbb{X}}(s_0)_{11} \in \mathcal{G}\mathcal{B}(U)$, equivalently, that $\widehat{\mathbb{M}}(s_0)_{22} \in \mathcal{G}\mathcal{B}(W)$, so that (5) holds.

2° $\exists \mathbb{M}_{22}^{-1} \in \mathrm{TIC}$: Since $\mathcal{P} \geq 0$ and $\mathbb{N}_{2*} = \mathbb{M}_{2*}$ (because $\mathbb{D} = \left[\begin{smallmatrix} * & * \\ 0 & I \end{smallmatrix}\right]$), we obtain from the (2,2)-block of (9.157) that

$$\gamma^2 (\mathbb{M}_{22}^t)^* \mathbb{M}_{22}^t - (\mathbb{N}_{12}^t)^* \mathbb{N}_{12}^t \geq -\pi_{[0,t)} S_{22} \geq \varepsilon^2 I \quad (t \geq 0). \tag{11.49}$$

It follows from Corollary 2.2.6, 1° and (11.49) that $\mathbb{M}_{22} \in \mathcal{G}\mathrm{TIC}_\infty(W)$ and $\|\mathbb{M}_{22}\|_{\mathrm{TIC}} \leq \gamma/\varepsilon$.

3° $\|\mathbb{D}_{12}^{\curvearrowleft}\|_{\mathrm{TIC}} \leq \gamma$: From $(\mathbb{M}_{22}^{-t})^* \cdot (11.49) \cdot \mathbb{M}_{22}^{-t}$ we obtain that

$$\gamma^2 I \geq (\mathbb{D}_{12}^{\curvearrowleft t})^* \mathbb{D}_{12}^{\curvearrowleft t} + \varepsilon^2 (\mathbb{M}_{22}^{-t})^* \mathbb{M}_{22}^{-t}, \tag{11.50}$$

hence $\|\mathbb{D}_{12}^{\curvearrowleft}\|_{\mathrm{TIC}} \leq \gamma$, by Lemma 2.1.14.

Remark: Given $w \in \mathrm{L}^2(\mathbf{R}_+;W)$, we have $\|\mathbb{D}_{12}^{\curvearrowleft} w\|_2^2 \leq \gamma^2 - \varepsilon^2 \|\mathbb{M}_{22}^{-1} w\|_2^2$, hence $\|\mathbb{D}_{12}^{\curvearrowleft} w\|_2 < \gamma$. If $\mathcal{U}_*^* = \mathcal{U}_{\exp}$, then $\Sigma_{\circlearrowleft}$ is exponentially stable, hence then $\mathbb{M}_{22} \in \mathrm{TIC}$, hence then $\|\mathbb{D}_{12}^{\curvearrowleft}\| \leq (\gamma^2 - \varepsilon^2/\|\mathbb{M}_{22}\|^2)^{1/2} < \gamma$.

4° $\mathbb{X}_{21}\mathbb{X}_{11}^{-1} \in \mathrm{TIC}$: Since $(\mathbb{D}_1^t)^* J_\gamma \mathbb{D}_1^t = (\mathbb{D}_{11}^t)^* \mathbb{D}_{11}^t \geq 0$ and $\mathcal{P} \geq 0$, we obtain from the (1,1)-block of (9.160) that

$$\begin{bmatrix} \mathbb{X}_{11}^t \\ \mathbb{X}_{21}^t \end{bmatrix}^* S \begin{bmatrix} \mathbb{X}_{11}^t \\ \mathbb{X}_{21}^t \end{bmatrix} \geq 0 \quad (t \geq 0). \tag{11.51}$$

Apply Lemma A.3.1(q) to obtain that there is $\delta > 0$ s.t. $(\mathbb{X}_{11}^t)^* \mathbb{X}_{11}^t \geq \delta (\mathbb{X}_{21}^t)^* \mathbb{X}_{21}^t$, i.e., $\delta^{-1} \geq (\mathbb{V}^t)^* \mathbb{V}^t$, for all $t > 0$, hence $\mathbb{V} \in \mathrm{TIC}$, by Lemma 2.1.14, where $\mathbb{V} := \mathbb{X}_{21}\mathbb{X}_{11}^{-1} = -\mathbb{M}_{22}^{-1}\mathbb{M}_{21}$ (by the (2,1)-block of $\mathbb{M}\mathbb{X} = I$).

(a2) 1° $\mathcal{U}_{\exp}$*: Suboptimality:* Since $\mathcal{P}$ is $\mathcal{U}_{\exp}$-stabilizing, $\Sigma_{\circlearrowleft}$ (and $\mathbb{B}_{\circlearrowleft}\tau$, by Lemma 6.1.10) is exponentially stable. From (11.10) we observe that $\mathbb{A}^{\curvearrowleft} x_0 \in \mathrm{L}^2$ for all $x_0 \in H$, hence $\Sigma^{\curvearrowleft}$ is exponentially stable, by Lemma A.4.5. Thus, $\left[\begin{array}{c|c} \overline{\mathbb{K}} & \overline{\mathbb{F}} \end{array}\right]$ is a H^∞-FI-pair, by Remark 11.2.5. By the remark in 3°, $\|\mathbb{D}_{12}^{\curvearrowleft}\|_{\mathrm{TIC}} < \gamma$ (since $\mathbb{M} = \mathbb{F}_{\circlearrowleft} + I \in \mathrm{TIC}$), i.e., $\left[\begin{array}{c|c} \overline{\mathbb{K}} & \overline{\mathbb{F}} \end{array}\right]$ is suboptimal.

2° $\mathcal{U}_{\mathrm{out}}$*: Suboptimality when* $\mathbb{M}_{12}, \mathbb{M}_{22} \in \mathrm{TIC}$: Because $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ is $\mathcal{U}_{\mathrm{out}}$-stabilizing, it is output-stabilizing, by Theorem 9.8.5. This and the stability of $\mathbb{M}_{12}$ (by the assumption), $\mathbb{D}_{12}^{\curvearrowleft} = \mathbb{N}_{12}\mathbb{M}_{22}^{-1}$ and $\mathbb{M}_{22}^{-1}$ (by 1°&2°), imply that $\mathbb{C}^{\curvearrowleft}$, $\mathbb{K}^{\curvearrowleft}$, $\mathbb{D}_{12}^{\curvearrowleft}$ and $\mathbb{F}_{12}^{\curvearrowleft}$ are stable (see (11.10)). Thus, $\left[\begin{array}{c|c} \overline{\mathbb{K}} & \overline{\mathbb{F}} \end{array}\right]$ is a H^∞-FI-pair, by Remark 11.2.5. By the remark in 2°, $\|\mathbb{D}_{12}^{\curvearrowleft}\|_{\mathrm{TIC}} < \gamma$ (since $\mathbb{M}_{22} \in \mathrm{TIC}$), i.e., $\left[\begin{array}{c|c} \overline{\mathbb{K}} & \overline{\mathbb{F}} \end{array}\right]$ is suboptimal.

3° *Case* $\|\mathbb{C}_1 x_0 + \mathbb{D}_{11} u + \mathbb{D}_{12} w\|_2 \geq \varepsilon \|u\|_2$: (Recall from (a1) that $\|\mathbb{D}_{12}^{\curvearrowleft}\| \leq \gamma$, hence $\left[\begin{array}{c|c} \overline{\mathbb{K}} & \overline{\mathbb{F}} \end{array}\right]$ is also here "almost suboptimal".) Since $\mathbb{C}_{\circlearrowleft}$ and $\mathbb{K}_{\circlearrowleft}$ are output stable (by Theorem 9.8.5), we have $\mathbb{D}_{\circlearrowleft}, \mathbb{F}_{\circlearrowleft} \in \mathrm{TIC}_\omega$ for all $\omega > 0$, by Lemma 6.1.11. This, (a1)2° and (11.10) imply that $\mathbb{C}^{\curvearrowleft}$, $\mathbb{D}^{\curvearrowleft}$, $\mathbb{K}^{\curvearrowleft}$ and $\mathbb{F}^{\curvearrowleft}$ are ω-stable for all $\omega > 0$.

By (a1), $\mathbb{D}_{12}^{\curvearrowleft}$ is stable. Consequently, $\mathrm{L}^2 \ni \mathbb{D}_{12}^{\curvearrowleft} w = \mathbb{D}_{12} w + \mathbb{D}_{11}\mathbb{F}_{12}^{\curvearrowleft} w$, by (11.9), hence $\mathbb{F}_{12}^{\curvearrowleft} w \in \mathrm{L}^2$, by (11.47), for all $w \in \mathrm{L}^2(\mathbf{R}_+;W)$. Thus, $\mathbb{F}_{12}^{\curvearrowleft} \in \mathrm{TIC}$, by Lemma 6.1.12.

By (11.10), $\mathbb{C}_1^{\curvearrowleft} = \mathbb{C}_{\circlearrowleft 1} - \mathbb{D}_{12}^{\curvearrowleft}\mathbb{K}_{\circlearrowleft 2}$ is stable, hence $\mathrm{L}^2 \ni \mathbb{C}_1^{\curvearrowleft} x_0 = \mathbb{C}_1 x_0 + \mathbb{D}_{11}\mathbb{K}_1^{\curvearrowleft} x_0$, hence $\mathbb{K}_1^{\curvearrowleft} x_0 \in \mathrm{L}^2$, for all $x_0 \in H$, so that $\mathbb{K}_1^{\curvearrowleft}$ is stable.

The stability of $\mathbb{C}_1^{\curvearrowleft}$, $\mathbb{D}_{12}^{\curvearrowleft}$, $\mathbb{K}_1^{\curvearrowleft}$ and $\mathbb{F}_{12}^{\curvearrowleft}$ imply that $\left[\begin{array}{c|c} \overline{\mathbb{K}} & \overline{\mathbb{F}} \end{array}\right]$ is a H^∞-FI-pair, by (11.8) and Remark 11.2.5.

(b) By Lemma 9.8.12(s1), also $(\mathcal{P}, S', \left[\begin{array}{c|c} \mathbb{K}' & \mathbb{F}' \end{array}\right])$ is a $\mathcal{U}_*^*$-stabilizing solution of the IARE, where

$$\left[\begin{array}{c|c} \mathbb{K}' & \mathbb{F}' \end{array}\right] := \left[\begin{array}{c|c} \widetilde{E}\mathbb{K} & \widetilde{E}\mathbb{F}+I-\widetilde{E} \end{array}\right], \ S' := \begin{bmatrix} \widetilde{S}_{11} & 0 \\ 0 & S'_{22} \end{bmatrix}, \tag{11.52}$$

$E := \left[\begin{smallmatrix} I & \widetilde{S}_{11}^{-1}\widetilde{S}_{12} \\ 0 & I \end{smallmatrix}\right] \in \mathcal{GB}(U \times W)$, $\widetilde{E} := E\widehat{\mathbb{X}}(s_0)^{-1} \in \mathcal{GB}$, and $S'_{22} := \widetilde{S}_{22} - \widetilde{S}_{21}\widetilde{S}_{11}^{-1}\widetilde{S}_{12} \ll 0$, because $S' = E^{-*}\widetilde{S}E^{-1} = \widetilde{E}^{-*}S\widetilde{E}^{-1}$. Since $S'_{22} \ll 0$ (and $\widetilde{S}$ remains invariant), the assumptions of (a) are satisfied by $(\mathcal{P}, S', \left[\begin{array}{c|c} \mathbb{K}' & \mathbb{F}' \end{array}\right])$.

Remark: We could replace $\widetilde{E}$ by $\begin{bmatrix} \widetilde{S}_{11}^{1/2} & 0 \\ 0 & (-\widetilde{S}_{22})^{1/2} \end{bmatrix}$ to obtain $S' = J_1$ above (this would slightly alter (11.48)). □

Next we note that $\mathbb{X}_{\natural 11} \in \mathcal{G}\mathrm{TIC}(U)$ is not superfluous in (FI3) (even $\mathbb{X}_{\natural 11} \in \mathcal{G}\mathrm{TIC}_\infty(U)$ is not sufficient):

Example 11.2.15 ($\mathbb{X}_{\natural 11} \in \mathcal{G}\mathrm{TIC}(U)$ is not superfluous) Let $\mathbb{D} := \left[\begin{smallmatrix} R & 2I \\ 0 & I \end{smallmatrix}\right] \in \mathrm{MTIC_d}(U \times U) \subset \mathrm{TIC}(U \times U)$, where $R := \tau^{-1}$ (here U may be any Hilbert space; we have taken $Z = W = U$ and $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{out}}$). Then $\mathbb{D}_{11}^*\mathbb{D}_{11} = I \gg 0$, hence Standing Hypotheses 11.1.1 and 11.2.1 are satisfied. Moreover, $\mathbb{D}^* J_1 \mathbb{D} = \mathbb{X}^* J_1 \mathbb{X}$, where

$$\mathbb{X} = \begin{bmatrix} \frac{4}{3}R + \frac{1}{3} & 2 \\ \frac{2}{3}R + \frac{2}{3} & 1 \end{bmatrix} \in \mathcal{G}\mathrm{MTIC_d} \subset \mathcal{G}\mathrm{TIC}. \tag{11.53}$$

However, (11.106)$= 3\pi_{[0,1)} - \pi_{[1,\infty)} \not\ll 0$, hence $\mathbb{D}$ is not minimax J-coercive (alternatively, this follows from Lemma 11.4.3(b), since $\mathbb{X}_{11} \notin \mathcal{G}\mathrm{TIC}(U)$, because $\widehat{\mathbb{X}}(\log 4 + \pi i) = 0$).

This also shows that condition $\mathcal{P} \geq 0$ is not superfluous in Lemmas 11.2.14 and 11.3.9(a). (Note that $\mathcal{P}$ can be computed from (8.46), once we choose a realization $\Sigma \in \mathrm{SOS}$ of $\mathbb{D}$ (e.g., the shift realization (6.11); since $\mathbb{D}$ is exponentially stable, this realization can be chosen to be exponentially stable, so that $\mathcal{P}$ becomes exponentially stabilizing and $\mathcal{U}_{\mathrm{out}} = \mathcal{U}_{\mathrm{exp}}$).) ◁

If both U and W are infinite-dimensional, then the dimensions of positive and negative eigenspaces of $J_1 \in \mathcal{B}(U \times W)$ do not determine those of $\mathbb{D}^* J \mathbb{D}$ unless we require $\widehat{\mathbb{M}}_{22}$ to be invertible somewhere:

Example 11.2.16 ($\mathbb{M}_{22} \in \mathcal{G}\mathrm{TIC}_\infty(W)$ is not superfluous in (FI7)) Let R be the right shift τ^{-1} on $\ell^2(\mathbf{N}) =: U =: W =: Z$, $L := R^*$, $P_0 := I - RL$, $Q := RL$. Set $\mathbb{D} := D := \begin{bmatrix} R & \sqrt{2}P_0 \\ 0 & I \end{bmatrix}$ to obtain $D^* J_1 D = \left[\begin{smallmatrix} I & 0 \\ 0 & 2P_0 - I \end{smallmatrix}\right] = \left[\begin{smallmatrix} I & 0 \\ 0 & P_0 - Q \end{smallmatrix}\right] = X^* J_1 X$, where $\mathbb{X} := X := \left[\begin{smallmatrix} R & P_0 \\ 0 & L \end{smallmatrix}\right] \in \mathcal{GB}(U \times W)$.

Then $\mathbb{D}_{11}^*\mathbb{D}_{11} = I \gg 0$, so that Standing Hypotheses 11.1.1 and 11.2.1 are satisfied, and so are the assumptions of Propositions 11.2.8 and 11.3.4 (set $\left[\begin{array}{c|c} \widetilde{\mathbb{K}} & \widetilde{\mathbb{F}} \end{array}\right] = \left[\begin{array}{c|c} 0 & 0 \end{array}\right]$).

Set $M := X^{-1} = \left[\begin{smallmatrix} L & 0 \\ P_0 & R \end{smallmatrix}\right]$, $N := DM = \begin{bmatrix} Q + \sqrt{2}P_0 & 0 \\ P_0 & R \end{bmatrix}$ to obtain a (J_1, J_1)-lossless r.c.f. of $\mathbb{D}$, so that (FI9) is satisfied. Then $N_{22} = R \notin \mathcal{GB}(W)$, hence (FI6)–(FI8) do not hold (hence none of (FI1)–(FI8) holds). ◁

The existence of a $\mathcal{U}_*^*$-stabilizing solution is not a sufficient condition; we have to know signature properties that guarantee that the solution is really a minimax control:

Example 11.2.17 ($S_{22} - S_{21}S_{11}^{-1}S_{12} \ll 0$ is necessary) To observe that the conditions on S are not superfluous in (FI5) (or in Theorem 11.1.3), set

$$A := -1,\ B := 0,\ C := 0,\ D := \begin{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 2 \end{bmatrix} \\ 0 & 1 \end{bmatrix} \in \mathcal{B}(\mathbf{C}\times\mathbf{C}, \mathbf{C}^2\times\mathbf{C}). \tag{11.54}$$

Then the CARE $-2\mathcal{P} = K^*SK$, $S = \begin{bmatrix} 1 & 0 \\ 0 & 4-\gamma^2 \end{bmatrix}$, $SK = 0$ has a unique solution $\mathcal{P} = 0$, which is exponentially stabilizing (unless $\gamma = 2$, in which case there is no solution of the CARE, and the solutions of the eCARE are given by $(\mathcal{P}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 \\ K_2 \end{bmatrix})$). Since

$$\|\mathbb{D}_{11}u + \mathbb{D}_{12}w\|_2^2 = \|u\|_2^2 + 4\|w\|_2^2, \tag{11.55}$$

the optimal H$^\infty$-FI-pair for Σ is given by $\begin{bmatrix} \mathbb{K} & \mathbb{F} \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}$, and $\gamma_0 = \gamma_{\text{FI}} = \gamma_{\text{SF}} = 2$. Indeed, Theorem 11.1.3(a) confirms this, since $S_{22} - S_{21}S_{11}^{-1}S_{12} = S_{22} = 4 - \gamma^2 \ll 0$ iff $\gamma > 2$. For $\gamma < 2$, the J-critical control is obviously a "min-min" control. $\triangleleft$

The following lemma will be needed for the Riccati equation form of the solution of the H$^\infty$ 4BP:

Lemma 11.2.18 ($\mathbb{M}_{22} \in \mathcal{G}$TIC $\Longrightarrow$ H$^\infty$-FI-pair) *Assume that $(\mathcal{P}, S, \begin{bmatrix} \mathbb{K} & \mathbb{F} \end{bmatrix})$ is an exponentially stabilizing solution of the IARE and $\mathcal{U}_*^* = \mathcal{U}_{\text{exp}}$ (resp. that $(\mathcal{P}, S, \begin{bmatrix} \mathbb{K} & \mathbb{F} \end{bmatrix})$ is a q.r.c.-SOS-stabilizing solution of the IARE and $\mathcal{U}_*^* = \mathcal{U}_{\text{out}}$).*

If $S_{22} \ll 0$ and $\mathbb{M}_{22} \in \mathcal{G}$TIC, then $\begin{bmatrix} \overline{\mathbb{K}} & \overline{\mathbb{F}} \end{bmatrix} := \begin{bmatrix} \mathbb{K}_1 & \mathbb{F}_{11} & \mathbb{F}_{12} \\ 0 & 0 & 0 \end{bmatrix}$ is a suboptimal H$^\infty$-FI-pair and $\mathcal{P} \geq 0$.

If $\mathbb{F}$ is UR, then (11.48) generates another suboptimal pair (use the lemma and Proposition 11.2.19(d1)), but these pairs need not be equal in general (the operator $\mathbb{M}$ above need not be equal to I).

Proof: (Here, as elsewhere, $\mathbb{X} := I - \mathbb{F}$, $\mathbb{M} := \mathbb{X}^{-1}$.)

1° $\begin{bmatrix} \overline{\mathbb{K}} & \overline{\mathbb{F}} \end{bmatrix} := \begin{bmatrix} \mathbb{K}_1 & \mathbb{F}_{11} & \mathbb{F}_{12} \\ 0 & 0 & 0 \end{bmatrix}$ *is a* H$^\infty$*-FI-pair:* Since $\mathcal{P}$ is $\mathcal{U}_{\text{exp}}$-stabilizing, the closed-loop system $\Sigma_\circlearrowleft$ (and $\mathbb{B}_\circlearrowleft\tau$, by Lemma 6.1.10) is exponentially stable. From (11.10) we observe that $\mathbb{A}^\curvearrowleft x_0 \in \mathrm{L}^2$ for all $x_0 \in H$, hence $\Sigma^\curvearrowleft$ is exponentially stable, by Lemma A.4.5. (For $\mathcal{U}_*^* = \mathcal{U}_{\text{out}}$, we observe from (11.10) that $\Sigma^\curvearrowleft \in$ SOS.) Thus, $\begin{bmatrix} \overline{\mathbb{K}} & \overline{\mathbb{F}} \end{bmatrix}$ is a H$^\infty$-FI-pair, by Remark 11.2.5.

2° $\begin{bmatrix} \overline{\mathbb{K}} & \overline{\mathbb{F}} \end{bmatrix}$ *is suboptimal:* By (11.11), we have

$$\mathbb{D}^\curvearrowleft := \mathbb{D}\begin{bmatrix} \mathbb{X}_{11} & \mathbb{X}_{12} \\ 0 & I \end{bmatrix}^{-1} = \mathbb{D}\mathbb{X}^{-1}\underline{\mathbb{M}}^{-1}, \tag{11.56}$$

where $\underline{\mathbb{M}} = \begin{bmatrix} I & 0 \\ \mathbb{M}_{21} & \mathbb{M}_{22} \end{bmatrix}$, $\mathbb{M} := \mathbb{X}^{-1}$, by Schur decomposition (Lemma A.1.1(d1)(A.9)). Therefore, from $(\mathbb{D}\mathbb{X}^{-1})^*J_\gamma\mathbb{D}\mathbb{X}^{-1} = S$, we obtain that

$$(\mathbb{D}_2^\curvearrowleft)^*J_\gamma\mathbb{D}_2^\curvearrowleft = \begin{bmatrix} 0 & I \end{bmatrix}\mathbb{D}^{\curvearrowleft *}J_\gamma\mathbb{D}^\curvearrowleft\begin{bmatrix} 0 \\ I \end{bmatrix} = \begin{bmatrix} 0 & I \end{bmatrix}\underline{\mathbb{M}}^*S\underline{\mathbb{M}}\begin{bmatrix} 0 \\ I \end{bmatrix} = \mathbb{M}_{22}^*S_{22}\mathbb{M}_{22}, \tag{11.57}$$

hence $(\mathbb{D}_2^{\frown})^* J_\gamma \mathbb{D}_2^{\frown} \ll 0$, because $\mathbb{M}_{22} \in \mathcal{G}\mathrm{TIC}$, by Lemma A.1.1(c1). By Lemma 11.2.4, it follows that $\left[\begin{array}{c|c} \overline{\mathbb{K}} & \overline{\mathbb{F}} \end{array}\right]$ is suboptimal.

3° $\mathcal{P} \geq 0$*:* This follows from Proposition 11.2.19(b2) (since $\gamma > \gamma_{\mathrm{FI}} \geq \gamma_0$, by 2°).

Remark: Parts 1°–2° hold also for the eFICP of Theorem 11.3.6 (i.e., for any $\Sigma \in \mathrm{WPLS}(U \times W, H, Y)$; we do not need Standing Hypothesis 11.1.1 nor 11.2.1).

If we do assume Standing Hypothesis 11.1.1, then Standing Hypothesis 11.2.1 becomes redundant: Since Standing Hypothesis 11.2.1 is not used in 1°–2°, as noted above, we obtain Standing Hypothesis 11.2.1 from Lemma 11.2.22(a) (with $\left[\begin{array}{c|c} \widetilde{\mathbb{K}} & \widetilde{\mathbb{F}} \end{array}\right] := \left[\begin{array}{c|c} \overline{\mathbb{K}} & \overline{\mathbb{F}} \end{array}\right]$). Indeed, by (d1) (resp. (c)) Lemma 9.12.3, $\mathcal{P}$ is an exponentially (resp. q.r.c.-SOS-)stabilizing solution for the CARE for $\Sigma_\flat$ and J_γ, hence $\mathbb{D}_\flat$ has a spectral factorization, by Corollary 9.9.11. □

Next we present a rather general necessity result that was used in the proofs of our main theorems:

Proposition 11.2.19 (General $\mathcal{U}_*^*$: Necessary conditions) *Assume that $\gamma > \gamma_0$ and that Z^{s} is reflexive. Then $\mathbb{D}$ is J_γ-coercive. Assume in addition that $\mathcal{U}_*^*(x_0) \neq \emptyset$ for each $x_0 \in H$. Then the following hold:*

(a1) There is a unique J_γ-critical input $\begin{bmatrix} u_{\mathrm{crit}}(x_0) \\ w_{\mathrm{crit}}(x_0) \end{bmatrix}$ for each $x_0 \in H$, and this input corresponds to the (unique) arguments of

$$\max_{w \in \mathrm{L}^2(\mathbf{R}_+;W)} \min_{u \in \mathcal{U}_{\mathrm{u}}(x_0,w)} \mathcal{J}(x_0,u,w). \tag{11.58}$$

(a2) If $(\Sigma, J) \in \mathrm{coerciveCARE}$ (see Remark 9.9.14), then there is a (SR) unique $\mathcal{U}_*^*$-stabilizing solution $(\mathcal{P}, S, K)$ of the CARE; the corresponding closed-loop control equals $\begin{bmatrix} u_{\mathrm{crit}}(x_0) \\ w_{\mathrm{crit}}(x_0) \end{bmatrix}$ for each $x_0 \in H$ (with no external input).

(b1) The J_γ-critical input (called the minimax control*) can be given in state feedback form iff the [e]IARE has a $\mathcal{U}_*^*$-stabilizing solution.*

(b2) Assume that a $\mathcal{U}_*^*$-stabilizing solution $(\mathcal{P}, S, \left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right])$ of the (e)IARE exists. Then $\mathcal{P} \geq 0$, $S \in \mathcal{GB}(U \times W)$ and $(\mathbb{X}^{t*} S \mathbb{X}^t)_{11} \geq \varepsilon_+^2 \mathrm{e}^{-2t\max\{0,\vartheta\}} I$ for all $t \geq 0$.

If, in addition, $\vartheta \leq 0$, then $(\mathbb{X}^{t*} S \mathbb{X}^t)_{11} \geq \varepsilon_+^2 I$ for all $t \geq 0$, and we can choose S and $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ so that $S = \left[\begin{smallmatrix} I & 0 \\ 0 & J_W \end{smallmatrix}\right]$, where $J_W = J_W^* = J_W^{-1} \in \mathcal{GB}(W)$.

(b3) If a $\mathcal{U}_*^*$-stabilizing solution $(\mathcal{P}, S, \left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right])$ exists s.t. $\mathbb{F} \in \mathrm{MTIC}_\infty$, then the CARE has a unique $\mathcal{U}_*^*$-stabilizing solution $(\mathcal{P}, S, K)$, and $S_{11} \geq \varepsilon_+^2 I$.

If, in addition, there is a suboptimal MTIC_∞ H^∞*-FI-pair (resp.* H^∞*-SF-operator), then $S_{22} - S_{21} S_{11}^{-1} S_{12} \ll 0$ (resp. $S_{22} \ll 0$).*

(c) Assume that a $\mathcal{U}_*^*$-stabilizing solution $(\mathcal{P}, S, \left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right])$ exists, there is a suboptimal H^∞*-FI-pair, and $\vartheta \leq 0$. Then $(\mathbb{X}^{t*} S \mathbb{X}^t)_{11} \geq \varepsilon_+^2 I$ for all $t \geq 0$,*

and $\widehat{\mathbb{S}} := \widehat{\mathbb{X}}^* S \widehat{\mathbb{X}}$ *satisfies*

$$\widehat{\mathbb{S}}_{11} \geq \varepsilon_+^2 I \quad \text{and} \quad \widehat{\mathbb{S}}_{22} - \widehat{\mathbb{S}}_{21}\widehat{\mathbb{S}}_{11}^{-1}\widehat{\mathbb{S}}_{12} \ll 0 \quad \text{on } \mathbf{C}_\omega^+, \tag{11.59}$$

for any $\omega \geq 0$ *s.t.* $\mathbb{X} \in \mathrm{TIC}_\omega$. *Moreover, there there is a* $\mathcal{U}_*^*$*-stabilizing solution* $(\mathcal{P}, J_1, \left[\begin{array}{c|c} \widetilde{\mathbb{K}} & \widetilde{\mathbb{F}} \end{array}\right])$ *of the IARE s.t.* $\widetilde{\mathbb{X}}_{11} \in \mathcal{G}\mathrm{TIC}_\infty$, $\|\widetilde{\mathbb{X}}_{11}^{-1}\|_{\mathrm{TIC}} \leq \varepsilon_+^{-1}$, *and* $\|\widetilde{\mathbb{X}}_{21}\widetilde{\mathbb{X}}_{11}^{-1}\|_{\mathrm{TIC}} \leq 1$.

Assume that the CARE has a SR $\mathcal{U}_*^*$*-stabilizing solution* $(\mathcal{P}, S, K)$ *and that* $\vartheta \leq 0$. *Then* $S_{11} \geq \varepsilon_+^2 I$ *and the following hold:*

(d1) $\mathbf{(S_{22} - S_{21}S_{11}^{-1}S_{12} \ll 0)}$ *If there is a suboptimal SR* H^∞*-FI-pair, then* $S_{22} - S_{21}S_{11}^{-1}S_{12} \ll 0$.

(d2) $\mathbf{(S_{22} \ll 0)}$ *If there is a suboptimal SR* H^∞*-SF-operator, then* $S_{22} \ll 0$.

(e) If $\|\pi_{[0,t)}\mathbb{D}_{11}u\|_2 \geq \varepsilon_+ \|\pi_{[0,t)}u\|_2$ *for all* $u \in \mathrm{L}^2(\mathbf{R}_+; U)$ *and* $t > 0$, *then the assumption "*$\vartheta \leq 0$*" can be removed everywhere in this theorem.*

(f) In (b3), (c), (d1) and (d2), the existence of a suboptimal H^∞*-FI-pair (resp.* H^∞*-SF-operator) is not needed if there is* $\mathbb{U} \in \mathrm{TIC}_\infty(W, U)$ *(resp. SR* $\mathbb{U} \in \mathrm{TIC}_\infty(W, U)$ *having* $\widehat{\mathbb{U}}(+\infty) = 0$*) s.t.* $\|\mathbb{D}_{11}\mathbb{U} + \mathbb{D}_{12}\|_{\mathrm{TIC}} < \gamma$ *(in (b3) we must also require that* $\mathbb{U} \in \mathrm{MTIC}_\infty(W, U)$*; in (c) that* $\mathbb{U} \in \mathrm{TIC}_\omega$*, and in (d1) that* $\mathbb{U}$ *is SR).*

See, e.g., Lemma 11.2.14 for the converses. Note that $\mathcal{U}_{\mathrm{exp}}$ and $\mathcal{U}_{\mathrm{out}}$ have Z^{s} reflexive (and $\vartheta = 0$, hence $\mathrm{e}^{-2t\vartheta} = 1$ in (b2)). If $(\mathcal{P}, S, \left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right])$ is as in (b2), then $\mathcal{P}$ is unique and S and $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ are unique modulo an invertible constant as in (9.114), by Theorem 9.9.1(a1)&(f1)&(f2). See Example 11.1.9 for the difference between (d1) and (d2).

Under the assumptions of Theorem 9.9.1(k), we can make J_W equal to $-I$ in (b2); this fact was used in Proposition 11.2.8. However, we do not know whether S reflects all signature properties of the problem for general $\mathcal{U}_*^*$, hence we cannot improve (b2) in the general case; see the notes on p. 481 for this problem.

Proof: By Theorem 11.7.2(b) (and Remark 8.3.4; note that Hypothesis 11.7.1 requires the reflexivity assumption), $\mathbb{D}$ is J_γ-coercive.

(a1) By Theorem 11.7.2(c), (11.58) exists and equals the J_γ-critical input.

(a2) Combine (a1) with the definition of coerciveCARE. (Note that under (4.) or (5.) of Remark 9.9.14, we have $\mathbb{F} \in \mathrm{MTIC}_\infty^{\mathrm{L}^1}$, so that (b3) applies. Under any of (1.)–(6.) of Remark 9.9.14, the solution is ULR, hence SR, so that then (d) applies.)

(b1) This follows from by Theorem 9.9.1(a1) (the eIARE is equivalent to the IARE, by (b2)).

(b2) 1° By Lemma 9.10.3, we have $S \in \mathcal{GB}(U \times W)$. By Theorem 9.9.1(f1), we have $\mathcal{P} = \mathbb{C}_{\mathrm{crit}}^* J_\gamma \mathbb{C}_{\mathrm{crit}}$. But $\mathcal{J}(x_0, u, 0) \geq 0$ for each x_0 and u, hence (11.58)≥ 0, i.e., $\mathbb{C}_{\mathrm{crit}}^* J_\gamma \mathbb{C}_{\mathrm{crit}} \geq 0$. Thus, $\mathcal{P} \geq 0$.

Let $t > 0$ and $u' \in \mathrm{L}^2([0,t); U)$ be given. Set $x_t := \mathbb{B}_1 \tau^t u$, $\widetilde{u} := u' + \tau^{-t} u$, where $u \in \mathcal{U}_*^*(x_t)$ is to be defined later. By Lemma 9.7.9, $\widetilde{u} \in \mathcal{U}_*^*(0)$. We first compute that

$$y_u := \mathbb{C}\mathbb{B}_1^t \widetilde{u} + \mathbb{D}_1 u = \pi_+ \mathbb{D}_1 (\pi_+ + \pi_-) \tau^t \widetilde{u} = \pi_+ \mathbb{D}_1 \tau^t \widetilde{u} \tag{11.60}$$

$$= \mathbb{D}_1 \tau^t \widetilde{u} - \pi_{[-t,0)} \mathbb{D} \tau^t \widetilde{u} = \mathbb{D}_1 \tau^t \widetilde{u} - \tau^t \mathbb{D}_1^t u'. \tag{11.61}$$

Consequently, (11.58) implies that

$$\langle \mathbb{B}_1^t u', \mathcal{P}\mathbb{B}_1^t u' \rangle = \max_{w \in \mathrm{L}^2(\mathbf{R}_+;U)} \min_{u \in \mathcal{U}_\mathrm{u}(x,w)} \mathcal{J}(\mathbb{B}_1^t u', u, w) \geq \min_{u \in \mathcal{U}_\mathrm{u}(x,0)} \mathcal{J}(\mathbb{B}_1^t u', u, 0) \tag{11.62}$$

$$= \min_{u \in \mathcal{U}_\mathrm{u}(x,0)} \langle y_u, J_\gamma y_u \rangle_{\mathrm{L}^2} = \min_{u \in \mathcal{U}_\mathrm{u}(x,0)} \langle \tau^{-t} y_u, \tau^{-t} J_\gamma y_u \rangle_{\mathrm{L}^2} \tag{11.63}$$

$$\stackrel{\dagger}{=} \min_{u \in \mathcal{U}_\mathrm{u}(x,0)} \left(\langle \mathbb{D}_1 \widetilde{u}, J_\gamma \mathbb{D}_1 \widetilde{u} \rangle - \langle \mathbb{D}_1^t u', J_\gamma \mathbb{D}_1^t u' \rangle \right) \tag{11.64}$$

$$= \min_{u \in \mathcal{U}_\mathrm{u}(x,0)} \langle \mathbb{D}_1 \widetilde{u}, J_\gamma \mathbb{D}_1 \widetilde{u} \rangle - \langle u', (\mathbb{D}_1^t)^* J_\gamma \mathbb{D}_1^t u' \rangle \tag{11.65}$$

(†: here both crossterms are negative, hence the minus sign though positive quadratic terms). But $\mathbb{X}^{t\,*} S \mathbb{X}^t = \mathbb{D}^{t\,*} J_\gamma \mathbb{D}^t + \mathbb{B}^{t\,*} \mathcal{P} \mathbb{B}^t$, by the eIARE, hence it follows from the above and Standing Hypothesis 11.2.1 that

$$\langle u', (\mathbb{X}^{t\,*} S \mathbb{X}^t)_{11} u' \rangle = \min_{u \in \mathcal{U}_\mathrm{u}(x,0)} \langle \mathbb{D}_1 \widetilde{u}, J_\gamma \mathbb{D}_1 \widetilde{u} \rangle \geq \varepsilon_+^2 \| \left[\begin{smallmatrix} \widetilde{u} \\ 0 \end{smallmatrix}\right] \|_{\mathcal{U}_*^*}^2 \tag{11.66}$$

$$\geq \varepsilon_+^2 \|\widetilde{u}\|_{\mathrm{L}^2_\vartheta}^2 \geq \varepsilon_+^2 \|u'\|_{\mathrm{L}^2_\vartheta}^2 \geq \varepsilon_+^2 \mathrm{e}^{-2t\max\{0,\vartheta\}} \|u'\|_2^2. \tag{11.67}$$

Thus, $(\mathbb{X}^{t\,*} S \mathbb{X}^t)_{11} \geq \varepsilon_+^2 \mathrm{e}^{-2t\max\{0,\vartheta\}}$ for all $t \geq 0$.

2° *Case* $\vartheta \leq 0$*:* Since $\mathrm{e}^{-2t\max\{0,\vartheta\}} = 1$, we now have $(\mathbb{X}^{t\,*} S \mathbb{X}^t)_{11} \geq \varepsilon_+^2 I$ for all $t \geq 0$.

For any $\omega > \omega_A$, we have $\mathbb{X} \in \mathrm{TIC}_\omega$; if also $\omega \geq 0$, then $(\widehat{\mathbb{X}}^* S \widehat{\mathbb{X}})_{11} \geq \varepsilon_+^2 I$ on $\mathbf{C}_\omega^+$, by Lemma 2.2.4. Choose $s_0 \in \mathbf{C}_\omega^+$. Then, $T := (\widehat{\mathbb{X}}^* S \widehat{\mathbb{X}})_{11}(s_0) \geq \varepsilon_+^2 I$.

By Lemma 11.3.14, there is $E_0 \in \mathcal{GB}(U \times W)$ s.t. $T = E_0^* \widetilde{S} E_0$, where $\widetilde{S} = \left[\begin{smallmatrix} I & 0 \\ 0 & J_W \end{smallmatrix}\right]$, $J_W = J_W^* = J_W^{-1}$. Set $E := E_0 \widehat{\mathbb{X}}(s_0)^{-1} \in \mathcal{GB}(U \times W)$. By Theorem 9.9.1(f2), also $(\mathcal{P}, \widetilde{S}, \left[\begin{array}{c|c} \widetilde{\mathbb{K}} & \widetilde{\mathbb{F}} \end{array}\right])$ is a $\mathcal{U}_*^*$-stabilizing solution of the IARE, where $\widetilde{S} := E^{-*} S E^{-1} = E_0^{-*} T E_0^{-1}$, $\widetilde{\mathbb{K}} = E\mathbb{K}$ and $(I - \widetilde{\mathbb{F}}) = E(I - \mathbb{F})$.

Remark: It seems that we could get $S = \left[\begin{smallmatrix} I & 0 \\ 0 & -I \end{smallmatrix}\right]$ (i.e., $J_W = I$) at least when both U and W are unseparable, by using Lemma B.3.16; however, we have not been able to establish this for general U and W (and hence obtain "(FI1)⇒(FIn)" for $n \in \{2,3,4,5\}$) without reduction to the stable case (under the assumptions of Theorem 11.2.7 or Proposition 11.2.8), where it is ultimately reduced to Lemma 11.4.3(a)&(c).

(b3) (Note in particular that the MTIC_∞ assumption allows us to avoid the "$\vartheta \leq 0$" assumption.)

1° *The CARE:* We have $\mathrm{MTIC}_\infty \subset \mathrm{ULR}$, by Theorem 2.6.4(f), hence we can redefine X to I so that we obtain a solution of the CARE, by Corollary 9.9.8 (still with $\mathbb{F} \in \mathrm{MTIC}_\infty$).

2° *Claim* $S_{11} \geq \varepsilon_+^2$*:* Let $u_0 \in U$. Set $u' := t^{-1/2} \pi_{[0,t)} u_0$. Let $t \to 0+$ in

(11.66) to obtain (by simple computations using Theorem 2.6.4(i3)) that

$$\langle X \begin{bmatrix} u_0 \\ 0 \end{bmatrix}, SX \begin{bmatrix} u_0 \\ 0 \end{bmatrix} \rangle_{U\times W} \geq \varepsilon_+^2 \mathrm{e}^0 \|u_0\|_U^2. \tag{11.68}$$

Since $u_0 \in U$ was arbitrary, this says that $(X^*SX)_{11} \geq \varepsilon_+^2 I$. By Lemma 11.3.14, we can redefine $\left[\ \mathbb{K}\ \middle|\ \mathbb{F}\ \right]$ and S so that $S = \begin{bmatrix} I & 0 \\ 0 & J_W \end{bmatrix}$ (cf. (b2)2°).

3° *The suboptimal case:* We obtain this from (11.78) as in 2° above (cf. the proofs of (d1)–(d2)).

Remark: In fact, (b3) holds even if we replace MTIC_∞ by $\mathrm{SMTIC}_\infty \cap \mathrm{SR}^{\mathrm{d}}$ or by $\mathcal{B}(U, \mathrm{L}^p_\omega(\mathbf{R}_+;Y)) \cap \mathrm{SR}^{\mathrm{d}}$ for some $\omega \in \mathbf{R}$ (since we only need that $\lim_{t\to 0+}(\mathbb{F}\chi_{\mathbf{R}_+}u_0)(t)$ exists and $\mathbb{F}^{\mathrm{d}} \in \mathrm{SR}$, and that $\mathbb{X}\begin{bmatrix} \mathbb{U} \\ I \end{bmatrix}$ also has same properties, where (the suboptimal) $\mathbb{U}$ (if any) is as in the proof of (c)).

(Indeed, then $\mathbb{F}, \mathbb{F}^{\mathrm{d}} \in \mathrm{SR}$, hence $X \in \mathcal{GB}$, by Proposition 6.2.8(a2), and we can work as above.)

(c) 1° Let $\left[\ \widetilde{\mathbb{K}}\ \middle|\ \widetilde{\mathbb{F}}\ \right]$ be a H^∞-FI-pair for Σ with closed-loop system $\Sigma^{\curvearrowleft}$ (we still denote by $\Sigma_{\circlearrowleft}$ the closed-loop system corresponding to K). Set $\mathbb{U} := \widetilde{\mathbb{F}}^{\curvearrowleft}_{12}$, so that $\mathbb{D}_2^{\curvearrowleft} = \mathbb{D}\begin{bmatrix} \mathbb{U} \\ I \end{bmatrix} \in \mathrm{TIC}(W,Y)$ (see Remark 11.2.5).

Let $t > 0$. Let $\pi_{[0,t)}w \in \mathrm{L}^2([0,t);W)$ be arbitrary and set $x_t := \mathbb{B}\tau^t \begin{bmatrix} \mathbb{U} \\ I \end{bmatrix} w$. Set $\pi_{[t,\infty)}w := \tau^{-t}\mathbb{K}_{\circlearrowleft 2}x_t \in \mathrm{L}^2([t,\infty);W)$, so that (use the fact that $\begin{bmatrix} u_{\mathrm{crit}}(x_t) \\ w_{\mathrm{crit}}(x_t) \end{bmatrix} = \mathbb{K}_{\circlearrowleft}x_t$)

$$\mathcal{J}(x_t, \pi_+\tau^t\mathbb{U}w, \pi_+\tau^t w) \geq \min_{u\in\mathcal{U}_{\mathrm{u}}} \mathcal{J}(x_t, u, \mathbb{K}_{\circlearrowleft 2}x_t) = \max_{\widetilde{w}} \min_{u\in\mathcal{U}_{\mathrm{u}}} \mathcal{J}(x_t, u, \widetilde{w}) = \langle x_t, \mathcal{P}x_t \rangle. \tag{11.69}$$

We have

$$\tau^t\pi_{[t,\infty)}\mathbb{D}_2^{\curvearrowleft}w = \pi_+\tau^t\mathbb{D}\begin{bmatrix} \mathbb{U} \\ I \end{bmatrix} w = \pi_+\mathbb{D}(\pi_+ + \pi_-)\tau^t \begin{bmatrix} \mathbb{U} \\ I \end{bmatrix} w = \mathbb{D}\pi_+\tau^t \begin{bmatrix} \mathbb{U} \\ I \end{bmatrix} w + \mathbb{C}x_t, \tag{11.70}$$

and $\mathbb{D}_2^{\curvearrowleft} = \pi_{[t,\infty)}\mathbb{D}_2^{\curvearrowleft} + \mathbb{D}_2^{\curvearrowleft\, t}$, hence

$$\langle \mathbb{D}_2^{\curvearrowleft}w, J_\gamma\mathbb{D}_2^{\curvearrowleft}w \rangle - \langle \mathbb{D}_2^{\curvearrowleft\, t}w, J_\gamma\mathbb{D}_2^{\curvearrowleft\, t}w \rangle = \langle \pi_{[t,\infty)}\mathbb{D}_2^{\curvearrowleft}w, J_\gamma\pi_{[t,\infty)}\mathbb{D}_2^{\curvearrowleft}w \rangle \tag{11.71}$$

$$= \langle \tau^t\pi_{[t,\infty)}\mathbb{D}_2^{\curvearrowleft}w, J_\gamma\tau^t\pi_{[t,\infty)}J_\gamma\mathbb{D}_2^{\curvearrowleft}w \rangle \tag{11.72}$$

$$= \langle \mathbb{C}x_t + \mathbb{D}\pi_+\tau^t \begin{bmatrix} \mathbb{U} \\ I \end{bmatrix} w, J_\gamma(\mathbb{C}x_t + \mathbb{D}\pi_+\tau^t \begin{bmatrix} \mathbb{U} \\ I \end{bmatrix} w) \rangle \tag{11.73}$$

$$= \mathcal{J}(x_t, \pi_+\tau^t\mathbb{U}w, \pi_+\tau^t w) \tag{11.74}$$

$$\geq \langle x_t, \mathcal{P}x_t \rangle = \langle \begin{bmatrix} \mathbb{U} \\ I \end{bmatrix} w, \mathbb{B}^{t\,*}\mathcal{P}\mathbb{B}^t \begin{bmatrix} \mathbb{U} \\ I \end{bmatrix} w \rangle, \tag{11.75}$$

by (11.69). From this and (9.160) we obtain that

$$\langle \mathbb{D}_2^{\curvearrowleft}w, J_\gamma\mathbb{D}_2^{\curvearrowleft}w \rangle \geq \langle \mathbb{X}^t \begin{bmatrix} \mathbb{U} \\ I \end{bmatrix} w, S\mathbb{X}^t \begin{bmatrix} \mathbb{U} \\ I \end{bmatrix} w \rangle. \tag{11.76}$$

If $\left[\ \widetilde{\mathbb{K}}\ \middle|\ \widetilde{\mathbb{F}}\ \right]$ is suboptimal, then there is $\varepsilon > 0$ s.t. $\langle \mathbb{D}_2^{\curvearrowleft}w, J_\gamma\mathbb{D}_2^{\curvearrowleft}w \rangle \leq -\varepsilon\|w\|_2^2$ for all $w \in \mathrm{L}^2(\mathbf{R}_+;W)$. Since $t > 0$ and $\pi_{[0,t)}w \in \mathrm{L}^2$ were arbitrary, we obtain from (11.76) that

$$\langle \mathbb{X}^t \begin{bmatrix} \mathbb{U}^t \\ I \end{bmatrix} w, S\mathbb{X}^t \begin{bmatrix} \mathbb{U}^t \\ I \end{bmatrix} w \rangle \leq -\varepsilon\|w\|_2^2 \quad (t \geq 0,\ w \in \mathrm{L}^2([0,t);W)), \tag{11.77}$$

i.e., that $((\mathbb{X}\begin{bmatrix} \mathbb{U} \\ I \end{bmatrix})^t)^* S (\mathbb{X}\begin{bmatrix} \mathbb{U} \\ I \end{bmatrix})^t \leq -\varepsilon I$ for all $t \geq 0$. By Lemma 2.2.4(a), this implies that

$$(\widehat{\mathbb{X}}\begin{bmatrix}\widehat{\mathbb{U}}\\ I\end{bmatrix})^* S\widehat{\mathbb{X}}\begin{bmatrix}\widehat{\mathbb{U}}\\ I\end{bmatrix} \le -\varepsilon I \tag{11.78}$$

on $\mathbf{C}^+_\omega$, for any $\omega \ge 0$ s.t. $\mathbb{X}, \mathbb{U} \in \mathrm{TIC}_\omega$.

2° *Implication "$\vartheta \le 0 \Rightarrow S' = J_1$" etc.:* Assume that $\vartheta \le 0$. Choose $\omega \ge 0$ s.t. $\mathbb{X} \in \mathrm{TIC}_\omega$ (note that $\mathbb{U} := \widetilde{\mathbb{F}}^{\circlearrowleft}_{12} \in \mathrm{TIC} \subset \mathrm{TIC}_\omega$, because $\mathbb{U}w \in \mathrm{L}^2_\vartheta \subset \mathrm{L}^2$ for all $w \in \mathrm{L}^2(\mathbf{R}_+;W)$, by (11.7)). Since we assumed that $\vartheta \le 0$, we have $\widehat{\mathbb{S}}_{11} \ge \varepsilon_+^2 I$ on $\mathbf{C}^+_\omega$, by (b2). This and (11.78) imply that $\widetilde{S} := \widehat{\mathbb{S}}(s_0)$ satisfies (i') (hence (i)–(vi)) of Lemma 11.3.13 for any $s_0 \in \mathbf{C}^+_\omega$. Consequently, (11.59) holds and $\widetilde{S} = \widetilde{X}^* J_1 \widetilde{X}$ for some $\widetilde{X} = \left[\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right] \in \mathcal{GB}$ (with $X_{11}, X_{22} \gg 0$), hence $S = E^* J_1 E$, where $E := \widetilde{X}\widehat{\mathbb{X}}(s_0)^{-1} \in \mathcal{GB}$.

By using this $\widetilde{E}$, we obtain $(\mathcal{P}, J_1, \left[\begin{array}{c|c} \widetilde{\mathbb{K}} & \widetilde{\mathbb{F}} \end{array}\right])$ as in (b2)2° above. Moreover, then $\widehat{\widehat{\mathbb{X}}}(s_0) = E\widehat{\mathbb{X}}(s_0) = \widetilde{X}$. By (b2), we have

$$\varepsilon_+^2 I \le (\widehat{\mathbb{X}}^* J_1 \widehat{\mathbb{X}})_{11} = \widehat{\mathbb{X}}_{11}^* \widehat{\mathbb{X}}_{11} - \widehat{\mathbb{X}}_{21}^* \widehat{\mathbb{X}}_{21} \tag{11.79}$$

on $\mathbf{C}^+_\omega$. This and the invertibility of $\widehat{\widehat{\mathbb{X}}}(s_0)_{11} = \widetilde{X}_{11}$, imply that $\widetilde{\mathbb{X}}_{11} \in \mathcal{G}\mathrm{TIC}_\omega(U)$, by Proposition 2.2.5(5). The rest follows from Lemma 11.2.21.

(d) By (b2), we have $((\mathbb{X}^t)^* S \mathbb{X}^t)_{11} \ge \varepsilon_+^2 I_U$, hence $(\widehat{\mathbb{X}}^* S \widehat{\mathbb{X}})_{11} \ge \varepsilon_+^2 I_U$ on $\mathbf{C}^+_\omega$ for ω big enough, by Lemma 2.2.4. Therefore, for any $u_0 \in U$, we have

$$\varepsilon_+^2 \|u_0\|^2 \le \langle \widehat{\mathbb{X}}(s)\left[\begin{smallmatrix} u_0 \\ 0 \end{smallmatrix}\right], S\widehat{\mathbb{X}}(s)\left[\begin{smallmatrix} u_0 \\ 0 \end{smallmatrix}\right]\rangle_{U\times W} \to \langle \left[\begin{smallmatrix} u_0 \\ 0 \end{smallmatrix}\right], S\left[\begin{smallmatrix} u_0 \\ 0 \end{smallmatrix}\right]\rangle_{U\times W}, \tag{11.80}$$

as $s \to +\infty$, by strong regularity. Thus, $S_{11} \ge \varepsilon_+^2 I$.

(d1) In (b2)2° we showed that $(\widehat{\mathbb{X}}^* S \widehat{\mathbb{X}})_{11}(r) \ge \varepsilon_+^2 I$ for big r; let $r \to +\infty$ to obtain that $S_{11} \ge \varepsilon_+^2 I$ (since $\widehat{\mathbb{X}}(+\infty) = I$). Analogously, from (11.78) we obtain that $(I\left[\begin{smallmatrix} \widetilde{U} \\ I \end{smallmatrix}\right])^* S (I\left[\begin{smallmatrix} \widetilde{U} \\ I \end{smallmatrix}\right]) \le -\varepsilon I$, where $\widetilde{U} := \widehat{\mathbb{U}}(+\infty)$. By Lemma 11.3.13(i')&(i), this means that $S_{22} - S_{21}S_{11}^{-1}S_{12} \le -\varepsilon I$.

(d2) Now we have $\widetilde{U} = 0$ in the proof of (d1), hence $S_{22} \le -\varepsilon I$.

Remark: In (d1) and (d2), we can use any $\varepsilon > 0$ s.t. $\langle \mathbb{D}^{\circlearrowleft}_{2} w, J_\gamma \mathbb{D}^{\circlearrowleft}_{2} w\rangle \le -\varepsilon \|w\|_2^2$ for all $w \in \mathrm{L}^2(\mathbf{R}_+;W)$ (where $\Sigma^{\curvearrowleft}$ is the closed-loop system corresponding to the suboptimal pair or operator, as in the proof of (c)).

Remark: In (d)–(d2), the assumption on the CARE may be replaced by the weaker assumption that the IARE has a SR solution $(\mathcal{P}, S, \left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right])$ s.t. $F = 0$.

(By Proposition 9.8.10, the only difference is that $\mathbb{D}$ need not be WR; obviously, the regularity of $\mathbb{D}$ is not needed in the above proofs.)

(e) (The assumption of (e) is rather common, since one usually has "$\mathcal{J} = \|u\|_2^2 + \|y\|_2^2$" or something similar.)

The assumption that $\vartheta \le 0$ was only used in (b2)2° (and later to justify referations to (b2)), to show that $(\mathbb{X}^{t*} S \mathbb{X}^t)_{11} \ge \varepsilon_+^2 I$; under the assumption of (e) this inequality follows directly from (11.66).

(f) In the proofs of (b3), (c), (d1) and (d2), we did not need $\mathbb{K}$ or $\mathbb{F}$, just a

$\mathbb{U}$ with the properties mentioned in (f). (Note that (f) and (e) are compatible.) $\square$

From the above proposition, one can conclude that the existence of a well-posed solution of the FICP in the I/O sense (i.e., of a suboptimal well-posed control law) implies a solution in the usual sense if the finite cost condition ($\mathcal{U}_*^* \neq 0$) is satisfied and Σ is smooth enough to guarantee that unique J_γ-critical control (of a J_γ-coercive system) corresponds to a WR state feedback operator (i.e., to a CARE):

Lemma 11.2.20 (I/O FICP ⇒ FICP) *Assume that there is* $\mathbb{U} \in \mathrm{TIC}_\infty$ *s.t.* $\|\mathbb{D}_{11}\mathbb{U} + \mathbb{D}_{12}\| < \gamma$ *and* $\begin{bmatrix} \mathbb{U} \\ I \end{bmatrix}[\mathrm{L}^2(\mathbf{R}_+;W)] \subset \mathcal{U}_{\exp}(0)$. *Assume also that* Σ *is optimizable and* $(\Sigma, J) \in$ coerciveCARE *(cf. Remark 9.9.14).*

Then there is a unique exponentially stabilizing solution $(\mathcal{P}, S, K)$ *of the CARE. Moreover,* $\mathcal{P} \geq 0$, $S_{11} \geq \varepsilon_+ I$, $S_{22} - S_{21}S_{11}^{-1}S_{12} \ll 0$, *and (11.39) generate a SR suboptimal* H^∞*-FI-pair (here* $K_1 := \begin{bmatrix} I & 0 \end{bmatrix} K$*).*

(This lemma will be used for the H^∞ 4BP.)

Proof: (As one observes from the proof, we could allow for $\mathcal{U}_{\mathrm{out}}$ instead of $\mathcal{U}_{\exp}$ if we would require (11.47) for some $\varepsilon > 0$ and replace "exponentially stabilizing" by "$\mathcal{U}_{\mathrm{out}}$-stabilizing".)

The assumptions $\begin{bmatrix} \mathbb{U} \\ I \end{bmatrix}[\mathrm{L}^2(\mathbf{R}_+;W)] \subset \mathcal{U}_*^*(0)$ and $\|\mathbb{D}_{11}\mathbb{U} + \mathbb{D}_{12}\| < \gamma$ imply that $\gamma > \|\mathbb{D}_{11}\mathbb{U} + \mathbb{D}_{12}\| \geq \gamma_0$ (we could replace "$\begin{bmatrix} \mathbb{U} \\ I \end{bmatrix}[\mathrm{L}^2(\mathbf{R}_+;W)] \subset \mathcal{U}_*^*(0)$" by "$\gamma > \gamma_0$" in the assumptions).

By Proposition 11.2.19(a2) the CARE for Σ and J_γ has a (unique) SR $\mathcal{U}_*^*$-stabilizing solution $(\mathcal{P}, S, K)$.

By Proposition 11.2.19(b2)&(f)&(c), we have that $\mathcal{P} \geq 0$ and (11.59) holds (the condition "$\mathbb{U} \in \mathrm{TIC}_\omega$" is redundant since $\mathbb{U} \in \mathrm{TIC}(W, U)$ (because $\mathbb{U}[\mathrm{L}^2(\mathbf{R}_+;W)] \subset \mathcal{U}_{\exp}(0) \subset \mathrm{L}^2$; use also (2.13))).

Consequently, Lemma 11.2.14(4.)&(b)&(a2) can be applied to obtain a SR (since $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ is SR) suboptimal state feedback pair $\left[\begin{array}{c|c} \overline{\mathbb{K}} & \overline{\mathbb{F}} \end{array}\right]$ as above. The existence of such a pair implies that $S_{11} \geq \varepsilon_+ I$ and $S_{22} - S_{21}S_{11}^{-1}S_{12} \ll 0$, by Proposition 11.2.19(d1).

$\square$

The following lemma was used above:

Lemma 11.2.21 *Let* $\omega \in \mathbf{R}$, $\mathbb{X} \in \mathcal{G}\mathrm{TIC}_\omega(U \times W)$, $\widehat{\mathbb{X}}_{11}(s_0) \in \mathcal{G}\mathcal{B}(U)$ *for some* $s_0 \in \mathbf{C}_\omega^+$ *and* $(\mathbb{X}^{t\,*} J_1 \mathbb{X}^t)_{11} \geq \varepsilon I$ *(on* $\mathrm{L}^2([0,t);U)$*) for all* $t > 0$. *Then* $\mathbb{X}_{11} \in \mathcal{G}\mathrm{TIC}_\omega(U)$, $\|\mathbb{X}_{11}^{-1}\|_{\mathrm{TIC}} \leq \varepsilon^{-1/2}$ *and* $\|\mathbb{X}_{21}\mathbb{X}_{11}^{-1}\|_{\mathrm{TIC}} \leq 1$.

Proof: By Lemma 2.2.4(a) we have

$$\varepsilon I \leq (\widehat{\mathbb{X}}^* J_1 \widehat{\mathbb{X}})_{11} = \widehat{\mathbb{X}}_{11}^* \widehat{\mathbb{X}}_{11} - \widehat{\mathbb{X}}_{21}^* \widehat{\mathbb{X}}_{21} \tag{11.81}$$

on $\mathbf{C}_\omega^+$, hence $\mathbb{X}_{11} \in \mathcal{G}\mathrm{TIC}_\omega$, by Proposition 2.2.5(5). But

$$\varepsilon I \leq (\mathbb{X}^{t\,*} J_1 \mathbb{X}^t)_{11} = \mathbb{X}^{t\,*}{}_{11} \mathbb{X}^t{}_{11} - \mathbb{X}^{t\,*}{}_{21} \mathbb{X}^t{}_{21} \tag{11.82}$$

implies that

$$\varepsilon \mathbb{X}^{t\,-*}_{11}\mathbb{X}^{t\,-1}_{11} + \mathbb{X}^{t\,-*}_{11}\mathbb{X}^{t\,*}_{21}\mathbb{X}^{t}_{21}\mathbb{X}^{t\,-1}_{11} \leq I. \tag{11.83}$$

By Lemma 2.1.14, it follows that $\mathbb{X}_{11}^{-1}, \mathbb{X}_{21}\mathbb{X}_{11}^{-1} \in \mathrm{TIC}$, $\|\mathbb{X}_{11}^{-1}\| \leq \varepsilon^{-1/2}$, and $\|\mathbb{X}_{21}\mathbb{X}_{11}^{-1}\| \leq 1$ □

The proofs of Theorem 11.2.7 and Proposition 11.2.8 were based on the fact that under their assumptions, we are able to reduce the problem to the (SOS-)stable case:

Lemma 11.2.22 ($\Sigma \Leftrightarrow \Sigma_\flat$) *Assume that* $\left[\begin{array}{c|c} \widetilde{\mathbb{K}} & \widetilde{\mathbb{F}} \end{array}\right] = \left[\begin{array}{c|cc} \widetilde{\mathbb{K}}_1 & \widetilde{\mathbb{F}}_{11} & \widetilde{\mathbb{F}}_{12} \\ 0 & 0 & 0 \end{array}\right]$ *is an admissible state feedback pair for* Σ *with closed-loop system* $\Sigma_\flat$*, and that* $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{exp}}$ *(or that* $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{out}}$ *and* $\left[\begin{array}{c|c} \widetilde{\mathbb{K}} & \widetilde{\mathbb{F}} \end{array}\right]$ *is q.r.c.-SOS-stabilizing).*

(a) (Drop Standing Hypothesis 11.2.1 for part (a).)

Then $\mathbb{D}_{\flat 11}^*\mathbb{D}_{\flat 11} \gg 0$ *iff Hypothesis 11.2.1 holds. A sufficient condition for this is that* $\mathbb{D}_\flat^* J_\gamma \mathbb{D}_\flat$ *has a spectral factorization.*

(b) We have, $\gamma > \gamma_0$ *iff*

$$\gamma > \sup_{w \in \mathrm{L}^2(\mathbf{R}_+;W)\setminus\{0\}} \inf_{u \in \mathrm{L}^2(\mathbf{R}_+;U)} \|\mathbb{D}_\flat \left[\begin{smallmatrix} u \\ w \end{smallmatrix}\right]\|_2 / \|w\|_2. \tag{11.84}$$

(c) There is a suboptimal H^∞*-FI-pair for* Σ *iff there is a suboptimal* H^∞*-FI-pair for* $\left[\begin{array}{c|c} \mathbb{A}_\flat & \mathbb{B}_\flat \\ \hline \mathbb{C}_\flat & \mathbb{D}_\flat \end{array}\right]$ *(the same holds for WR suboptimal* H^∞*-SF-operators if* $\widetilde{\mathbb{F}}$ *is SR and* $\widetilde{F} = 0$*).*

Recall from Theorem 8.4.5(g2) that if $\left[\begin{array}{c|c} \widetilde{\mathbb{K}} & \widetilde{\mathbb{F}} \end{array}\right]$ is [[exponentially] strongly] q.r.c.-stabilizing, then $\mathcal{U}_{\mathrm{out}} = \mathcal{U}_{\mathrm{sta}}[= \mathcal{U}_{\mathrm{str}}[= \mathcal{U}_{\mathrm{exp}}]]$.

Proof: (a) 1° *The equivalence, case* $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{exp}}$ *(resp.* $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{out}}$*):* From (11.10) (and (6.133)), we easily observe that $\left[\begin{array}{c|c} \widetilde{\mathbb{K}}_1 & \mathbb{F}_{11} \end{array}\right]$ is an admissible state feedback pair for $\Sigma_{11} := \left[\begin{array}{c|c} \mathbb{A} & \mathbb{B}_1 \\ \hline \mathbb{C} & \mathbb{D}_1 \end{array}\right]$, with closed loop system $\Sigma_{\flat 11} := \left[\begin{array}{c|c} \mathbb{A}_\flat & \mathbb{B}_{\flat 1} \\ \hline \mathbb{C}_\flat & \mathbb{D}_{\flat 1} \end{array}\right]$, which is exponentially (resp. SOS-)stable, because so is $\Sigma_\flat$.

By Lemma 11.2.2, Hypothesis 11.2.1 is satisfied iff Σ_{11} is I-coercive. By Theorem 8.4.5(d) (resp. and (g1)), Σ_{11} is I-coercive iff $\Sigma_{\flat 11}$ is I-coercive, i.e., iff $\mathbb{D}_{\flat 11} \gg 0$ (by Lemma 8.4.11(b1)).

2° The latter claim follows from Lemma 11.3.12.

(b)&(c) *I Case* $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{exp}}$*:*

*I.*1° *"*$\gamma_0 = \gamma_{\flat,0}$*":* Set $\widetilde{\mathbb{M}} := (I - \widetilde{\mathbb{F}})^{-1} =: \begin{bmatrix} \widetilde{\mathbb{M}}_{11} & \widetilde{\mathbb{M}}_{12} \\ 0 & I \end{bmatrix} \in \mathrm{TIC}_\infty(U \times W)$. By Theorem 8.4.5(e)&(c1), $\mathcal{U}_{\mathrm{exp}}^{\Sigma_\flat}(x_0) = \mathrm{L}^2(\mathbf{R}_+; U \times W)$ for all $x_0 \in H$ and $\mathcal{U}_{\mathrm{exp}}(0) = \widetilde{\mathbb{M}}[\mathcal{U}_{\mathrm{exp}}^{\Sigma_\flat}(0)]$. Thus, if $\left[\begin{smallmatrix} u_\flat \\ w \end{smallmatrix}\right] \in \mathrm{L}^2(\mathbf{R}_+; U \times W)\,\mathcal{U}_{\mathrm{exp}}^{\Sigma_\flat}(0))$, then $u := \begin{bmatrix} \widetilde{\mathbb{M}}_{11} & \widetilde{\mathbb{M}}_{12} \end{bmatrix} \left[\begin{smallmatrix} u_\flat \\ w \end{smallmatrix}\right] \in \mathcal{U}_{\mathrm{u}}(0,w)$ and $\|\mathbb{D}\left[\begin{smallmatrix} u \\ w \end{smallmatrix}\right]\| = \|\mathbb{D}_\flat \left[\begin{smallmatrix} u_\flat \\ w \end{smallmatrix}\right]\|$.

Therefore, $f(w) := \min_{u\in\mathcal{U}_{\mathrm{u}}(0,w)} \mathcal{J}(0,u,w) \le \min_{u_\flat\in\mathcal{U}_{\mathrm{u}}^{\Sigma_\flat}(0,w)} \mathcal{J}_\flat(0,u,w) =: f_\flat(w)$. Since w was arbitrary, $f \ll 0$ if $f_\flat \ll 0$. Exchange the roles of Σ and $\Sigma_\flat$ to obtain the converse.

I.2° Suboptimal H^∞*-FI-pairs:* If $\left[\begin{array}{c|c} \mathbb{K}_\natural & \mathbb{F}_\natural \end{array}\right]$ is a suboptimal H^∞-FI-pair for $\Sigma_\flat$ (i.e., it is exponentially stabilizing and its second row equals zero, by Remark 11.2.5), then also (6.193) is exponentially stabilizing and has a zero second row. Since $\mathbb{D}^\curvearrowleft$ is the same for these two pairs, (6.193) is a suboptimal H^∞-FI-pair for Σ. Exchange the roles of Σ and $\Sigma_\flat$ to obtain the converse.

I.3° Suboptimal H^∞*-SF-operators:* The proof of 2° applies except that we have to use Proposition 6.6.18(f) (which shows that any WR $\begin{bmatrix} (K_\natural)_1 \\ 0 \end{bmatrix}$ for $\Sigma_\flat$ corresponds to the WR state feedback operator $\begin{bmatrix} K_1 \\ 0 \end{bmatrix} := \begin{bmatrix} \widetilde{K}_1 \\ 0 \end{bmatrix} + \begin{bmatrix} (K_\natural)_1 \\ 0 \end{bmatrix}$ for Σ).

II Case $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{out}}$*:* The proofs is analogous to that above. In particular, *I.1°* applies since again $\mathcal{U}_{\mathrm{out}}(0) = \widetilde{\mathbb{M}}[\mathcal{U}_{\mathrm{out}}^{\Sigma_\flat}(0)] = \mathrm{L}^2(\mathbf{R}_+; U\times W)$ (see Theorem 8.4.5(g1) and its proof).

II.1° Suboptimal H^∞*-FI-pairs —* $\Sigma_\flat \Rightarrow \Sigma$*:* Assume then that $\left[\begin{array}{c|c} \mathbb{K}_\natural & \mathbb{F}_\natural \end{array}\right]$ is a suboptimal H^∞-FI-pair for $\Sigma_\flat$. Define $\left[\begin{array}{c|c} \mathbb{K}' & \mathbb{F}' \end{array}\right]$ and $\Sigma'_\circlearrowleft$ by (6.193). Let $x_0 \in H$ and $w \in \mathrm{L}^2(\mathbf{R}_+; W)$.

By Theorem 8.4.5(g1)&(e), $\mathbb{M}_\natural \mathbb{K}_\natural x_0 + \mathbb{M}_\natural \left[\begin{smallmatrix} 0 \\ w \end{smallmatrix}\right] \in \mathcal{U}_{\mathrm{out}}^{\Sigma_\flat}(x_0) = \mathrm{L}^2$, hence $\mathbb{K}_\flat x_0 + \widetilde{\mathbb{M}}\left[\begin{smallmatrix} u_\flat \\ w \end{smallmatrix}\right] \in \mathcal{U}_{\mathrm{out}}(x_0)$, where $\mathbb{M}_\natural := I - \mathbb{F}_\natural$ and $\widetilde{\mathbb{M}} := I - \widetilde{\mathbb{F}}$. But $\mathbb{K}_\flat x_0 + \widetilde{\mathbb{M}}\left[\begin{smallmatrix} u_\flat \\ w \end{smallmatrix}\right] = \mathbb{K}'_\circlearrowleft x_0 + \mathbb{M}'\left[\begin{smallmatrix} 0 \\ w \end{smallmatrix}\right]$; because x_0 and w were arbitrary, $\left[\begin{array}{c|c} \mathbb{K}' & \mathbb{F}' \end{array}\right]$ is a H^∞-FI-pair for Σ. Since y is the same for both pairs, by Theorem 8.4.5(g1)&(c1), the pair $\left[\begin{array}{c|c} \mathbb{K}' & \mathbb{F}' \end{array}\right]$ is suboptimal for $\Sigma_\flat$.

II.2° Suboptimal H^∞*-FI-pair —* $\Sigma \Rightarrow \Sigma_\flat$*:* For the converse, assume that $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ is a suboptimal H^∞-FI-pair for Σ with closed-loop system $\Sigma_\circlearrowleft$. Define $\left[\begin{array}{c|c} \mathbb{K}_\natural & \mathbb{F}_\natural \end{array}\right]$ by (6.180) and set $\mathbb{M}_\natural := (I - \mathbb{F}_\natural)^{-1}$, $\mathbb{M} := (I - \mathbb{F})^{-1}$. By (6.183), we have for any $x_0 \in H$ and $w \in \mathrm{L}^2(\mathbf{R}_+; W)$ that

$$\begin{bmatrix} u_{\flat\circlearrowleft} \\ w \end{bmatrix} := \begin{bmatrix} \mathbb{M}_\natural \mathbb{K}_\natural & \mathbb{M}_\natural \end{bmatrix} \begin{bmatrix} x_0 \\ 0 \\ w \end{bmatrix} = \begin{bmatrix} \widetilde{\mathbb{X}}\mathbb{K}_\circlearrowleft - \widetilde{\mathbb{K}} & \widetilde{\mathbb{X}}\mathbb{M} \end{bmatrix} \begin{bmatrix} x_0 \\ 0 \\ w \end{bmatrix} = -\widetilde{\mathbb{K}} x_0 + \widetilde{\mathbb{X}}\left[\begin{smallmatrix} u_\circlearrowleft \\ w \end{smallmatrix}\right], \tag{11.85}$$

where $\left[\begin{smallmatrix} u_\circlearrowleft \\ w \end{smallmatrix}\right] := \mathbb{K}_\circlearrowleft x_0 + \mathbb{M}\left[\begin{smallmatrix} 0 \\ w \end{smallmatrix}\right] \in \mathcal{U}_{\mathrm{out}}(x_0)$, so that $\left[\begin{smallmatrix} u_{\flat\circlearrowleft} \\ w \end{smallmatrix}\right] \in \mathcal{U}_{\mathrm{out}}^{\Sigma_\flat}(x_0) = \mathrm{L}^2$, by Theorem 8.4.5(g1)&(c1). As in *II.1°*, we see that $\left[\begin{array}{c|c} \mathbb{K}_\natural & \mathbb{F}_\natural \end{array}\right]$ is suboptimal for $\Sigma_\flat$.

II.3° Suboptimal $\mathcal{U}_{\mathrm{out}}$*-SF pairs:* This goes as in *I.3°*. □

Notes

The “short-hand-notation” or “extended FICP” formulation of Definition 11.2.3 and Lemma 11.2.4 was given in [S98d] for the stable case. The principle to reduce the $\mathcal{U}_{\exp}$ problem to the stable case (Lemma 11.2.22) is old. Lemma 11.2.18 and its proof are close to [S98d, Theorem 7.2].

Condition (FI9) was used by Michael Green [Green] to solve a problem close to the FICP; the results of [Green] were extended to $\mathrm{MTIC}^{\mathrm{L}^1}_{\exp}(\mathbf{C}^n, \mathbf{C}^m)$ I/O maps in [CG97]. Hidenori Kimura and others have produced analogous results for (possibly nonlinear) finite-dimensional systems by using the conjugation method [KK]. See also the notes on pp. 628 and 669.

Naturally, if we strengthen (FI1) by requiring that the “minimax” control can be given in [regular] state feedback form, then (FI1) becomes equivalent to the IARE [and CARE] having a $\mathcal{U}_*^*$-stabilizing solution (see Proposition 11.2.19 and Theorem 9.9.1).

The reduction the stable case and the coverage of several equivalent conditions has made the proofs “unnecessarily complex”. The main results can be obtained much more directly, as explained below.

The sufficiency of (FI5) (for (FI2)) is shown in Lemma 11.2.13 (and in Lemma 11.2.14; see Proposition 11.2.9 for final details). This proof is direct and constructive: we open the disturbance feedback from the closed-loop system corresponding to the CARE, and it appears that the resulting pair is a H^∞-FI-pair (i.e., it stabilizes the system in the desired way (depending on $\mathcal{U}_*^*$)) and suboptimal.

The necessity of (FI5) is shown in Proposition 11.2.19(a2)&(d1) (assuming that $(\Sigma, J) \in$ coerciveCARE; note that this is always the case in discrete time for $\mathcal{U}_{\mathrm{exp}}$; this is the case in continuous time too if we make the assumptions of Theorem 11.2.7 or Proposition 11.2.8(a2), so that any unique J_γ-critical control is necessarily of SR state feedback form).

Also this necessity proof is direct and constructive: we first show that if $\gamma > \gamma_0$, then $\mathbb{D}$ is J_γ-coercive over $\mathcal{U}_*^*$, so that the CARE has a $\mathcal{U}_*^*$-stabilizing solution $\mathcal{P}$; we also show that the corresponding closed-loop input is the minimax input, from which we deduce that $\mathcal{P} \geq 0$. Then we assume the existence of a causal suboptimal control law $w \mapsto u$ (e.g., of a suboptimal H^∞-FI-pair) and use the minimax property to show that this implies the signature condition $S_{11} \gg 0$ and $S_{22} - S_{21}S_{11}^{-1}S_{12} \ll 0$ of (FI5).

However, in the proofs of our main results (see the proof of Proposition 11.2.8), we have reduced our solution to the stable case for two reasons:
1. this way we were able to extend the equivalence to (FI1), i.e., we were able to show that if there are any suboptimal controls for each x_0 and w, then there is a (causal!) suboptimal state feedback controller (H^∞-FI-pair) (cf. also the remark in the proof of Proposition 11.2.19(b2));
2. this allowed us to use the techniques of [S98d] to provide the formula for all solutions; this formula will be needed for the H^∞ 4BP.

The results 1. and 2. were established in [S98d] for stable WPLSs over separable Hilbert spaces (for $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{out}}$), and we use essentially the same methods except for the treatment of the unseparable case; see the next section for details. Theorem 11.1.5 illustrates a case where it was not possible to use this reduction to the stable case; this explains the missing “(i)” (or “(FI1)”).

The results of this section and Section 11.3 also contain [partial] sufficiency results under further alternative conditions, some of which are needed for the H^∞ 4BP.

11.3 The H^∞ FICP: stable case

All the world's a stage, And all the men and women merely players. They have their exits and their entrances, And one man in his time plays many parts, His acts being seven ages.

— William Shakespeare (1564–1616), "As You Like It"

In this section, we shall solve the H^∞ FICP in the stable case, parameterize all solutions, and present some additional results. The stable case is interested in its own right, but it is also useful for the H^∞ 4BP and for the proofs of the results in previous sections, although we were able to prove part of them (e.g., the equivalence of (FI2)–(FI5)) directly in the unstable case.

In addition to Standing Hypotheses 11.0.1 and 11.1.1, we shall assume the following:

Standing Hypothesis 11.3.1 (Stable case) *Throughout this section, we assume that* $\gamma > 0$, $\Sigma \in \mathrm{SOS}$, $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{out}}$ *and* $\mathbb{D}_{11}^*\mathbb{D}_{11} \gg 0$.

(It follows that $\vartheta = 0$ and Z^{s} is reflexive, by Definition 8.3.2.)

Recall from Lemma 8.3.3, that if Σ is [[exponentially] strongly] stable, then $[[\mathcal{U}_{\mathrm{exp}} =]\mathcal{U}_{\mathrm{str}} =]\mathcal{U}_{\mathrm{sta}} = \mathcal{U}_{\mathrm{out}}$.

Since $\mathbb{D}$ is stable, the norm $\|\cdot\|_{\mathcal{U}_{\mathrm{out}}}$ is equivalent to $\|\cdot\|_2$. Consequently, Standing Hypothesis 11.3.1 is equivalent to Hypothesis 11.2.1 with the additional conditions $\Sigma \in \mathrm{SOS}$ and $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{out}}$. In particular, Hypothesis 11.2.1 holds.

Lemma 11.3.2 *An admissible state feedback pair* $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right] = \left[\begin{array}{c|cc} \mathbb{K}_1 & \mathbb{F}_{11} & \mathbb{F}_{12} \\ 0 & 0 & 0 \end{array}\right]$ *is a* H^∞*-FI-pair iff* $\mathbb{K}^{\curvearrowleft}$ *and* $\mathbb{F}_{12}^{\curvearrowleft}$ *are stable.*

However, in the theorem and proposition below, we shall make such assumptions that the existence of a H^∞-FI-pair implies the existence of stable H^∞-FI-pair.

Proof: This follows from Definition 11.1.2, because now $\mathcal{U}_{\mathrm{out}}(x_0) = \mathrm{L}^2(\mathbf{R}_+; U \times W)$ for all $x_0 \in H$ (note that $\mathbb{F}^{\curvearrowleft} = \left[\begin{smallmatrix} * & * \\ 0 & I \end{smallmatrix}\right]$). □

Now we state the main result of this section, the stable counterpart of Theorem 11.2.7:

Theorem 11.3.3 ($\widetilde{\mathcal{A}}$: Stable FICP) *Assume that* $\mathbb{D} \in \widetilde{\mathcal{A}}$. *Then the following are equivalent:*

(FI1s) $\gamma > \gamma_0$; *i.e.,* $\inf_{u \in \mathcal{U}_{\mathrm{u}}(0,\cdot)} \mathcal{J}(0,u,\cdot) \ll 0$;

(FI1$\frac{1}{2}$s) $\|\mathbb{D}_{11}\mathbb{U} + \mathbb{D}_{12}\| < \gamma$ *for some* $\mathbb{U} \in \mathrm{TIC}(W,U)$;

(FI2s) $\gamma > \gamma_{\mathrm{FI}}$; *i.e., there is a suboptimal* H^∞*-FI-pair for* Σ;

(FI3s) $\mathbb{D}^* J_\gamma \mathbb{D} = \mathbb{X}^* J_1 \mathbb{X}$ *where* $\mathbb{X}, \mathbb{X}_{11} \in \mathcal{G}\mathrm{TIC}$;

(FI4s) the IARE has a stable, P-SOS-stabilizing solution $(\mathcal{P}, S, \left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right])$, *and* $\mathcal{P} \geq 0$, *and* $\widetilde{S} := (\widehat{\mathbb{X}}^* S \widehat{\mathbb{X}})(s_0)$ *safisfies* $\widetilde{S}_{11} \gg 0$ *and* $\widetilde{S}_{22} - \widetilde{S}_{21}\widetilde{S}_{11}^{-1}\widetilde{S}_{12} \ll 0$ *for some (equivalently, all)* $s_0 \in \mathbf{C}^+$.

(FI5s) *the CARE has a UR stable, P-SOS-stabilizing solution* $(\mathcal{P}, S, K)$, *and* $\mathcal{P} \geq 0$, $S_{11} \gg 0$ *and* $S_{22} - S_{21}S_{11}^{-1}S_{12} \ll 0$.

Moreover, (a)–(f) of Proposition 11.3.4 apply, and any solutions ($\mathbb{X}$ *or* $\mathbb{F}$) *of (FI3)–(FI5) belong to* $\widetilde{\mathcal{A}}$.

In particular, then there is a suboptimal H^∞*-FI-pair (resp.* H^∞*-SF-operator) iff the CARE has a (necessarily unique) stable, P-SOS-stabilizing solution* $(\mathcal{P}, S, K)$, *and* $S_{11} \gg 0$ *and* $S_{22} - S_{21}S_{11}^{-1}S_{12} \ll 0$ *(resp.* $S_{11} \gg 0$ *and* $S_{22} \ll 0$*).*

If this is the case, then (11.39) generates (resp. K_1 *=(11.40) is) a ULR stable, r.c.-SOS-stabilizing suboptimal* H^∞*-FI-pair (resp.* H^∞*-SF-operator).*

Note from (d1) that "stable, P-SOS-stabilizing" means "exponentially stabilizing" when Σ is exponentially stable. Naturally, $\mathbb{X} := I - \mathbb{F}$ in (FI4s).

Condition (FI1$\frac{1}{2}$s) can be considered as the "frequency-domain stable H^∞ FICP". Without the requirement $\mathbb{U} \in \mathrm{TIC}$, (FI1$\frac{1}{2}$s) would always be equivalent to (FI1s), by Lemma 11.3.10.

We conclude from the theorem that $\mathbb{D} \in \widetilde{\mathcal{A}}$ implies that $\gamma_0 = \gamma_{\mathrm{FI}}$.

Proof of Theorem 11.3.3: Assume that some of (FI1s)–(FI5s) holds. As obvious from the proof of Proposition 11.3.4, this implies that (FI1s) holds, hence $\mathbb{D}$ is J_γ-coercive, by Lemma 11.3.10, hence $\mathbb{D}^* J_\gamma \mathbb{D}$ has a spectral factorization $\mathbb{X}_\diamond^* S_\diamond \mathbb{X}_\diamond$ with $\mathbb{X}_\diamond \in \widetilde{\mathcal{A}}$ (because $\mathbb{D} \in \widetilde{\mathcal{A}}$). Thus, the assumptions of the proposition are satisfied.

Since $\mathbb{X}_\diamond \in \widetilde{\mathcal{A}} \subset \mathrm{ULR}$, the assumptions of (a) are satisfied and thus we obtain the other claims (with $\mathbb{F} = I - E\mathbb{X}_\diamond \in \widetilde{\mathcal{A}}$ for some $E \in \mathcal{GB}(U \times W)$); in particular, K and $\mathbb{F}$ are necessarily ULR. □

Since (FI1s) does not imply any of (FI3s)–(FI5s) for general strongly stable WPLSs, by Example 11.3.7(a), we made the $\widetilde{\mathcal{A}}$ assumption above, and we shall make a weaker spectral factorization assumption in the following stable variant of Proposition 11.2.8 (this assumption is necessary for (FI3s)–(FI5s)):

Proposition 11.3.4 (Stable FICP) *Assume that* $\mathbb{D} \in \mathrm{TIC}$ *and* $\mathbb{D}^* J_\gamma \mathbb{D} = \mathbb{X}_\diamond^* S_\diamond \mathbb{X}_\diamond$ *for some* $\mathbb{X}_\diamond \in \mathcal{G}\mathrm{TIC}(U \times W)$ *and* $S_\diamond \in \mathcal{GB}(U \times W)$.

Then (FI1s)–(FI4s) are equivalent and implied by (FI5s). Also the following hold:

(a) *Assume that* $\mathbb{D}$ *is WR and* $\mathbb{X}_\diamond$ *is UR. Then (FI1s)–(FI5s) are equivalent, and the CARE has a unique stable, P-SOS-stabilizing solution* $(\mathcal{P}, S, K)$.

There is a suboptimal H^∞*-FI-pair (resp.* H^∞*-SF-operator) iff* $S_{11} \gg 0$ *and* $S_{22} - S_{21}S_{11}^{-1}S_{12} \ll 0$ *(resp.* $S_{11} \gg 0$ *and* $S_{22} \ll 0$*); if this is the case, then (11.39) generate (resp.* K_1 *=(11.40) is) a UR stable, r.c.-SOS-stabilizing suboptimal* H^∞*-FI-pair (resp.* H^∞*-SF-operator).*

(b1) *The condition on* $\widetilde{S}$ *in (FI4s) is independent on the choice of* S *and* $\begin{bmatrix} \mathbb{K} \mid \mathbb{F} \end{bmatrix}$ *(and* $s_0 \in \mathbf{C}^+$*), and* $\mathcal{P}$ *is unique. Condition* $\mathbb{X}_{\natural 11} \in \mathcal{G}\mathrm{TIC}(U)$ *in (FI3) is independent on* $\mathbb{X}_\natural$ *(by (c1)).*

(b2) *A stable, P-SOS-stabilizing solution of the CARE is unique.*

(b3) A solution of (FI3s) or (FI4s) is unique modulo an invertible constant.

(c1) If (FI1s) holds, $\mathbb{D}^* J_\gamma \mathbb{D} = \mathbb{X}^* J_1 \mathbb{X}$ *and* $\mathbb{X} \in \mathcal{G}\mathrm{TIC}$, *then* $\mathbb{X}_{11} \in \mathcal{G}\mathrm{TIC}(U)$ *and* $\|\mathbb{X}_{21}\mathbb{X}_{11}^{-1}\|_{\mathrm{TIC}} < 1$.

(c2) Any solution of (FI5s) is unique and a solution of (FI4s).

(c3) If $\mathbb{X}$ *and* $\mathbb{F}$ *are as in (FI3s) and (FI4s), respectively, then* $\mathbb{X} := E(I - \mathbb{F}) = E'\mathbb{X}_\diamond$ *for some* $E, E' \in \mathcal{G}\mathcal{B}(U \times W)$.

(d1) Assume that Σ *is exponentially stable. Then "stable, P-SOS-stabilizing" has the following equivalent forms: "exponentially stabilizing" and "I/O-, input-, output- or internally stabilizing".*

(d2) Assume that Σ *is strongly stable. Then "stable, P-SOS-stabilizing" has the following equivalent forms: "stable, strongly stabilizing", "internally stabilizing (i.e.,* $\mathbb{A}$*-stabilizing)".*

(e) Any UR solution of (FI3s) can be redefined s.t. $X = \begin{bmatrix} X_{11} & X_{12} \\ 0 & X_{22} \end{bmatrix}$, $X_{11}, X_{22} \in \mathcal{G}\mathcal{B}$.

(f) Even without the above spectral factorization assumption (on $\mathbb{X}_\diamond$ *and* $S_\diamond$*), we have (FI5s)⇒(FI4s)⇔(FI3s)⇒(FI2s)⇒(FI1*$\frac{1}{2}$*s)⇒(FI1s), and (FIns)⇔(FIn) for* $n = 1, 2, 3, 4, 5$.

(g) If (FI1s) holds, then $\mathbb{A}_\circlearrowleft$, $\mathbb{C}_\circlearrowleft$, $\mathbb{K}_\circlearrowleft$ *and* $\mathcal{P}$ *are given by (8.43)–(8.46).*

By Example 11.2.16, condition $\mathbb{X}_{11} \in \mathcal{G}\mathrm{TIC}(U)$ is not redundant in general; i.e., $\mathbb{D}^* J_\gamma \mathbb{D} = \mathbb{X}^* J_1 \mathbb{X}$ and $\mathbb{X} \in \mathcal{G}\mathrm{TIC}(U \times W)$ do not imply any of (FI1s)–(FI5s)).

Recall from Lemma 5.2.1(d) that the spectral factorization assumption could be formulated by "$\mathbb{D}^* J_\gamma \mathbb{D} = \mathbb{Y}^* \mathbb{X}_\diamond$ for some $\mathbb{Y}, \mathbb{X}_\diamond \in \mathcal{G}\mathrm{TIC}(U \times W)$" (since necessarily $\mathbb{Y} = S\mathbb{X}_\diamond$ for some $S = S^* \in \mathcal{G}\mathcal{B}(U \times W)$). By Proposition 11.2.8(d), under this assumption also conditions (FI1)–(FI4), (FI6) and (FI7) are equivalent to any of (FI1s)–(FI4s) (naturally, this refers to the $\mathcal{U}_{\mathrm{out}}$ forms of (FI1)–(FI7) (not to $\mathcal{U}_{\mathrm{exp}}$ forms unless Σ is exponentially stable)).

Even if the above spectral factorization assumption (on $\mathbb{X}_\diamond$ and $S_\diamond$) does not hold, we have (FI*n*s)⇔(FI*n*) for $n = 1, 2, 3, 4, 5$, as one observes from Corollary 9.9.11 (and Lemma 11.3.9(b)).

Proof of Proposition 11.3.4: 1° *(FI3s)⇒(FI2s)&(FI4s):* By Corollary 9.9.11, the pair (9.140) defines a stable and P-r.c.-SOS-stabilizing solution of the IARE. By Lemma 11.2.18, $\mathcal{P} \geq 0$ and $\left[\begin{array}{c|c} \overline{\mathbb{K}} & \overline{\mathbb{F}} \end{array}\right]$ is a suboptimal H$^\infty$-FI-pair, hence (FI2s) holds; from Proposition 11.2.19(c) we obtain (FI4s).

2° *(FI2s)⇒(FI1*$\frac{1}{2}$*s):* By (11.8)–(11.9) we can take $\mathbb{U} := \mathbb{F}_{12}^{\curvearrowleft}$.

3° *(FI1*$\frac{1}{2}$*s)⇒(FI1s):* This follows from Lemma 11.3.10.

4° *(FI1s)⇔(FI3s):* By Lemma 11.3.10, (FI1s) holds iff $\mathbb{D}$ is minimax J_γ-coercive. By Lemma 11.4.3(a)&(c), the latter is equivalent to (FI3s) (and any J_1-spectral factor $\mathbb{X}$ of $\mathbb{D}^* J_\gamma \mathbb{D}$ satisfies $\mathbb{X}_{11} \in \mathcal{G}\mathrm{TIC}(U)$ and $\|\mathbb{X}_{21}\mathbb{X}_{11}^{-1}\|_{\mathrm{TI}} < 1$).

5° *(FI4s)⇒(FI2s):* This is contained in Lemma 11.3.9.

6° *(FI5s)⇒(FI4s):* This is given in the proof of Lemma 11.3.8.

(a) 1° *Uniqueness:* A solution of (FI5s) (if any) is $\mathcal{U}_*^*$-stabilizing (by Proposition 9.8.11(iii)&(ii)), hence unique.

2° *The CARE:* By Corollary 9.9.11 (and Theorem 9.9.10(a1)–(b)), the spectral factorization $\mathbb{D}^* J_\gamma \mathbb{D} = \mathbb{X}_\diamond^* S_\diamond \mathbb{X}_\diamond$ defines a UR J_γ-critical stable, r.c.-P-SOS-stabilizing and $\mathcal{U}_*^*$-stabilizing solution $(\mathcal{P}, S_\diamond, \begin{bmatrix} \mathbb{K}_\diamond & | & \mathbb{F}_\diamond \end{bmatrix})$ of the IARE.

All stable, P-SOS-stabilizing solutions of the IARE are given by (9.114). One of them, say $(\mathcal{P}, S, K)$, is a solution of the CARE, since $I - F_\diamond \in \mathcal{GB}$, by Proposition 6.3.1(b1) (note that $\mathbb{F}_\diamond = I - \mathbb{X}_\diamond$, $\mathbb{F} = I - E\mathbb{X}_\diamond$ and $\mathbb{K} = E\mathbb{K}_\diamond$ for some $E \in \mathcal{GB}$, hence K inherits the uniform regularity of $\mathbb{X}_\diamond$).

3° *(FI2s)⇒(FI5s):* This follows from Proposition 11.2.19(d1).

4° *"iff $S_{11} \gg 0 \gg S_{22}$:* "Only if" follows from Proposition 11.2.19(d2); "if" follows from Lemma 11.3.8.

(b1) This follows from (b3).

(b2) This is Theorem 9.8.12(d)&(s3) (with Proposition 9.8.11).

(b3) For (FI4s), this follows from Theorem 9.8.12(d)&(s1) (with Proposition 9.8.11) (i.e., all triples are given by (9.114)).

If $\mathbb{X}^* J_1 \mathbb{X} = \widetilde{\mathbb{X}}^* J_1 \widetilde{\mathbb{X}}$ and $\mathbb{X}, \widetilde{\mathbb{X}} \in \mathcal{G}$TIC, then $\widetilde{\mathbb{X}} = E\mathbb{X}$ for some $E \in \mathcal{GB}$ s.t. $E^* J_1 E = J_1$ (so not all invertible E's will do).

(c1) This follows from Lemma 11.4.3(b) (note also that $\mathbb{X}_{21}\mathbb{X}_{11}^{-1} = -\mathbb{M}_{22}^{-1}\mathbb{M}_{21}$).

(c2) (Here $\begin{bmatrix} \mathbb{K} & | & \mathbb{F} \end{bmatrix}$ refers to the pair generated by K.) Uniqueness follows from Theorem 9.8.12(e)&(s1), the rest follows from (11.59).

(c3) The claim on $\mathbb{X}_\diamond$ follows from Lemma 6.4.5(a). If $\begin{bmatrix} \mathbb{K} & | & \mathbb{F} \end{bmatrix}$ is as in (FI4s), then $\widetilde{\mathbb{X}} := E(I - \mathbb{F}) \in \mathcal{G}$TIC satisfies $\widetilde{\mathbb{X}}^* J_1 \widetilde{\mathbb{X}} = \mathbb{D}^* J_\gamma \mathbb{D}$ for some $E \in \mathcal{GB}$, by Lemma 11.3.13(i)&(iii), hence $\widetilde{\mathbb{X}}$ satisfies (FI3s), by (c1). Therefore, $E := \mathbb{X}(I - \mathbb{F})^{-1} \in \mathcal{GB}$, by (b3). (Remark: the pair $\begin{bmatrix} \mathbb{K} & | & \mathbb{F} \end{bmatrix}$ is necessarily stable and r.c.-SOS-stabilizing.)

(d1)&(d2) These follow from Proposition 9.8.11(d2)&(d3) (and (a)2°). (Note that "$\mathcal{U}_{\text{out}}$-stabilizing", "$\mathcal{U}_{\text{sta}}$-stabilizing" and "stable and strongly r.c.-stabilizing" are also equivalent forms in (d1) and (d2); so is also "exponentially stable and exponentially r.c.-stabilizing" in (d1).)

(e) Set $S := X^* J_1 X$. By Proposition 6.3.1(b1), we have $X, X_{11} \in \mathcal{GB}$. It follows from (c1) and Lemma 11.3.13(vi)&(iii') that $S = \widetilde{X}^* J_1 \widetilde{X}$, where $\widetilde{X}$ is as in (e). Replace $\mathbb{X}$ by $\widetilde{X} X^{-1} \mathbb{X}$ to complete the proof.

(f) In the above proofs of implications (FI5s)⇒(FI4s)⇔(FI3s)⇒(FI2s)⇒(FI1$\frac{1}{2}$s)⇒(FI1s) we did not use the spectral factorization assumption.

The claim "(FI*n*s)⇔(FI*n*)" is trivial for $n = 1, 2, 3$; for $n = 4, 5$ we obtain this from Corollary 9.9.11 (and Lemma 11.3.9(b)). (Here we referred to $\mathcal{U}_{\text{out}}$ forms of (FI1)–(FI5); they are equivalent to $\mathcal{U}_{\text{exp}}$ forms if Σ is exponentially stable).

(g) By Lemma 11.3.10, (FI1s) implies that $\mathbb{D}$ is minimax J_γ-coercive, hence J_γ-coercive, i.e., the Toeplitz operator $\pi_+ \mathbb{D}^* J_\gamma \mathbb{D} \pi_+$ is invertible. By (a)1°, a pair $\begin{bmatrix} \mathbb{K} & | & \mathbb{F} \end{bmatrix}$ corresponding to (FI4s) or (FI5s) is J_γ-critical. Thus, Proposition 8.3.10 applies to $\Sigma_\circlearrowleft$ and $\mathcal{P}$. □

We have shown above that the spectral factorization condition (FI3s) is sufficient (and necessary when $\mathbb{D} \in \widetilde{\mathcal{A}}$) for the existence of a suboptimal control

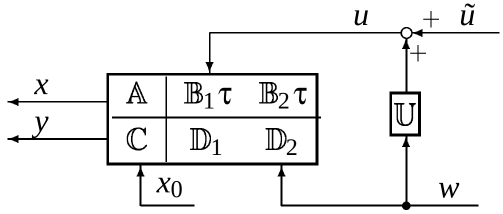

Figure 11.2: Dynamic feedforward compensator

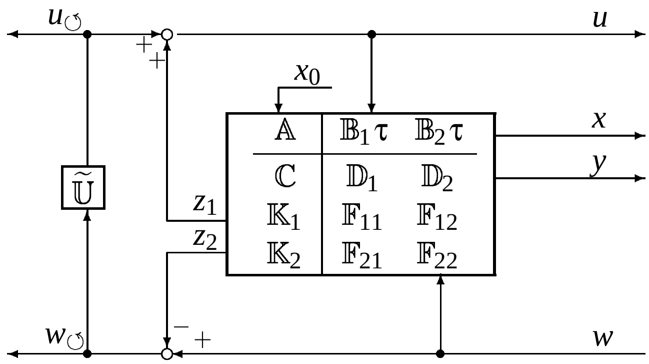

Figure 11.3: Parameterization of all suboptimal compensators

law, and that such a law can be realized as a state feedback controller. For the solution of the H^∞ four-block problem (see Chapter 12), we shall need the following standard parametrization of all suboptimal TIC controllers:

Corollary 11.3.5 (All suboptimal controllers) *If $\mathbb{D}^* J_\gamma \mathbb{D} = \mathbb{X}^* J_1 \mathbb{X}$ for some $\mathbb{X} \in \mathrm{TIC}(U \times W)$ s.t. $\mathbb{X}_{11} \in \mathcal{G}\mathrm{TIC}(U)$, then all solutions $\mathbb{U} \in \mathrm{TIC}(W,U)$ for $\|\mathbb{D}_{11}\mathbb{U} + \mathbb{D}_{12}\|_{\mathrm{TIC}} < \gamma$ are given by Theorem 11.3.6 below.*

The problem "$\|\mathbb{D}_{11}\mathbb{U}+\mathbb{D}_{12}\| < \gamma$" is illustrated in Figure 11.2 and its solution in Figure 11.3 (recall that $y = \left[\begin{smallmatrix} z \\ w \end{smallmatrix}\right]$). Indeed, from Figure 11.3 we observe that $\left[\begin{smallmatrix} \widetilde{\mathbb{U}} \\ I \end{smallmatrix}\right] w_{\circlearrowleft} = \mathbb{X}\left[\begin{smallmatrix} u \\ w \end{smallmatrix}\right]$, i.e., $\left[\begin{smallmatrix} \mathbb{U}_1 \\ \mathbb{U}_2 \end{smallmatrix}\right] w_{\circlearrowleft} = \left[\begin{smallmatrix} u \\ w \end{smallmatrix}\right]$, hence $u = \mathbb{U}w$ (when $\mathbb{U}_2$ is invertible, and this is the case then $\|\widetilde{\mathbb{U}}\| \le 1$). Thus, each suboptimal controller can be realized as a state feedback plus a subunitary dynamic parameter.

Proof of Corollary 11.3.5: (Recall our hypothesis that $\mathbb{D}_{11}^*\mathbb{D}_{11} \gg 0$, hence $\mathbb{D}_1^* J_\gamma \mathbb{D}_1 \gg 0$, as required in Theorem 11.3.6.)

Indeed, we have

$$(\mathbb{D}\left[\begin{smallmatrix} \mathbb{U} \\ I \end{smallmatrix}\right])^* J_\gamma (\mathbb{D}\left[\begin{smallmatrix} \mathbb{U} \\ I \end{smallmatrix}\right]) = (\mathbb{D}_{11}\mathbb{U}+\mathbb{D}_{12})^*(\mathbb{D}_{11}\mathbb{U}+\mathbb{D}_{12}) - \gamma^2 I \ll 0 \tag{11.86}$$

iff $\|\mathbb{D}_{11}\mathbb{U}+\mathbb{D}_{12}\| < \gamma$, by Lemma A.3.1(d). $\square$

The formulae in the actual proof become simpler for the more general extended FICP below:

Theorem 11.3.6 (All suboptimal controllers) *Assume that $J = J^* \in \mathcal{B}(Y)$, and that $\mathbb{D} := \begin{bmatrix} \mathbb{D}_1 & \mathbb{D}_2 \end{bmatrix} \in \mathrm{TIC}(U \times W, Y)$ has a (spectral) factorization $\mathbb{D}^* J \mathbb{D} = \mathbb{X}^* J_1 \mathbb{X}$ s.t. $\mathbb{X}, \mathbb{X}_{11} \in \mathcal{G}\mathrm{TIC}$, and $\mathbb{D}_1^* J \mathbb{D}_1 \gg 0$. Set $\mathbb{M} := \mathbb{X}^{-1}$.*

Then all solutions $\mathbb{U} \in \mathrm{TIC}(W,U)$ *to* $(\mathbb{D}\left[\begin{smallmatrix}\mathbb{U}\\ I\end{smallmatrix}\right])^* J\mathbb{D}\left[\begin{smallmatrix}\mathbb{U}\\ I\end{smallmatrix}\right] \ll 0$ *(equivalently, to* $(w \mapsto \mathcal{J}(0,\mathbb{U}w,w)) \ll 0$*) are given by*

$$\mathbb{U} := \mathbb{U}_1\mathbb{U}_2^{-1}, \quad \begin{bmatrix}\mathbb{U}_1\\ \mathbb{U}_2\end{bmatrix} = \mathbb{M}\begin{bmatrix}\widetilde{\mathbb{U}}\\ I\end{bmatrix}, \quad (\widetilde{\mathbb{U}} \in \mathrm{TIC}(W,U),\ \|\widetilde{\mathbb{U}}\|_{\mathrm{TIC}} < 1) \tag{11.87}$$

(by $\|\widetilde{\mathbb{U}}\|_{\mathrm{TIC}} \le 1$ *we get all solutions to* $(\mathbb{D}\left[\begin{smallmatrix}\mathbb{U}\\ I\end{smallmatrix}\right])^* J\mathbb{D}\left[\begin{smallmatrix}\mathbb{U}\\ I\end{smallmatrix}\right] \le 0$*).*

Two alternative formulations of (11.87) are given by (all these three formulae produce the same $\mathbb{U}$ *for any given* $\widetilde{\mathbb{U}} \in \mathrm{TIC}(W,U)$ *s.t.* $\|\widetilde{\mathbb{U}}\| < 1$ *(or* $\|\widetilde{\mathbb{U}}\| \le 1$*)):*

(2.) $\mathbb{U} = \mathbb{Q}_2^{-1}\mathbb{Q}_1$, $\begin{bmatrix}\mathbb{Q}_2 & -\mathbb{Q}_1\end{bmatrix} = \begin{bmatrix}I & -\widetilde{\mathbb{U}}\end{bmatrix}\mathbb{X}$.

(3.) $\mathbb{U} = \mathcal{F}_\ell(\mathbb{T},\widetilde{\mathbb{U}})$*, where*

$$\mathbb{T} := \begin{bmatrix}I & 0\\ \mathbb{X}_{21} & \mathbb{X}_{22}\end{bmatrix}\begin{bmatrix}\mathbb{X}_{11} & \mathbb{X}_{12}\\ 0 & I\end{bmatrix}^{-1} = \begin{bmatrix}\mathbb{M}_{11} & \mathbb{M}_{12}\\ 0 & I\end{bmatrix}\begin{bmatrix}I & 0\\ \mathbb{M}_{21} & \mathbb{M}_{22}\end{bmatrix}^{-1}. \tag{11.88}$$

Moreover, we have the following:

$$\mathbb{U} = (\mathbb{M}_{11}\widetilde{\mathbb{U}} + \mathbb{M}_{12})(\mathbb{M}_{21}\widetilde{\mathbb{U}} + \mathbb{M}_{22})^{-1} = (\mathbb{X}_{11} - \widetilde{\mathbb{U}}\mathbb{X}_{21})^{-1}(-\mathbb{X}_{12} + \widetilde{\mathbb{U}}\mathbb{X}_{22}), \tag{11.89}$$

$$\widetilde{\mathbb{U}} = (\mathbb{M}_{11} - \mathbb{U}\mathbb{M}_{21})^{-1}(-\mathbb{M}_{12} + \mathbb{U}\mathbb{M}_{22}) = (\mathbb{X}_{11}\mathbb{U} + \mathbb{X}_{12})(\mathbb{X}_{21}\mathbb{U} + \mathbb{X}_{22})^{-1}, \tag{11.90}$$

$$\widetilde{\mathbb{U}}_2^{-1} := (\mathbb{X}_{21}\mathbb{U} + \mathbb{X}_{22})^{-1} = \mathbb{U}_2 = \mathbb{M}_{21}\widetilde{\mathbb{U}} + \mathbb{M}_{22} \in \mathcal{G}\mathrm{TIC}, \tag{11.91}$$

$$\mathbb{Q}_2^{-1} := (\mathbb{X}_{11} - \widetilde{\mathbb{U}}\mathbb{X}_{21})^{-1} = \mathbb{M}_{11} - \mathbb{U}\mathbb{M}_{21} \in \mathcal{G}\mathrm{TIC}, \tag{11.92}$$

$$\mathbb{U} = \begin{bmatrix}I & 0\end{bmatrix}\begin{bmatrix}\mathbb{X}_{11} & \mathbb{X}_{12}\\ 0 & I\end{bmatrix}^{-1}\begin{bmatrix}\widetilde{\mathbb{U}}\mathbb{U}_2^{-1}\\ I\end{bmatrix}. \tag{11.93}$$

If $\mathbb{D} \in \widetilde{\mathcal{A}}$*, then* $\mathbb{X} \in \widetilde{\mathcal{A}}$*, hence then* $\mathbb{U} \in \widetilde{\mathcal{A}} \Leftrightarrow \widetilde{\mathbb{U}} \in \widetilde{\mathcal{A}}$ *(this also holds with* $\mathrm{TIC}_{\mathrm{exp}}$ *in place of* $\widetilde{\mathcal{A}}$*).*

As shown in [S98d], much (but not all) of Proposition 11.3.4 also holds for the above more general “extended” H$^\infty$ FICP (“eFICP”) in the stable case.

We shall meet the above more general “extended” H$^\infty$ FICP (“eFICP”) again in connection with the H$^\infty$ four-block problem (this corresponds to the (dual of) (Factor2Z) part of Theorem 12.3.7). However, we shall then reduce that problem to a FICP (see Lemma 12.4.8). It seems that this is the easiest way to show that the hypotheses of eFICP theory are satisfied, hence we have no use for a direct eFICP theory and will not treat it further.

Proof of Theorem 11.3.6: (The theorem and this proof hold even without Standing Hypotheses 11.1.1 and 11.2.1.)

We prove the parametrization of all controllers and obtain the other formulae on the way.

1° *Sufficiency:* Let $\widetilde{\mathbb{U}}, \mathbb{U}_1, \mathbb{U}_2, \mathbb{U}$ be as in the statement of the theorem. The $(2,1)$-block of equation $\mathbb{M}\mathbb{X} = I$ implies that $\mathbb{M}_{21}\mathbb{X}_{11} + \mathbb{M}_{22}\mathbb{X}_{21} = 0$, i.e., that $\mathbb{M}_{22}^{-1}\mathbb{M}_{21} = -\mathbb{X}_{21}\mathbb{X}_{11}^{-1}$ (we have $\mathbb{M}_{22} \in \mathcal{G}\mathrm{TIC}$, by Lemma A.1.1(c1)).

But $\|\mathbb{X}_{21}\mathbb{X}_{11}^{-1}\| < 1$, by Lemma 11.4.3(c), hence $\|\mathbb{M}_{22}^{-1}\mathbb{M}_{21}\widetilde{\mathbb{U}}\| \le \|\mathbb{M}_{22}^{-1}\mathbb{M}_{21}\| < 1$, hence $\mathbb{M}_{22}^{-1}\mathbb{M}_{21}\widetilde{\mathbb{U}} + I \in \mathcal{G}\mathrm{TIC}(W)$, equivalently, $\mathcal{G}\mathrm{TIC}(W) \ni \mathbb{M}_{21}\widetilde{\mathbb{U}} + \mathbb{M}_{22} =: \mathbb{U}_2$.

From $\begin{bmatrix} 0 & I \end{bmatrix} \mathbb{X} \begin{bmatrix} \mathbb{U}_1 \\ \mathbb{U}_2 \end{bmatrix} = \begin{bmatrix} 0 & I \end{bmatrix} \mathbb{X}\mathbb{M} \begin{bmatrix} \widetilde{\mathbb{U}} \\ I \end{bmatrix}$ we obtain that $I = \mathbb{X}_{21}\mathbb{U}_1 + \mathbb{X}_{22}\mathbb{U}_2 = (\mathbb{X}_{21}\mathbb{U} + \mathbb{X}_{22})\mathbb{U}_2$, hence (11.91) holds.

Now $\mathbb{D}_1\mathbb{U} + \mathbb{D}_2 = \mathbb{D}\begin{bmatrix} \mathbb{U} \\ I \end{bmatrix} = \mathbb{D}\begin{bmatrix} \mathbb{U}_1 \\ \mathbb{U}_2 \end{bmatrix}\mathbb{U}_2^{-1} = \mathbb{D}\mathbb{M}\begin{bmatrix} \widetilde{\mathbb{U}} \\ I \end{bmatrix}\mathbb{U}_2^{-1}$, hence

$$(\mathbb{D}_1\mathbb{U} + \mathbb{D}_2)^* J(\mathbb{D}_1\mathbb{U} + \mathbb{D}_2) = \mathbb{U}_2^{-*}\begin{bmatrix} \widetilde{\mathbb{U}} \\ I \end{bmatrix}^* J_1 \begin{bmatrix} \widetilde{\mathbb{U}} \\ I \end{bmatrix}\mathbb{U}_2^{-1} = \mathbb{U}_2^{-*}[\widetilde{\mathbb{U}}^*\widetilde{\mathbb{U}} - I]\mathbb{U}_2^{-1}$$

is $\ll 0$ [≤ 0] iff $\|\widetilde{\mathbb{U}}\| < 1$ [≤ 1] (see Lemma A.3.1(b)&(d)).

2° *Rest of the formulae:* Assume (11.87).

(3.) The formula $\mathbb{U} = \mathcal{F}_\ell(\mathbb{T}, \widetilde{\mathbb{U}})$ can now be verified by a direct computation using the formula $\mathbb{M}\mathbb{X} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$. We note from Lemma A.3.1(d1) that

$$\mathbb{T} = \begin{bmatrix} \mathbb{X}_{11}^{-1} & -\mathbb{X}_{11}^{-1}\mathbb{X}_{12} \\ \mathbb{X}_{21}\mathbb{X}_{11}^{-1} & \mathbb{X}_{22} - \mathbb{X}_{21}\mathbb{X}_{11}^{-1}\mathbb{X}_{12} \end{bmatrix} = \begin{bmatrix} \mathbb{M}_{11} - \mathbb{M}_{12}\mathbb{M}_{22}^{-1}\mathbb{M}_{21} & \mathbb{M}_{12}\mathbb{M}_{22}^{-1} \\ -\mathbb{M}_{22}^{-1}\mathbb{M}_{21} & \mathbb{M}_{22}^{-1} \end{bmatrix}; \tag{11.94}$$

in particular, $\mathbb{T}_{11}, \mathbb{T}_{22} \in \mathcal{G}\mathrm{TIC}$.

(2.) From $\begin{bmatrix} I & -\widetilde{\mathbb{U}} \end{bmatrix} \mathbb{X}\mathbb{M} \begin{bmatrix} \widetilde{\mathbb{U}} \\ I \end{bmatrix} = 0$ we get that $(\mathbb{X}_{11} - \widetilde{\mathbb{U}}\mathbb{X}_{21})^{-1}(\mathbb{X}_{12} - \widetilde{\mathbb{U}}\mathbb{X}_{22}) = -(\mathbb{M}_{11}\widetilde{\mathbb{U}} + \mathbb{M}_{12})(\mathbb{M}_{21}\widetilde{\mathbb{U}} + \mathbb{M}_{22})^{-1}$ (clearly $\mathbb{X}_{11} - \widetilde{\mathbb{U}}\mathbb{X}_{21} = (I - \widetilde{\mathbb{U}}\mathbb{X}_{21}\mathbb{X}_{11}^{-1})\mathbb{X}_{11} \in \mathcal{G}\mathrm{TIC}$), which is equal to $-\mathbb{U}$. Formulation (11.89) follows from this and (11.87), and from (11.89) we get (2.). But (2.) implies that

$$\mathbb{Q}_2 \begin{bmatrix} I & -\mathbb{U} \end{bmatrix} \mathbb{M} = \begin{bmatrix} \mathbb{Q}_2 & -\mathbb{Q}_1 \end{bmatrix} \mathbb{M} = \begin{bmatrix} I & -\widetilde{\mathbb{U}} \end{bmatrix}, \tag{11.95}$$

hence $\mathbb{Q}_2(\mathbb{M}_{11} - \mathbb{U}\mathbb{M}_{21}) = I$, so that (11.92) holds. The formula (11.90) can be obtained in a similar way.

By applying $\mathbb{M} = \begin{bmatrix} \mathbb{X}_{11} & \mathbb{X}_{12} \\ 0 & I \end{bmatrix}^{-1} \begin{bmatrix} I & 0 \\ \mathbb{M}_{21} & \mathbb{M}_{22} \end{bmatrix}$ (from (A.9)) to $\mathbb{U} = \mathbb{U}_1\mathbb{U}_2^{-1}$ one obtains (11.93).

The claim "$\mathbb{U} \in \widetilde{\mathcal{A}} \Leftrightarrow \widetilde{\mathbb{U}} \in \widetilde{\mathcal{A}}$" follows from (11.89), (11.90), and Lemma 8.4.10.

3° *Necessity assuming that* $\mathbb{X}_{21}\mathbb{U} + \mathbb{X}_{22} \in \mathcal{G}\mathrm{TIC}$: Let $(\mathbb{D}_1\mathbb{U} + \mathbb{D}_2)^* J(\mathbb{D}_1\mathbb{U} + \mathbb{D}_2) \ll 0$ [≤ 0]. Define $\begin{bmatrix} \widetilde{\mathbb{U}}_1 \\ \widetilde{\mathbb{U}}_2 \end{bmatrix} := \mathbb{X}\begin{bmatrix} \mathbb{U} \\ I \end{bmatrix}$, so that $\widetilde{\mathbb{U}}_2 = \mathbb{X}_{21}\mathbb{U} + \mathbb{X}_{22}$. Assume that $\widetilde{\mathbb{U}}_2 \in \mathcal{G}\mathrm{TIC}$.

Then

$$\widetilde{\mathbb{U}}_1^*\widetilde{\mathbb{U}}_1 - \widetilde{\mathbb{U}}_2^*\widetilde{\mathbb{U}}_2 = \begin{bmatrix} \widetilde{\mathbb{U}}_1 \\ \widetilde{\mathbb{U}}_2 \end{bmatrix}^* J_1 \begin{bmatrix} \widetilde{\mathbb{U}}_1 \\ \widetilde{\mathbb{U}}_2 \end{bmatrix} = \begin{bmatrix} \mathbb{U} \\ I \end{bmatrix}^* \mathbb{X}^* J_1 \mathbb{X} \begin{bmatrix} \mathbb{U} \\ I \end{bmatrix} \tag{11.96}$$

$$= \begin{bmatrix} \mathbb{U} \\ I \end{bmatrix}^* \mathbb{D}^* J \mathbb{D} \begin{bmatrix} \mathbb{U} \\ I \end{bmatrix} = (\mathbb{D}_1\mathbb{U} + \mathbb{D}_2)^* J(\mathbb{D}_1\mathbb{U} + \mathbb{D}_2) \ll 0 \ \ [\le 0], \tag{11.97}$$

hence the norm of $\widetilde{\mathbb{U}} := \widetilde{\mathbb{U}}_1\widetilde{\mathbb{U}}_2^{-1}$ (a r.c.f.) is < 1 [≤ 1].

Moreover, $\begin{bmatrix} \mathbb{U}_1 \\ \mathbb{U}_2 \end{bmatrix} := \mathbb{M}\begin{bmatrix} \widetilde{\mathbb{U}} \\ I \end{bmatrix} = \mathbb{M}\begin{bmatrix} \widetilde{\mathbb{U}}_1 \\ \widetilde{\mathbb{U}}_2 \end{bmatrix}\widetilde{\mathbb{U}}_2^{-1} = \begin{bmatrix} \mathbb{U} \\ I \end{bmatrix}\widetilde{\mathbb{U}}_2^{-1}$, hence $\mathbb{U}_1\mathbb{U}_2^{-1} = \mathbb{U}$, so $\mathbb{U}$ is of the form claimed above.

4° *Necessity completed:* Let Υ be the set of all solutions $\mathbb{U} \in \mathrm{TIC}(W,U)$ for which $(w \mapsto \mathcal{J}(0,\mathbb{U}w,w)) \le 0$, and Υ_0 those of $\mathbb{U} \in \Upsilon$ for which $\mathbb{X}_{21}\mathbb{U}+\mathbb{X}_{22} \in \mathcal{G}\mathrm{TIC}$ (as in 3°). We shall show that $\Upsilon_0 = \Upsilon$; this together with 3° implies the necessity of (11.87).

Let $\mathbb{U} \in \Upsilon$ be arbitrary. Set $\mathbb{U}_0 := \mathbb{M}_{12}\mathbb{M}_{22}^{-1} \in \Upsilon_0$, $s := \inf[0,1] \setminus E_0$, where $E_0 := \{t \in [0,1] \,|\, \mathbb{U}_t := \mathbb{U}_0 + t(\mathbb{U} - \mathbb{U}_0) \in \Upsilon_0\}$. Assuming $s < \infty$, we will derive a contradiction, thus showing that $E_0 = [0,1]$, hence $(\mathbb{U} =)\mathbb{U}_1 \in \Upsilon_0$.

If $\mathcal{J}(0,u,w) \le 0$ and $\mathcal{J}(0,\widetilde{u},w) \le 0$, and $f(r) := \mathcal{J}(0,u+r(\widetilde{u}-u),w)$, then $f''(r) = 2\langle \widetilde{u}-u, (\mathbb{D}_1^* J \mathbb{D}_1)\widetilde{u}-u\rangle \ge 0$ for all $r \in [0,1]$, hence f has no maximum on $(0,1)$, so that $f \le 0$. This convexity leads us to conclude that $\mathbb{U}_t \in \Upsilon$ for all $t \in [0,1]$.

For each $t \in E_0$, define $\widetilde{\mathbb{U}}_t$ as in 3°, so that $\|\widetilde{\mathbb{U}}_t\| \le 1$. We have $(\mathbb{X}_{21}\mathbb{U}_t + \mathbb{X}_{22})^{-1} = \mathbb{M}_{21}\widetilde{\mathbb{U}}_t + \mathbb{M}_{22} \in \mathcal{G}\mathrm{TIC}(W)$ for all $t \in E_0$, by 1°, hence for all $t < s$. Since $\|\mathbb{M}_{21}\widetilde{\mathbb{U}}_t + \mathbb{M}_{22}\| \le \|\mathbb{M}_{21}\| + \|\mathbb{M}_{22}\|$ for all $t \in E_0$, we have $\mathbb{X}_{21}\mathbb{U}_t + \mathbb{X}_{22} \in \mathcal{G}\mathrm{TIC}(W)$ for $t = s$ too, by Lemma A.3.3(A3). On the other hand, $\mathcal{G}\mathrm{TIC}$ is open, hence s cannot be the infimum of E_0, QED. □

Minimax J-coercivity does not guarantee that the J-critical (i.e., minimax) control would be given by a state feedback controller:

Example 11.3.7 (Minimax J-coercive $\not\Rightarrow \exists \left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$, $\exists \mathbf{X}^* S \mathbf{X}$)

(a) **($\nexists \left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$, $\nexists$ CARE)** Let $\gamma > 0$. There is a strongly stable system $\Sigma = \left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D} \end{array}\right] \in \mathrm{WPLS}(\mathbf{C}^2 \times \mathbf{C}^2, \mathrm{L}^2(\mathbf{R}_+;\mathbf{C}^6), \mathbf{C}^4 \times \mathbf{C}^2)$ s.t. $\mathbb{D}_{11}^*\mathbb{D}_{11} \gg 0$, $\mathbb{D} = \left[\begin{smallmatrix} * & * \\ 0 & I \end{smallmatrix}\right]$, $\gamma > \gamma_0$ (i.e., $\mathbb{D}$ is minimax J_γ-coercive, hence J_γ-coercive) but the unique J_γ-critical ("minimax") control for Σ over $\mathcal{U}_{\mathrm{out}} = \mathcal{U}_{\mathrm{sta}} = \mathcal{U}_{\mathrm{str}}$ is not of (well-posed) state feedback form; it is ill-posed in both open-loop and closed-loop forms.

Condition (FI1s) of Theorem 11.3.3 holds but (FI3s)–(FI5s) do not; in fact, $\mathbb{D}^* J_\gamma \mathbb{D}$ does not have a spectral factorization. Analogously, condition (FI1) of Theorem 11.2.7 holds but (FI3)–(FI5) do not (this corresponds to case $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{out}} = \mathcal{U}_{\mathrm{sta}} = \mathcal{U}_{\mathrm{str}}$).

(b) **(Unstable $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$)** Let $\gamma > 0$. There is a strongly stable UHPR system $\Sigma = \left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D} \end{array}\right] \in \mathrm{WPLS}(\mathbf{C}^2 \times \mathbf{C}^2, \mathrm{L}^2(\mathbf{R}_+;\mathbf{C}^6), \mathbf{C}^4 \times \mathbf{C}^2)$ s.t. $\mathbb{D}_{11}^*\mathbb{D}_{11} \gg 0$, $\mathbb{D} = \left[\begin{smallmatrix} * & * \\ 0 & I \end{smallmatrix}\right]$, $\gamma > \gamma_0$ (i.e., $\mathbb{D}$ is minimax J_γ-coercive, hence J_γ-coercive) but the unique J_γ-critical control for Σ over $\mathcal{U}_{\mathrm{out}} = \mathcal{U}_{\mathrm{sta}} = \mathcal{U}_{\mathrm{str}}$ corresponds to an unstable state feedback pair $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$, whose closed-loop form $\left[\begin{array}{c|c} \mathbb{K}_\circlearrowleft & \mathbb{F}_\circlearrowleft \end{array}\right]$ is also unstable, since $\widehat{\mathbb{X}}, \widehat{\mathbb{X}}^{-1}$ are unbounded at $\pm i$.

As in (a), conditions (FI1) and (FI1s) hold but (FI3)–(FI5) and (FI3s)–(FI5s) do not, since $\mathbb{D}^* J_\gamma \mathbb{D}$ does not have a spectral factorization. However, in this case the CARE and the IARE have a (unique and UHPR) $\mathcal{U}_{\mathrm{out}}$-stabilizing solution $\mathcal{P} \ge 0$ (which is neither stable nor SOS-stabilizing).

Nevertheless, $\widehat{\mathbb{D}}$ and $\widehat{\mathbb{X}}^{\pm 1}$ are holomorhic at infinity, hence uniformly half-plane-regular.

(c) **($\exists \left[\begin{array}{c|c} \mathbb{K} & \mathbb{F}\end{array}\right]$, $\nexists$ CARE, although $\mathbb{D} \in$ ULR)** Let $\gamma > 0$. There is a strongly stable ULR system $\Sigma = \left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D}\end{array}\right] \in \mathrm{WPLS}(\mathbf{C}\times\mathbf{C}, \mathrm{L}^2(\mathbf{R}_+;\mathbf{C}^2), \mathbf{C}\times\mathbf{C})$ s.t. $\mathbb{D}_{11}^*\mathbb{D}_{11} \gg 0$, $\mathbb{D} = \left[\begin{smallmatrix} \mathbb{D}_{11} & 0 \\ 0 & I\end{smallmatrix}\right]$, $\gamma > \gamma_{\mathrm{FI}}$ (hence, $\mathbb{D}$ is minimax J_γ-coercive, hence J_γ-coercive).

Moreover, conditions (FI1s)–(FI4s) of Theorem 11.3.3 hold but (FI5s) does not, since the spectral factor of $\mathbb{D}^* J_\gamma \mathbb{D}$ is not WR; in particular, the CARE does not have a stabilizing solution. Analogously, conditions (FI1)–(FI4) of Theorem 11.2.7 hold but (FI5) does not. (This corresponds to case $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{out}} = \mathcal{U}_{\mathrm{sta}} = \mathcal{U}_{\mathrm{str}}$.)

$\triangleleft$

(Note that Σ satisfies (Standing) Hypotheses 11.1.1, 11.2.1 and 11.3.1 (for $\mathcal{U}_{\mathrm{out}}$, $\mathcal{U}_{\mathrm{sta}}$ and $\mathcal{U}_{\mathrm{str}}$).)

The anomalies in (a) and (c) cannot happen in discrete time, since in discrete time a unique J_γ-critical control is always of state feedback form and corresponds to a DARE, by Theorem 14.1.6 (recall that discrete-time maps are always "regular" due to bounded input and output operators). However, the anomaly in (b) happens in discrete time too, mutatis mutandis (with $\widehat{\mathbb{X}}, \widehat{\mathbb{X}}^{-1} \in \mathrm{H}^2(\mathbf{D};\mathcal{B}) \setminus \mathrm{H}^\infty(\mathbf{D};\mathcal{B})$) unless, e.g., $\widehat{\mathbb{D}}$ is exponentially stable.

In the above examples, we have $\mathcal{U}_*^* \in \{\mathcal{U}_{\mathrm{out}}, \mathcal{U}_{\mathrm{sta}}, \mathcal{U}_{\mathrm{str}}\}$. An expert on the area considers it almost sure that the I/O map of Example 8.4.13 can be modified so that it is exponentially stable (so that we can let Σ be an exponentially stable realization of $\mathbb{D}$); if this is the case, then the anomalies of (a) and (c) (but not that in (b)) also exist in case $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{exp}}$. As mentioned above with $\mathcal{U}_{\mathrm{exp}}$ in discrete time (but the Cayley transform of $\mathrm{TIC}_{\mathrm{exp}}$ covers much more than $\mathrm{tic}_{\mathrm{exp}}$).

Proof of Example 11.3.7: (a) (We assume here that $\gamma = \sqrt{2}$ as in Example 8.4.13. For general $\gamma > 0$, one has to replace $\mathbb{D}$ by $\begin{bmatrix} (\gamma/\sqrt{2})I & 0 \\ 0 & I\end{bmatrix}\mathbb{D}$ and $\mathbb{X}$ by $(\gamma/\sqrt{2})\mathbb{X}$.)

Let $\heartsuit\mathbb{D} := \mathbb{D}_0$ and $J_\gamma := \widetilde{J} = \begin{bmatrix} I & 0 \\ 0 & -\gamma^2 I\end{bmatrix}$, where $\mathbb{D}_0$ and $\widetilde{J}$ are the maps from of Example 8.4.13(b), so that $\mathbb{D} \in \mathrm{TIC}(\mathbf{C}^2\times\mathbf{C}^2, \mathbf{C}^4\times\mathbf{C}^2)$ is minimax J_γ-coercive (equivalently, $\gamma > \gamma_0$, by Lemma 11.3.10), and $\widehat{\heartsuit}\widehat{\mathbb{D}}$ has a $\mathcal{G}\mathrm{H}^2$-factorization (see Definition 9.15.1) $(\widehat{\heartsuit}\widehat{\mathbb{X}})^* J_1 (\widehat{\heartsuit}\widehat{\mathbb{X}})$ s.t. $\widehat{\mathbb{X}}^{\pm 1} \in \mathrm{H}(\mathbf{C}^+;\mathcal{B}(U)) \setminus \mathrm{H}^\infty(\mathbf{C}^+_\omega;\mathcal{B}(U))$ for all $\omega \in \mathbf{R}$, as noted in Example 8.4.13.

Let Σ be the strongly stable (shift) realization (13.46) of $\mathbb{D}$. Due to minimax J_γ-coercivity, there is a unique J_γ-critical (i.e., minimax) input for each $x_0 \in H$ over $\mathcal{U}_{\mathrm{out}}$. By Lemma 8.3.3, $\mathcal{U}_{\mathrm{str}} = \mathcal{U}_{\mathrm{sta}} = \mathcal{U}_{\mathrm{out}}$.

If $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F}\end{array}\right]$ is any J_γ-critical state feedback pair for Σ, then $I - \mathbb{F} = E\mathbb{X}$ for some $E \in \mathcal{G}\mathcal{B}(U)$), by Lemma 9.15.4 (and Lemma 9.15.2); but then $\mathbb{F}, \mathbb{F}_{\circlearrowleft} \notin \mathrm{TIC}_\infty$, i.e., the corresponding "controller" is non-well-posed in both its open-loop and closed-loop forms!

By Corollary 9.9.11, $\mathbb{D}^* J_\gamma \mathbb{D}$ does not have a spectral factorization (although it is J_γ-coercive), since otherwise the J_γ-critical control could be given state feedback form. We do not know whether (FI2s) (equivalently, (FI2)) holds.

However, by Theorem 8.3.9 (see also Section 9.7; Remark 9.7.7 in particular), this J_γ-critical control can be written in WPLS form, i.e., as non-well-posed state feedback (where there are no well-posed maps between an external (closed-loop) input "$u_\circlearrowleft$" and the internal (open-loop) control "u").

(By Theorem 11.4.11(i) of [Sbook], one can apply Cayley transform to Σ to obtain a strongly stable wpls "$\heartsuit\Sigma$" (whose semigroup is a contraction). The above $\mathcal{G}\mathrm{H}^2$-factorization defines the unique (modulo E) J_γ-critical (well-posed and admissible) state feedback pair "$\heartsuit \left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$" for $\heartsuit\Sigma$, but, as noted above, its continuous-time equivalent is not well-posed.)

(b) The proof of (a) applies mutatis mutandis, just use (c) instead of (b) of Example 8.4.13.

(c) 1° *The proof:* (We shall assume that $\gamma = 1$. For general $\gamma > 0$, one has to replace $\mathbb{D}$ by $\left[\begin{smallmatrix} \gamma I & 0 \\ 0 & I \end{smallmatrix}\right]\mathbb{D}$ and $\mathbb{X}$ by $\gamma\mathbb{X}$.)

Let $\mathbb{D}_{11}$ (resp. $\mathbb{X}_{11}$) be the element $\mathbb{D}$ (resp. $\mathbb{X}_{11}$) of Proposition 9.13.1(c1). Then $\mathbb{X} := \left[\begin{smallmatrix} \mathbb{X}_{11} & 0 \\ 0 & I \end{smallmatrix}\right]$ satisfies $\mathbb{X}^* J_1 \mathbb{X} = \mathbb{D}^* J_\gamma \mathbb{D}$ and $\mathbb{X}, \mathbb{X}_{11} \in \mathcal{G}\mathrm{TIC}$ (in particular, (FI3s) holds), but $\mathbb{D} \in \mathrm{ULR}$ (since $\mathbb{D}_{11} \in \mathrm{ULR}$) and $\mathbb{X}^{\pm 1} \notin \mathrm{WR}$.

Let Σ be the shift realization (6.11). Then (c) is satisfied, by Proposition 11.3.4 (indeed, by Theorem 9.9.1(c)&(e2)&(f1), the eIARE has a unique output-stabilizing solution and this solution is not WR (because $\mathbb{F} = I - \mathbb{X}$ is not WR), hence it is not a solution of the [e]CARE).

2° *Remark:* If the reader is not happy with the fact that $\mathbb{D}_{12} \neq 0$ (which means that $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right] = \left[\begin{array}{c|c} 0 & 0 \end{array}\right]$ is a suboptimal H^∞-FI-pair, so that the FICP is trivial), (s)he may take first some rational (or $\mathrm{MTIC}^{\mathrm{L}^1}$) $\widehat{\mathbb{D}}_0$ s.t. $\mathbb{D}_0$ is minimax J_γ-coercive, so that $\mathbb{X}_0^* J_1 \mathbb{X}_0 = \mathbb{D}_0^* J_\gamma \mathbb{D}_0$ for some rational (or $\mathrm{MTIC}^{\mathrm{L}^1}$) $\widehat{\mathbb{X}}_0$, and then use $\begin{bmatrix} \mathbb{D}_{11} & 0 \\ 0 & \mathbb{D}_0 \end{bmatrix}$ in place of $\mathbb{D}$, so that $\mathbb{X}$ is replaced by $\begin{bmatrix} \mathbb{X}_{11} & 0 \\ 0 & \mathbb{X}_0 \end{bmatrix}$ (when J_γ is replaced by $\begin{bmatrix} I & 0 \\ 0 & J_\gamma \end{bmatrix}$); here $\mathbb{D}_{11}$ refers to the original $\mathbb{D}_{11}$, so that the dimension of Z is increased by one.

This way the FICP becomes nontrivial but (c) is unchanged except for the dimensions and the fact that $\mathbb{D}_{12} \neq 0$. □

The main part of this section ends here; the rest consists on auxiliary lemmas that were used above; some of the lemmas also have further use in Chapter 12.

From a solution of the CARE we obtain a suboptimal pair or operator as follows:

Lemma 11.3.8 (CARE ⇒ H^∞-FI-pair) *Assume that the CARE has a UR stable, P-SOS-stabilizing solution $(\mathcal{P}, S, K)$ s.t. $\mathcal{P} \geq 0$, $S_{11} \gg 0$ and $S_{22} - S_{21} S_{11}^{-1} S_{12} \ll 0$ (resp. and $S_{22} \ll 0$).*

Then (11.39) generate a UR suboptimal H^∞-FI-pair (resp. $K_1 =$(11.40) is a UR suboptimal H^∞-SF-operator), which is stable and r.c.-SOS-stabilizing.

(By Proposition 9.8.11(iii)&(ii), a stable, P-SOS-stabilizing solution of the CARE is $\mathcal{U}_*^*$-stabilizing, hence $\mathcal{P}$, S and K are unique.)

Proof: By Proposition 9.8.10, the assumptions of Lemma 11.3.9(a) (resp. and (b)) are satisfied for $s_0 = +\infty$. □

Lemma 11.3.8 can be extended to cover the IARE (thus we need not assume any regularity):

Lemma 11.3.9 (IARE $\Rightarrow$ H^∞-FI-pair) *Assume that the IARE has a stable, P-SOS-stabilizing solution $(\mathcal{P}, S, \left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right])$ s.t. $\mathcal{P} \geq 0$. Then*

(a) If $S_{22} \ll 0$ and any of (1.)–(4.) of Lemma 11.2.14 holds (for $\alpha = 0$), then $\left[\begin{array}{c|cc} \mathbb{K}_1 & \mathbb{F}_{11} & \mathbb{F}_{12} \\ 0 & 0 & 0 \end{array}\right]$ is a stable and r.c.-SOS-stabilizing suboptimal H^∞-FI-pair, and $\mathbb{X}_{11} \in \mathcal{G}\mathrm{TIC}(U)$.

(b) If $\widetilde{S} := (\widehat{\mathbb{X}}^* S \widehat{\mathbb{X}})(s_0)$ safisfies $\widetilde{S}_{11} \gg 0$ and $\widetilde{S}_{22} - \widetilde{S}_{21}\widetilde{S}_{11}^{-1}\widetilde{S}_{12} \ll 0$ for some (equivalently, all) $s_0 \in \mathbf{C}^+$ ($s_0 = +\infty$ can be allowed for $\mathbb{X} \in \mathrm{UR}$), then (11.48) is a stable and r.c.-SOS-stabilizing suboptimal H^∞-FI-pair.

Here, as elsewhere, $\mathbb{X} := I - \mathbb{F}$.

Proof: (By Theorem 9.8.12(s4) and Proposition 9.8.11(iii)&(ii), $\widetilde{S}$ (and $\mathcal{P}$) is independent on the choice of a stable, P-SOS-stabilizing solution, and such a solution is $\mathcal{U}_*^*$-stabilizing and r.c.-SOS-stabilizing.)

(a) By Lemma 11.3.15, $\left[\begin{array}{c|c} \overline{\mathbb{K}} & \overline{\mathbb{F}} \end{array}\right] := \left[\begin{array}{c|cc} \mathbb{K}_1 & \mathbb{F}_{11} & \mathbb{F}_{12} \\ 0 & 0 & 0 \end{array}\right]$ is a stable and r.c.-SOS-stabilizing H^∞-FI-pair. By Proposition 9.8.11(iii)&(ii), $\mathcal{P}$ is $\mathcal{U}_{\mathrm{out}}$-stabilizing. By Lemma 11.2.14(a), $\left[\begin{array}{c|c} \overline{\mathbb{K}} & \overline{\mathbb{F}} \end{array}\right]$ is suboptimal (recall that $\mathbb{M} = \mathbb{F}_{\circlearrowleft} + I \in \mathrm{TIC}$) and $\mathbb{M}_{22}^{-1} \in \mathrm{TIC}(W)$; since $\vartheta = 0$ and $\mathbb{F} \in \mathrm{TIC}$, we can take $\alpha = 0$. By Lemma A.1.1(c1), it follows that $\mathbb{X}_{11} \in \mathcal{G}\mathrm{TIC}(U)$.

(b) 1° *"Some $s_0 \in \mathbf{C}^+$" suffices:* This follows from Lemma 11.2.14(b)&(a) (see the proof of (a) above).

2° *"Equivalently, all $s_0 \in \mathbf{C}^+$":* This follows from (11.59), since $\gamma > \gamma_{\mathrm{FI}} \geq \gamma_0$, and $\mathcal{P}$ is $\mathcal{U}_*^*$-stabilizing (and $\vartheta = 0$). □

Lemma 11.3.10 *The following are equivalent:*

(i) $\gamma > \gamma_0$.

(ii) $\mathbb{D}$ is minimax J_γ-coercive.

(iii) $\|\mathbb{D}_{11}\mathbb{U} + \mathbb{D}_{12}\|_{\mathcal{B}(\mathrm{L}^2(\mathbf{R}_+;W),\mathrm{L}^2(\mathbf{R}_+;Z))} < \gamma$ for some $\mathbb{U} \in \mathcal{B}(\mathrm{L}^2(\mathbf{R}_+;W),\mathrm{L}^2(\mathbf{R}_+;U))$.

See Definition 11.4.1 for minimax J_γ-coercivity. The above result allows us to use this property in the proof of implication (FI1s)$\Rightarrow$(FI3s) (see Lemma 11.4.3(a)).

Proof: 1° *(i)$\Leftrightarrow$(ii):* Set $\mathbb{T} := \pi_+\mathbb{D}_1^* J_\gamma \mathbb{D}_1 \pi_+ \gg 0$ (on $\mathrm{L}^2(\mathbf{R}_+;U)$; see Standing Hypothesis 11.3.1). By Fréchet differentiation (or by completing the square or by applying (8.19) suitably), we see that $u_{\min} := -\mathbb{T}^{-1}\pi_+\mathbb{D}_1^* J_\gamma \mathbb{D}_2 \pi_+ w =: \mathbb{U}w$ minimizes $\mathcal{J}(0, \left[\begin{smallmatrix} \cdot \\ w \end{smallmatrix}\right])$, for any $w \in \mathrm{L}^2(\mathbf{R}_+;W)$ (we have added redundant π_+'s above and below to make the computations easier). Combine this with Lemma 11.2.4 to observe that $\gamma > \gamma_0$ iff

$$\min_{u \in \mathcal{U}_{\mathrm{u}}(0,w)} \mathcal{J}(0,u,w) = \langle \mathbb{D}\pi_+ \left[\begin{smallmatrix} \mathbb{U} \\ I \end{smallmatrix}\right] w, J_\gamma \mathbb{D}\pi_+ \left[\begin{smallmatrix} \mathbb{U} \\ I \end{smallmatrix}\right] w \rangle \le -\varepsilon \|w\|_2^2 \tag{11.98}$$

for some $\varepsilon > 0$, i.e., iff

$$0 \gg \begin{bmatrix} \mathbb{U} \\ I \end{bmatrix}^* \pi_+ \mathbb{D}^* J_\gamma \mathbb{D} \pi_+ \begin{bmatrix} \mathbb{U} \\ I \end{bmatrix} = \cdots = \pi_+ \mathbb{D}_2^* J_\gamma (I - P_1) \mathbb{D}_2 \pi_+ \tag{11.99}$$

(the equality follows by a straightforward computation (as in Lemma 2.6 of [S98d])), where $P_1 := \pi_+ \mathbb{D}_1 \mathbb{T}^{-1} \mathbb{D}_1^* J_\gamma \pi_+ = P_1^2$. But (11.99) holds iff $\mathbb{D}$ is minimax J_γ-coercive, by (11.106) and Definition 11.4.1.

2° *(i)⇔(iii):* For a given $\mathbb{U} \in \mathcal{B}$, condition (iii) holds iff $(\mathbb{D}_{11}\mathbb{U} + \mathbb{D}_{12})^*(\mathbb{D}_{11}\mathbb{U} + \mathbb{D}_{12}) \ll 0$ (by Lemma A.3.1(d)), i.e., iff (11.98) holds for this $\mathbb{U}$. This implies that (11.98) holds for the minimizing $\mathbb{U}$; take the minimizing $\mathbb{U}$ to obtain the converse. □

Lemma 11.3.11 ((FI3s) ⇒ $X_{21} = 0$) *Assume that (FI3s) is satisfied by some SR $\mathbb{X}$ with $X \in \mathcal{GB}(U \times W)$. Then we can choose $\mathbb{X}$ s.t. $X_{11}, X_{22} \in \mathcal{GB}$ and $X_{21} = 0$.*

Proof: By Proposition 11.3.4(f), (FI4s) has a solution $(\mathcal{P}, \widetilde{S}, \left[\begin{array}{c|c} \widetilde{\mathbb{K}} & \widetilde{\mathbb{F}} \end{array}\right])$ and (FI2s) holds. By Theorem 9.8.12(s1), we can have $\widetilde{F} = 0$. But then $\widetilde{S}_{11} \gg 0$ and $\widetilde{S}_{22} - \widetilde{S}_{21}\widetilde{S}_{11}^{-1}\widetilde{S}_{12} \ll 0$, by Proposition 11.2.19(d1) (it was remarked in the proof how we may use the IARE instead of the CARE).

By Lemma 11.3.13(i)&(iii'), we can make redefine $\left[\begin{array}{c|c} \widetilde{\mathbb{K}} & \widetilde{\mathbb{F}} \end{array}\right]$ s.t. $X_{11}, X_{22} \in \mathcal{GB}$, $X_{21} = 0$ and $S = J_1$, where $X := I - \widetilde{F}$. Set $\mathbb{X} := I - \widetilde{\mathbb{F}} \in \mathcal{G}\mathrm{TIC}(U \times W)$. By the proof of Proposition 11.3.4(f), we have $\mathbb{X}^* J_1 \mathbb{X} = \mathbb{D}^* J \mathbb{D}$. By Proposition 11.3.4(c1), $\mathbb{X}_{11} \in \mathcal{G}\mathrm{TIC}(U)$, hence also this new $\mathbb{X}$ satisfies (FI3s). □

Lemma 11.3.12 (SpF ⇒ $\mathbb{D}_{11}$ I-coercive) *(Drop Standing hypothesis 11.2.1 for the moment). If $\mathbb{D}^* J_\gamma \mathbb{D}$ has a spectral factorization, then $\mathbb{D}_{11}$ is I-coercive (as required in Hypotheses 11.2.1 and 11.2.1).*

Proof: Let $\mathbb{D}^* J_\gamma \mathbb{D} = \mathbb{X}^* S \mathbb{X}$, $\mathbb{X} \in \mathcal{G}\mathrm{TIC}(U \times W)$, $S \in \mathcal{GB}(U \times W)$. Set $\varepsilon := \|\mathbb{X}^{-1} S^{-1} \mathbb{X}^{-*} \mathbb{D}^* J_\gamma\|^{-1} > 0$. Then

$$\|v\|_2 = \|\mathbb{X}^{-1} S^{-1} \mathbb{X}^{-*} \mathbb{D}^* J_\gamma \mathbb{D} v\|_2 \le \varepsilon_+^{-1} \|\mathbb{D} v\|_2, \tag{11.100}$$

i.e., $\|\mathbb{D}v\|_2 \ge \varepsilon_+ \|v\|_2$, for all $v \in \mathrm{L}^2(\mathbf{R}_+; \mathcal{B}(U \times W))$. Consequently,, $\|\mathbb{D}_{11} u\|_2 = \|\mathbb{D}\left[\begin{smallmatrix} u \\ 0 \end{smallmatrix}\right]\|_2 \ge \varepsilon_+ \|u\|_2$ for all $u \in \mathrm{L}^2(\mathbf{R}_+; \mathcal{B}(U))$. □

We have already used the following lemma several times:

Lemma 11.3.13 ($\mathbf{S_{11} \gg 0}$ and $\mathbf{S_{22} - S_{21} S_{11}^{-1} S_{12} \ll 0}$) *Let $S = S^* \in \mathcal{B}(U \times W)$ and $\gamma > 0$. Then the following are equivalent:*

(i) $S_{11} \gg 0$ *and* $S_{22} - S_{21}S_{11}^{-1}S_{12} \ll 0$.

(i') $S_{11} \gg 0$ *and* $\begin{bmatrix} F \\ I \end{bmatrix}^* S \begin{bmatrix} F \\ I \end{bmatrix} \ll 0$ *for some* $F \in \mathcal{B}(W,U)$.

(ii) $\widetilde{S}_{11} \gg 0$ *and* $\widetilde{S}_{22} - \widetilde{S}_{21}\widetilde{S}_{11}^{-1}\widetilde{S}_{12} \ll 0$ *for some (hence all)* $F \in \mathcal{B}(W,U)$, *where* $\widetilde{S} := \begin{bmatrix} I & F \\ 0 & I \end{bmatrix}^* S \begin{bmatrix} I & F \\ 0 & I \end{bmatrix}$.

(ii') $S_{11} = \widetilde{S}_{11} \gg 0$ *and* $\widetilde{S}_{22} \ll 0$ *for some* $F \in \mathcal{B}(W,U)$, *where* $\widetilde{S} := \begin{bmatrix} I & F \\ 0 & I \end{bmatrix}^* S \begin{bmatrix} I & F \\ 0 & I \end{bmatrix}$.

(ii") $\widetilde{S}_{11} \gg 0$ *and* $\widetilde{S}_{22} - \widetilde{S}_{21}\widetilde{S}_{11}^{-1}\widetilde{S}_{12} \ll 0$ *for some (hence all)* $Z := \begin{bmatrix} Z_{11} & Z_{12} \\ 0 & Z_{22} \end{bmatrix} \in \mathcal{GB}(U \times W)$ *s.t.* $Z_{11} \in \mathcal{GB}(U)$, *where* $\widetilde{S} := Z^*SZ$.

(iii) $S = X^*J_\gamma X$ *for some* $X = \begin{bmatrix} X_{11} & X_{12} \\ 0 & X_{22} \end{bmatrix} \in \mathcal{GB}(U \times W)$ *s.t.* $X_{11} \in \mathcal{GB}$.

(iii') $S = \begin{bmatrix} X_{11} & X_{12} \\ 0 & X_{22} \end{bmatrix}^* J_\gamma \begin{bmatrix} X_{11} & X_{12} \\ 0 & X_{22} \end{bmatrix}$, *where* $X_{11} \gg 0$, $X_{22} \gg 0$.

(iv) $S = X^*J_\gamma X$ *for some* $X \in \mathcal{GB}(U \times W)$ *s.t.* $X_{11} \in \mathcal{GB}(U)$ *and* $\begin{bmatrix} H & I \end{bmatrix} X \begin{bmatrix} I \\ 0 \end{bmatrix} = 0$ *for some* $H \in \mathcal{B}(U,W)$ *s.t.* $\|H\| < \gamma^{-1}$.

(v) $S_{11} \gg 0$ *and* $S = X^*J_\gamma X$ *for some* $X \in \mathcal{GB}(U \times W)$ *s.t.* $X_{11} \in \mathcal{GB}(U)$.

(vi) $S = X^*J_\gamma X$, *where* $X, X_{11} \in \mathcal{GB}$ *and* $\|X_{21}X_{11}^{-1}\| < \gamma^{-1}$.

(vi') $M^*SM = J_\gamma$, *where* $M, M_{22} \in \mathcal{GB}$ *and* $\|M_{22}^{-1}M_{21}\| < \gamma^{-1}$.

(vii) $\widetilde{S}_{11} \gg 0$ *and* $\widetilde{S}_{22} - \widetilde{S}_{21}\widetilde{S}_{11}^{-1}\widetilde{S}_{12} \ll 0$, *where* $\widetilde{S} := \mathrm{diag}(I_{U'}, S, -I_{W'}) \in \mathcal{B}((U' \times U) \times (W \times W'))$, *for some (hence all) Hilbert spaces* W' *and* U'.

(viii) $S_{11} \gg 0$, $S \in \mathcal{GB}(U \times W)$ *and* $(S^{-1})_{22} \ll 0$.

Moreover,

(a) If $\dim U < \infty$, *then one more equivalent condition is that* $S_{11} \gg 0$ *and* $S = X^*J_\gamma X$ *for some* $X \in \mathcal{GB}$.

(b1) If (i) holds and $S = X^*J_\gamma X$ *for some* $X \in \mathcal{GB}(U \times W)$, *then* X *is as in (vi).*

(b2) If (i) holds, $S = X^*TX$, $X \in \mathcal{GB}(U \times W)$, $T_{11} \gg 0$ *and* $T_{22} \ll 0$, *then* $X_{11} \in \mathcal{GB}(U)$.

(b3) If (i) holds and $S = X^*J_\gamma X$ *for some* $X \in \mathcal{GB}(U \times W)$ *s.t.* $X_{21} = 0$, *then* $X_{11}, X_{22} \in \mathcal{GB}$.

(c) If (i) holds, then any $F \in \mathcal{B}(W,U)$ *is as in (ii).*

(d) If $S_{11} \geq \varepsilon^2 I$, $\varepsilon > 0$, $S_{22} - S_{21}S_{11}^{-1}S_{12} \ll 0$, $S = X^*J_\gamma X$ *and* $X \in \mathcal{GB}(U \times W)$, *then* $X_{11} \in \mathcal{GB}(U)$ *and* $\|X_{11}^{-1}\| \leq \varepsilon^{-1}$.

(e) If $S_{11} \gg 0$, $S_{22} \ll 0$, $X \in \mathcal{GB}(U \times W)$ *and* $S = X^*J_\gamma X$, *then* $X_{11}, X_{22} \in \mathcal{GB}$, $\|X_{21}X_{11}^{-1}\| < \gamma^{-1}$ *and* $\|M_{11}^{-1}M_{12}\| = \|X_{12}X_{22}^{-1}\| < \gamma$, *where* $M := X^{-1}$.

(f) If X *and* S *are as in (iii), then* $X_{22}^*X_{22} = \gamma^{-2}(S_{12}^*S_{11}^{-1}S_{12} - S_{22})$.

We note that "$S = X^*J_\gamma X$, $X, X_{11} \in \mathcal{GB}$" is not sufficient for (i)–(vi): take $X := \begin{bmatrix} \gamma I & I \\ I & 0 \end{bmatrix}$, so that $X, X_{11} \in \mathcal{GB}(U \times W)$ but $S_{11} := (X^*J_\gamma X)_{11} = 0 \not\gg 0$. By Example 11.2.16 (take $S = X^*J_1X$), condition "$\dim U < \infty$" is not superfluous in (a) and condition "$X_{11} \in \mathcal{GB}$" is not superfluous in (iii), nor in (v).

Proof: 1° *(iii')⇒(iii)⇒(i):* Obviously, (iii')⇒(iii). If (iii) holds, then $S_{11} = \widetilde{X}_{11}^* \widetilde{X}_{11} \gg 0$ and $S_{22} - S_{21} S_{11}^{-1} S_{12} = -\gamma^2 \widetilde{X}_{22}^* \widetilde{X}_{22} \ll 0$.

2° *(i)⇒(ii')&(iii'):* Assume (i). Set

$$X_{11} := S_{11}^{1/2} \gg 0,\; X_{12} := X_{11}^{-1} S_{12},\; X_{22} := \gamma^{-1} (S_{21} S_{11}^{-1} S_{12} - S_{22})^{1/2} \gg 0. \quad (11.101)$$

Then

$$S = \begin{bmatrix} I & S_{11}^{-1} S_{12} \\ 0 & I \end{bmatrix}^* \begin{bmatrix} S_{11} & 0 \\ 0 & S_{22}' \end{bmatrix} \begin{bmatrix} I & S_{11}^{-1} S_{12} \\ 0 & I \end{bmatrix} = X^* J_\gamma X, \quad (11.102)$$

where $S_{22}' := S_{22} - S_{21} S_{11}^{-1} S_{12} \ll 0$, $X = \left[\begin{smallmatrix} X_{11} & X_{12} \\ 0 & X_{22} \end{smallmatrix}\right] \in \mathcal{GB}(U \times W)$, $X_{11} \gg 0$, $X_{22} \gg 0$. Thus, (ii') and (iii') hold.

3° *(ii')⇔(i'):* Condition (i') is a reformulation of (ii'), because, obviously, $\widetilde{S}_{11} = S_{11}$ and $\widetilde{S}_{22} = \left[\begin{smallmatrix} F \\ I \end{smallmatrix}\right]^* S \left[\begin{smallmatrix} F \\ I \end{smallmatrix}\right]$ in (ii').

4° *(ii')⇒(ii):* Assume (ii'). Since $\widetilde{S}_{21} = \widetilde{S}_{12}^*$ and $\widetilde{S}_{11}^{-1} \gg 0$, we have $-\widetilde{S}_{21} \widetilde{S}_{11}^{-1} \widetilde{S}_{12} \le 0$, hence (ii) holds.

5° *(ii)⇒(iii):* Assume (ii). By 1°, we have $\widetilde{S} = \widetilde{X}^* J_\gamma \widetilde{X}$ for some $\widetilde{X} = \left[\begin{smallmatrix} \widetilde{X}_{11} & \widetilde{X}_{12} \\ 0 & \widetilde{X}_{22} \end{smallmatrix}\right] \in \mathcal{GB}$. Take $X := \widetilde{X} \left[\begin{smallmatrix} I & F \\ 0 & I \end{smallmatrix}\right]^{-1}$ to obtain (iii).

6° *(iii)⇔(iv):* Trivially (iii)⇒(iv) (take $H = 0$). Assume (iv). Set $X' := \left[\begin{smallmatrix} I & 0 \\ 0 & \gamma \end{smallmatrix}\right] X$, $H' := \gamma H$, so that $X'^* J_1 X' = S$, $\|H'\| < 1$ and $\begin{bmatrix} H' & I \end{bmatrix} X' \left[\begin{smallmatrix} I \\ 0 \end{smallmatrix}\right] = \gamma 0 = 0$. Apply Lemma 12.4.11 to obtain $Z' = \left[\begin{smallmatrix} Z_{11}' & Z_{12}' \\ 0 & Z_{22}' \end{smallmatrix}\right] \in \mathcal{GB}$ s.t. $S = Z'^* J_1 Z' = Z^* J_\gamma Z$, where $Z := \left[\begin{smallmatrix} I & 0 \\ 0 & \gamma \end{smallmatrix}\right] Z'$.

7° *(iii)⇒(vi)⇒(iv):* Trivially, (iii) implies (vi). If (vi) holds, then $H := -X_{21} X_{11}^{-1}$ satisfies (iv).

8° *(v)⇔(vi):* Since $S_{11} = X_{11}^* (I - \gamma^2 X_{11}^{-*} X_{21}^* X_{21} X_{11}^{-1}) X_{11}$, (v) is a reformulation of (vi), by Lemma A.3.1(e2).

9° *(vi)⇔(vi'):* Condition (vi') is a reformulation of (vi) (through $M = X^{-1}$, note that $X_{11} \in \mathcal{GB}(U) \Leftrightarrow M_{22} \in \mathcal{GB}(W)$, and that in either case we have $X_{21} X_{11}^{-1} = -M_{22}^{-1} M_{21}$, by Lemma A.3.1(c1)).

10° *(ii): "(hence all)"* Assume (ii) for some F. Let $S' := \left[\begin{smallmatrix} I & F' \\ 0 & I \end{smallmatrix}\right]^* S \left[\begin{smallmatrix} I & F' \\ 0 & I \end{smallmatrix}\right]$. Then $\widetilde{S} = \left[\begin{smallmatrix} I & F-F' \\ 0 & I \end{smallmatrix}\right]^* S' \left[\begin{smallmatrix} I & F-F' \\ 0 & I \end{smallmatrix}\right]$, hence $S_{11}' \gg 0$ and $S_{22}' - S_{21}' (S_{11}')^{-1} S_{12} \ll 0$, by "(i)⇔(ii)".

11° *(ii")⇔(iii) and "(hence all)"* By "(i)⇔(iii)", we have $\widetilde{S} = X^* J_1 X$ for some $X = \left[\begin{smallmatrix} X_{11} & X_{12} \\ 0 & X_{22} \end{smallmatrix}\right] \in \mathcal{GB}$ s.t. $X_{11} \in \mathcal{GB}$. Consequently, $S = (XZ^{-1})^* J_1 X Z^{-1}$, hence also S satisfies the condition in (iii) (since also XZ^{-1} is of the required form, by Lemma A.1.1(b)). The converse is analogous.

12° *(vii)⇔(i):* Obviously, $\widetilde{S}_{11} \gg 0 \Leftrightarrow S_{11} \gg 0$ (we have made the partition so that $\widetilde{S}_{11} \in \mathcal{B}(U' \times U)$). But

$$\widetilde{S}_{22}' := \widetilde{S}_{22} - \widetilde{S}_{21} \widetilde{S}_{11}^{-1} \widetilde{S}_{12} = \operatorname{diag}(S_{22}', -I_{W'}), \quad (11.103)$$

where $S_{22}' = S_{22} - S_{21} S_{11}^{-1} S_{12}$, hence $\widetilde{S}_{22}' \ll 0 \Leftrightarrow S_{22}' \ll 0$.

13° *(i)⇔(viii):* If (i) holds, then $S \in \mathcal{GB}$ and $(S^{-1})_{22}^{-1} = S_{22} - S_{21} S_{11}^{-1} S_{12} \ll 0$, by Lemma A.1.1(d1). If (viii) holds, then $(S_{22} - S_{21} S_{11}^{-1} S_{12})^{-1} \ll 0$, hence then $S_{22} - S_{21}S_{11}^{-1}S_{12} \ll 0$, by Lemma A.1.1(i) holds, by Lemma A.3.1(b1).

(a) By (i) and (iii), the condition of (a) is necessary (regardless of U). Conversely, if S and X are as in (a), then $0 \ll S_{11} = X_{11}^*X_{11} - X_{21}^*X_{21}$, hence $X_{11}^*X_{11} \gg 0$, hence $X_{11} \in \mathcal{GB}(U)$ (if $\dim U < \infty$), so that (v) holds.

(b1) By (b2), $X_{11} \in \mathcal{GB}(U)$. Since $0 \ll S_{11} = X_{11}^*X_{11} - \gamma^2 X_{21}^*X_{21}$, we obtain $\|X_{21}X_{11}^{-1}\| < \gamma^{-1}$ as in 8°.

(b2) (Remark: We cannot replace $T_{22} \ll 0$ by $T_{22} - T_{21}T_{11}^{-1}T_{12} = 1-2 \ll 0$; set $T = \begin{bmatrix} 2 & 2 \\ 2 & 1 \end{bmatrix}$, $X = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$ to obtain a counter-example. On the other hand, if $\dim U < \infty$ or $\dim W < \infty$, then $T_{22} \leq 0$ would suffice, by Lemma A.1.1(c1) (and the fact that $X_{11}^*X_{11} \gg 0$ as shown below).)

Apply Lemma A.3.1(q) to the (1,1)-block of $S = X^*TX$ to obtain that $X_{11}^*X_{11} \gg 0$.

By Lemma A.1.1(c1), $S, T \in \mathcal{GB}(U \times W)$ and $(S^{-1})_{22} = (S_{22} - S_{21}S_{11}^{-1}S_{12})^{-1}$, hence $(S^{-1})_{22} \ll 0$, by Lemma A.3.1(b1); similarly, $(T^{-1})_{11} = (T_{11} - T_{12}T_{22}^{-1}T_{21})^{-1} \gg 0$. Set $M := X^{-1}$ to obtain that $S^{-1} = MT^{-1}M^*$ and hence

$$0 \ll -(S^{-1})_{22} \leq \begin{bmatrix} 0 & I \end{bmatrix} MT^{-1}M^* \begin{bmatrix} 0 \\ I \end{bmatrix} = \begin{bmatrix} M_{21} & M_{22} \end{bmatrix} (-T^{-1}) \begin{bmatrix} M_{21}^* \\ M_{22}^* \end{bmatrix}. \tag{11.104}$$

By Lemma A.3.1(q) (interchange the rows and also the columns and recall that $-(T^{-1})_{11}) \ll 0$), we have $M_{22}M_{22}^* \gg 0$. By Lemma A.1.1(c2), this implies that also X_{11} is right-invertible, i.e., $X_{11}X_{11}^* \gg 0$, hence X_{11} is invertible.

(b3) By (b2), $X_{11} \in \mathcal{GB}(U)$, hence $X_{22} \in \mathcal{GB}(W)$, by Lemma A.1.1(b2).

(c) Now $\widetilde{S}_{11} = S_{11} \gg 0$ and $\widetilde{S} = E^*X^*J_\gamma XE$, where X is as in (v) and $E = \begin{bmatrix} I & F \\ 0 & I \end{bmatrix}$, so that $(XE)_{11} = X_{11} \in \mathcal{GB}(U)$, Therefore, also $\widetilde{S}$ satisfies (v), hence it satisfies (i) too (in the place of S), so that F is as in (ii).

(d) By (v) and (b1), $X_{11} \in \mathcal{GB}(U)$. Because $\varepsilon^2 I \leq S_{11} = X_{11}^*X_{11} - \gamma^2 X_{21}^*X_{21}$, we have $\|X_{11}^{-1}\| < \varepsilon$, by Lemma A.3.1(c1)(ii)&(1').

(e) By (ii') and (b), X is as in (vi). Assume, w.l.o.g., that $\gamma = 1$ (cf. 6°). By duality, $X_{22} \in \mathcal{GB}(W)$ and $\|X_{12}X_{22}^{-1}\| < 1$ (apply (b) to $-S_{\mathrm{d}} = X_{\mathrm{d}}^*J_1X_{\mathrm{d}}$, where $S_{\mathrm{d}} := \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} S \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$, $X_{\mathrm{d}} := \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} X \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$ (note that $J_1 = -\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} J_1 \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$)). From the $(1,2)$-block of $MX = I$ we obtain that $M_{11}^{-1}M_{12} = -X_{12}X_{22}^{-1}$.

(f) From $X^*J_\gamma X = S$ we obtain that $X_{11}^*X_{11} = S_{11}$,

$$X_{12} = X_{11}^{-*}S_{12}, \text{ and } \gamma^2 X_{22}^*X_{22} = X_{12}^*X_{12} - S_{22}, \tag{11.105}$$

from which we obtain (f). □

We have also used the following:

Lemma 11.3.14 *Let $X, S \in \mathcal{GB}(U \times W)$, $S = S^*$ and $(X^*SX)_{11} \gg 0$. Then $X^*SX = \widetilde{X}^*\widetilde{S}\widetilde{X}$ for some $\widetilde{X}, \widetilde{S} \in \mathcal{GB}(U \times W)$ s.t. $\widetilde{S} = \widetilde{S}^* = \begin{bmatrix} I & 0 \\ 0 & J_W \end{bmatrix}$, where $J_W = J_W^* = J_W^{-1} \in \mathcal{GB}(W)$.*

Proof: Set $T := X^*SX$. By (A.5), $T = E_a^*T'E_a$, where $T' := \begin{bmatrix} T_{11} & 0 \\ 0 & T_{22}' \end{bmatrix}$, $T_{22}' := T_{22} - T_{21}T_{11}^{-1}T_{12}$, $E_a := \begin{bmatrix} I & T_{11}^{-1}T_{12} \\ 0 & I \end{bmatrix}$. By Lemma 2.4.4, $T_{22}' = G^*J_WG$, where $J_W = J_W^* = J_W^{-1} \in \mathcal{GB}(W)$ and $G \in \mathcal{GB}(W)$. Thus, $T' = E_b^* \widetilde{S} E_b$, where $E_b := \begin{bmatrix} T_{11}^{-1/2} & 0 \\ 0 & G^{-1} \end{bmatrix}$. Consequently, $X^*SX = \widetilde{X}^*\widetilde{S}\widetilde{X}$, where $\widetilde{X} := E_b E_a X \in \mathcal{GB}(U \times W)$. □

Lemma 11.3.15 (H^∞-FI-pair) *Assume that* $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ *is a stable and SOS-stabilizing state-feedback pair for* Σ*, and* $I - \mathbb{F}_{11} \in \mathcal{G}\mathrm{TIC}(U)$*. Then* $\left[\begin{array}{c|c} \overline{\mathbb{K}} & \overline{\mathbb{F}} \end{array}\right] := \left[\begin{array}{c|cc} \mathbb{K}_1 & \mathbb{F}_{11} & \mathbb{F}_{12} \\ 0 & 0 & 0 \end{array}\right]$ *is a stable and r.c.-SOS-stabilizing* H^∞*-FI-pair.*

Proof: From (11.9) we observe that $\Sigma^{\curvearrowleft} \in$ SOS, i.e., that $\left[\begin{array}{c|c} \overline{\mathbb{K}} & \overline{\mathbb{F}} \end{array}\right]$ is SOS-stabilizing; since $\left[\begin{array}{c|c} \overline{\mathbb{K}} & \overline{\mathbb{F}} \end{array}\right]$ is also stable, it is a r.c.-SOS-stabilizing state feedback pair, by Lemma 6.6.17(b). By Lemma 11.3.2, it follows that $\left[\begin{array}{c|c} \overline{\mathbb{K}} & \overline{\mathbb{F}} \end{array}\right]$ is a H^∞-FI-pair. □

Notes

The stable H^∞ FICP was solved by Olof Staffans in [S98d], which proves the implications "(FI3s)⇒(FI2s)⇒(FI1$\frac{1}{2}$s)⇒(FI1s)" and "(FI3s)⇒(FI5s)" (the latter in the case of a regular WPLSs and spectral factors), thus establishing the equivalence of (FI1s)–(FI3s) (and the equality $\gamma_0 = \gamma_{\mathrm{FI}}$) for Σ s.t. $(\mathbb{D}, J_\gamma) \in$ SpF for all $\gamma > 0$. Also Lemma 11.3.10, much of Proposition 11.3.4 and most of Corollary 11.3.5 and Theorem 11.3.6 are from [S98d], and so are most of the corresponding proofs. Cf. the notes on p. 673.

This spectral factorization approach is rather old, see, e.g., [Francis], [Green] and [CG97], which all contain the equivalence of (FI1s)–(FI3s) in some sense for rational transfer functions ([CG97] for $\mathrm{MTIC}^{\mathrm{L}^1}_{\mathrm{exp}}(\mathbf{C}^n, \mathbf{C}^m)$ I/O maps).

It seems that many of the results of this and previous sections also hold for the extended FICP described in Theorem 11.3.6 (part of this is shown in [S98d]), but, e.g., the implication "(FI5s)⇒(FI2s)" would need additional assumptions.

11.4 Minimax J-coercivity

In some ways we are more confused than ever, but we feel that we are confused on a higher level and about more important things.

Here we state some minimax results that are needed for the solution of H^∞ and Nehari problems. Here $\mathbb{D} \in \mathrm{TI}(U \times W, Y)$, $J = J^* \in \mathcal{B}(Y)$, and $J_1 := \left[\begin{smallmatrix} I & 0 \\ 0 & -I \end{smallmatrix}\right] \in \mathcal{GB}(U \times W)$ (in practical applications we will have $\mathbb{D} \in \mathrm{TIC}$, but TI formulations allow us to obtain the dual results more neatly). Recall that $\|\cdot\|_{\mathrm{TI}} := \|\cdot\|_{\mathcal{B}(\mathrm{L}^2,\mathrm{L}^2)}$.

Definition 11.4.1 (Minimax J-coercivity, $\mathbb{F}$) *A map* $\mathbb{D} = \begin{bmatrix} \mathbb{D}_1 & \mathbb{D}_2 \end{bmatrix} \in \mathrm{TI}$ *is* minimax J-coercive *iff* $\mathbb{F} := \pi_+ \mathbb{D}^* J \mathbb{D} \pi_+ \in \mathcal{B}(\pi_+ \mathrm{L}^2)$ *satisfies* $\mathbb{F}_{11} \gg 0$ *and* $\mathbb{F}_{22} - \mathbb{F}_{21}\mathbb{F}_{11}^{-1}\mathbb{F}_{12} \ll 0$ *(on* $\pi_+\mathrm{L}^2$*).*

$\mathbb{D} \in \mathrm{TI}$ *is* co-minimax J-coercive *iff* $\mathbb{F} := \pi_- \mathbb{D}^* J \mathbb{D} \pi_- \in \mathcal{B}(\pi_- \mathrm{L}^2)$ *satisfies (equivalently,* $\widetilde{\mathbb{F}} := \pi_+ \mathbb{D}^{\mathrm{d}} J (\mathbb{D}^{\mathrm{d}})^* \pi_+ = \boldsymbol{\text{Я}} \mathbb{F} \boldsymbol{\text{Я}}$ *satisfies)* $\mathbb{F}_{11} \gg 0$ *and* $\mathbb{F}_{22} - \mathbb{F}_{21}\mathbb{F}_{11}^{-1}\mathbb{F}_{12} \ll 0$.

As noted in [S98d, Lemma 3.2], the minimax J-coercivity of $\mathbb{D} \in \mathrm{TIC}$ means that $\mathcal{J}(x_0, \left[\begin{smallmatrix} u \\ w \end{smallmatrix}\right])$ is uniformly convex w.r.t. u, and $\min_{u \in \mathrm{L}^2(\mathbf{R}_+;U)} \mathcal{J}(x_0, \left[\begin{smallmatrix} u \\ w \end{smallmatrix}\right])$ is uniformly concave w.r.t. w. By Lemma 11.3.10, an equivalent condition is that $\gamma > \gamma_0$ when the assumptions of previous section are satisfied. Similarly, cominimax J-coercivity is "roughly equivalent" to the solvability of the Nehari problem, see Theorem 11.8.3. We write the negative term out for later use:

$$0 \gg \mathbb{F}_{22} - \mathbb{F}_{21}\mathbb{F}_{11}^{-1}\mathbb{F}_{12} = \pi_+\mathbb{D}_2^* J \mathbb{D}_2 \pi_+ - \pi_+ \mathbb{D}_2^* J \mathbb{D}_1 \pi_+ (\pi_+ \mathbb{D}_1^* J \mathbb{D}_1 \pi_+)^{-1} \pi_+ \mathbb{D}_1^* J \mathbb{D}_2 \pi_+. \tag{11.106}$$

By combining (A.11) and Lemma A.3.1(b1) we obtain

Lemma 11.4.2 *If* $\mathbb{D} = \begin{bmatrix} \mathbb{D}_1 & \mathbb{D}_2 \end{bmatrix} \in \mathrm{TI}$ *is minimax J-coercive, then* $\mathbb{F} \in \mathcal{GB}(\pi_+\mathrm{L}^2)$ *and* $(\mathbb{F}^{-1})_{22} = (\mathbb{F}_{22} - \mathbb{F}_{21}\mathbb{F}_{11}^{-1}\mathbb{F}_{12})^{-1} \ll 0$ *(in particular,* $\mathbb{D}$ *is J-coercive).*

Similarly, if $\mathbb{D} = \begin{bmatrix} \mathbb{D}_1 & \mathbb{D}_2 \end{bmatrix} \in \mathrm{TI}$ *is co-minimax J-coercive, then* $\mathbb{F} \in \mathcal{GB}(\pi_-\mathrm{L}^2)$ *and* $(\mathbb{F}^{-1})_{22} = (\mathbb{F}_{22} - \mathbb{F}_{21}\mathbb{F}_{11}^{-1}\mathbb{F}_{12})^{-1} \ll 0$. □

Note that $\mathbb{E} := \mathbb{D}^* J \mathbb{D}$ may satisfy $\mathbb{E}_{11} \gg 0$ and $\mathbb{E}_{22} - \mathbb{E}_{21}\mathbb{E}_{11}^{-1}\mathbb{E}_{12} \ll 0$ and still $\mathbb{F}$ may be noninvertible (take $\mathbb{D} = \begin{bmatrix} I & 0 \\ \tau(1) & I \end{bmatrix}$ and $J = J_1$; similarly $\mathbb{D}^{\mathrm{d}}$ (which is of Nehari form with $\mathcal{G} = \tau(1)$) is not co-minimax J-coercive).

Minimax J-coercivity can also be formulated in terms of a spectral factorization (if $\mathbb{D}^* J \mathbb{D}$ is regular enough to have one):

Lemma 11.4.3 (SpF) *Let* $\mathbb{D} = \begin{bmatrix} \mathbb{D}_1 & \mathbb{D}_2 \end{bmatrix} \in \mathrm{TI}(U \times W, Y)$*,* $J = J^* \in \mathcal{B}(Y)$*, and* $\gamma > 0$.

(a) Assume that $\mathbb{D}^* J \mathbb{D} = \mathbb{Y}^* \mathbb{Z}$*, where* $\mathbb{Y}, \mathbb{Z} \in \mathcal{G}\mathrm{TIC}$.

Then $\mathbb{D}$ *is minimax J-coercive iff* $\mathbb{D}^* J \mathbb{D} = \mathbb{X}^* J_\gamma \mathbb{X}$ *with* $\mathbb{X}, \mathbb{X}_{11} \in \mathcal{G}\mathrm{TIC}$ *and* $\|\mathbb{X}_{21}\mathbb{X}_{11}^{-1}\|_{\mathrm{TI}} < \gamma$.

(b) Assume that $\mathbb{D}^* J \mathbb{D} = \mathbb{X}^* J_\gamma \mathbb{X}$*, where* $\mathbb{X} \in \mathcal{G}\mathrm{TIC}$*. Then* $\mathbb{D}$ *is minimax J-coercive iff* $\mathbb{X}_{11} \in \mathcal{G}\mathrm{TIC}$ *and* $\|\mathbb{X}_{21}\mathbb{X}_{11}^{-1}\|_{\mathrm{TI}} < \gamma$.

(c) Assume that $\mathbb{D}^* J\mathbb{D} = \mathbb{X}^* J_\gamma \mathbb{X}$, where $\mathbb{X}, \mathbb{X}_{11} \in \mathcal{G}\mathrm{TIC}$.

Then $\mathbb{D}$ is minimax J-coercive iff $\mathbb{D}_1^* J\mathbb{D}_1 \gg 0$ iff $\mathbb{F}_{11} \gg 0$ iff $\mathbb{F}_{22} - \mathbb{F}_{21}\mathbb{F}_{11}^{-1}\mathbb{F}_{12} \ll 0$ iff $\|\mathbb{X}_{21}\mathbb{X}_{11}^{-1}\|_{\mathrm{TI}} < \gamma$ iff $\|(\mathbb{X}^{-1})_{22}^{-1}(\mathbb{X}^{-1})_{21}\|_{\mathrm{TI}} < \gamma$.

(d) Assume that $\mathbb{D}^* J\mathbb{D} = \mathbb{X}^* S\mathbb{X}$, $\mathbb{X} \in \mathcal{G}\mathrm{TIC}(U \times W) \cap \mathrm{UR}$, $X = I$ and $S \in \mathcal{G}\mathcal{B}(U \times W)$. If $\mathbb{D}$ is minimax J-coercive, then $S_{11} \gg 0$ and $S_{22} - S_{21}S_{11}^{-1}S_{12} \ll 0$.

We used (a) and Lemma 11.3.10 to show that (FI1s) implies (FI3s) in Proposition 11.3.4.

Proof: (We take $\gamma = 1$ by using $\begin{bmatrix} I & 0 \\ 0 & \gamma \end{bmatrix} \mathbb{X}$ instead of $\mathbb{X}$.)

Let $\mathbb{F} := \pi_+ \mathbb{D}^* J\mathbb{D}\pi_+ \in \mathcal{B}(\pi_+ \mathrm{L}^2)$, so that $\mathbb{F}_{11} \gg 0$ and $(\mathbb{F}^{-1})_{22} \ll 0$ iff $\mathbb{D}$ is minimax J-coercive, by Lemma 11.4.2.

(a) 1° *"If":* Since now $\mathbb{F} = \pi_+ \mathbb{X}^* \pi_+ J_1 \pi_+ \mathbb{X}\pi_+$, and $\pi_+ \mathbb{X}\pi_+ \in \mathcal{G}\mathcal{B}(\mathrm{L}^2(\mathbf{R}_+; U \times W))$, we obtain "If" from Lemma 11.3.13(vi)&(i).

2° *"Only if":* Let $\mathbb{D}$ be minimax J-coercive and $\mathbb{D}^* J\mathbb{D} = \mathbb{Y}^*\mathbb{Z}$, $\mathbb{Y}, \mathbb{Z}$. Then we have $\mathbb{D}^* J\mathbb{D} = \mathbb{X}^* J_1 \mathbb{X}$ for some $\mathbb{X} \in \mathcal{G}\mathrm{TIC}$, by Lemma 11.4.8. The $(1,1)$-block of $\pi_+ \mathbb{D}^* J\mathbb{D}\pi_+ = \pi_+ \mathbb{X}^* J_1 \mathbb{X}\pi_+$ implies that

$$0 \ll \mathbb{F}_{11} = \pi_+ \mathbb{D}_1^* J\mathbb{D}_1 \pi_+ = \pi_+ (\mathbb{X}_{11}^* \mathbb{X}_{11} - \mathbb{X}_{21}^* \mathbb{X}_{21})\pi_+, \tag{11.107}$$

hence $\pi_+ \mathbb{X}_{11}^* \mathbb{X}_{11} \pi_+ \gg 0$, i.e., $\mathbb{X}_{11}^* \mathbb{X}_{11} \gg 0$, by Lemma 6.4.6.

By Lemma A.1.1(c), the [left-]invertibility of $\mathbb{X}_{11}$ is equivalent to that of $\mathbb{M}_{22}$, where $\mathbb{M} := \mathbb{X}^{-1}$. Therefore, $\mathbb{M}_{22}^* \mathbb{M}_{22} \gg 0$, so by Lemma 2.2.3, it is enough to prove that $\mathbb{M}_{22}\pi_+ \mathbb{M}_{22}^* \gg 0$ on $\mathrm{L}^2(\mathbf{R}_+; W)$ to obtain $\mathbb{M}_{22} \in \mathcal{G}\mathrm{TIC}$ (hence $\mathbb{X}_{11} \in \mathcal{G}\mathrm{TIC}$). We shall do it.

Clearly $\mathbb{F}^{-1} = \pi_+ \mathbb{M}\pi_+ J_1 \mathbb{M}^* \pi_+$, hence $0 \gg (\mathbb{F}^{-1})_{22} = \mathbb{M}_{21}\pi_+ \mathbb{M}_{21}^* - \mathbb{M}_{22}\pi_+ \mathbb{M}_{22}^*$ on $\pi_+ \mathrm{L}^2$, by Lemma 11.4.2, which implies that $\mathbb{M}_{22}\pi_+ \mathbb{M}_{22}^* \gg 0$ on $\pi_+ \mathrm{L}^2$, therefore $\mathbb{M}_{22} \in \mathcal{G}\mathrm{TIC}$, i.e., $\mathbb{X}_{11} \in \mathcal{G}\mathrm{TIC}$.

From (11.107) and Lemma 6.4.6 we get $0 \ll \mathbb{X}_{11}^* \mathbb{X}_{11} - \mathbb{X}_{21}^* \mathbb{X}_{21}$, equivalently (by Lemma A.3.1(e2)), $\|\mathbb{X}_{21}\mathbb{X}_{11}^{-1}\|_{\mathrm{TI}} < 1$.

(b) "If" follows from (a) ($\mathbb{Y} := J_1\mathbb{X}$, $\mathbb{Z} := \mathbb{Z}$). Conversely, if $\mathbb{D}$ is minimax J-coercive, then the proof of (a) shows that an arbitrary J_1-spectral factor of $\mathbb{D}^* J\mathbb{D}$ is as in (B).

(c) From $(\pi_+ \mathbb{X}_{11}^{-1} \pi_+)(\pi_+ \mathbb{X}_{11} \pi_+) = \pi_+ = (\pi_+ \mathbb{X}_{11} \pi_+)(\pi_+ \mathbb{X}_{11}^{-1} \pi_+)$ we see that
$(\pi_+ \mathbb{X}_{11} \pi_+), (\pi_+ \mathbb{X}_{11}^{-1} \pi_+) \in \mathcal{G}\mathcal{B}(\pi_+ \mathrm{L}^2)$, hence $0 \ll \mathbb{F}_{11}$ is equivalent to

$$0 \ll (\pi_+ \mathbb{X}_{11}^{-*} \pi_+)\mathbb{F}_{11}(\pi_+ \mathbb{X}_{11}^{-1} \pi_+) = \pi_+ - \mathbb{X}_{11}^{-*}\mathbb{X}_{21}^* \mathbb{X}_{21} \mathbb{X}_{11}^{-1})\pi_+ \tag{11.108}$$

(because $\mathbb{F}_{11} = \pi_+ (\mathbb{X}_{11}^* \mathbb{X}_{11} - \mathbb{X}_{21}^* \mathbb{X}_{21})\pi_+$), i.e., $1 > \|\mathbb{X}_{21}\mathbb{X}_{11}^{-1}\|_{\mathrm{TI}}$ (cf. 1°). By (b), this is equivalent to the minimax J-coercivity of $\mathbb{D}$.

On the other hand, $0 \ll \mathbb{F}_{11} = \pi_+ \mathbb{D}_1^* J\mathbb{D}_1 \pi_+ \Leftrightarrow 0 \ll \mathbb{D}_1^* J\mathbb{D}_1$. The last "iff" follows from $\mathbb{M}_{22}^{-1}\mathbb{M}_{21} = -\mathbb{X}_{12}\mathbb{X}_{11}^{-1}$ and the third from Lemma A.1.1(c1).

(d) By (a), $\mathbb{D}^* J\mathbb{D} = \mathbb{Z}^* J_1 \mathbb{Z}$ for some $\mathbb{Z} \in \mathcal{G}\mathrm{TIC}(U \times W)$ s.t. $\mathbb{Z}_{11} \in \mathcal{G}\mathrm{TIC}(U)$ and $\|\mathbb{Z}_{21}\mathbb{Z}_{11}^{-1}\|_{\mathrm{TI}} < 1$. By Lemma 6.4.5(a), $\mathbb{Z} = Z\mathbb{X}$ and $S = Z^* J_1 Z$ for some $Z \in \mathcal{G}\mathcal{B}$, hence $\mathbb{Z}$ is UR. By Proposition 6.3.1(b1), $Z_{11} \in \mathcal{G}\mathcal{B}(U)$. Thus, $\|Z_{21}Z_{11}^{-1}\|_{\mathrm{TI}} < 1$. By Lemma 11.3.13(vi)&(i), $S_{11} \gg 0$ and $S_{22} - S_{21}S_{11}^{-1}S_{12} \ll 0$.

$\square$

Corollary 11.4.4 (MTI SpF) *Let $\mathbb{D} \in \widetilde{\mathcal{A}}(U \times W, Y)$ and $J = J^* \in \mathcal{B}(Y)$.*

Then $\mathbb{D}$ is minimax J-coercive iff $\mathbb{D}^* J \mathbb{D} = \mathbb{X}^* J_1 \mathbb{X}$, where $\mathbb{X}, \mathbb{X}_{11} \in \mathcal{G}\mathrm{TIC}$ and $\|\mathbb{X}_{21}\mathbb{X}_{11}^{-1}\|_{\mathrm{TI}} < 1$.

Naturally, the weaker assumption $\mathbb{D} \in \mathrm{TIC}(U \times W, Y)$, $(\mathbb{D}, J) \in \mathrm{SpF}$ would be sufficient, with essentially the same proof.

Proof: If $\mathbb{D}$ is minimax J-coercive, then $\pi_+ \mathbb{D}^* J \mathbb{D} \pi_+$ is invertible, by Lemma 11.4.2, so the existence of a factorization follows from Theorem 8.4.12(ii). The rest of the claims and the converse is obtained from Lemma 11.4.3. $\square$

Corollary 11.4.5 *Let $\mathbb{D} = \begin{bmatrix} \mathbb{D}_1 & \mathbb{D}_2 \end{bmatrix} \in \mathrm{TI}$, $\mathbb{X}, \mathbb{M} \in \mathrm{TIC}$, $\mathbb{M} = \mathbb{X}^{-1}$, $J = J^* \in \mathcal{B}$ and $\mathbb{D}^* J \mathbb{D} = \mathbb{X} J_1 \mathbb{X}^*$.*

The operator $\mathbb{D}$ is co-minimax J-coercive iff $\mathbb{X}_{11} \in \mathcal{G}\mathrm{TIC}$ and $\|\mathbb{X}_{11}^{-1}\mathbb{X}_{12}\| < 1$. If $\mathbb{X}_{11} \in \mathcal{G}\mathrm{TIC}$, then $\mathbb{D}$ is minimax J-coercive iff $\|\mathbb{X}_{11}^{-1}\mathbb{X}_{12}\|_{\mathrm{TI}} < 1$ iff $\mathbb{F}_{11} \gg 0$ iff $\mathbb{D}_1^* J \mathbb{D}_1 \gg 0$ iff $\mathbb{F}_{22} - \mathbb{F}_{21}\mathbb{F}_{11}^{-1}\mathbb{F}_{12} \ll 0$ (here $\mathbb{F} := \pi_- \mathbb{D}^* J \mathbb{D} \pi_-$).

Proof: Apply Lemma 11.4.3 to $((\mathbb{D}^{\mathrm{d}})^*)^* J((\mathbb{D}^{\mathrm{d}})^*) = (\mathbb{X}^{\mathrm{d}})^* J_1 \mathbb{X}^{\mathrm{d}}$. $\square$

For $\mathbb{D}$ of Nehari type, the condition $\|\mathbb{X}_{11}^{-1}\mathbb{X}_{12}\| < 1$ is redundant:

Corollary 11.4.6 (Nehari-form) *Let $\mathbb{D} = \left[\begin{smallmatrix} I & \mathbb{G} \\ 0 & I \end{smallmatrix}\right] \in \mathrm{TIC}$, $\mathbb{X} \in \mathcal{G}\mathrm{TIC}$, $J = J^* \in \mathcal{B}$ and $\mathbb{D}^* J \mathbb{D} = \mathbb{X} J_1 \mathbb{X}^*$. The operator $\mathbb{D}$ is co-minimax J-coercive iff $\mathbb{F}_{22} - \mathbb{F}_{21}\mathbb{F}_{11}^{-1}\mathbb{F}_{12} \ll 0$ iff $\mathbb{X}_{11} \in \mathcal{G}\mathrm{TIC}$ (here $\mathbb{F} := \pi_- \mathbb{D}^* J \mathbb{D} \pi_-$). Moreover, in that case always $\|\mathbb{X}_{11}^{-1}\mathbb{X}_{12}\| < 1$.*

Note that $\|\mathbb{X}_{11}^{-1}\mathbb{X}_{12}\| = \|\mathbb{M}_{12}\mathbb{M}_{22}^{-1}\|$, where $\mathbb{M} := \mathbb{X}^{-1}$, because the $(1,2)$-block of equation $\mathbb{X}\mathbb{M} = I$ is $\mathbb{X}_{11}\mathbb{M}_{12} + \mathbb{X}_{12}\mathbb{M}_{22} = 0$.

Proof: The first "iff" follows from Definition 11.4.1 & Lemma 11.4.2, because always $\mathbb{F}_{11} = \pi_- \gg 0$. Thus the latter "iff" and the inequality follow from Corollary 11.4.5. $\square$

Corollary 11.4.7 (Nehari-form) *Let $\mathbb{D} = \left[\begin{smallmatrix} I & \mathbb{G} \\ 0 & I \end{smallmatrix}\right] \in \mathrm{TIC}(U \times W)$, $\mathbb{Y}, \mathbb{Z} \in \mathcal{G}\mathrm{TIC}(U \times W)$, $J = J_\gamma \in \mathcal{B}(U \times W)$ and $\mathbb{D}^* J \mathbb{D} = \mathbb{Y}\mathbb{Z}^*$.*

Then the operator $\mathbb{D}$ is co-minimax J-coercive iff $\mathbb{F}_{22} - \mathbb{F}_{21}\mathbb{F}_{12} \ll 0$ iff $\mathbb{D}^* J \mathbb{D} = \mathbb{X} J_1 \mathbb{X}^*$ for some $\mathbb{X} \in \mathcal{G}\mathrm{TIC}(U \times W)$ having $\mathbb{X}_{11} \in \mathcal{G}\mathrm{TIC}(U)$. Moreover, in that case always $\|\mathbb{X}_{11}^{-1}\mathbb{X}_{12}\| < 1$.

Proof: By Lemma 11.4.2 and Corollary 11.4.6, we only have to obtain the third condition assuming the first.

Let $\mathbb{D}$ be co-minimax J-coercive. Set $\mathbb{E} := \boldsymbol{Я}\mathbb{D}^* J \mathbb{D}\boldsymbol{Я} = (\mathbb{Y}^{\mathrm{d}})^* \mathbb{Z}^{\mathrm{d}}$, so that $\mathbb{F} := \pi_- \boldsymbol{Я}\mathbb{E}\boldsymbol{Я}\pi_-$ is as in Definition 11.4.1, hence $\mathbb{F}_{11} \gg 0$ and $\mathbb{F}_{22} -$

$\mathbb{F}_{21}\mathbb{F}_{11}^{-1}\mathbb{F}_{12} \ll 0$ on $\mathrm{L}^2(\mathbf{R}_-;U\times W)$. We obviously have $\mathbb{E}_{11} = I \gg 0$. Moreover, $\mathbb{G}$ of Lemma 11.4.8 satisfies

$$\boldsymbol{\mathcal{R}}\pi_+\mathbb{G}\pi_+\boldsymbol{\mathcal{R}} = \boldsymbol{\mathcal{R}}\pi_+\mathbb{E}_{22}\pi_+\boldsymbol{\mathcal{R}} - \boldsymbol{\mathcal{R}}\pi_+\mathbb{E}_{21}\mathbb{E}_{12}\pi_+\boldsymbol{\mathcal{R}} = \boldsymbol{\mathcal{R}}\pi_+\mathbb{E}_{22}\pi_+\boldsymbol{\mathcal{R}} - \boldsymbol{\mathcal{R}}\pi_+\mathbb{E}_{21}\pi_+\mathbb{E}_{12}\pi_+\boldsymbol{\mathcal{R}} \tag{11.109}$$

$$= \boldsymbol{\mathcal{R}}\pi_+\mathbb{E}_{22}\pi_+\boldsymbol{\mathcal{R}} - \pi_-\boldsymbol{\mathcal{R}}\mathbb{E}_{21}\boldsymbol{\mathcal{R}}\pi_-\boldsymbol{\mathcal{R}}\mathbb{E}_{12}\boldsymbol{\mathcal{R}}\pi_- = \mathbb{F}_{22} - \mathbb{F}_{21}\mathbb{F}_{12} \ll 0 \tag{11.110}$$

on $\mathrm{L}^2(\mathbf{R}_-;U\times W)$, hence the assumptions of Lemma 11.4.8 are satisfied, by Lemma 6.4.6. We conclude that $\mathbb{E} = \widetilde{\mathbb{X}}^* J_1 \widetilde{\mathbb{X}}$ for some $\widetilde{\mathbb{X}} \in \mathcal{G}\mathrm{TIC}(U\times W)$, hence $\mathbb{D}^* J\mathbb{D} = \boldsymbol{\mathcal{R}}\widetilde{\mathbb{X}}^* J_1 \widetilde{\mathbb{X}}\boldsymbol{\mathcal{R}} = \mathbb{X}J_1\mathbb{X}^*$, where $\mathbb{X} := \widetilde{\mathbb{X}}^{\mathrm{d}} \in \mathcal{G}\mathrm{TIC}$. The rest follows from Corollary 11.4.6. □

In the proof of Lemma 11.4.3(a), we used the following:

Lemma 11.4.8 *Assume that* $\mathbb{E} = \mathbb{E}^* \in \mathrm{TI}(U\times W)$, $\mathbb{E}_{11} \gg 0$, $\mathbb{G} := \mathbb{E}_{22} - \mathbb{E}_{21}\mathbb{E}_{11}^{-1}\mathbb{E}_{12} \ll 0$, $\mathbb{E} = \mathbb{Y}^*\mathbb{Z}$ *for some* $\mathbb{Z},\mathbb{Y} \in \mathcal{G}\mathrm{TIC}(U\times W)$ *and* $\gamma > 0$.

Then $\mathbb{E} = \mathbb{X}^* J_\gamma \mathbb{X}$ *for some* $\mathbb{X} \in \mathcal{G}\mathrm{TIC}(U\times W)$.

(One could go on to obtain further results as in Lemma 11.4.3.)

Proof: By Lemma 5.2.1(d), $\mathbb{E} = \mathbb{X}^* S\mathbb{X}$ for some $\mathbb{X} \in \mathcal{G}\mathrm{TIC}(U\times W)$ and $S = S^* \in \mathcal{G}\mathcal{B}(U\times W)$. By Lemma 6.4.7(a), we have $\mathbb{E}_{11} = \mathbb{R}^*\mathbb{R}$ and $-\mathbb{G} = \mathbb{T}^*\mathbb{T}$ for some $\mathbb{R} \in \mathcal{G}\mathrm{TIC}(U)$ and $\mathbb{T} \in \mathcal{G}\mathrm{TIC}(W)$. By the Schur decomposition (A.5), we have

$$S = \mathbb{X}^{-*}\mathbb{E}\mathbb{X}^{-1} = \mathbb{X}^{-*}\begin{bmatrix} I & 0 \\ \mathbb{E}_{21}\mathbb{E}_{11}^{-1} & I \end{bmatrix}^* \begin{bmatrix} \mathbb{E}_{11} & 0 \\ 0 & \mathbb{G} \end{bmatrix} \begin{bmatrix} I & 0 \\ \mathbb{E}_{21}\mathbb{E}_{11}^{-1} & I \end{bmatrix} \mathbb{X}^{-1} \tag{11.111}$$

$$= \mathbb{X}^{-*}\begin{bmatrix} I & 0 \\ \mathbb{E}_{21}\mathbb{E}_{11}^{-1} & I \end{bmatrix}^* \begin{bmatrix} \mathbb{R} & 0 \\ 0 & \mathbb{T} \end{bmatrix}^* \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} \mathbb{R} & 0 \\ 0 & \mathbb{T} \end{bmatrix} \begin{bmatrix} I & 0 \\ \mathbb{E}_{21}\mathbb{E}_{11}^{-1} & I \end{bmatrix} \mathbb{X}^{-1}, \tag{11.112}$$

i.e., $S = \mathbb{U}^* J_1 \mathbb{U}$, where $\mathbb{U} := \begin{bmatrix} \mathbb{R} & 0 \\ 0 & \mathbb{T} \end{bmatrix} \begin{bmatrix} I & 0 \\ \mathbb{E}_{21} & \mathbb{E}_{11}^{-1} & I \end{bmatrix} \mathbb{X}^{-1} \in \mathcal{G}\mathrm{TI}(U\times W)$. Consequently, $S = E^* J_\gamma E$ for some $E \in \mathcal{G}\mathcal{B}(U\times W)$, by Theorem 2.4.5. □

We have thus generalized Lemma 5.4 of [S98d] to Hilbert spaces of arbitrary dimensions (see Lemma 11.4.3(a)). Because the assumed separability of the Hilbert spaces used in [S98d] was needed only in the proof of [S98d, Lemma 5.4] (as mentioned at the end of [S98d, Section 1]), the above lemma shows that the separability assumptions are not needed in [S98d]:

Corollary 11.4.9 *All separability assumptions in [S98d] can be removed.* □

Notes

Lemma 11.4.3(a) is roughly Lemma 5.4 of [S98d], and the proofs (including Lemma 11.4.8) use same methods; the difference is that we do not need any separability assumptions, due to Theorem 2.4.5.

11.5 The discrete-time H^∞ ficp

A walking shadow, a poor player,
that struts and frets his hour upon the stage,
And then is heard no more.

— William Shakespeare (1564–1616), "Macbeth"

As in other discrete-time sections, our references to continuous-time results, definitions and hypotheses refer to their discrete-time forms (cf. Theorems 13.3.13 and 11.5.2).

Recall that throughout this section and Section 11.6, we assume (in addition to Standing Hypothesis 11.0.1) that Standing Hypothesis 11.1.1 holds, i.e., we consider the system

$$\begin{cases} x_{n+1} = Ax_n + B_1 u_n + B_2 w_n, \\ z_n = C_1 x_n + D_{11} u_n + D_{12} w_n \end{cases} \qquad (n \in \mathbf{N}) \tag{11.113}$$

(and $w_n = I w_n$) with initial state $x_0 \in H$, disturbance input $w \in \ell^2(\mathbf{N};W)$, control input $u \in \ell^2(\mathbf{N};U)$ and objective output $z \in \ell^2(\mathbf{N};Z)$ (and second output equal to the disturbance input w_n; cf. (12.31)); here $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \in \mathcal{B}(H \times U \times W, H \times Z \times W)$ are the generators of $\Sigma \in \mathrm{wpls}(U \times W, H, Z \times W)$ (see Lemma 13.3.3) and $C_2 = 0$, $D_{21} = 0$, $D_{22} = I$. Condition (i) below says that

$$\gamma_0 := \sup_{\|w\|_{\ell^2}=1} \inf\{\|z\|_{\ell^2} \mid u \in \ell^2 \text{ is s.t. } x \text{ is stable (i.e., } x \in \ell^2)\}; \tag{11.114}$$

thus, γ_0 equals $\inf \|w \mapsto z\|_{\mathcal{B}(\ell^2)}$, the infimal disturbance-to-output norm $\|w \mapsto z\|$, over all control laws that make the system exponentially stable ($u \in \ell^2(\mathbf{N};U)$ s.t. $x \in \ell^2$). By (ii), infimum of over (causal) state-feedback (plus feedthrough) controllers

$$u(t) = K_1 x(t) + F_{12} w(t) \tag{11.115}$$

is as low. By (iii) and (a), a control law achieving a performance below γ_0 is found iff the DARE (11.117) has a nonnegative exponentially (or power) stabilizing solution satisfying the signature condition (for S). Moreover, such a solution determines one possible choice of K_1 and F_{12}, by (11.118). Finally, if $S_{22} \ll 0$, we can take $F_{12} = 0$ to obtain the pure state-feedback controller (11.119).

Theorem 11.5.1 ($\mathcal{U}_{\mathbf{exp}}$: $\mathbf{H}^\infty$ ficp) *Assume that $\gamma > 0$ and that there is $\varepsilon > 0$ s.t.*

$$(z - A)x_0 = Bu_0 \Longrightarrow \|C_1 x_0 + D_{11} u_0\|_Y \geq \varepsilon(\|x_0\|_H + \|u_0\|_U) \quad (x_0 \in H,\ u_0 \in U,\ z \in \partial\mathbf{D}). \tag{11.116}$$

Then (i)–(iii) are equivalent:

(i) $\gamma > \gamma_0 := \sup_{w:\mathbf{N}\to W,\ \|w\|_{\ell^2}=1} \inf_{u \in \mathcal{U}_{\mathrm{u}}(0,w)} \|\mathbb{D}_{11} u + \mathbb{D}_{12} w\|_{\ell^2}$, *and* (A, B_1) *is exponentially stabilizable;*

(ii) $\gamma > \gamma_{\mathrm{FI}}$, *i.e., there is a suboptimal* H^∞*-FI-pair for* Σ*;*

(iii) the DARE

$$\begin{cases} \mathcal{P} = A^*\mathcal{P}A + C_1^*C_1 - K^*SK, \\ S = \begin{bmatrix} D_{11}^*D_{11} & D_{11}^*D_{12} \\ D_{12}^*D_{11} & D_{12}^*D_{12} - \gamma^2 I \end{bmatrix} + B^*\mathcal{P}B, \\ K = -S^{-1}\left(\begin{bmatrix} D_{11}^* \\ D_{12}^* \end{bmatrix} C_1 + B^*\mathcal{P}A \right), \end{cases} \tag{11.117}$$

has a solution $\mathcal{P} \in \mathcal{B}(H)$ *s.t.* $\mathcal{P} \geq 0$, $S_{11} \gg 0$, $S_{22} - S_{21}S_{11}^{-1}S_{12} \ll 0$ *and* $\rho(A + BK) < 1$.

Moreover, the following hold:

(a) Assume that $(\mathcal{P}, S, K)$ *satisfies (iii). Then*

$$\left(\begin{array}{c|cc} -S_{11}^{-1}(D_{11}^*C_1 + B_1^*\mathcal{P}A) & 0 & -S_{11}^{-1}S_{12} \\ 0 & 0 & 0 \end{array} \right); \tag{11.118}$$

is a suboptimal (exponentially stabilizing) H^∞*-FI-pair for* Σ.

There is a suboptimal H^∞*-SF-operator iff* $S_{22} \ll 0$*; if this is the case, then*

$$K_1 := \begin{bmatrix} I & 0 \end{bmatrix} K = -(S_{11} - S_{12}S_{22}^{-1}S_{21})^{-1}(D_{11}^*C_1 + B_1^*\mathcal{P}A - S_{12}S_{22}^{-1}(D_{12}^*C_1 + B_2^*\mathcal{P}A)) \tag{11.119}$$

is a suboptimal (exponentially stabilizing) H^∞*-SF-operator for* Σ.

(b) If (i)–(iii) hold, then the assumptions of Proposition 11.2.8 (also those of (a1)) are satisfied and (FI1)–(FI5) hold.

One more equivalent condition is that $\|\mathbb{D}_{11}\mathbb{U} + \mathbb{D}_{12}\|_{\mathcal{B}(\ell^2(\mathbf{N};W),\ell^2(\mathbf{N};Z))} < \gamma$ for some $\mathbb{U} : \ell^2(\mathbf{N};W) \to \ell^2(\mathbf{N};U)$, and (A, B_1) is exponentially stabilizable (obviously this is stronger than (i) and weaker than (ii)).

We recall from Section 11.1 that $\mathcal{U}_\mathrm{u}$ refers to the controls that make the state and output "$\mathcal{U}_*^*$-stable", i.e.,

$$\mathcal{U}_\mathrm{u}(x_0, w) := \{u \in \ell^2(\mathbf{N};U) \mid \left[\begin{smallmatrix} u \\ w \end{smallmatrix}\right] \in \mathcal{U}_*^*(x_0)\}, \tag{11.120}$$

(in Theorem 11.5.1, $\mathcal{U}_*^* := \mathcal{U}_\mathrm{exp}$, so that "$\left[\begin{smallmatrix} u \\ w \end{smallmatrix}\right] \in \mathcal{U}_*^*(x_0)$" can be replaced by "$\mathbb{B}\tau\left[\begin{smallmatrix} u \\ w \end{smallmatrix}\right] \in \ell^2$", i.e., by the assumption that u makes the state trajectory belong to ℓ^2), and that by γ_FI (resp. γ_SF) we denote the infimum of the norm $\|w \mapsto z\|_{\ell^2(\mathbf{N};W) \to \ell^2(\mathbf{N};Z)}$ (i.e., of $\|\mathbb{D}_{12}^{\circlearrowleft}\|_\mathrm{TIC}$; see (11.8)) over all $\mathcal{U}_*^*$-stabilizing state feedback pairs (resp. $\mathcal{U}_*^*$-stabilizing state feedback operators) for Σ of form

$$\left(\begin{array}{c|cc} K_1 & F_{11} & F_{12} \\ 0 & 0 & 0 \end{array} \right) \tag{11.121}$$

(i.e., we allow state feedback through the first input (the control u) only). Thus, γ_SF requires that $F_{11} = 0 = F_{12}$. Trivially, $\infty \geq \gamma_\mathrm{SF} \geq \gamma_\mathrm{FI} \geq \gamma_0 \geq 0$ (cf. (11.132))

Proof of Theorem 11.5.1: 0.1° *Remark: Assumption (11.116):* By Proposition 15.2.2(c), this implies that $\mathbb{D}_{11}$ (with realization $\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B}_1 \\ \hline \mathbb{C}_1 & \mathbb{D}_{11} \end{array}\right]$) is I-coercive. If (A, B_1) is exponentially stabilizable, then (11.116) is equivalent to the I-coercivity of $\mathbb{D}_{11}$, by Proposition 15.2.2(e) (see the proposition for several sufficient and some equivalent conditions).

1° We get "(i)⇒(iii)" from Proposition 11.6.2, "(iii)⇒(ii)" from Lemma 11.6.3, and "(ii)⇒(i)" from (11.12) and Lemma 11.6.4. (See also remark 0.3° of the roof of Theorem 11.1.3. Recall from Lemma 13.3.7(iv) that $\rho(A+BK) < 1$ iff $\mathcal{P}$ (that is, K) is exponentially stabilizing.)

Also (a) follows from Proposition 11.6.2, and Lemma 11.6.3. By Proposition 11.2.9, (b) holds. □

Practically all our H^∞ FICP results hold also in their discrete-time forms:

Theorem 11.5.2 (Discrete form of $\mathbf{H^\infty}$ FICP results) *All results of Sections 11.1–11.4 and 11.8–11.9 hold also in their discrete-time forms (i.e., after the changes listed in Theorem 13.3.13), except that in Theorems 11.1.3, 11.1.4 and 11.1.6, assumption (2.) can be removed but equation (11.39) must be replaced by (11.118) and equation (11.17) by (11.119).*

When applying the above results, do not forget (the discrete-time forms of) Standing Hypotheses 11.0.1, 11.1.1, 11.2.1 and 11.3.1 (the last two of which are only assumed for the results of corresponding sections). Note that we have written explicitly the simplified discrete-time forms of some major results in this section and Section 11.6.

Recall from Lemma 14.3.5 that we can have $\widetilde{\mathcal{A}} = \mathrm{tic}_{\mathrm{exp}}$ or $\widetilde{\mathcal{A}} = \ell^1*$; this is particularly useful in Theorems 11.2.7 and 11.3.3. Therefore, in discrete time, we may allow for general (exponentially stabilizable) WPLSs in the $\mathcal{U}_{\mathrm{exp}}$ case of Theorem 11.2.7, while Example 11.3.7(b) (its discrete-time variant) shows that (FI1) does not imply any of (FI3)–(FI5) when $\mathcal{U}_*^* \in \{\mathcal{U}_{\mathrm{out}}, \mathcal{U}_{\mathrm{sta}}, \mathcal{U}_{\mathrm{str}}\}$.

Proof of Theorem 11.5.2: Recall that these changes include having the DARE (11.117) in place of the $[B_{\mathrm{w}}$-]CARE.

This follows roughly by applying (13.63) also to the proofs (recall from Lemma 14.3.5 that $\widetilde{\mathcal{A}} := \mathrm{tic}_{\mathrm{exp}}$ satisfies Standing Hypothesis 11.0.1).

(An alternative proof of the I/O part of Sections 11.8–11.9 is obtained by using the Cayley transform (Theorem 13.2.3).)

Note that (11.17) and (11.17) used the fact that "$S = D^*J_\gamma D$", which is not necessarily true under our discrete-time assumptions (or any reasonable counterparts, cf. Example 14.2.9). However, assumption (2.) can be removed from Theorems 11.1.3, 11.1.4 and 11.1.6, since we can use Theorem 14.1.6 (and Lemma 9.9.7(c2)) instead of Theorems 9.9.6 and Theorem 9.2.9 in the proofs. □

From the above theorem and Theorem 11.2.8(f)&(d), we deduce that if the system is q.r.c.-SOS-stabilizable through u (as in Theorem 11.2.8), then any of the factorization conditions (FI6)–(FI8) on p. 633 are sufficient for (ii), i.e., for the existence of a γ-suboptimal controller (over $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{out}}$). If, in addition, the q.r.c.-SOS-stabilized I/O map has, e.g., ℓ^1 impulse response, then these conditions are also necessary. If (A, B_1) is exponentially stabilizable and (A, C) is detectable, then (FI1)–(FI8) become equivalent (and (FI9) if $\dim U < \infty$ or $\dim W < \infty$; use Theorem 6.7.15(c3) and $\widetilde{\mathcal{A}} = \mathrm{tic}_{\mathrm{exp}}$ in Theorem 11.2.7).

Notes

Much of the notes of previous sections also apply to the discrete-time H^∞ ficp. For example, the equivalence of (ii) and (iii) (and part (a)) in Theorem 11.5.1 is given in [IOW] under the same and in [GL] under stronger simplifying assumptions, although both books assume that Σ is finite-dimensional. However, discrete-time H^∞ problems are more rarely treated than their continuous-time counterparts, partially because they are more complicated (as long as the input and output operators are bounded; in our generality the continuous-time results are much more involved).

If we delete "(i)" from Theorem 11.5.1, then its proof can be obtained rather directly from Proposition 11.6.2(a1)&(d1) and Lemma 11.2.13, as noted in the notes on p. 652.

11.6 H^∞ ficp: proofs

By the time the fool has learned the game, the players have dispersed.
— Ghanaian Proverb

In addition to Standing Hypotheses 11.0.1 and 11.1.1, we assume the following:

Standing Hypothesis 11.6.1 (H^∞ full-information control problem (ficp)) *Throughout this section, we make the following assumptions: Hypothesis 14.0.1 is satisfied (with $U \mapsto U \times W$ and $Y \mapsto Z \times W$), $\gamma > 0$, and there is $\varepsilon_+ > 0$ s.t. $\|\mathbb{D}_{11} u\|_{\ell^2} \geq \varepsilon_+ \|\left[\begin{smallmatrix} u \\ 0 \end{smallmatrix}\right]\|_{\mathcal{U}_*^*}$ for all $u \in \mathcal{U}_{\mathrm{u}}(0,0)$.*

(See the remarks below Hypothesis 11.2.1, which is the continuous-time counterpart of this hypothesis.)

Due to bounded generators, the discrete-time variant of Proposition 11.2.19 becomes simpler:

Proposition 11.6.2 (Necessary conditions) *Assume that $\gamma > \gamma_0$ and that Z^{s} is reflexive. Then $\mathbb{D}$ is J_γ-coercive. Assume in addition that $\mathcal{U}_*^*(x_0) \neq \emptyset$ for each $x_0 \in H$. Then the following hold:*

- *(a1) The DARE has a unique $\mathcal{U}_*^*$-stabilizing solution $(\mathcal{P}, S, K)$, $\mathcal{P} \geq 0$, $S \in \mathcal{GB}(U \times W)$ and $S_{11} \geq \varepsilon_+^2 I$.*
- *(a2) For each $x_0 \in H$, the corresponding closed-loop second output $\begin{bmatrix} u_{\mathrm{crit}}(x_0) \\ w_{\mathrm{crit}}(x_0) \end{bmatrix} := \mathbb{K}_\circlearrowleft x_0$ is the unique J_γ-critical input (called the* minimax control*), and this input corresponds to the (unique) arguments of*

$$\max_{w \in \ell^2(\mathbf{N};W)} \min_{u \in \mathcal{U}_{\mathrm{u}}(x_0,w)} \mathcal{J}(x_0,u,w). \tag{11.122}$$

- *(b2) The IARE has a $\mathcal{U}_*^*$-stabilizing solution $(\mathcal{P}, \widetilde{S}, \left[\begin{array}{c|c} \widetilde{\mathbb{K}} & \widetilde{\mathbb{F}} \end{array}\right])$ s.t. $S = \left[\begin{smallmatrix} I & 0 \\ 0 & J_W \end{smallmatrix}\right]$, where $J_W = J_W^* = J_W^{-1} \in \mathcal{GB}(W)$.*
- *(d1)* $(\mathbf{S_{22} - S_{21} S_{11}^{-1} S_{12} \ll 0})$ *If there is a suboptimal* H^∞*-FI-pair, then $S_{22} - S_{21} S_{11}^{-1} S_{12} \ll 0$, and the IARE has a $\mathcal{U}_*^*$-stabilizing solution $(\mathcal{P}, J_1, \left[\begin{array}{c|c} \widetilde{\mathbb{K}} & \widetilde{\mathbb{F}} \end{array}\right])$ s.t. $\widetilde{\mathbb{X}}_{11}, \widetilde{\mathbb{X}}_{22} \in \mathcal{G}\mathrm{tic}_\infty$, $\widetilde{X}_{21} = 0$, $\|\widetilde{\mathbb{X}}_{11}^{-1}\|_{\mathrm{tic}} \leq \varepsilon_+^{-1}$, and $\|\widetilde{\mathbb{X}}_{21} \widetilde{\mathbb{X}}_{11}^{-1}\|_{\mathrm{tic}} \leq 1$.*
- *(d2)* $(\mathbf{S_{22} \ll 0})$ *If there is a suboptimal* H^∞*-SF-operator, then $S_{22} \ll 0$.*
- *(f) In (d1) (resp. (d2)), the existence of a suboptimal* H^∞*-FI-pair (resp.* H^∞*-SF-operator) is not needed if there is $\mathbb{U} \in \mathrm{tic}_\infty(W,U)$ s.t. $\|\mathbb{D}_{11}\mathbb{U} + \mathbb{D}_{12}\|_{\mathrm{tic}} < \gamma$ (resp. and $\widehat{\mathbb{U}}(0) = 0$).*

Proof: Most of this follows as in the proof of Proposition 11.2.19. We explain the least obvious changes below.

(a1)&(a2) The unique J_γ-critical control corresponds to the unique $\mathcal{U}_*^*$-stabilizing solution of the DARE, by Theorem 14.1.6. Substitute $t = 1$ into (11.66) (whose continuous-time proof applies, mutatis mutandis) to obtain

$$\langle u_0', S_{11} u_0' \rangle_U \geq \varepsilon_+^2 \|u'\|_{\ell_\vartheta^2}^2 = \varepsilon_+^2 \|u_0'\|_U^2, \tag{11.123}$$

where $u_0' \in U$ is arbitrary. Thus, $S_{11} \geq \varepsilon_+^2 I$.

(d1) Substitute $t = 1$ into (11.77) to observe that

$$\left\langle \begin{bmatrix} \widetilde{U} \\ I \end{bmatrix} w_0, S \begin{bmatrix} \widetilde{U} \\ I \end{bmatrix} w_0 \right\rangle_{U \times W} \leq -\varepsilon \|w_0\|_W^2 \quad (w \in W), \tag{11.124}$$

where $\widetilde{U} \in \mathcal{B}(W,U)$ is the feedthrough operator of $\mathbb{U} := \widetilde{\mathbb{F}}_{12}^{\circlearrowleft}$ (the suboptimal closed-loop $w \mapsto u$ map). By Lemma 11.3.13(i')&(i)&(iii'), this means that $S_{22} - S_{21} S_{11}^{-1} S_{12} \leq -\varepsilon I$ and that

$$S = \widetilde{X}^* J_1 \widetilde{X}, \quad \text{where } \widetilde{X} = \begin{bmatrix} \widetilde{X}_{11} & \widetilde{X}_{12} \\ 0 & \widetilde{X}_{22} \end{bmatrix}, \ \widetilde{X}_{11} \gg 0 \gg \widetilde{X}_{22}. \tag{11.125}$$

By Theorem 9.9.1(f1), also $(\mathcal{P}, J_1, \left(\ \widetilde{X} K, I - \widetilde{X} \ \right))$ is a $\mathcal{U}_*^*$-stabilizing solution of the IARE. Since $\widetilde{X}_{11}, \widetilde{X}_{22} \in \mathcal{GB}$, we have $\widetilde{\mathbb{X}}_{11}, \widetilde{\mathbb{X}}_{22} \in \mathcal{G}\mathrm{TIC}_\infty$. The rest of (d1) follows from (11.77) as in the proofs of Proposition 11.2.19(c) and Lemma 11.2.21 (we do not need to study $\widetilde{\widetilde{\mathbb{X}}}$, since we already know that $\widetilde{\mathbb{X}}_{11} \in \mathcal{G}\mathrm{tic}_\infty$).

(d2) Now $\widetilde{U} = 0$ in (11.124), hence $S_{22} \leq -\varepsilon I$.

Remark: We did not have to assume that $\vartheta \leq 0$. This follows from the inequality $\| \left[\begin{smallmatrix} u \\ 0 \end{smallmatrix}\right] \|_{\ell_\vartheta^2} \geq \|u_0\|_U \ \ (u \in \ell_\vartheta^2)$, whose continuous-time analogy does not hold.

To obtain an analogous result in continuous time, we have to let $t \to 0+$, which requires additional regularity, as in Proposition 11.2.19(b3). □

Also the sufficiency part becomes simpler:

Lemma 11.6.3 (General $\mathcal{U}_*^*$: DARE $\Rightarrow$ ficp) *Assume that DARE has a $\mathcal{U}_*^*$-stabilizing solution $(\mathcal{P}, S, K)$ s.t. $\mathcal{P} \geq 0$, $S_{11} \gg 0$ and $S_{22} - S_{21} S_{11}^{-1} S_{12} \ll 0$.*

Then the assumptions of Lemma 11.2.14 are satisfied (including (4.)). In particular, if $\mathcal{U}_*^* = \mathcal{U}_{\text{exp}}$ (or $\mathcal{U}_*^* = \mathcal{U}_{\text{out}}$ and $(I - \mathbb{F})^{-1} \in \mathrm{tic}$), then (11.48) is a suboptimal H^∞*-FI-pair for Σ, with generators (11.118). If, in addition, $S_{22} \ll 0$, then (11.119) is a suboptimal* H^∞*-SF-operator for Σ.* □

(The proofs of Lemmas 11.2.13 and 11.2.14 apply mutatis mutandis; note that this lemma also holds without the assumption on the existence of ε_+ (see Standing Hypothesis 11.6.1).)

If we wish that some kind of controller $x_0, w \mapsto u$ stabilizes Σ exponentially (cf. (ii)), then so does a strict state feedback controller (i.e., a H^∞-SF-operator); conversely, such a controller is sufficient for our "H^∞ Finite Cost Condition" $\mathcal{U}_{\mathrm{u}}(x_0, w) \neq \emptyset$ for all x_0, w:

Lemma 11.6.4 (FCC $\Leftrightarrow$ $(\mathbf{A},\mathbf{B_1})$ exp. stab.) *Let $\mathcal{U}_*^* = \mathcal{U}_{\text{exp}}$. Then the following are equivalent:*

(i) $\left(\begin{array}{c|c} A & B_1 \end{array} \right)$ *is exponentially stabilizable (or optimizable);*

(ii) $\mathcal{U}_{\text{u}}(x_0,w) \neq \emptyset$ *for all* $x_0 \in H$ *and* $w \in \ell^2(\mathbf{N};W)$*;*

(iii) $\mathcal{U}_{\text{u}}(x_0,0) \neq \emptyset$ *for all* $x_0 \in H$*;*

(iv) there is a H^∞*-FI-pair for* Σ*;*

(v) there is a H^∞*-SF-operator for* Σ*.*

Proof: 1° *"(v)⇒(iv)" and "(ii)⇒(iii)":* These are trivial.

2° *"(iv)⇒(ii)":* Given $x_0 \in H$ and $w \in \ell^2(\mathbf{N};W)$, we have $\mathbb{A}x_0 + \mathbb{B}_1\tau u + \mathbb{B}_2\tau w \in \ell^2$, where $u := \mathbb{X}_{11}^{-1}\mathbb{K}_1 x_0 - \mathbb{B}_1\mathbb{X}_{11}^{-1}\mathbb{X}_{12}w \in \ell^2(\mathbf{N};U)$, by (11.9).

3° *"(iii)⇔(i)⇒(v)":* Obviously, (iii) says that $\left(\begin{array}{c|c} A & B_1 \end{array} \right)$ is optimizable. By Proposition 13.3.14, $\left(\begin{array}{c|c} A & B_1 \end{array} \right)$ is optimizable iff it is estimatable (hence we have (i)⇔(iii)), and in either case it has an exponentially stabilizing state feedback operator K_1 (i.e., $\rho(A+B_1K_1) < 1$). But then $\begin{bmatrix} K_1 \\ 0 \end{bmatrix}$ is obviously exponentially stabilizing for Σ (because the corresponding closed-loop state-to-state map is also given by $A+B_1K_1$); hence it is a H^∞-SF-operator, by Remark 11.2.5. $\square$

Thus, if $\mathcal{U}_*^* = \mathcal{U}_{\text{exp}}$, then we must assume that $\left(\begin{array}{c|c} A & B_1 \end{array} \right)$ is exponentially stabilizable, so that the problem can be reduced to the exponentially stable case, by Lemma 11.2.22 (which obviously holds in discrete time too).

(See the notes on p. 677.)

11.7 The abstract H^∞ FICP

You can not win the game, and you are not allowed to stop playing.

— The Third Law Of Thermodynamics (this is what we wish to make for the destiny of the second player (nature/disturbance))

In this section, we solve the H^∞ FICP in the abstract setting of Section 8.1. These results were applied to WPLSs and wpls's in previous sections.

For simplicity of notation, we replace the space U of inputs by $U \times W$, where $u \in U$ is considered as "control" and $w \in W$ is considered as "disturbance"; otherwise we share the notation of Section 8.1:

Standing Hypothesis 11.7.1 *Throughout this section, we shall assume that U, W, X, Y^{s} and Z^{s} are reflexive Banach spaces, that Y and Z are TVSs, and that the embeddings $Y^{\mathrm{s}} \subset Y$ and $Z^{\mathrm{s}} \subset Z$ are continuous. We also assume that $\begin{bmatrix} A & B_1 & B_2 \\ C & D_1 & D_2 \end{bmatrix} \in \mathcal{B}(X \times U \times W, Z \times Y)$ and $J = J^* \in \mathcal{B}(Y^{\mathrm{s}}, Y^{\mathrm{s}*})$.*

Note that $U \times W$ has now taken the role of U; we also set

$$\mathcal{J}(x,u,w) := \mathcal{J}(x,\left[\begin{smallmatrix} u \\ w \end{smallmatrix}\right]) := \langle D\left[\begin{smallmatrix} u \\ w \end{smallmatrix}\right], JD\left[\begin{smallmatrix} u \\ w \end{smallmatrix}\right]\rangle_{Y^{\mathrm{s}}} \quad (\left[\begin{smallmatrix} u \\ w \end{smallmatrix}\right] \in \mathcal{U}(x)). \tag{11.126}$$

In applications one usually takes $U = \mathrm{L}^2(\mathbf{R}_+;U_0)$, $W = \mathrm{L}^2(\mathbf{R}_+;W_0)$ for some Hilbert spaces U_0, W_0 (cf. Remark 8.3.4).

As in previous sections, in the H^∞ FICP one wishes to find a control law $X \times W \ni (x,w) \mapsto u_{x,w} \in U$ s.t. $\left[\begin{smallmatrix} u_{x,w} \\ w \end{smallmatrix}\right]$ is a "stabilizing" input for the given "initial state" x, for each "disturbance" w, i.e., $\left[\begin{smallmatrix} u_{x,w} \\ w \end{smallmatrix}\right] \in \mathcal{U}(x)$ for all $x \in X$ and all $w \in W$. We denote the set of these "admissible controls" by

$$\mathcal{U}_{\mathrm{u}}(x,w) := \{u \in U \mid \left[\begin{smallmatrix} u \\ w \end{smallmatrix}\right] \in \mathcal{U}(x)\} = \{u \in U \mid Cx + Du \in Y^{\mathrm{s}} \ \& \ Ax + Bu \in Z^{\mathrm{s}}\}. \tag{11.127}$$

(We set $A = 0 = B$ or $Z^{\mathrm{s}} = Z$ if we only wish to require the output to be stable.)

Moreover, this control law should be "suboptimal" (see p. 613)., However, to keep the notation simple, we study here the more general *extended Full-Information* H^∞ *Control Problem (*H^∞ *eFICP)*, where the suboptimality condition is replaced by the condition that there is $\varepsilon > 0$ s.t. $\mathcal{J}(0, u_{x,w}, w) \leq -\varepsilon \|w\|^2$. As noted on p. 613, this condition is equivalent to suboptimality in the (special case) setting of (11.1) (with $\mathbb{D} := D = \begin{bmatrix} D_1 & D_2 \end{bmatrix}$).

In the following theorem, we first show that the existence of a suboptimal control law implies J-coercivity, and then we show how the unique J-critical control is of maximin form. In Propositions 11.6.2 and 11.2.19, these facts were used to show to necessity of the signature condition and the existence of a unique stabilizing solution to the Riccati equation.

Theorem 11.7.2 (H^∞ eFICP) *Set $\mathcal{U}_\mathrm{u}(x,w) := \{u \in U \mid \left[\begin{smallmatrix} u \\ w \end{smallmatrix}\right] \in \mathcal{U}(x)\}$. Assume that there is $\varepsilon_\mathrm{u} > 0$ s.t.*

$$\langle D_1 u, J D_1 u\rangle \geq \varepsilon_\mathrm{u} \| \left[\begin{smallmatrix} u \\ 0 \end{smallmatrix}\right] \|_D^2 \qquad (u \in \mathcal{U}_\mathrm{u}(0,0)). \tag{11.128}$$

Then the following hold:

(a) We have (i)⇔(ii), where

(i) There is $\varepsilon_\mathrm{w} > 0$ s.t. $\inf_{u\in\mathcal{U}_\mathrm{u}(0,w)} \mathcal{J}(0,u,w) \leq -\varepsilon_\mathrm{w}\|w\|^2$ for all $w \in W$.
(ii) There is $F^\curvearrowleft \in \mathcal{B}(W,U)$ s.t. $F^\curvearrowleft w \in \mathcal{U}_\mathrm{u}(0,w)$ for all $w \in W$, $D^\curvearrowleft := D\left[\begin{smallmatrix} F^\curvearrowleft \\ I \end{smallmatrix}\right] \in \mathcal{B}(W, Y^\mathrm{s})$, and $D^{\curvearrowleft *} J D^\curvearrowleft \ll 0$ (on W).

(b) If (i) (or (ii)) holds, then D is J-coercive.

(c) Assume that (i) (or (ii)) holds and $\mathcal{U}(x) \neq \emptyset$ for all $x \in X$. Then there are $F^\curvearrowleft \in \mathcal{B}(W,U)$, $K_u^\curvearrowleft \in \mathcal{B}(X,U)$, $K_\mathrm{crit}^\mathrm{w} \in \mathcal{B}(X,W)$ s.t.

$$\begin{bmatrix} K_u^\curvearrowleft & F^\curvearrowleft \end{bmatrix} \left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathcal{U}_\mathrm{u}(x,w) \qquad (x \in X,\ w \in W), \tag{11.129}$$

$$\mathcal{J}(x, K_u^\curvearrowleft x + F^\curvearrowleft w, w) = \min_{u\in\mathcal{U}_\mathrm{u}(x,w)} \mathcal{J}(x,u,w) \quad (x \in X,\ w \in W), \quad \textit{and} \tag{11.130}$$

$$\min_{u\in\mathcal{U}_\mathrm{u}(x,K_\mathrm{crit}^\mathrm{w} x)} \mathcal{J}(x,u,K_\mathrm{crit}^\mathrm{w} x) = \max_{w\in W} \min_{u\in\mathcal{U}_\mathrm{u}(x,w)} \mathcal{J}(x,u,w) \quad (x \in X). \tag{11.131}$$

Moreover, then for any $x \in X$, the unique J-critical input is given by Kx, where $K := \left[\begin{smallmatrix} K_1 \\ K_2 \end{smallmatrix}\right] := \left[\begin{smallmatrix} K_u^\curvearrowleft + F^\curvearrowleft K_\mathrm{crit}^\mathrm{w} \\ K_\mathrm{crit}^\mathrm{w} \end{smallmatrix}\right]$, and the corresponding cost is

$$\mathcal{J}(x, K_1 x, K_2 x) = \max_{w\in W} \min_{u\in\mathcal{U}_\mathrm{u}(x,w)} \mathcal{J}(x,u,w) \geq \min_{u\in\mathcal{U}_\mathrm{u}(x,0)} \langle Cx + D_1 u, J(Cx + D_1 u)\rangle. \tag{11.132}$$

In the FICP (see (11.1), we have $J \geq 0$ on $\mathrm{Ran}(\begin{bmatrix} C & D_1 \end{bmatrix})$, so that $\mathcal{J}(x, K_1 x, K_2 x) \geq 0$ for all $x \in X$, by (11.132); this leads to a nonnegative J-critical cost operator $\mathcal{P}$ (see Proposition 11.2.19). Condition (11.128) is the standard nonsingularity assumption, and it is necessary for the J-coercivity stated in the theorem.

Here $F^\curvearrowleft$ and $K_u^\curvearrowleft$ refer to a system where only u is controlled ("the optimal state feedback through the control input u"), whereas the J-critical "maximin state feedback" K controls both u and w. By (11.132), (i) implies the J-critical control corresponds to "the best control u under the worst disturbance w" (or "the maximin control $\left[\begin{smallmatrix} u \\ w \end{smallmatrix}\right]$").

By slightly modifying the proof of (a), one can show that conditions (i) and (ii) hold iff D is J-coercive and $\min_{u\in\mathcal{U}_\mathrm{u}(0,w)} \mathcal{J}(0,u,w) \leq 0$ for all $w \in W$. Mere J-coercivity is not sufficient (take, e.g., $D = \left[\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{smallmatrix}\right]^\mathrm{T}$, $J = \mathrm{diag}(1,1,-\gamma)$, $\gamma < 1$, so that $D^* J D = \left[\begin{smallmatrix} 1 & 0 \\ 0 & (1-\gamma) \end{smallmatrix}\right] \gg 0$), we must also know that the cost function is concave in w (it is convex in u, by (11.128)). If $\mathcal{U}(x) \neq \emptyset$ for all $x \in H$ ("the Finite Cost

Condition is satisfied"), then a fourth equivalent condition is the existence of a J-critical K (as above) satisfying a suitable signature condition.

However, such additional conditions can be obtained in more useful forms when working with causal systems, therefore we have placed a further treatment in Sections 11.2 and 11.6.

Proof of Theorem 11.7.2: (a) 1° *(ii)⇒(i):* Assume (ii). Then there is $\varepsilon_{\mathrm{w}} > 0$ s.t. $\mathcal{J}(0, F^{\curvearrowleft}w, w) = \langle D^{\curvearrowleft}w, JD^{\curvearrowleft}w\rangle \le -\varepsilon_{\mathrm{w}}\|w\|^2$ for all $w \in W$, hence (i) holds.

2° *(i)⇒(ii):* Assume (i). Then $\mathcal{U}_{\mathrm{u}}(0,w) \neq \emptyset$ for all $w \in W$ (since $\inf\emptyset = +\infty$). The map D_1 with the system $\Sigma_{\mathrm{u}} := \left[\begin{array}{c|c} B_2 & B_1 \\ \hline D_2 & D_1 \end{array}\right]$ is J-coercive (this corresponds to substitutions $X \mapsto W$, $\mathcal{U}_*^* \mapsto \mathcal{U}_{\mathrm{u}}(0,\cdot)$ and $\mathcal{J}(w,u) \mapsto \mathcal{J}(0,u,w)$), by (11.128); indeed, we obviously have $\|u\|_{D_1} = \|\left[\begin{smallmatrix} u \\ 0 \end{smallmatrix}\right]\|_D$. Therefore, is a unique J-critical control $F^{\curvearrowleft}w$ for Σ_{u} for each $w \in W$, where

$$\left[\begin{array}{c} B^{\curvearrowleft} \\ \hline D^{\curvearrowleft} \\ F^{\curvearrowleft} \end{array}\right] \in \mathcal{B}(W, Z^{\mathrm{s}} \times Y^{\mathrm{s}} \times U) \tag{11.133}$$

is as in Theorem 8.1.10; in particular, $F^{\curvearrowleft}w \in \mathcal{U}_{\mathrm{u}}(0,w)$ and $\langle D_2 w + D_1 F^{\curvearrowleft}w, Du\rangle = 0$ for all $u \in \mathcal{U}_{\mathrm{u}}(0,w)$ and $w \in W$, and

$$\mathcal{J}(0, F^{\curvearrowleft}w, w) = \min_{u \in \mathcal{U}_{\mathrm{u}}(0,w)} \mathcal{J}(0,u,w) \le -\varepsilon_{\mathrm{w}}\|w\|^2 \text{ for each } w \in W. \tag{11.134}$$

Since $D^{\curvearrowleft} := D_2 + D_1 F^{\curvearrowleft} \in \mathcal{B}(W, Y^{\mathrm{s}})$, we obtain (ii) from (11.134) (because $\mathcal{J}(0, F^{\curvearrowleft}w, u) = \langle D^{\curvearrowleft}w, JD^{\curvearrowleft}w\rangle$). (Intuitively, $F^{\curvearrowleft}$ is the "optimal stabilizing controller" $w \mapsto u$.)

(b) Assume (i). We have $\begin{bmatrix} D_1 & D^{\curvearrowleft} \end{bmatrix} = DE$, where $E := \left[\begin{smallmatrix} I & F^{\curvearrowleft} \\ 0 & I \end{smallmatrix}\right]$, where $F^{\curvearrowleft}$ is as in (a)2°. Because $\left[\begin{smallmatrix} F^{\curvearrowleft}w \\ w \end{smallmatrix}\right] \in \mathcal{U}(0)$ for all $w \in W$, we have $E\left[\begin{smallmatrix} u \\ w \end{smallmatrix}\right] \in \mathcal{U}(0) \Leftrightarrow u \in \mathcal{U}_{\mathrm{u}}(0,0)$ & $w \in W$, by Lemma 8.1.4; thus, $\mathcal{U}(0) = E[\mathcal{U}_{\mathrm{u}}(0,0) \times W]$. From this and (a)2° we see that the assumptions of Lemma 11.7.3 are satisfied for $\left[\begin{smallmatrix} A & BE \\ C & DE \end{smallmatrix}\right]$, so that DE is J-coercive. By Lemma 8.2.4(c), it follows that also D is J-coercive.

(c) 1° $\mathcal{U}_{\mathrm{u}}(x,w) \neq \emptyset$ *for all x and w:* Given $x \in X$ and $w \in W$, there are $\begin{bmatrix} u' \\ w' \end{bmatrix} \in \mathcal{U}(x)$ and $u'' \in \mathcal{U}_{\mathrm{u}}(0, w - w')$, by the assumptions, so that $\left[\begin{smallmatrix} u' + u'' \\ w \end{smallmatrix}\right] \in \mathcal{U}(x)$, i.e., $u' + u'' \in \mathcal{U}_{\mathrm{u}}(x,w)$. Thus, $\mathcal{U}_{\mathrm{u}}(x,w) \neq \emptyset$ for all x and w.

2° *Equations (11.129) and (11.130):* These are obtained as in (a)2° (with substitutions $X \mapsto \left[\begin{smallmatrix} X \\ W \end{smallmatrix}\right]$, $x \mapsto \left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$, $\mathcal{U}_*^*(\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]) \mapsto \mathcal{U}_{\mathrm{u}}(x,w)$ and $\mathcal{J}(\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right], u) \mapsto \mathcal{J}_{\mathrm{u}}(\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right], u) := \mathcal{J}(x,u,w)$). (Obviously, $F^{\curvearrowleft}$ is the same as in (a)2°.)

3° *Equation (11.131):* We also obtain in 2° that

$$\Sigma_{\mathrm{ext}}^{\curvearrowleft} := \left[\begin{array}{c|c} A^{\curvearrowleft} & B^{\curvearrowleft} \\ \hline C^{\curvearrowleft} & D^{\curvearrowleft} \end{array}\right] := \left[\begin{array}{c|c} A + B_1 K_u^{\curvearrowleft} & B_2 + B_1 F^{\curvearrowleft} \\ \hline C + D_1 K_u^{\curvearrowleft} & D_2 + D_1 F^{\curvearrowleft} \end{array}\right] \in \mathcal{B}(X \times W, Z \times Y^{\mathrm{s}}) \tag{11.135}$$

and $D^{\curvearrowleft *} J D^{\curvearrowleft} \ll 0$.

Apply Corollary 8.2.7 and Theorem 8.1.10 to $\Sigma_{\mathrm{ext}}^{\curvearrowleft}$ and $-J$ (with cost function $=: -\mathcal{J}^{\curvearrowleft}$) to obtain that there is $K_{\mathrm{crit}}^{\mathrm{w}} \in \mathcal{B}(X, W)$ s.t.

$$\mathcal{J}^{\curvearrowleft}(x, K_{\mathrm{crit}}^{\mathrm{w}} x) = \max_{w \in W} \mathcal{J}^{\curvearrowleft}(x,w) = \max_{w \in W} \min_{u \in \mathcal{U}(x,w)} \mathcal{J}(x,u,w) \quad (x \in X). \tag{11.136}$$

4° "$\begin{bmatrix} K_u^{\curvearrowleft}+F^{\curvearrowleft}K_{\rm crit}^{\rm w} \\ K_{\rm crit}^{\rm w}\end{bmatrix} x$ *is the unique J-critical control for any $x \in X$":* Let $x \in X$. By (b), D is J-coercive, so that there is a unique J-critical control for x, by Theorem 8.2.5. Thus, we only have to show that this control is given by $\left[\begin{smallmatrix} u \\ w \end{smallmatrix}\right] := \begin{bmatrix} K_u^{\curvearrowleft}+F^{\curvearrowleft}K_{\rm crit}^{\rm w} \\ K_{\rm crit}^{\rm w}\end{bmatrix} x$.

Let $\begin{bmatrix} \eta \\ \eta' \end{bmatrix} \in \mathcal{U}(0)$ be arbitrary. Set $\eta'' := \eta - F^{\curvearrowleft}\eta' = \eta - u_{\rm crit}(0,\eta')$. Apply Lemma 8.1.7(ii) to $\mathcal{J}_{\rm u}$, to $\mathcal{J}_{\rm w}$ and then again to $\mathcal{J}_{\rm u}$ to obtain (here $u_{\rm crit}(x',w') := \begin{bmatrix} K_u^{\curvearrowleft} & F^{\curvearrowleft} \end{bmatrix} \begin{bmatrix} x' \\ w' \end{bmatrix}$ for all x',w') that $\mathcal{J}(x,u+\eta,w+\eta')$

$$= \mathcal{J}(x,K_u^{\curvearrowleft}x + F^{\curvearrowleft}(w+\eta') + \eta'', w+\eta') \tag{11.137}$$
$$= \mathcal{J}_{\rm u}(\left[\begin{smallmatrix} x \\ w+\eta' \end{smallmatrix}\right], u_{\rm crit}(x,w+\eta')) + \mathcal{J}_{\rm u}(0,\eta'') \tag{11.138}$$
$$= \mathcal{J}^{\curvearrowleft}(x,w+\eta') + \mathcal{J}_{\rm u}(0,\eta'') = \mathcal{J}^{\curvearrowleft}(x,w) + \mathcal{J}^{\curvearrowleft}(0,\eta') + \mathcal{J}_{\rm u}(0,\eta'') \tag{11.139}$$
$$= \mathcal{J}(x,u,w) + \mathcal{J}_{\rm u}(\begin{bmatrix} 0 \\ \eta' \end{bmatrix}, u_{\rm crit}(0,\eta')) + \mathcal{J}_{\rm u}(0,\eta'') \tag{11.140}$$
$$= \mathcal{J}(x,u,w) + \mathcal{J}_{\rm u}(\begin{bmatrix} 0 \\ \eta' \end{bmatrix}, u_{\rm crit}(0,\eta') + \eta'') = \mathcal{J}(x,u,w) + \mathcal{J}(0,\eta,\eta'). \tag{11.141}$$

By "(ii)$\Rightarrow$(i)" of Lemma 8.1.7, $\left[\begin{smallmatrix} u \\ w \end{smallmatrix}\right]$ is J-critical for x.

5° *Equation (11.132):* This follows from (11.130) and (11.131) (the inequality follows by taking $w = 0$). □

The following result for "uncoupled" H^∞ control was used above:

Lemma 11.7.3 *Assume that $\mathcal{U}(0) = U_1 \times W$ for some $U_1 \subset U$, $D_2^*JD_2 \leq 0$, $\langle D_1u, D_2w\rangle = 0$ for all $u \in U_1$, $w \in W$, $\varepsilon_{\rm u} > 0$, and $\langle D_1u, JD_1u\rangle \geq \varepsilon_{\rm u}(\|u\|_U^2 + \|B_1u\|_{Z^{\rm s}}^2 + \|D_1u\|_{Y^{\rm s}}^2)$ for all $u \in U$. Then D is J-coercive iff $D_2^*JD_2 \ll 0$.*

Proof: Since $\mathcal{U}(0)$ is a subspace of $U \times W$, U_1 is a subspace of U. Since $B_2[W] \subset Z^{\rm s}$ and $D_2[W] \subset Y^{\rm s}$, we have $B_2 \in \mathcal{B}(W,Z^{\rm s})$ and $D_2 \in \mathcal{B}(W,Y^{\rm s})$, by Lemma A.3.6 (so that $D_2^*JD_2$ is well defined on W).

1° *"Only if":* If D is J-coercive, then, for each nonzero $w \in W$, there is a nonzero $w' \in W$ s.t. $\langle D_2w', JD_2w\rangle \geq \varepsilon'\|w\|\|w'\|$, where ε' is as in (8.12) (take $u = 0$). Consequently, then $D_2^*JD_2 \ll 0$. by Lemma A.3.4(c1)(xi)&(c4)&(b1).

2° *"If":* Assume that $D_2^*JD_2 \ll -\varepsilon_{\rm w}I$. Given $u \in U_1$ and $w \in W$, we have

$$\langle D\left[\begin{smallmatrix} u \\ -w \end{smallmatrix}\right], JD\left[\begin{smallmatrix} u \\ w \end{smallmatrix}\right]\rangle = \varepsilon_{\rm u}(\|u\|^2 + \|B_1u\|^2 + \|D_1u\|^2) + \varepsilon_{\rm w}\|w\|^2 \tag{11.142}$$
$$\geq \varepsilon'(\|u\|^2 + \|w\|^2 + \|\left[\begin{smallmatrix} u \\ w \end{smallmatrix}\right]\|^2 + \|B_1u\|^2 + \|B_2\|^2\|w\|^2 + \|D_1u\|^2 + \|D_2\|^2\|w\|^2), \tag{11.143}$$

where $\varepsilon' := \min\{\varepsilon_{\rm u}, \varepsilon_{\rm w}\}$, from which one easily obtains (8.12). □

Notes

For finite-dimensional WPLSs, the idea to first minimize w.r.t. u and then maximize w.r.t. w is very old, although we do not know any references to the exact technique used in this section. See the previous sections for applications to WPLSs.

11.8 The Nehari problem

If you think the problem is bad now, just wait until we've solved it.
— Arthur Kasspe

In the Nehari problem, one wishes to determine the norm of the Hankel operator $\pi_+\mathbb{G}\pi_-$ of a given map $\mathbb{G} \in \mathrm{TIC}(W,U)$, or alternatively, to find a *suboptimal approximation* $-\mathbb{U}^* \in \mathrm{TIC}^*(W,U)$ of $\mathbb{G}$.

Given $\gamma > 0$, one has $\|\pi_+\mathbb{G}\pi_-\| < \gamma$ iff $\inf_{\mathbb{U}\in\mathrm{TIC}(U,W)} \|\mathbb{G}+\mathbb{U}^*\| < \gamma$, i.e., iff $d(\mathbb{G},\mathrm{TIC}^*) < \gamma$ (or $d(\widehat{\mathbb{G}},\mathrm{H}^\infty(\mathbf{C}^-;\mathcal{B}(W,U))) < \gamma$), by the Nehari Theorem 11.8.3. As the theorem shows, one can reduce the problem of finding such an operator $\mathbb{U}$ to a co-spectral factorization problem, and such a factorization leads to a parameterization of all solutions $\mathbb{U}$ of $\|\mathbb{G}+\mathbb{U}^*\| < \gamma$.

(Sometimes the problem is formulated so that $\mathbb{G}$ belongs to MTI or some other noncausal decomposing class. Then one may reduce the problem to the above one by replacing $\mathbb{G}$ with its causal part $\mathbb{G}_+$, where $\mathbb{G} = \mathbb{G}_+ + \mathbb{G}_-$, $\mathbb{G}_+, \mathbb{G}_-^* \in \mathrm{TIC}$.)

In this section, we take a very brief look at the aspects of the Nehari problem that closely resemble those of the stable H^∞ FICP.

Standing Hypothesis 11.8.1 *Throughout Sections 11.8–11.9, we assume the following:* $\Sigma := \left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{G} \end{array}\right] \in \mathrm{WPLS}(W,H,U)$ *is stable and* $\gamma > 0$.

Recall from Definition 6.1.6 that any $\mathbb{G} \in \mathrm{TIC}(W,U)$ has a realization Σ (for some H).

We shall show that the following conditions are equivalent (under some additional assumptions):

Definition 11.8.2 *We define* $\mathbb{D} := \begin{bmatrix} I & \mathbb{G} \\ 0 & I \end{bmatrix} \in \mathrm{TIC}(U \times W)$*, and we define the following conditions:*

(i) There is $\mathbb{U} \in \mathrm{TIC}(U,W)$ *s.t.* $\|\mathbb{G}+\mathbb{U}^*\|_{\mathcal{B}(\mathrm{L}^2)} < \gamma$ *(i.e.,* $d(\mathbb{G},\mathrm{TIC}^*) < \gamma$*).*

(ii) The Hankel norm $\|\pi_+\mathbb{G}\pi_-\| = \rho(\mathbb{B}\mathbb{B}^*\mathbb{C}^*\mathbb{C})^{1/2}$ *of* $\mathbb{G}$ *is less than* γ*.*

(iii) The I/O map $\mathbb{D}$ *is co-minimax* J_γ*-coercive [Def11.4.1].*

(iv) The Toeplitz operator $\mathbb{F} := \pi_-\mathbb{D}^*J\mathbb{D}\pi_-$ *satisfies* $\mathbb{F}_{22} - \mathbb{F}_{21}\mathbb{F}_{11}^{-1}\mathbb{F}_{12} \ll 0$*.*

(v) The I/O map $\mathbb{D} := \begin{bmatrix} I & \mathbb{G} \\ 0 & I \end{bmatrix} \in \mathrm{TIC}$ *has a (co-spectral) factorization* $\mathbb{D}^*J_\gamma\mathbb{D} = \mathbb{X}J_1\mathbb{X}^*$ *with* $\mathbb{X},\mathbb{X}_{11} \in \mathcal{G}\mathrm{TIC}$*.*

We have $\pi_+\mathbb{G}\pi_- = \mathbb{C}\mathbb{B}$, by Definition 6.1.1, hence

$$\rho(\mathbb{B}\mathbb{B}^*\mathbb{C}^*\mathbb{C})^{1/2} = \rho(\mathbb{B}^*\mathbb{C}^*\mathbb{C}\mathbb{B})^{1/2} = \|\mathbb{C}\mathbb{B}\| = \|\pi_+\mathbb{G}\pi_-\|, \tag{11.144}$$

by Lemma A.3.3(s2), Lemma A.3.1(c6) and Definition 6.1.1. Note that $\mathbb{B}\mathbb{B}^*$ is the reachability Gramian and $\mathbb{C}^*\mathbb{C}$ is the observability Gramian of Σ. Except for this observation (and the equality in (ii)), our claims only concern the I/O map $\mathbb{G}$, not the realization Σ.

If $\mathbb{D} \in \mathrm{MTIC}$ and $\dim U \times W < \infty$, or if $\mathbb{D} \in \mathrm{MTIC}_{T\mathbf{Z}}$, then (i)–(v) are equivalent:

Theorem 11.8.3 (Nehari) *We have (v)⇒(i)⇒(ii)⇔(iii)⇔(iv). If U and W are separable, then (i)–(iv) are equivalent.*

If $\mathcal{A}^* = \boldsymbol{Я}\mathcal{A}\boldsymbol{Я} = \mathcal{A} \underset{a}{\subset}$ TI, $\mathbb{G} \in \mathcal{A}$, and $\mathcal{A}(U \times W)$ admits spectral factorization (e.g., (α) or (β) of Theorem 8.4.9 holds), then (i)–(v) are equivalent.

If (v) holds, then all solutions to (i) are the ones given by Theorem 11.9.4 (if $\mathbb{G} \in \mathcal{A}$, then $\mathbb{X} \in \mathcal{A}$, by Theorem 11.9.3, hence then we can take $\mathbb{U} \in \mathcal{A}$, by Theorem 11.9.4, and all solutions $\mathbb{U} \in \mathcal{A}$ correspond to all parameters $\widetilde{\mathbb{U}} \in \mathcal{A}$, as in Theorem 11.3.6).

By "$\mathcal{A}(U \times W)$ admits spectral factorization" we mean that if $\mathbb{E} \in \mathcal{A}(U \times W)$ and $\pi_+ \mathbb{E} \pi_+$ is (boundedly) invertible on $\mathrm{L}^2(\mathbf{R}_+; U \times W)$ (this condition is necessary for any $\mathcal{A} \underset{a}{\subset}$ TI), then $\mathbb{E} = \mathbb{X}^* S \mathbb{X}$ for some $\mathbb{X} \in \mathcal{G}\mathrm{TIC}(U \times W)$, $S \in \mathcal{G}\mathcal{B}(U \times W)$.

Proof: 1° The first chain of implications follows from Theorems 11.9.4, 11.9.2 and 11.9.1 and Definition 11.4.1.

2° Implication (ii)⇒(i) follows from Theorem 11.9.2.

3° By 1° and Theorem 11.9.3, (i)–(v) are equivalent. (From the proof of Theorem 8.4.9 we observe that each of assumptions (α) and (β) is sufficient for the admissibility assumption; obviously the former condition is satisfied, i.e., $\mathcal{A} \underset{a}{\subset}$ TI and $\mathbb{E}^*, \boldsymbol{Я}\mathbb{E}\boldsymbol{Я} \in \mathcal{A}$ for each $\mathbb{E} \in \mathcal{A}$.) □

One observes directly from the definition that $\mathbb{D}$ is minimax J_γ-coercive. However, we have no use for this fact.

Notes

Above, we have primarily given frequency-domain results. One traditionally also establishes the connection to two particular Riccati equations having nonnegative solutions; see, e.g., [IOW] for the finite-dimensional case (both continuous and discrete time), [CZ] for the case with bounded B and C, and [CZ94] for the case of smooth Pritchard–Salamon systems. The above references only treat the exponentially stable case; see [CO98] for WPLSs with $\mathbb{B}$, $\mathbb{C}$ and $\mathbb{D}$ stable and B, C bounded (but $\mathbb{A}$ possibly unstable; note also that $\mathbb{A}$ could be allowed to be unstable through Sections 11.8–11.9).

For $\dim U \times Y < \infty$, the Nehari problem has an extended version, the *Hankel norm approximation problem*, whose I/O form was presented and solved in [AAK] (the *Adamjan–Arov–Krein theorem*). An up-to-date state-space solution for exponentially stable analytic and PS-systems is given in [Sasane] (partially also in [SC]).

Since the Nehari Riccati equation theory does not follow from the theory of Chapter 9, we omit the state-space part corresponding to the above results; it might be worth of a separate study (where results such as Theorem 8.3.9 should be rewritten in this "noncausal" setting (e.g., we have $\mathbb{N}^* J \mathbb{N} = S$, where $\mathbb{N} := \mathbb{D}\mathbb{X}^{-*} \notin \mathrm{TIC} \cup \mathrm{TIC}^*$ in general)). Part of the state-space theory (including a realization of the spectral factor) is given in [SM], due to Olof Staffans.

The results of this section are well known in the generality treated in the above references (excluding [SM]), and the implication "(i)⇔(ii)" is contained in [Treil85] in the separable case.

11.9 The proofs for Section 11.8

HOW TO PROVE IT, PARTS 1–2:
Proof by intimidation: 'Trivial'.
Proof by vigorous handwaving: Works well in a classroom or seminar setting.
Proof by cumbersome notation: Best done with access to at least four alphabets and special symbols.

FURTHER PROOF TECHNIQUES:
Blatant assertion.
Changing all the 2's to n's.
Mutual consent.
Lack of a counterexample.

Now we can prove the implications compiled into Theorem 11.8.3:

Theorem 11.9.1 *(iii)⇔(ii).*

Proof: We have

$$\mathbb{F} := \pi_+ \mathbb{D}^* J \mathbb{D} \pi_+ = \begin{bmatrix} \pi_- & \pi_- \mathbb{G} \\ \mathbb{G}^* \pi_- & \pi_- \mathbb{G}^* \mathbb{G} \pi_- - \gamma^2 \pi_- \end{bmatrix}, \tag{11.145}$$

hence the co-minimax J-coercivity of $\mathbb{D}$ is equivalent to $0 \gg \pi_- \mathbb{G}^* \mathbb{G} \pi_- - \gamma^2 \pi_- - \mathbb{G}^* \pi_- \mathbb{G} = -(\gamma^2 \pi_- - \pi_- \mathbb{G}^* \pi_+ \mathbb{G} \pi_-)$, hence to $\gamma^2 \pi_- \gg \pi_- \mathbb{G}^* \pi_+ \mathbb{G} \pi_-)$, hence to $\|\pi_+ \mathbb{G} \pi_-\| < \gamma$, by Lemma A.1.1(d). □

Theorem 11.9.2 *We have (i)⇒(ii). The converse is true for separable U and W.*

Proof: 1° "(i)⇒(ii)": Because $\pi_+ \mathbb{U}^* \pi_- = 0$, we have

$$\|\pi_+ \mathbb{G} \pi_-\| = \|\pi_+ (\mathbb{G} + \mathbb{U}^*) \pi_-\| \leq \|\mathbb{G} + \mathbb{U}^*\| \tag{11.146}$$

for $\mathbb{U} \in \text{TIC}$, hence (i) implies (ii).

2° "(ii)⇒(i)": By, e.g., the Theorem on p. 57 of [Treil85] (and Theorem 13.2.3), this is true at least in the separable case, and (in the sepable case) we also have that

$$\|\pi_+ \mathbb{G} \pi_-\| = \inf_{\widehat{\mathbb{U}}(-\cdot) \in \mathrm{H}^\infty} \|\widehat{\mathbb{G}} + \widehat{\mathbb{U}}\|_\infty = \inf_{\mathbb{U} \in \text{TIC}} \|\mathbb{G} + \mathbb{U}^*\|_{\mathcal{B}(\mathrm{L}^2)} = \min_{\mathbb{U} \in \text{TIC}} \|\mathbb{G} + \mathbb{U}^*\|_{\mathcal{B}(\mathrm{L}^2)}. \tag{11.147}$$

□

Theorem 11.9.3 *Assume that $\mathcal{A}^* = \mathbf{Я} \mathcal{A} \mathbf{Я} = \mathcal{A} \underset{a}{\subset} \text{TI}$, that $\mathbb{G} \in \mathcal{A}$, and that $\mathcal{A}(U \times W)$ admits spectral factorization. Then (iii) implies (v) (with $\mathbb{X} \in \mathcal{GA}$).*

Proof: Assume (iii). Then, by Lemma 11.4.2, $\pi_- \mathbb{D}^* J \mathbb{D} \pi_- \in \mathcal{GB}(\mathrm{L}^2(\mathbf{R}_-; U \times W))$, i.e., $\pi_+ \mathbb{D}^\mathrm{d} J (\mathbb{D}^\mathrm{d})^* \pi_+ \in \mathcal{GB}(\mathrm{L}^2(\mathbf{R}_+; U \times W))$. Consequently, there are $\mathbb{X}^\mathrm{d} \in \mathcal{G}\text{TIC}(U \times W) \cap \mathcal{A}(U \times W)$ and $S \in \mathcal{GB}(U \times W)$ s.t. $(\mathbb{X}^\mathrm{d})^* S \mathbb{X}^\mathrm{d} = \mathbb{D}^\mathrm{d} J (\mathbb{D}^\mathrm{d})^*$, i.e., $\mathbb{X} S \mathbb{X}^* = \mathbb{D}^* J \mathbb{D}$.

By Corollary 11.4.7, we can have $S = J_1$ and $\mathbb{X}_{11} \in \mathcal{G}\mathrm{TIC}(U)$. $\square$

All solutions to $\|\mathbb{G}+\mathbb{U}^*\| < \gamma$ are given by the standard formula (cf. Theorem 3.2 of [CO98]):

Theorem 11.9.4 (All solutions) *Let the I/O map* $\mathbb{D} := \begin{bmatrix} I & \mathbb{G} \\ 0 & I \end{bmatrix} \in \mathrm{TIC}$ *have a co-spectral factorization* $\mathbb{D}^* J_\gamma \mathbb{D} = \mathbb{X} J_1 \mathbb{X}^*$ *with* $\mathbb{X}, \mathbb{X}_{11} \in \mathcal{G}\mathrm{TIC}$ *(i.e., let (v) hold). Then* $\mathbb{U}^* := \mathbb{M}_{21}^* \mathbb{M}_{22}^{-*} = -\mathbb{X}_{11}^{-*} \mathbb{X}_{21}^*$ *satisfies* $\|\mathbb{G}+\mathbb{U}^*\| < \gamma$*, where* $\mathbb{M} := \mathbb{X}^{-1}$*; in particular, (i) holds.*

Moreover, in that case, all solutions to $\|\mathbb{G}+\mathbb{U}^*\| < \gamma$ *[*$\leq \gamma$*] are given by* $\mathbb{U}^* := \mathbb{U}_1^* \mathbb{U}_2^{-*}$, $\begin{bmatrix} \mathbb{U}_1^* \\ \mathbb{U}_2^* \end{bmatrix} = \mathbb{M}^* \begin{bmatrix} \widetilde{\mathbb{U}}^* \\ I \end{bmatrix}$, $\widetilde{\mathbb{U}} \in \mathrm{TIC}$ *and* $\|\widetilde{\mathbb{U}}\| < 1$ *[*≤ 1*].*

Thus, all solutions are given by $\mathbb{U} = \mathbb{U}_2^{-1} \mathbb{U}_1$ (clearly a l.c.f.), $\begin{bmatrix} \mathbb{U}_1 & \mathbb{U}_2 \end{bmatrix} = \begin{bmatrix} \widetilde{\mathbb{U}} & I \end{bmatrix} \mathbb{M}$, $\widetilde{\mathbb{U}} \in \mathrm{TIC}$ and $\|\widetilde{\mathbb{U}}\| < 1$ [≤ 1].

Proof: By Corollary 11.4.6, we have $\|\mathbb{X}_{11}^{-1} \mathbb{X}_{12}\| < 1$, therefore, the proof of Theorem 11.3.6 applies, with substitutions $\mathbb{U} \mapsto \mathbb{U}^*$, $\widetilde{\mathbb{U}} \mapsto \widetilde{\mathbb{U}}^*$, $\mathbb{X} \mapsto \mathbb{X}^*$, $\mathbb{M} \mapsto \mathbb{M}^*$, $\begin{bmatrix} \mathbb{X}_{11} & \mathbb{X}_{12} \\ \mathbb{X}_{21} & \mathbb{X}_{22} \end{bmatrix} \mapsto \begin{bmatrix} \mathbb{X}_{11}^* & \mathbb{X}_{21}^* \\ \mathbb{X}_{12}^* & \mathbb{X}_{22}^* \end{bmatrix}$, $\begin{bmatrix} \mathbb{M}_{11} & \mathbb{M}_{12} \\ \mathbb{M}_{21} & \mathbb{M}_{22} \end{bmatrix} \mapsto \begin{bmatrix} \mathbb{M}_{11}^* & \mathbb{M}_{21}^* \\ \mathbb{M}_{12}^* & \mathbb{M}_{22}^* \end{bmatrix}$, $\mathbb{U}_k \mapsto \mathbb{U}_k^*$, $\widetilde{\mathbb{U}}_k \mapsto \widetilde{\mathbb{U}}_k^*$, $\mathbb{Q}_k \mapsto \mathbb{Q}_k^*$ $(k = 1,2)$, $(W,U) \mapsto (U,W)$ and $\mathrm{TIC} \mapsto \mathrm{TIC}^*$ (or $\mathrm{TIC} \mapsto \mathrm{TI}$, since causality is not needed in the proof). $\square$

(See the notes on p. 686.)