Chapter 10

Quadratic Minimization ($\min \mathcal{J}$)

Alas, I am dying beyond my means.

— Oscar Wilde (1856–1900) [as he sipped champagne on his deathbed]

Throughout this chapter we assume that Standing Hypotheses 9.0.1 and 10.6.6 hold (i.e., $\Sigma = \left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D} \end{array}\right] \in \mathrm{WPLS}(U,H,Y)$, $\mathcal{U}_*^*$ is reasonable and "$\widetilde{\mathcal{A}}_+$" denotes some ULR class admitting positive spectral factorization). Hypothesis 10.1.1 will be assumed through Section 10.1.

We strongly recommend the reader to start by reading the introduction to this chapter (p. 31), where also the main results (particularly the LQR problem but also the real lemmas and the H^2 problem) are explained.

We shall first present some minimization results for cost functions $\int_0^\infty (\|x\|_H^2 + \|u\|_U^2)\,dm$, $\int_0^\infty (\|y\|_H^2 + \|u\|_U^2)\,dm$ and their variants in Section 10.1. These allow one to simplify significantly the Riccati equation theory, and, for $\int_0^\infty (\|x\|_H^2 + \|u\|_U^2)\,dm$, any nonnegative solution of the LQR-CARE becomes unique, exponentially stabilizing and minimizing (and also a converse holds), so that one only has to find a nonnegative solution without any stabilization requirements.

Section 10.2 contains a more detailed study on minimization for general WPLSs and cost functions. In Section 10.3, we present several conditions that are equivalent to positive J-coercivity over $\mathcal{U}_{\mathrm{out}}$ or over $\mathcal{U}_{\mathrm{exp}}$ and show how they are implied by or equivalent to various classical assumptions for minimization problems in the literature. The H^2 problem is solved in Section 10.4.

In Section 10.5, we present the Bounded Real Lemma and the Positive Real Lemma, which allow one to use the Riccati equation or the Riccati inequality to verify whether $\|\mathbb{D}\|_{\mathrm{TIC}} \le \gamma$ or $\mathrm{Re}\langle \mathbb{D}\cdot, \cdot\rangle \gg 0$, respectively. We give our results for sufficiently regular systems and sketch corresponding more general results (the latter lack necessity unless we accept IAREs in place of CAREs).

In Section 10.6, we present the equivalence between the uniform positivity of the Popov operator ($\mathbb{D}^* J \mathbb{D} \gg 0$), I-spectral factorization ($\mathbb{D}^* J \mathbb{D} = \mathbb{X}^* \mathbb{X}$) and stabilizing solutions of the Riccati equation or of the Riccati inequality under varying assumptions. These are used for the minimization results in other sections. We also give sufficient conditions for different versions of the equivalence.

In Section 10.7, we show how any solutions of positive Riccati equations are WR and admissible, even stabilizing under suitable conditions.

The proofs can most easily be read in the order Section 10.7 $\rightarrow$ Section 10.6 $\rightarrow$ Section 10.2 $\rightarrow$ the other sections.

10.1 Minimizing $\int_0^\infty (\|y\|_H^2 + \|u\|_U^2)\,dm$ (LQR)

In spite of the cost of living, it's still popular.
— Kathleen Norris (1880–1960)

Here we study the Linear Quadratic Regulator (LQR) problem for cost functions $\int_0^\infty (\|x\|_H^2 + \|u\|_U^2)\,dm$ and $\int_0^\infty (\|y\|_H^2 + \|u\|_U^2)\,dm$; or more generally, $\mathcal{J}(x_0,u) := \langle y, Qy\rangle_{L^2} + \langle u, Ru\rangle_{L^2}$ for some $Q,R \gg 0$; here $x := \mathbb{A}x_0 + \mathbb{B}\tau u$, $y := \mathbb{C}x_0 + \mathbb{D}u$. More general minimization problems will be studied in Section 10.2, which also provides further definitions, results and explanations. See also the presentation in the introduction (p. 31).

Given an initial state $x_0 \in H$, we minimize the cost function over a set $\mathcal{U}_*^*(x_0)$ of admissible controls. We are mainly interested in the classical cases $\mathcal{U}_*^* = \mathcal{U}_{\text{exp}} := \{u \in L^2(\mathbf{R}_+;U) \,\big|\, x,y \in L^2\}$ and $\mathcal{U}_*^* = \mathcal{U}_{\text{out}} := \{u \in L^2(\mathbf{R}_+;U) \,\big|\, y \in L^2\}$ (see Definition 8.3.2). Note that $\mathcal{J}(x_0,u) = +\infty$ for $u \notin \mathcal{U}_{\text{out}}(x_0)$.

To formulate the system and cost function as before, we augment Σ by the extra row $\left[\begin{array}{c|c} 0 & I \end{array}\right]$ when we wish to apply the results of the other sections:

Standing Hypothesis 10.1.1 (LQR) *Throughout this section, we assume that $\mathbb{D}$ is UR or $\dim U < \infty$ and $\mathbb{D}$ is WR, and that $J = \left[\begin{smallmatrix} Q & 0 \\ 0 & R \end{smallmatrix}\right] \in \mathcal{B}(Y \times U)$ and $J \gg 0$.*

By minimization and CAREs (and IAREs), we refer to the augmented system $\Sigma_{\text{aug}} := \left[\begin{smallmatrix} \Sigma \\ 0 \mid I \end{smallmatrix}\right] \in \text{WPLS}(U,H,Y \times U)$ and to the operator J.

Thus, the maps "C" and "D" in the CARE are replaced by $\left[\begin{smallmatrix} C \\ 0 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} D \\ I \end{smallmatrix}\right]$, etc., and the cost function becomes $\mathcal{J}(x_0,u) := \langle y, Qy\rangle_{L^2} + \langle u, Ru\rangle_{L^2}$.

For general $\mathbb{D}$'s (WR or even irregular), one can apply the results of Section 10.2 to obtain results similar to those in this section. Therefore we omit the most general case and study the (rather general) case that $\mathbb{D}$ and (at least) $\mathbb{X}$ are UR. This allows us to rewrite the CARE as follows and guarantee that any solution is admissible (see Theorem 10.1.4):

Definition 10.1.2 (LQR-CARE) *We call $(\mathcal{P},S,K)$ (or $\mathcal{P}$) a nonnegative solution of the* LQR-CARE *iff* $0 \le \mathcal{P} \in \mathcal{B}(H)$, $S \in \mathcal{B}(U)$, $K \in \mathcal{B}(H_1,U)$,

$$\begin{cases} K^*SK = A^*\mathcal{P} + \mathcal{P}A + C^*QC, \\ \quad S = R + D^*QD + \lim_{s\to+\infty} B_{\text{w}}^*\mathcal{P}(s-A)^{-1}B, \\ \quad K = -S^{-1}(B_{\text{w}}^*\mathcal{P} + D^*QC), \end{cases} \tag{10.1}$$

and $\lim_{s\to+\infty} B_{\text{w}}^*\mathcal{P}(s-A)^{-1}B \ge 0$ *or* $S \gg 0$. *We use prefices and suffices (e.g., "PB-") as in Definition 9.8.1.*

If Σ is ULR, then we call $\mathcal{P}$ (or $(\mathcal{P},S,K)$) a nonnegative solution of the LQR-B_{w}^*-CARE *iff* $0 \le \mathcal{P} \in \mathcal{B}(H, \text{Dom}(B_{\text{w}}^*))$ *and $\mathcal{P}$ satisfies*

$$K^*SK = A^*\mathcal{P} + \mathcal{P}A + C^*QC, \tag{10.2}$$

where $S := R + D^*QD$, *and* $K = -S^{-1}(B_{\text{w}}^*\mathcal{P} + D^*QC)$.

(One often has $D = 0$, so that the state feedback becomes $K = -S^{-1}B_{\text{w}}^*\mathcal{P}$.)

Thus, the LQR-CARE is the CARE for Σ_{aug} and J with the additional conditions that $S \geq D_{\mathrm{aug}}^* J D_{\mathrm{aug}}$ or $S \gg 0$ and that the limit in S converges uniformly (i.e., in $\mathcal{B}(U)$; see Lemma A.3.1(h)). The LQR-B_{w}^*-CARE is the corresponding B_{w}^*-CARE (cf. Definition 9.2.6). If $\dim U < \infty$, then any WR K (or $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$) is UR, by Lemma 6.3.2(a1)&(a2).

The advantage of the LQR-CARE is that any nonnegative solution is SOS-stabilizing (in particular, it is admissible):

Lemma 10.1.3

(a) The nonnegative solutions of the LQR-CARE are exactly the UR nonnegative admissible solutions of the CARE (for Σ_{aug}).

Moreover, all of them are SOS-stabilizing and have $\mathbb{M}^{\pm 1} \in \mathrm{UR}$, $M \in \mathcal{GB}(U)$, $\mathbb{N} \in \mathrm{WR}$, $S \gg 0$ and $\lim_{s \to +\infty} B_{\mathrm{w}}^* \mathcal{P}(s-A)^{-1} B \geq 0$.

(b) Any solution of the LQR-CARE is a solution of the CARE.

(c) $\mathcal{P}$ is a UR $\mathcal{U}_*^*$-stabilizing solution of the eCARE iff $\mathcal{P}$ is a nonnegative $\mathcal{U}_*^*$-stabilizing solution of the LQR-CARE.

(d1) The nonnegative solutions of the LQR-B_{w}^*-CARE are exactly the nonnegative solutions of the B_{w}^*-CARE (for Σ_{aug}), hence all of them are nonnegative solutions of the LQR-CARE.

(d2) If Hypothesis 9.2.1 holds (for corresponding $\mathcal{U}_*^*$), then the $\mathcal{U}_*^*$-stabilizing solutions of the LQR-B_{w}^*-CARE are exactly the nonnegative $\mathcal{U}_*^*$-stabilizing solutions of the LQR-CARE, hence exactly the UR $\mathcal{U}_*^*$-stabilizing solutions of the eCARE.

Note from Proposition 9.8.10 that the UR nonnegative admissible solutions of the CARE are exactly the UR nonnegative admissible solutions of the IARE modulo (9.114).

By Theorem 9.2.9 and the above lemma, the $\mathcal{U}_*^*$-stabilizing solutions of the LQR-B_{w}^*-CARE are exactly the $\mathcal{U}_*^*$-stabilizing solutions of the LQR-CARE if Hypothesis 9.2.1 holds.

Proof: Obviously, any solution of the LQR-CARE is a solution of the CARE (even UR, by Lemma 9.11.5(e)).

(a)&(b) By Proposition 10.7.4, a nonnegative solution of the LQR-CARE is a UR SOS-stabilizing solution of the CARE with $S \gg 0$.

Conversely, any UR admissible nonnegative solution $(\mathcal{P}, S, K)$ of the CARE has a uniform (not merely weak) limit in S, either because $\dim U < \infty$ (so that weak=uniform) or by Lemma 9.11.5(e). Thus, $(\mathcal{P}, S, K)$ satisfies the LQR-CARE (and $\lim_{s \to +\infty} B_{\mathrm{w}}^* \mathcal{P}(s-A)^{-1} B = S - (D^* Q D + R) \geq 0$, by Proposition 9.11.4(c2); in particular, $S \geq D^* Q D + R \geq R \gg 0$).

Since $\mathbb{X} =: \mathbb{M}^{-1}$ is necessarily UR for such a solution, we have $\mathbb{M}^{\pm 1} \in \mathrm{UR}$, $M \in \mathcal{GB}(U)$ and $\mathbb{N} = \mathbb{D}\mathbb{M} \in \mathrm{WR}$, by Proposition 6.3.1(b1) and Lemma 6.2.5.

(c) By (a), a nonnegative $\mathcal{U}_*^*$-stabilizing solution of the LQR-CARE is a nonnegative ULR $\mathcal{U}_*^*$-stabilizing solution of the eCARE. Conversely, a $\mathcal{U}_*^*$-stabilizing solution $(\mathcal{P}, S, \left[\begin{array}{c|c} K & F \end{array}\right])$ of the eCARE has $S \gg 0$, by Lemma

9.10.3, and $\mathcal{P} \geq 0$, since $\mathcal{J} \geq 0$ (see Theorem 9.9.1), and $X \in \mathcal{GB}(U)$, by Proposition 6.3.1(b1), hence it is equivalent to a UR $\mathcal{U}_*^*$-stabilizing solution $(\mathcal{P}, S, K')$ of the CARE, by Remark 9.8.2, hence of the LQR-CARE, by (a).

(d1) By Proposition 9.2.7, the solutions of the LQR-B_{w}^*-CARE are admissible ULR solutions of the CARE and the IARE, hence of the LQR-CARE too, by (a).

(d2) By (d1) and Proposition 9.2.7(a)&(b), a $\mathcal{U}_*^*$-stabilizing solution of the LQR-B_{w}^*-CARE is a URL $\mathcal{U}_*^*$-stabilizing solution of the CARE; the converse follows from Theorem 9.2.9, and the second equivalence follows from (c). □

Now we state the connection between the LQR-CARE and UR minimizing state feedback operators:

Theorem 10.1.4 ($\min_u \int_0^\infty (\|y\|_Y^2 + \|u\|_U^2)$)

(a1) **($\mathcal{U}_*^*$)** *There is a minimizing UR state feedback operator iff there is a [nonnegative] $\mathcal{U}_*^*$-stabilizing solution of the LQR-CARE.*

(a2) **(Uniqueness)** *Any $\mathcal{U}_*^*$-stabilizing solution of the LQR-CARE, minimizing control or minimizing state feedback operator is unique.*

(b1) **($\mathcal{U}_{\mathrm{out}}$)** *There is a UR minimizing state feedback operator over $\mathcal{U}_{\mathrm{out}}$ iff there is a minimal nonnegative solution of the LQR-CARE and this solution satisfies (PB) for $\mathcal{U}_{\mathrm{out}}$.*

(b2) **($\mathcal{U}_{\mathrm{exp}}$)** *There is a UR minimizing state feedback operator over $\mathcal{U}_{\mathrm{exp}}$ iff there is a maximal nonnegative solution of the LQR-CARE and this solution is exponentially stabilizing.*

(b3) **(Smooth Σ)** *If Hypothesis 9.2.1 holds (for the corresponding choice of $\mathcal{U}_*^*$), then we can replace the LQR-CARE by LQR-B_{w}^*-CARE everywhere in this theorem.*

(b4) **(Smooth Σ: $\mathcal{U}_{\mathrm{out}}$)** *Assume that Hypothesis 9.2.1 holds for $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{out}}$. Then the following are equivalent:*

(i) **(Min)** *there is a $\langle y, Qy\rangle + \langle u, Ru\rangle$-minimizing control $u_{\min}(x_0)$ over $u \in \mathrm{L}^2(\mathbf{R}_+; U)$ for each $x_0 \in H$;*

(ii) **(FCC)** *for each $x_0 \in H$ there is $u \in \mathrm{L}^2(\mathbf{R}_+; U)$ s.t. $y \in \mathrm{L}^2$.*

(iii) The LQR-B_{w}^*-CARE has a nonnegative solution.

(iv) The LQR-B_{w}^*-CARE has a smallest nonnegative solution, and that solution corresponds to a (unique) ULR minimizing state feedback operator over $\mathcal{U}_{\mathrm{out}}$.

(We can replace LQR-B_{w}^*-CARE by LQR-CARE in (iii) and (iv).) If, in addition, $D = 0$, then the LQR-B_{w}^*-CARE becomes

$$(B_{\mathrm{w}}^*\mathcal{P})R^{-1}B_{\mathrm{w}}^*\mathcal{P} = A^*\mathcal{P} + \mathcal{P}A + C^*QC \tag{10.3}$$

(in either case, we only require that $0 \ll \mathcal{P} \in \mathcal{B}(H, \mathrm{Dom}(B_{\mathrm{w}}^*))$).

(b5) We have above (i)⇔(ii)⇐(iii)⇐(iv)⇐(v) in general.

(b6) **(Smooth $\boldsymbol{\Sigma}$: $\boldsymbol{\mathcal{U}_{\mathrm{exp}}}$)** *Assume that $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{exp}}$, and that (1.) Hypothesis 9.2.1 holds, or that (2.) $\mathbb{A}B \in \mathrm{L}^2([0,T);\mathcal{B}(U,H))$, $C_{\mathrm{w}}\mathbb{A} \in \mathrm{L}^1([0,T);\mathcal{B}(H,Y))$ and $C_{\mathrm{w}}\mathbb{A}B \in \mathrm{L}^1([0,T);\mathcal{B}(U,Y))$. Then the following are equivalent:*

(i) There is a [unique] minimizing control for each $x_0 \in H$.

(ii) There is a [unique] exponentially stabilizing solution $(\mathcal{P},S,K)$ of the LQR-CARE.

(iii) $\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \end{array}\right]$ is optimizable, and $\mathbb{D}_{\mathrm{aug}}$ is I-coercive, i.e., there is $\varepsilon > 0$ s.t.

$$(ir-A)x_0 = Bu_0 \Longrightarrow \|C_{\mathrm{w}}x_0 + Du_0\|_Y + \|u_0\|_U \geq \varepsilon\|x_0\|_H \quad (x_0 \in H,\ u_0 \in U,\ r \in \mathbf{R}). \tag{10.4}$$

Let $(\mathcal{P},S,K)$ be as in (ii). Then $S = R + D^*QD \gg 0$, and K is ULR and the unique minimizing state feedback operator. In case (2.), we have $\mathbb{B}\tau, \mathbb{D}, \mathbb{F} \in \mathrm{MTIC}_\infty^{\mathrm{L}^1}$ and $\mathbb{B}_\circlearrowleft\tau, \mathbb{N}, \mathbb{M} \in \mathrm{MTIC}_\omega^{\mathrm{L}^1} \subset \mathrm{UHPR}$ for some $\omega < 0$.

(c1) **($\min_u \int_0^\infty (\|x\|_H^2 + \|u\|_U^2)$: unique $\boldsymbol{\mathcal{P}}$)**
Assume that Σ is estimatable (e.g., $C \in \mathcal{B}(H,Y)$ and $C^*C \gg 0$).

Then there is at most one nonnegative solution of the LQR-CARE. Such a solution (if any) is strictly minimizing over $\mathcal{U}_{\mathrm{out}}$, $\mathcal{U}_{\mathrm{sta}}$, $\mathcal{U}_{\mathrm{str}}$ and $\mathcal{U}_{\mathrm{exp}}$, and exponentially q.r.c.-stabilizing.

Moreover, such a solution defines an exponential normalized q.r.c.f. (even r.c.f. if $C \in \mathcal{B}(H,Y)$) $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$, where $\widehat{\mathbb{N}}(s) := D + (C+DK)(s-A-BK)^{-1}B$ and $\widehat{\mathbb{M}}(s) := I + K(s-A-BK)^{-1}B$ are exponentially q.r.c. and stable, $\mathbb{M}^{\pm 1} \in \mathrm{UR}$ and $\mathbb{N}^*Q\mathbb{N} + \mathbb{M}^*R\mathbb{M} = I$.

(c2) Assume that Σ is strongly top row–detectable.

Then there is at most one nonnegative solution of the LQR-CARE. Such a solution (if any) is strictly minimizing over $\mathcal{U}_{\mathrm{out}}$, $\mathcal{U}_{\mathrm{sta}}$ and $\mathcal{U}_{\mathrm{str}}$.

(d) A CARE solution $((\mathcal{P},S,K))$ of the form mentioned in any of (a1)–(c2) (except in (b4)(iii)) is unique (of that form) and $\mathcal{U}_*^*$-stabilizing, and the same K is the unique minimizing state feedback operator.

(Note that (2.) of (b6) is implied by the "Parabolic system assumption" Hypothesis 9.5.1, by Lemma 9.5.2.)

Thus, when minimizing over $\mathcal{U}_{\mathrm{out}}$, we only have to find a minimal solution and check the condition (PB) for that solution only, by (b1). Analogously, when minimizing over $\mathcal{U}_{\mathrm{exp}}$, instead of looking for an exponentially stabilizing solution, it suffices to look for a maximal solution and check the exponential stability of $\mathbb{A}_\circlearrowleft$ for that solution only. (If none exists or (at least) one exists but does not satisfy (PB), then the minimizing control (if any) cannot be given in the state feedback form.)

By (b3)–(b6), the LQR-CARE can be replaced by the "LQR-B_{w}^*-CARE" if Σ is smooth enough, and in this case any minimizing control is necessarily of state feedback form (and ULR). See Theorems 9.2.10–9.2.12 for further analogous results. See also Corollary 9.5.10 for the case where $\mathbb{A}$ is analytic.

By (c1)&(c2), estimatability or strong detectability implies that a nonnegative solution of the LQR-CARE is unique and minimizing. The same holds when Σ is exponentially q.r.c.-stabilizable or strongly stable, by Theorem 10.1.6.

Proof of Theorem 10.1.4: (a1) This follows from Lemma 10.1.3(c), and Corollary 9.9.2(a2)&(e1)&(e2).

(a2) By Theorem 9.9.1(f2) (Lemma 10.1.3(c)), a $\mathcal{U}_*^*$-stabilizing solution is unique. By Lemma 8.3.8, a minimizing control is unique; consequently, a compatible state feedback operator is unique (on H_B), by Lemma 8.3.17(b).

(b1)&(b2) If we drop the minimality/maximality condition, then the equivalence follows from (a1) and Theorem 9.8.5 (for (b1) we used the fact that $\mathcal{P}$ is SOS-stabilizing, by Lemma 10.1.3(a)).

But the minimality in (b1) (resp. maximality in (b2)) is necessary, by Theorem 9.9.1(a2).

(b3) This follows from Lemma 10.1.3(d2). (Note that in (b1) (resp. (b2)) we must have Hypothesis 9.2.1 for $\mathcal{U}_{\text{out}}$ (resp. for $\mathcal{U}_{\text{exp}}$) etc.)

(b4) By Theorem 9.2.10(b), (ii) implies (iv); the rest follows from (b5).

(b5) Since $\mathbb{D}_{\text{aug}}$ is positively J-coercive over $\mathcal{U}_{\text{out}}$, we have (i)$\Leftrightarrow$(ii), by Theorem 8.4.3 and Lemma 10.2.2.

Implication "(iv)$\Rightarrow$(iii)" is trivial. and "(iii)$\Rightarrow$(ii)" follows from the fact that any nonnegative solution is SOS-stabilizing, by Lemma 10.1.3(d1)&(a).

(b6) This follows from Corollary 10.2.9 (case (1.)) or Corollary 10.2.10 (case (2.)), Lemma 10.1.3 and Proposition 10.3.2(ii') (note that $J \gg 0$, hence $\mathbb{D}$ is positively J-coercive iff it is [positively] I-coercive).

(Obviously, $u = 0$ is the unique minimizing control for $x_0 = 0$, hence any minimizing control (for any $\mathcal{U}_*^*$) is unique, by Lemma 8.3.8.)

(c1) This follows from Proposition 10.7.3(d3) except for the last claim, which is from Theorem 10.1.6 (if $C \in \mathcal{B}(H,Y)$, then any exponentially stabilizing state feedback operator is exponentially r.c.-stabilizing, by Lemmas 6.6.25 and 6.6.26, hence then "q.r.c." becomes "r.c.").

(N.B. the last equation is equivalent to $\widehat{\mathbb{N}}(s)^* Q \widehat{\mathbb{N}}(s) + \widehat{\mathbb{M}}(s)^* R \widehat{\mathbb{M}}(s) = I$ on $i\mathbf{R}$.)

(c2) (By top row–detectability we mean that some admissible output injection pair $\left[\begin{smallmatrix}\mathbb{H}\\ \mathbb{G}\end{smallmatrix}\right]$ makes $\left[\begin{array}{c|cc}\mathbb{A}_\sharp & \mathbb{H}_\sharp & \mathbb{B}_\sharp\end{array}\right]$ strongly stable. This obviously holds if Σ is strongly detectable. Actually, it suffices to assume that $\Delta^{\mathrm{S}}\Sigma$ is top row–detectable, as one observes from the proof.)

1° Let $\mathcal{P}$ be a nonnegative solution of the LQR-CARE, hence SOS-stabilizing, by Lemma 10.1.3(a). By 2°, $\mathbb{A}_\circlearrowleft$ is strongly stable, hence $\mathcal{P}$ is $\mathcal{U}_{\text{str}}$-stabilizing, by Theorem 9.8.5, hence unique and strictly minimizing over $\mathcal{U}_{\text{str}}$, by (a2).

Let $\mathcal{P}'$ be the J-critical cost operator over $\mathcal{U}_{\text{out}}$ (recall that $\mathbb{D}$ is J-coercive over $\mathcal{U}_{\text{out}}$). Then $\mathcal{P}' = \mathcal{P}$, by uniqueness (use the discrete version of (c2)). By Lemma 8.3.3, $\mathcal{P}$ is (strictly) minimizing over $\mathcal{U}_{\text{sta}}$ too.

2° $\mathbb{A}_\circlearrowleft$ *is strongly stable:* We prove this in discrete time (note that state feedback and output injection pairs can be discretized and also stability is preserved under discretization in both directions).

As at the end of the proof of Lemma 6.7.11(a), we note that also $\Sigma_{\circlearrowleft}$ has the above detectability property (note that $\widetilde{H} := -\mathtt{H}D - B$ implies that $\widetilde{\mathbb{H}}_\sharp = -\mathbb{H}_\sharp D - \mathbb{B}_\sharp$, hence it is stable too; by assumption, so is $\left[\begin{array}{c|cc} \mathbb{A}_\sharp & \mathbb{H}_\sharp & \mathbb{B}_\sharp \end{array}\right]$). Thus, $\mathbb{A}_{\circlearrowleft} + \mathbb{H}'_\sharp \tau \mathbb{C}_{\circlearrowleft}$ and $\mathbb{H}'_\sharp$ become stable for the output injection pair $\left(\frac{H'}{0}\right)$, where $H' := \begin{bmatrix} \mathtt{H} & \widetilde{H} \end{bmatrix}$. Consequently, also $\mathbb{A}_{\circlearrowleft}$ is stable (since $\mathbb{H}'_\sharp \tau^t \mathbb{C}_{\circlearrowleft} x_0 \to 0$, as $t \to \infty$, for all $x_0 \in H$; the proof of this is analogous to that of Lemma 6.6.8(a)).

(d) Since the solutions mentions above are $\mathcal{U}^*_*$-stabilizing, they contain the minimizing K. □

Remark 10.1.5 *The cost is finite for $u \in \mathcal{U}_{\text{out}}(x_0)$ only, hence minimization over all (measurable) $u : \mathbf{R} \to U$ corresponds to minimization over $\mathcal{U}_{\text{out}}$ (this was applied in Theorem 10.1.4(b3)&(b4)).*

In Theorem 10.1.4(b1) we showed that such a minimizing control is generated by an UR state feedback operator iff there is a (necessarily smallest nonnegative) $\mathcal{U}_{\text{out}}$-stabilizing solution of the LQR-CARE. □

In the strongly stable case there is at most one solution of the LQR-CARE (see also Theorem 10.1.4(c1)&(c2)):

Theorem 10.1.6 ($\mathcal{U}_{\text{out}}$: LQR⇔r.c.f.⇔CARE) *Assume that Σ is strongly stable or exponentially q.r.c.-stabilizable. Then the following are equivalent:*

(i) **(*K*)** *There is a [unique] UR minimizing state feedback operator K over $\mathcal{U}_{\text{out}}$.*

(ii) **(CARE)** *The LQR-CARE has a [unique] nonnegative solution $\mathcal{P}$.*

(iii) **(R.c.f.)** *There is a q.r.c.f. $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$ with $\mathbb{M} \in \mathrm{UR}$ and $\mathbb{N}^* Q \mathbb{N} + \mathbb{M}^* R \mathbb{M} = I$.*

Moreover, the following holds:

(a1) The above conditions imply (Crit1+WR)–(Crit4+WR) of Theorem 10.2.14, hence (a1)–(g3) of Theorem 9.9.10 apply (for Σ_{aug} and J).

(a2) The solutions K of (i) and $\mathcal{P}$ of (ii) are unique, UR, strongly q.r.c.-stabilizing, strictly minimizing over $\mathcal{U}_{\text{out}}$, $\mathcal{U}_{\text{sta}}$ and $\mathcal{U}_{\text{str}}$, and equal to those of Theorem 9.1.7 (so is the solution of (iii) too if we require that $M = I$).

If Σ is exponentially q.r.c.-stabilizable, then K and $\mathcal{P}$ are exponentially q.r.c.-stabilizing and strictly minimizing over $\mathcal{U}_{\text{exp}}$ too.

(b1) **(B^*_{w}-CARE)** *Assume that Hypothesis 9.2.1 holds and $D^* J D \gg 0$.*

Then (i)–(iii) have solutions, we may replace the LQR-CARE by the LQR-B^*_{w}-CARE, and $\mathbb{D}, \mathbb{F}, \mathbb{N}, \mathbb{M}^{\pm 1} \in \mathrm{ULR}$.

(b2) Assume that Σ is strongly stable and satisfies Hypothesis 10.6.1(3.) (resp. (6.)).

Then (i)–(iii) have solutions (resp. and (b1) applies).

(b3) Assume that Σ has a UR (resp. URL) exponentially q.r.c.-stabilizing state feedback operator $\widetilde{K}$ s.t. the resulting closed-loop system $\left[\begin{array}{c|c} \mathbb{A}_\flat & \mathbb{B}_\flat \\ \hline \mathbb{C}_{\mathrm{aug},\flat} & \mathbb{D}_{\mathrm{aug},\flat} \end{array}\right]$ satisfies Hypothesis 10.6.1(3.) (resp. (6.)).

Then (i)–(iii) have solutions (resp. and (b1) applies).

(b4) Assume that $\mathbb{D} \in \widetilde{\mathcal{A}}_+$ and Σ is strongly stable (resp. that $\mathbb{D}$ is exponentially q.r.c.-stabilizable in $\widetilde{\mathcal{A}}_+$).

Then (i)–(iii) have solutions with $\mathbb{N}, \mathbb{M} \in \widetilde{\mathcal{A}}_+$.

(See Standing Hypothesis 10.6.6 for $\widetilde{\mathcal{A}}_+$.) Note that (iii) is equivalent to the existence of a (J,I)-inner q.r.c.f. $\mathbb{D}_{\mathrm{aug}} = \mathbb{N}_{\mathrm{aug}}\mathbb{M}^{-1}$ with $\mathbb{M} \in \mathrm{UR}$ (it follows that $\mathbb{N}_{\mathrm{aug}} = \left[\begin{smallmatrix} \mathbb{N} \\ \mathbb{M} \end{smallmatrix}\right]$). For stable (d), this becomes equivalent to the existence of an I-spectral factorization $\mathbb{D}^* Q \mathbb{D} + R = \mathbb{X}^* \mathbb{X}$, by Lemma 6.4.8(a).

If we drop the stability/stabilizability assumptions of the theorem, then K and $\mathcal{P}$ must be assumed to be strongly q.r.c.-stabilizing (or q.r.c.-SOS-stabilizing and q.r.c.-SOS-P-stabilizing) in (i) and (ii), and Σ must be assumed to be strongly q.r.c.-stabilizable in (iii) (or q.r.c.-SOS-stabilizable), by Theorems 9.1.7 and 9.9.10 (or Corollary 10.2.12, which weakens (b3)).

Recall from Theorem 6.7.15(c1) that if Σ is estimatable, then any output-stabilizing state feedback operator is exponentially q.r.c.-stabilizing.

Proof of Theorem 10.1.6: 1° *(i)$\Leftrightarrow$(ii), (a2):* This follows from Theorem 10.1.4(b1)&(a2) and Proposition 10.7.3(d2)&(d3).

2° *(iii)$\Leftrightarrow$(ii):* This follows from Lemma 6.5.7(c) and Theorem 9.1.7 (indeed, a solution of (ii) is a q.r.c.-stabilizing solution of the CARE, by (a2) and Lemma 10.1.3; a solution of (iii) can be chosen s.t. $X = I$, by (a2); on the other hand, $\mathbb{M}^{\pm 1} \in \mathrm{UR}$ for any solution of (ii) or (iii), by Lemma 10.1.3 and 1°).

(It follows that $\mathbb{N}_{\mathrm{aug}}$ is (J,S)-inner, where $S \gg 0$ is from the (LQR-)CARE; replace $\mathbb{M}$ by $\mathbb{M}S^{-1/2}$ to get $S = I$.)

(a1) This follows from (a2) and Lemma 10.1.3.

(b1)–(b4) These follow from Theorem 10.2.14(b1)–(b4) or Corollary 10.2.15(b1) (resp. (b2)) (since now $\mathbb{X}$ and hence $\mathbb{M} := \mathbb{X}^{-1}$ (by Proposition 6.3.1(b1)) is necessarily UR (resp. ULR and so is $\mathbb{D}$)), Lemma 10.1.3 and Proposition 9.2.7(c).

For (b4) we note that if $\mathbb{D} \in \widetilde{\mathcal{A}}_+$ (resp. $\mathbb{N}, \mathbb{M} \in \widetilde{\mathcal{A}}_+$), then $\mathbb{D}_{\mathrm{aug}} \in \widetilde{\mathcal{A}}_+$ (resp. $\mathbb{D}_{\mathrm{aug},\flat} = \left[\begin{smallmatrix} \mathbb{N} \\ \mathbb{M} \end{smallmatrix}\right] \in \widetilde{\mathcal{A}}_+$). Thus, we could equivalently pose the assumption on Σ_{aug}. $\square$

Notes

One often calls any minimization problems (with a quadratic cost function) *Linear Quadratic Regular* (*LQR*) problems, and the problems of this section (those satisfying Hypothesis 10.1.1) are then called *standard LQR problems* or something similar.

Since the LQR problem is perhaps the most popular subject in infinite-dimensional control theory, we can only try to give a picture of some most general current results. The earlier history of infinite-dimensional Riccati equations is documented in the notes to Section 6 of [CZ].

Most of our results are known for several special cases (see, e.g., Section 6.2 of [CZ] for WPLSs with bounded B and C and Theorems 3.3 and 3.4 of [PS87] for Pritchard–Salamon systems; these treat both $\mathcal{U}_{\text{out}}$ and $\mathcal{U}_{\text{exp}}$ to some extent).

For $\mathcal{U}_*^* = \mathcal{U}_{\text{out}}$, the case covered by Proposition 9.7.6 was solved in [FLT] (with a "generalized CARE" and possibly non-well-posed feedback; for parabolic systems these issues are well settled and the results very general, see [LT00a]). A similar result for general WPLSs was given in [Zwart], and the regular stable minimization problem was solved independently in [S97b] and [WW] (with roughly the implication "(iii)⇒(i)&(ii)" of Theorem 10.1.6 for stable WPLSs with $\mathcal{U}_*^* = \mathcal{U}_{\text{out}}$). Theorem 10.1.4(c2) generalizes Theorem 3.3.2 of [Oostveen] (which assumed bounded B and C); see [Oostveen] for its physical applications. See also the notes on p. 571

10.2 General minimization (LQR)

Incidis in Scyllam, cupens vitare Charybdim.

— Homer

In this section we shall present some applications of the CARE theory to minimization problems, where one wishes to find a minimizing control over some set $\mathcal{U}_*^*$ of allowable controls, in state feedback form (i.e., to find a [regular] state feedback operator K or a pair $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ that produces a minimizing control). Thus, this is a generalization of Section 10.1.

Since this section is rather technical, lengthy and boring (due to the reasons explained below Corollary 10.2.3), a casual reader might wish to just have a glance at Subsections 10.2.1–10.2.10 (or less) and then proceed to the next section.

Most results become essentially simpler under Hypothesis 9.2.1, as illustrated in Section 9.2 (in particular, in Theorems 9.2.10–9.2.12), or in their discrete-time forms, as illustrated in Section 15.1.

Therefore, in this section we have the emphasis on general WPLS results and on results for MTIC I/O maps. This makes several classical results rather complicated, and almost each piece of simplicity must be obtained at the cost of generality. Consequently, we give certain results under some different assumptions, and leave it to the reader to extra- and interpolate further results under other kind of assumptions, using the results of Sections 8.3–8.4, Chapter 9, and the rest of this chapter.

We use the word "minimizing" in the same way as the word "J-critical":

Definition 10.2.1 (Minimizing $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ and K) *We call $u_{\min} \in \mathcal{U}_*^*(x_0)$ a* [strictly] minimizing control *(over $\mathcal{U}_*^*$) for $x_0 \in H$ (and Σ and J) if $\mathcal{J}(x_0, u_{\min}) \leq \mathcal{J}(x_0, u)$ for all $u \in \mathcal{U}_*^*(x_0)$ [and $u_{\min}$ is unique].*

Let $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ be an admissible state feedback pair for Σ with closed-loop system $\Sigma_\circlearrowleft$. Then we call $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ minimizing *(over $\mathcal{U}_*^*$ for Σ and J) iff $\mathbb{K}_\circlearrowleft x_0$ is minimizing for each $x_0 \in H$. In this case, we say that the minimizing control is of* state feedback form.

We call a WR admissible state feedback operator $K \in \mathcal{B}(H_1, U)$ minimizing *if $\left[\begin{array}{c|c} K & 0 \end{array}\right]$ generate minimizing pair $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ for Σ.*

Consequently, $u_{\text{crit}} \in \mathrm{L}^2(\mathbf{R}_+; U)$ is [strictly] minimizing for x_0 over $\mathcal{U}_{\text{out}}$ (resp. $\mathcal{U}_{\exp}$) iff $y := \mathbb{C}x_0 + \mathbb{D}u \in \mathrm{L}^2$ (resp. and $x := \mathbb{A}x_0 + \mathbb{B}\tau u \in \mathrm{L}^2$) and u [strictly] minimizes the *cost function* $\mathcal{J}(x_0, u) := \int_0^\infty \langle y(t), Jy(t)\rangle_Y \, dt$ among such controls. See Definition 9.1.4 (or Definition 6.6.10) for K, $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$, and $\Sigma_\circlearrowleft$.

Note that "minimizing over $\mathcal{U}_*^*$" does not affect the other attributes (prefices) and vice versa, i.e., "minimizing exponentially stabilizing state feedback pair over $\mathcal{U}_{\text{out}}$" produces a control that is minimizing over all elements of $\mathcal{U}_{\text{out}}(x_0)$ for each $x_0 \in H$, not just over those produced by exponentially stabilizing state feedback.

Lemma 10.2.2 (Min $\Leftrightarrow$ J-crit. & pos.) *Let $x_0 \in H$. A control $u \in \mathcal{U}_*^*(x_0)$ is [strictly] minimizing for x_0 iff u is J-critical for x_0 and $\langle \mathbb{D}\eta, J\mathbb{D}\eta\rangle \geq 0$ [> 0] for all $\eta \in \mathcal{U}_*^*(0) \setminus \{0\}$.*

Thus, if there is a [strictly] minimizing control for any x_0, then 0 is a [strictly] minimizing control for $x_0 = 0$ (since $\mathcal{J}(0,\eta) = \langle \mathbb{D}\eta, J\mathbb{D}\eta\rangle$).

If there is a minimizing control for all x_0, then we may call the J-critical cost operator $\mathcal{P} = \mathcal{P}^* \in \mathcal{B}(H)$ (the one for which $\mathcal{J}(x_0, u_{\min}(x_0)) = \langle x_0, \mathcal{P}x_0\rangle$ $(x_0 \in H)$, as in Theorem 8.3.9(b1)) the *minimal cost operator*. Recall that it is equal to the (unique) $\mathcal{U}_*^*$-stabilizing solution of the Riccati equation (if any, i.e., if a minimizing control is of state feedback form).

Proof of Lemma 10.2.2: Proof 1: Corollary 8.1.8 and Remark 8.3.4.

Proof 2: "If": This follows from Lemma 8.3.7(ii). "Only if": This follows from Lemma 8.3.6 and Lemma 8.3.7(ii). □

At their best, our minimization results for general cost functions look like the following one:

Corollary 10.2.3 (Minimization for bounded B) *Assume that $B \in \mathcal{B}(U,H)$, $\dim U < \infty$. Then the following are equivalent:*

(i) there is a unique minimizing control for each $x_0 \in H$;

(ii) $\mathcal{J}(0,\cdot) \geq 0$, $D^*JD \gg 0$, and the "B-CARE"

$$(B^*\mathcal{P} + D^*JC)^*(D^*JD)^{-1}(B^*\mathcal{P} + D^*JC) = A^*\mathcal{P} + \mathcal{P}A + C^*JC \qquad (10.5)$$

has a (unique) $\mathcal{U}_*^*$-stabilizing solution $\mathcal{P}$.

(iii) there is a unique minimizing state feedback operator for Σ.

Moreover, if (ii) holds, then minimizing state feedback is given by $u_{\min}(t) = K_{\mathrm{L,s}}x(t)$ a.e., where $K := -(D^*JD)^{-1}(B^*\mathcal{P} + D^*JC)$ is URL, the minimal cost is $\mathcal{J}(x_0, u_{\min}) = \langle x_0, \mathcal{P}x_0\rangle$ etc. as in Proposition 9.9.1. If $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{exp}}$, then also the following is equivalent to (i):

(iv) Σ is positively J-coercive and optimizable.

(Condition $\mathcal{J}(0,\cdot) \geq 0$ is redundant at least for $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{exp}}$, by Corollary 10.2.6. The operator $K_{\mathrm{L,s}}$ can be replaced by any of its extensions, such as $K_{\mathrm{L,w}}$, K_{s} and K_{w}.)

We recall from Theorem 9.8.5 that "$\mathcal{U}_{\mathrm{exp}}$-stabilizing" means "exponentially stabilizing", whereas "$\mathcal{U}_{\mathrm{out}}$-stabilizing" is rather complicated for general unstable Σ. If Σ is q.r.c.-SOS-stabilizable (e.g., jointly stabilizable and detectable) and J-coercive, then the latter case can be slightly simplified, as in Corollary 10.2.12–Theorem 10.2.14.

Proof of Corollary 10.2.3: By Proposition 10.3.2(e2)&(i)&(v), we have "(ii)⇒(iv)⇒(i)" for $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{exp}}$. The rest follows from Lemma 10.2.2 and Theorem 9.9.6 (see also Theorem 9.9.1(a2)&(e2) etc.). □

Instead of B being bounded, we can assume that Hypothesis 9.2.1 holds and that $D^*JD \gg 0$, by Theorem 9.2.9 (then B^* must be replaced by B_{w}^*, and we must, additionally, require $\mathcal{P}[H] \subset \mathrm{Dom}(B_{\mathrm{w}}^*)$ in (ii)). Since the generators of a wpls (a discrete-time system) are bounded, Corollary 10.2.3 always holds in discrete time (we just have to replace equations for $\mathcal{P}$, D^*JD and K by the DARE; this follows from Theorem 15.1.2, as in the above proof). Unfortunately, for general WPLSs, the situation is not as nice (in continuous time):

(1.) If B is unbounded, then $B^*\mathcal{P}$ is not defined in general, and even $B_w^*\mathcal{P}$ need not always be bounded (it is defined at least on $\mathrm{Dom}(A_{\min})$ when $\mathbb{D}$ is WR, by Theorem 9.7.3(b)).

Thus, the "B-CARE" (10.5) must be replaced by the more complicated CARE (for which D^*JD is replaced by the limit S).

(2.) Even worse, the minimizing state feedback need not be regular (even when $\mathbb{D}$ were ULR, by Proposition 9.13.1(c1)), hence the CARE must be replaced by the IARE (or one must use discretization) and the unique minimizing state feedback operator must be replaced by an essentially unique minimizing state feedback pair.

(3.) Still worse: for general WPLSs we do not even know whether (i) implies (ii) and (iii), i.e., whether a minimizing control is of state feedback form (it is of the generalized, non-well-posed sense of Definition 8.3.15, by Theorem 8.3.9); thus, instead of the IARE we must use the "generalized IARE" of Theorem 9.7.1 (or the "CARE on $\mathrm{Dom}(A_{\min})$" of Theorem 9.7.3 if $\mathbb{D}$ is regular).

(4.) If $\dim U = \infty$, then the IARE (resp. CARE) must be replaced by eIARE (eCARE), since we only know that $S > 0$ (even though B were bounded).

We shall present below some results for general WPLS and then use different assumptions to get rid of some of the above problems.

By using Hypothesis 9.2.1, we get rid of (2.), (3.) and most of (1.); some corresponding results are given Section 9.2; see, e.g., Theorems 9.2.10–9.2.12.

A set of weaker assumptions is given in Hypothesis 10.6.1; they allow us to get rid of (2.) and (3.) (and a part of (1.)), as shown in several results below and in Section 10.1. Also Corollaries 10.2.10 and 9.5.10 present similar results under different assumptions.

For stable (or suitably stabilizable) systems, the positive spectral factorization result of Lemma 6.4.7(a) can be used to avoid problem (3.) for general WPLSs; see Corollaries 10.2.6–10.2.13.

Sometimes we use J-coercivity or analogous assumptions to overcome (4.). Under sufficient regularity assumptions we have $S = D^*JD$, hence then we can make S invertible just by assuming that $D^*JD \gg 0$.

In Section 10.1, we give some analogous results for the more specific LQR problem, and in Section 10.6, we give further minimization and [e]IARE and [e]CARE results.

Naturally, one can obtain several analogous theorems by combining the results of this and the previous chapter in different ways; we hope that the following results provide a helpful guideline.

Standard coercivity assumptions guarantee the existence of a unique minimizing control:

Lemma 10.2.4 *Let Z^{s} be reflexive, and let $\mathbb{D}$ be positively J-coercive, i.e., let there be $\varepsilon > 0$ s.t.*

$$\langle \mathbb{D}u, J\mathbb{D}u\rangle \geq \varepsilon \|u\|^2_{\mathcal{U}_*^*} \quad (u \in \mathcal{U}_*^*(0)). \tag{10.6}$$

If $x_0 \in H$ is s.t. $\mathcal{U}_*^*(x_0) \neq \emptyset$, then there is a unique minimizing control for x_0. □

(This follows from Theorem 8.2.5, Lemma 8.2.3 and Lemma 10.2.2 through Remark 8.3.4. Recall that Z^{s} is reflexive for $\mathcal{U}_{\mathrm{exp}}$ and $\mathcal{U}_{\mathrm{out}}$.)

Thus, if $\mathcal{U}_*^*(x_0) \neq \emptyset$ for all $x_0 \in H$ (i.e., the *Finite Cost Condition* holds) and $\mathbb{D}$ is positively J-coercive (see Section 10.3 for several equivalent conditions for $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{out}}$ and $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{exp}}$), then there is a unique minimizing control for each $x_0 \in H$ (provided that Z^{s} is reflexive). The minimizing control and corresponding state, output and cost are then given in WPLS form by Theorem 8.3.9

However, as noted above, without further assumptions we do not know whether the unique minimizing control is of state feedback form. We give below different formulations of sufficient conditions.

We now recall the basic minimization results from Chapter 9 (do not forget our definitions: we require the solutions of Riccati equations to be self-adjoint):

Corollary 10.2.5 (Minimizing control⇔eCARE/eIARE) *Let $\mathcal{J}(0,\cdot) \geq 0$. Then the following hold:*

(a) There is a minimizing state feedback pair $\begin{bmatrix} \mathbb{K} \mid \mathbb{F} \end{bmatrix}$ for Σ iff the eIARE has a $\mathcal{U}_*^*$-stabilizing solution $(\mathcal{P}, S, \begin{bmatrix} \mathbb{K} \mid \mathbb{F} \end{bmatrix})$.

(b) Let $\mathbb{D}$ be WR. Then there is a minimizing WR state feedback operator K for Σ iff the eCARE

$$\begin{cases} K^*SK = A^*\mathcal{P} + \mathcal{P}A + C^*JC, \\ \quad S = D^*JD + \underset{s\to+\infty}{\text{w-lim}}\, B_{\mathrm{w}}^*\mathcal{P}(s-A)^{-1}B, \\ SK = -(B_{\mathrm{w}}^*\mathcal{P} + D^*JC), \end{cases} \tag{10.7}$$

has a $\mathcal{U}_*^*$-stabilizing solution $(\mathcal{P}, S, \begin{bmatrix} K \mid 0 \end{bmatrix})$.

If such a solution exists and $x_0 \in H$, then the minimizing control $u_{\min}(x_0)$ is given by $u_{\min}(x_0)(t) = K_{\mathrm{w}}x(t)$ a.e., where $x = \mathbb{A}x_0 + \mathbb{B}\tau u_{\min}(x_0)$ is the corresponding state.

(c) Let $\mathcal{P}$ be a $\mathcal{U}_*^*$-stabilizing solution of the eIARE (resp. eCARE, hence of both).

Then $\mathcal{P}$ is unique, $S \geq 0$, and $\begin{bmatrix} \mathbb{K} \mid \mathbb{F} \end{bmatrix}$ is the pair determined by $\mathcal{P}$ (resp. generated by $\begin{bmatrix} K \mid 0 \end{bmatrix}$). The control $\mathbb{K}_{\circlearrowleft}$ is strictly minimizing iff $S > 0$.

Also parts (f1)–(k) of Theorem 9.9.1 apply to $(\mathcal{P}, S, \begin{bmatrix} \mathbb{K} \mid \mathbb{F} \end{bmatrix})$; in particular, the minimal cost is given by $\mathcal{J}(x_0, u_{\min}(x_0)) = \langle x_0, \mathcal{P}x_0\rangle$.

If $\mathcal{J}(\cdot,\cdot) \geq 0$ and $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{out}}$, then $\mathcal{P}$ is the smallest nonnegative output-stabilizing solution of the eIARE (resp. eCARE).

(d) If $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{exp}}$, then "$\mathcal{J}(0,\cdot) \geq 0$" can be dropped from this corollary if we require the solutions of the eIARE and the eCARE to satisfy $S \geq 0$.

In particular, (b)–(d) apply to the ($\mathcal{U}_*^*$-stabilizing) solutions of the eCARE (for Σ) mentioned in other results of this chapter.

Proof: (a)–(c) This is a combination of Theorem 9.9.1(a2)&(f1)–(k) and Corollary 9.9.2.

(d) If there is a minimizing control for any $x_0 \in H$, then $\mathcal{J}(0,\cdot) \geq 0$, by Lemma 10.2.2, hence then $S \geq 0$, by (c). Conversely, any solution with $S \geq 0$ is minimizing, by Theorem 9.9.1(k). □

By combining (a)&(b)&(d) of Corollary 10.2.5 with Theorem 9.8.5, we obtain:

Corollary 10.2.6 ($\mathcal{U}_{\mathrm{exp}}$-min $\Leftrightarrow$ eIARE) *There is a minimizing state feedback pair over $\mathcal{U}_{\mathrm{exp}}$ iff the eIARE has an exponentially stabilizing solution with $S \geq 0$.*

Let $\mathbb{D}$ be WR. Then there is a minimizing WR state feedback operator over $\mathcal{U}_{\mathrm{exp}}$ iff the eCARE has an exponentially stabilizing solution with $S \geq 0$. □

If $\mathcal{P}$ is such a solution and $S \gg 0$ (equivalently, $\mathbb{D}$ is J-coercive over $\mathcal{U}_{\mathrm{exp}}$, by Proposition 9.9.12), then $\mathcal{P}$ is the greatest admissible solution of the eIARE having $S \geq 0$, by Corollary 15.1.3. Naturally, also Corollary 10.2.5(c) applies.

Next we note some "anomalies" (see Section 9.13 for more):

Example 10.2.7 1. We may have $\mathcal{P} \ll 0$ (this is trivial; see Example 9.13.13).

2. All minimizing controls need not be of the state feedback form even though one of them is (see Example 9.13.6). ◁

To start with a simple case, we first generalize Theorem 16.3.3 of [LR] (see Proposition 10.3.1(d) for (10.87) and positive J-coercivity):

Corollary 10.2.8 *Assume that Σ is optimizable and estimatable, $\mathbb{D}$ is positively J-coercive over $\mathcal{U}_{\mathrm{out}}$, and (1.), (2.) or (4.) of Hypothesis 9.2.2 holds. Then there is a unique exponentially stabilizing solution $(\mathcal{P}, S, K)$ of the B_{w}^*-CARE. Moreover,*

(a) $\mathcal{J}(x_0, u) = \langle x_0, \mathcal{P} x_0 \rangle$ for all $x_0 \in H$;

(b) For each $x_0 \in H$, there is a strictly minimizing control over $\mathcal{U}_{\mathrm{out}} = \mathcal{U}_{\mathrm{exp}}$, namely that given by the state feedback operator K.

(c) K is ULR and exponentially stabilizing.

(d) If (10.87) holds for some $\varepsilon > 0$, then $\mathcal{P}$ is the unique nonnegative solution of the B_{w}^*-CARE.

(e) $\mathcal{P}$ is the greatest solution of the B_{w}^*-CARE.

See Proposition 10.3.1 for necessary and sufficient conditions for positive J-coercivity over $\mathcal{U}_{\mathrm{out}}$.

Proof: By Lemma 8.3.3, we have $\mathcal{U}_{\mathrm{out}} = \mathcal{U}_{\mathrm{exp}}$. By Lemma 9.2.17, we have $D^* J D \gg 0$. By Corollary 10.2.9, $(\mathcal{P}, S, K)$ exists and claims (b) and (c) hold; we then obtain (a) from Theorem 9.9.1. Claim (d) follows from Proposition 10.7.3(d3) and claim (e) follows from Corollary 9.2.11. □

Under sufficient regularity, minimization over $\mathcal{U}_{\mathrm{exp}}$ is easy:

Corollary 10.2.9 ($\mathcal{U}_{\text{exp}}$: Unique minimum $\Leftrightarrow$ B_{w}^*-CARE$\Leftrightarrow$ J-coercive) *Assume that Hypothesis 9.2.1 holds for $\mathcal{U}_*^* = \mathcal{U}_{\text{exp}}$, and that $D^*JD \gg 0$. Then the following are equivalent:*

(i) There is a unique minimizing control over $\mathcal{U}_{\text{exp}}(x_0)$ for each $x_0 \in H$.

(ii) The B_{w}^*-CARE (9.13) has an exponentially stabilizing solution $\mathcal{P} = \mathcal{P}^* \in \mathcal{B}(H, \text{Dom}(B_{\text{w}}^*))$.

(iii) Σ is optimizable and $\mathbb{D}$ is positively J-coercive over $\mathcal{U}_{\text{exp}}$ (i.e., any (hence all) of Proposition 10.3.2(i)–(iii) holds).

If (ii) holds, then $K := -(D^*JD)^{-1}(B_{\text{w}}^*\mathcal{P} + D^*JC)$ is the unique minimizing state feedback operator over $\mathcal{U}_{\text{exp}}$ and ULR.

If $J \geq 0$, then the word "unique" is redundant in (i). If $\pi_{[0,1)}\mathbb{A}B \in \mathrm{L}^1([0,1]; \mathcal{B}(U,H))$ then "$D^*JD \in \mathcal{GB}(U)$" is redundant in (iii).

Further results are given in Theorems 9.2.10–9.2.12. See Corollary 9.5.10 for applications of this and the following corollary for parabolic problems.

Recall that if $\mathcal{P}$ is a solution of the B_{w}^*-CARE (which requires that $D^*JD \in \mathcal{GB}(U)$, by Definition 9.2.6), then $A_{\circlearrowleft} := A + BK$ generates a C_0-semigroup $\mathbb{A}_{\circlearrowleft}$; the additional requirement in (ii) is that this semigroup satisfies $\|\mathbb{A}_{\circlearrowleft}(t)\| \leq M\mathrm{e}^{-\varepsilon t}$ ($t \geq 0$) for some $\varepsilon > 0$ and $M < \infty$.

If $J \gg 0$, then $\mathbb{D}$ is positively J-coercive over $\mathcal{U}_{\text{exp}}$ iff there is $\varepsilon > 0$ s.t.

$$(ir - A)x_0 = Bu_0 \implies \|C_{\text{w}}x_0 + Du_0\|_Y \geq \varepsilon\|x_0\|_H \quad (x_0 \in H,\ u_0 \in U,\ r \in \mathbf{R}). \tag{10.8}$$

(Note that $(ir - A)x_0 = Bu_0 \Rightarrow x_0 \in H_B \subset \text{Dom}(C_{\text{w}})$.) The same holds for Corollary 10.2.10.

Proof of Corollary 10.2.9: Set $\mathcal{U}_*^* := \mathcal{U}_{\text{exp}}$.

1° *(i)$\Leftrightarrow$(ii)$\Leftrightarrow$(iii):* By Theorem 9.2.16, (i) and (iii) imply (ii). If (ii) holds, then we obtain Theorems 9.2.16 and 9.9.1(k)&(a2) that (i) holds, $\mathbb{D}$ is J-coercive and $\mathcal{J}(0,\cdot) \geq 0$, hence $\mathbb{D}$ is positively J-coercive, hence also (iii) holds.

2° *Proposition 10.3.2(i)–(iii):* By Proposition 10.3.2(c)&(g1), any of Proposition 10.3.2(i)–(iii) implies Proposition 10.3.2(i). Conversely, if (iii) holds, then (ii) holds, hence then Σ has an exponentially stabilizing ULR K, hence Proposition 10.3.2(i)–(iii) are equivalent (hence they all hold), by Proposition 10.3.2(e2) (since $D^*JD \gg 0$).

3° The uniqueness of K follows from Proposition 6.6.18(g). The "$J \geq 0$" claim follows from Lemma 9.3.7(1+). If $\pi_{[0,1)}\mathbb{A}B \in \mathrm{L}^1([0,1]; \mathcal{B}(U,H))$, then "$D^*JD \in \mathcal{GB}(U)$" is redundant in (iii) by Proposition 10.3.2(e2). □

We now establish Corollary 10.2.9 under "weaker" assumptions (this forces us to replace the B_{w}^*-CARE by the CARE):

Corollary 10.2.10 ($\mathcal{U}_{\text{exp}}$: Unique minimum $\Leftrightarrow$ CARE $\Leftrightarrow$ J-coercive) *Assume that $\mathbb{A}B \in \mathrm{L}^2([0,1); \mathcal{B}(U,H))$, $C_{\text{w}}\mathbb{A} \in \mathrm{L}^1([0,1); \mathcal{B}(H,Y))$, and $C_{\text{w}}\mathbb{A}B \in \mathrm{L}^1([0,1); \mathcal{B}(U,Y))$.*

Then the following are equivalent:

(i) there is a [unique] minimizing control over $\mathcal{U}_{\exp}(x_0)$ for each $x_0 \in H$, and $D^*JD \gg 0$;

(ii) there is a [unique] exponentially stabilizing solution $(\mathcal{P}, S, K)$ of the CARE, and $S \gg 0$ (or $D^*JD \gg 0$);

(iii) $\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \end{array}\right]$ is optimizable, and $\mathbb{D}$ is positively J-coercive over $\mathcal{U}_{\exp}$ (i.e., any (hence all) of Proposition 10.3.2(i)–(iii) holds).

Moreover, any solution of (ii) is as in Theorem 9.2.18 (in particular, $S = D^*JD \gg 0$), and K is ULR and the unique minimizing state feedback operator over $\mathcal{U}_{\exp}$. If $J \geq 0$, then the word "unique" is redundant in (i).

For L^1 in place of L^2 above, we have an analogous corollary of Theorem 9.2.18 (thus, then we must assume positive J-coercivity and we may remove "$D^*JD \gg 0$" from (i) and (ii)).

Proof: 1° *(i)⇒(ii)&(iii):* Assume (i). By Corollary 9.2.19, (ii) and (iii) are satisfied (and $S = D^*JD \gg 0$) except for the word "positively", which is obtained from Proposition 9.9.12(b).

2° *(iii)⇒(i):* This follows from Corollary 9.2.19, since $D^*JD \gg 0$, by Lemma 9.2.17.

3° *(ii)⇒(i):* If $D^*JD \gg 0$, then this is trivial. If $S \gg 0$, then we obtain (iii) from Corollary 9.2.19, and Proposition 9.9.12(b), hence (i) holds, by 2°.

4° *Proposition 10.3.2(i)–(iii):* Use Proposition 10.3.2(e2) in 2° of the proof of Corollary 10.2.9.

5° *Final claims:* By Corollary 9.2.19, any solution of (ii) is as in Theorem 9.2.18 The uniqueness of K follows from Proposition 6.6.18(g). The "$J \geq 0$" claim follows from Lemma 9.3.7(1+). □

For general WPLSs, the above problem becomes rather tricky, and we cannot say much more than in Corollary 10.2.5. To make our result neater, we assume that $J \geq 0$ and start with the case $\dim U < \infty$:

Theorem 10.2.11 ($\mathcal{U}_{\exp}$: Unique $\left[\begin{array}{c|c} \mathbb{K}_{\min} & \mathbb{F}_{\min} \end{array}\right]$ ⇔ IARE ⇔ J-coercive) *Assume that $J \geq 0$, $\dim U < \infty$ and $\mathcal{U}_*^* = \mathcal{U}_{\exp}$. Then (i)–(iii) are equivalent.*

(i) There is a unique (modulo (9.114)) minimizing state feedback pair for Σ.

(ii) The IARE has an exponentially stabilizing solution.

(iii) Σ is exponentially stabilizable and [positively] J-coercive.

Moreover, the following hold:

(a) If (ii) holds, then this exponentially stabilizing solution is the greatest nonnegative admissible solution and strictly minimizing.

(b) If Σ has an SR (resp. ULR) exponentially stabilizing state feedback operator with closed-loop system $\left[\begin{array}{c|c} \mathbb{A}_\flat & \mathbb{B}_\flat \\ \hline \mathbb{C}_\flat & \mathbb{D}_\flat \end{array}\right]$ satisfying Hypothesis 10.6.1(1.) (resp. (6.)), then the IARE is equivalent to the CARE (resp. and to the B_w^*-CARE), and also (i')–(iii') below become equivalent to (i)–(iii).

Thus, then there is a unique WR (resp. ULR) minimizing state feedback operator iff any (hence all) of these hold.

(c) **($\dim U = \infty$)** *Drop the assumption that* $\dim U < \infty$. *Then all of the above holds if we add to (i) and (i') the requirement "and* $\mathbb{D}$ *is J-coercive" (or "and* $\langle \mathbb{D}u, J\mathbb{D}u\rangle \geq \varepsilon \|u\|_2^2$ *for some* $\varepsilon > 0$ *and all* $u \in \mathcal{U}_{\mathrm{exp}}(0)$*").*

The original (i) holds iff the eIARE has an exponentially stabilizing solution with $S > 0$.

(We believe that implication (iii)$\Rightarrow$(i) does not hold in the indefinite case.)

See Proposition 10.3.2 for equivalent conditions for (positive) J-coercivity. For (b) we note that when $\dim U < \infty$ (but not in case (c)), the concept WR (resp. WLR) is equivalent to SR (resp. SLR) and to UR (resp. ULR) (for state feedback operators and pairs and solutions of the AREs), by Lemma 6.3.2(a1)&(a2).

By (a), it suffices to find a maximal nonnegative admissible solution (if any) and check whether it is exponentially stabilizing (if none/not, then (i)–(iii) do not hold).

In addition to the equivalence between (i)–(iii), we have an equivalence between the following, weaker conditions corresponding to a minimizing, possibly non-well-posed "state feedback" (still under the assumptions that $J \geq 0$ and $\dim U < \infty$):

(i') There is a unique minimizing control for Σ over $\mathcal{U}_{\mathrm{exp}}(x_0)$ for each $x_0 \in H$.

(ii') The discretized IARE has an exponentially stabilizing solution.

(iii') Σ is optimizable and [positively] J-coercive over $\mathcal{U}_{\mathrm{exp}}$.

(Any of these is equivalent to the existence of a unique minimizing control in WPLS form, see Theorems 8.3.9.) If $\mathbb{D}$ is SR and $D^*JD > 0$, then (ii') becomes equivalent to the $\mathrm{Dom}(A_{\mathrm{crit}})$-CARE (9.67) having an exponentially stabilizing solution (cf. Remark 9.7.7(b2)).

(Modify the proof of Proposition 9.9.12 to observe that the discrete "S" is $\gg 0$ to obtain (i')$\Leftrightarrow$(iii'). Equivalence (i')$\Leftrightarrow$(ii') follows from the above proposition (in its discrete-time form) and Theorem 14.1.6. If $\mathbb{D}$ is SR and $D^*JD > 0$, then $\mathbb{D}$ (and hence also $\mathbb{D}^{\mathrm{d}}$) is UR and $D^*JD \gg 0$ (since $\dim U < \infty$), so that the last equivalence follows from Remark 9.7.7(b2).)

If Hypothesis 9.2.1 holds for $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{exp}}$ and $D^*JD \gg 0$, then any of (i')–(iii') is equivalent to (i)–(iii), as well as to the B_{w}^*-CARE having an exponentially stabilizing solution, by Corollary 10.2.9, even for general J and U. The situation in discrete time is about the same as for bounded B (with the exception that D^*JD must be replaced by $S = D^*JD + B^*\mathcal{P}B$).

Proof of Theorem 10.2.11: 1° *(i)$\Leftrightarrow$(ii):* This follows from Theorem 9.9.1(a2)&(e1)&(f2) (for "(i)$\Rightarrow$(ii)" we note that $S > 0$ implies that $S \gg 0$; for "(ii)$\Rightarrow$(i)" we note that $\mathcal{P} = \mathbb{C}_{\circlearrowleft}^* J\mathbb{C}_{\circlearrowleft} \gg 0$ and $J \geq 0$ imply that $S \geq 0$, hence $S \gg 0$).

2° *(ii)$\Rightarrow$(iii):* This follows from Proposition 9.9.12, since $J \geq 0$ implies that J-coercivity is equivalent to positive J-coercivity (and the invertibility of S implies that of $\mathbb{T}^t$).

3° *(iii)$\Rightarrow$(i):* Assume (iii). Let $\left[\begin{array}{c|c} \mathbb{K}' & \mathbb{F}' \end{array}\right]$ be exponentially stabilizing for Σ with closed-loop system $\Sigma_\flat$, so that $\Sigma_\flat$ is positively J-coercive over $\mathcal{U}_{\mathrm{exp}}^{\Sigma_\flat} = \mathcal{U}_{\mathrm{out}}^{\Sigma_\flat}$, by Theorem 8.4.5(d), i.e., $\mathbb{D}_\flat^* J\mathbb{D}_\flat \gg 0$, by Lemma 8.4.11(a2).

By Corollary 10.2.13(b) (whose proof is independent of this), the IARE for $\Sigma_\flat$ has an exponentially stabilizing, minimizing solution, hence so has the IARE for Σ, by Lemma 9.12.3(d1). The uniqueness claim follows from Theorem 9.9.1(f2).

(a) Since $J \geq 0$, we have $S \geq 0$, hence $S \gg 0$ (because $S \in \mathcal{GB}(U)$, by 1°) and the solution is the greatest nonnegative solution, by Theorem 9.9.1(a2)&(e1)&(f2).

(b) Let $\widetilde{K}$ be a SR (resp. ULR) state feedback operator for Σ with closed-loop system $\Sigma_\flat$ having $\mathbb{D}_\flat \in \widetilde{\mathcal{A}}_+$. Let $\left[\begin{array}{c|c} \widetilde{\mathbb{K}} & \widetilde{\mathbb{F}} \end{array}\right]$ be the corresponding pair.

Assume (iii'). Then $\Sigma_\flat^1 := \left[\begin{array}{c|c} \mathbb{A}_\flat & \mathbb{B}_\flat \\ \hline \mathbb{C}_\flat & \mathbb{D}_\flat \end{array}\right]$ is (exponentially stable and) positively J-coercive over $\mathcal{U}_{\mathrm{exp}}^{\Sigma_\flat} = \mathcal{U}_{\mathrm{out}}^{\Sigma_\flat}$, by Theorem 8.4.5(d). By Theorem 10.6.3(f1) (resp. (a)), there is a (unique) exponentially stabilizing solution $(\mathcal{P}, S, K_\natural)$ of the CARE (resp. and the B_{w}^*-CARE) for $\Sigma_\flat^1$. Set $K := \widetilde{K} + K_\natural$. By Proposition 9.12.4(b), $(\mathcal{P}, S, K)$ is an exponentially stabilizing solution of the CARE (resp. and the B_{w}^*-CARE, by Proposition 9.3.5(a). (By (a), K is strictly minimizing.)

(c) Parts 1°–3° still hold except that now "$S \gg 0$" in 1° must be deduced from Lemma 9.9.7(c3)&(c4) (note that necessarily $\mathcal{P} = \mathbb{C}_\circlearrowleft^* J \mathbb{C}_\circlearrowleft \gg 0$). The last claim follows from Corollary 10.2.5(a)&(c).

The proofs of (a) and (b) do not use the assumption $\dim U < \infty$ (this is why we write "SR" instead of "WR" or "UR" in (b)). □

For most of the rest of this section, we shall present result for $\mathcal{U}_{\mathrm{out}}$ (recall that $\mathcal{U}_{\mathrm{exp}} = \mathcal{U}_{\mathrm{str}} = \mathcal{U}_{\mathrm{sta}} = \mathcal{U}_{\mathrm{out}}$ when Σ is estimatable or exponentially q.r.c.-stabilizable, by Lemma 8.3.3).

When minimizing over $\mathcal{U}_{\mathrm{out}}$, we cannot we cannot reduce the problem to the stable case as in Theorem 10.2.11, (unless we choose to use Theorem 8.4.5(f)), as illustrated in Example 9.13.2. However, if the system is q.r.c.-SOS-stabilizable (e.g., SOS-stable), then the reduction will succeed. This fact will be applied in most results below. We first give two results on the IARE and then two on the CARE.

Corollary 10.2.12 ($\mathcal{U}_{\mathrm{out}}$: Unique $\left[\begin{array}{c|c} \mathbb{K}_{\min} & \mathbb{F}_{\min} \end{array}\right]$ ⇔ IARE ⇔ J-coercive⇔ r.c.f.) *Then the following conditions are equivalent to each other and stronger than the conditions (Crit1)–(Crit4) of Theorem 9.9.10:*

- *(Crit1+)* **(Minimizing $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$)** *There is a [strictly] minimizing q.r.c.-SOS-stabilizing state feedback pair for Σ over $\mathcal{U}_{\mathrm{out}}$, and $\mathbb{D}$ is J-coercive over $\mathcal{U}_{\mathrm{out}}$.*
- *(Crit2+)* **(IARE)** *The IARE (9.111) has a q.r.c.-SOS-P-stabilizing solution $\mathcal{P} = \mathcal{P}^* \in \mathcal{B}(H)$ s.t. $S \gg 0$.*
- *(Crit3+)* **(J-coercivity)** *Σ is positively J-coercive over $\mathcal{U}_{\mathrm{out}}$ and q.r.c.-SOS-stabilizable.*
- *(Crit4+)* **(R.c.f.)** *The map $\mathbb{D}$ has a (J,I)-inner q.r.c.f. $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$, and Σ is q.r.c.-SOS-stabilizable.*

If any of these has a solution, then that solution solves the other conditions and (Crit1)–(Crit4), and (a1)–(g3) of Theorem 9.9.10 apply.

See Theorem 10.2.14 for corresponding CAREs.

We recall that $\mathbb{D}$ is positively J-coercive over $\mathcal{U}_{\text{out}}$ iff there is $\varepsilon > 0$ s.t.

$$\langle \mathbb{D}u, J\mathbb{D}u\rangle \geq \varepsilon\left(\|u\|_2^2 + \|\mathbb{D}u\|_2^2\right) \qquad (u \in \mathcal{U}_{\text{out}}(0)). \tag{10.9}$$

Proof: 1° *(Crit1+)⇔(Crit2+)⇔(Crit4+):* Except for (Crit3+), This follows from Theorem 9.9.10(e2) (use normalization $\mathbb{N}' := \mathbb{N}S^{-1/2}$, $\mathbb{M}' := \mathbb{M}S^{-1/2}$ for implication (Crit2+)&(Crit4)⇒(Crit4+)).

2° *(Crit1+)⇔(Crit3+):* Obviously, (Crit1+) implies (Crit3+) (*positive* J-coercivity follows from Lemma 10.2.2). For the converse, assume (Crit3+) and let $\Sigma_\flat$ be the corresponding closed-loop system. By Lemma 8.4.11(c), $\mathbb{D}_\flat$ is J-coercive over $\mathcal{U}_{\text{out}}$, hence there is a minimizing q.r.c.-SOS-stabilizing pair for $\left[\begin{array}{c|c} \mathbb{A}_\flat & \mathbb{B}_\flat \\ \hline \mathbb{C}_\flat & \mathbb{D}_\flat \end{array}\right]$, by (the proof of) Corollary 10.2.13. By Theorem 8.4.5(g1)&(a), there is a minimizing q.r.c.-SOS-stabilizing pair for Σ too.

3° The final claims follow from Theorem 9.9.10(a1)–(a3). □

In the stable case, one more equivalent condition is the uniform positivity of the Popov operator (which is obtained, e.g., by adding $\varepsilon\|u\|^2$ to the nonnegative cost function):

Corollary 10.2.13 ($\mathcal{U}_{\text{out}}$: $\left[\begin{array}{c|c} \mathbb{K}_{\min} & \mathbb{F}_{\min} \end{array}\right]$ — Stable case) *Let $\Sigma \in$ SOS. The following conditions are equivalent to and have the same solutions as (Crit1+)–(Crit4+) of Corollary 10.2.12:*

(Crit1SOS+) There is a stable uniformly minimizing SOS-stabilizing feedback pair $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ for Σ over $\mathcal{U}_{\text{out}}$;

(Crit2SOS+) the IARE has a stable P-SOS-stabilizing solution having $S \gg 0$;

(Crit3SOS+) $\mathbb{D}^* J\mathbb{D} \gg 0$;

(Crit4SOS+) $\mathbb{D}^* J\mathbb{D} = \mathbb{X}^*\mathbb{X}$ for some $\mathbb{X} \in \mathcal{G}\mathrm{TIC}(U)$.

Moreover,

(a) If Σ satisfies Hypothesis 10.6.1(1.) (resp. (6.)), then we can replace the IARE by the CARE (resp. B_{w}^*-CARE) above.

(b) If Σ is exponentially stable, then solutions of (Crit1SOS+)–(Crit2SOS+) are exactly the minimizing pairs over $\mathcal{U}_{\exp}$, equivalently, the exponentially (equivalently, I/O-)stabilizing solutions of the IARE having $S \gg 0$.

Recall from Lemma 8.4.11(a2) that (Crit3SOS+) holds iff $\mathbb{D}$ is positively J-coercive over $\mathcal{U}_{\text{out}}$.

By a "uniformly minimizing" $u_{\min}$ we mean here that $\mathcal{J}(x_0, u_{\min}(x_0) + \eta) - \mathcal{J}(x_0, u_{\min}) \geq \varepsilon\|\eta\|_2^2$ for all $\eta \in \mathrm{L}^2(\mathbf{R}_+; U)$. If $\dim U < \infty$, then we could replace "uniformly minimizing" by "strictly minimizing" in (Crit2SOS+), because then $S \gg 0 \Leftrightarrow S > 0$ (cf. the proof below). The minimizing pair is given by (9.140), i.e., by

$$\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right] := \left[\begin{array}{c|c} -\pi_+\mathbb{X}^{-*}\mathbb{D}^* J\mathbb{C} & I - \mathbb{X} \end{array}\right], \tag{10.10}$$

where $\mathbb{X}$ is as in (Crit4SOS+).

If Σ is [strongly] stable, then "SOS-" [and "P-"] can be removed from conditions (Crit2SOS+) and (Crit3SOS+) [and "strongly" can be added], by Corollary 6.6.9.

Proof of Theorem 10.2.13: By Theorem 10.6.3(f2)&(b), (Crit2SOS+)–(Crit4SOS+) and (Crit2+) are equivalent (and the (unique) solutions $\mathcal{P}$ of (Crit2+) are exactly the ones of (Crit2SOS+)).

Finally, we obtain (Crit1+)⇔(Crit1SOS+), i.e., the fact that the control $\mathbb{K}_{\circlearrowleft}x_0$ is uniformly minimizing iff $S \gg 0$, by setting $u_{\circlearrowleft} := \mathbb{M}^{-1}\eta$ in Theorem 9.9.10(e1), because $S \gg 0$ iff $\mathbb{M}^{-*}S\mathbb{M}^{-1} \gg 0$ (note that $\mathbb{M} = (I-\mathbb{F})^{-1} \in \mathcal{G}$TIC).

(a) This follows from Theorem 10.6.3(b)&(f1).

(b) This follows from Theorem 9.9.10(c1)&(c2). □

That was the case with IAREs, but we would like to use the CAREs instead of IAREs. Therefore, we shall assume further regularity. One way for this is to use the class $\widetilde{\mathcal{A}}_+$ (see Standing Hypothesis 10.6.6 and (b3) below) $\widetilde{\mathcal{A}}_+$ to formulate conditions under which a unique minimizing control necessarily corresponds to a ($\mathcal{U}_*^*$-stabilizing) solution of the CARE, hence to a WR state feedback operator.

Theorem 10.2.14 ($\mathcal{U}_{\mathrm{out}}$: Unique $K_{\min}$ ⇔ CARE ⇔ J-coercive ⇔ r.c.f.) *Let $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{out}}$. Assume that $\mathbb{D}$ is WR. Then the following conditions are equivalent to each other and stronger than the conditions (Crit1+)–(Crit4+) of Corollary 10.2.12:*

(Crit1+WR) **(Minimizing K)** *There is a WR [strictly] minimizing q.r.c.-SOS-stabilizing state feedback operator for Σ, and $\mathbb{D}$ is J-coercive.*

(Crit2+WR) **(CARE)** *The CARE has a q.r.c.-SOS-P-stabilizing solution having $S \gg 0$.*

(Crit4+WR) **(WR r.c.f.)** *The map $\mathbb{D}$ has a (J,I)-inner q.r.c.f. $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$ with $\mathbb{X} := \mathbb{M}^{-1}$ being WR with $X \in \mathcal{G}\mathcal{B}(U)$, and Σ is q.r.c.-SOS-stabilizable.*

Moreover, the following hold:

(a) If any of (Crit1+WR)–(Crit4+WR) has a solution, then that solution solves the other conditions and (Crit1+)–(Crit4+) of Corollary 10.2.12 and (Crit1)–(Crit4), and then (a1)–(g3) of Theorem 9.9.10 apply.

(b1) Assume that Σ has a SR (resp. URL) q.r.c.-SOS-stabilizing state feedback operator $\widetilde{K}$ s.t. the resulting closed-loop system $\left[\begin{array}{c|c} \mathbb{A}_\flat & \mathbb{B}_\flat \\ \hline \mathbb{C}_\flat & \mathbb{D}_\flat \end{array}\right]$ satisfies Hypothesis 10.6.1(1.) (resp. (6.)).

Then (Crit7+WR) (resp. and (Crit6+WR)) is equivalent to (Crit1+WR)–(Crit4+WR):

(Crit6+WR) **(B_{w}^*-CARE)** *The B_{w}^*-CARE has q.r.c.-SOS-P-stabilizing solution.*

(Crit7+WR) $\mathbb{D}$ is positively J-coercive (equivalently, $\mathbb{D}_\flat^* J \mathbb{D}_\flat \gg 0$).

(b2) **(B_w^*-CARE)** *Assume that Hypothesis 9.2.1 holds and* $D^*JD \gg 0$.

Then conditions (Crit1+WR)–(Crit6+WR) are equivalent.

(b3) **(MTIC)** *Assume that* Σ *is q.r.c.-SOS-stabilizable in* $\widetilde{\mathcal{A}}_+$*, then (Crit1+WR)–(Crit4+WR) are equivalent to (Crit7+WR), and imply that* $\mathbb{N}, \mathbb{M} \in \widetilde{\mathcal{A}}_+$.

(b4) In (b1)–(b3), conditions (Crit1+WR)–(Crit4+WR) and (Crit1+)–(Crit4+) (and (Crit1)–(Crit4) in (b2)) are all equivalent to each other.

(c) If Σ *is exponentially [q.]r.c.-stabilizable, then any I/O-stabilizing or input-stabilizing solution of the CARE having* $S \gg 0$ *is exponentially [q.]r.c.-stabilizing and minimizing over* $\mathcal{U}_{\text{exp}} = \mathcal{U}_{\text{str}} = \mathcal{U}_{\text{sta}} = \mathcal{U}_{\text{out}}$. *(See Theorem 6.7.15 for further reductions.)*

If Σ is estimatable, then any minimizing pair over $\mathcal{U}_{\text{out}}$ is exponentially q.r.c.-stabilizing and minimizing over $\mathcal{U}_{\text{exp}} = \mathcal{U}_{\text{str}} = \mathcal{U}_{\text{sta}} = \mathcal{U}_{\text{out}}$, by Theorem 9.9.1(d). Therefore, then we get further equivalent conditions from the results for $\mathcal{U}_{\text{exp}}$ in the first part of this section.

Proof: The equivalence and (a) follow from Corollary 10.2.12 and Theorem 9.9.10(d1) (note that the $\mathbb{X}$ of (d1) is replaced by $S^{1/2}\mathbb{X}$).

(a) This follows from the above and Corollary 10.2.12.

(b1) 1° *(1.):* By Proposition 9.12.4(c) the (unique) solutions of (Crit2+WR) correspond 1-1 to the q.r.c.-SOS-P-stabilizing solutions of the CARE for $\left[\begin{array}{c|c} \mathbb{A}_\flat & \mathbb{B}_\flat \\ \hline \mathbb{C}_\flat & \mathbb{D}_\flat \end{array}\right]$. By Corollary 10.2.15(b1) (and (Crit2stable+WR) and (Crit2stable+WR')), such a solution exists iff $\mathbb{D}_\flat^* J \mathbb{D}_\flat \gg 0$, i.e., iff $\mathbb{D}$ is positively J-coercive (by Lemma 8.4.11(b1)).

2° *(5.):* This is contained in 1° except for the B_w^*-CARE claim which can be obtained as in the proof of Proposition 9.3.5.

(Note that Hypothesis 9.2.1 would not necessarily guarantee that the minimizing K is q.r.c.-stabilizing.)

(b2) By the above, we have (Crit1+WR)–(Crit4+WR)⇒(Crit1-4+). By Theorem 9.9.10(d2), (Crit6+WR) is weaker than any of the above, and by Theorem 9.2.9, (Crit6+WR) implies (Crit2+WR).

(b3) This is contained in (b1) except for the fact that $\mathbb{N}, \mathbb{M} \in \widetilde{\mathcal{A}}_+$, which follows from Lemma 10.6.7(b).

(b4) For (b2), this was noted in the proof of (b2). Obviously, (Crit3+) implies (Crit7+WR), hence also the claims on (b1) and (b3) hold.

(c) By Theorem 6.7.15(b1), any I/O-stabilizing solution of the CARE is exponentially [q.]r.c.-stabilizing. The rest follows from (Crit2+WR) (and (a)) and Lemma 8.3.3. □

The stable case is a bit simpler:

Corollary 10.2.15 ($\mathcal{U}_{\text{out}}$: $K_{\min}$ — Stable case) *Let* $\mathcal{U}_*^* = \mathcal{U}_{\text{out}}$. *Assume that* $\mathbb{D}$ *is WR and* Σ *is strongly stable. Then the following conditions are equivalent to each other and to (Crit1+WR)–(Crit4+WR).*

(Crit1stable+WR) **(Minimizing K)** *There is a WR [strictly] minimizing stable, stabilizing state feedback operator for* Σ*, and* $\mathbb{D}$ *is* J*-coercive.*

(Crit2stable+WR) **(CARE)** *The CARE has a stable, stabilizing solution having $S \gg 0$.*

(Crit2stable+WR') **(CARE)** *The CARE has a solution having $\mathbb{M}$ stable and $S \gg 0$.*

(Crit4stable+WR) **(WR SpF)** *We have $\mathbb{D}^* J\mathbb{D} = \mathbb{X}^*\mathbb{X}$ for some WR $\mathbb{X} \in \mathcal{G}\mathrm{TIC}$ having $X \in \mathcal{GB}(U)$.*

Moreover, the following hold:

(a) If any of (Crit1stable+WR)–(Crit4stable+WR) has a solution, then that solution solves the other conditions, (Crit1+WR)–(Crit4+WR) of Theorem 10.2.14, (Crit1+)–(Crit4+) of Corollary 10.2.12 and (i)–(iii) of Theorem 9.1.7, and then (a1)–(g3) of Theorem 9.9.10 apply. In particular, such solutions are stable and strongly r.c.-stabilizing.

(b1) **(MTIC)** *If Σ satisfies Hypothesis 10.6.1(1.) (see Lemma 10.6.2), then (Crit1stable+WR)–(Crit4stable+WR) are equivalent to (Crit1+)–(Crit4+) and to*

(Crit7stable+WR) $\mathbb{D}^* J\mathbb{D} \gg 0$.

(b2) **(B_{w}^*-CARE)** *If Σ satisfies Hypothesis 10.6.1(6.), then (Crit1stable+WR)–(Crit7stable+WR) are equivalent:*

(Crit6stable+WR) $D^*JD \gg 0$ and (the B_{w}^*-CARE)

$$(B_{\mathrm{w}}^*\mathcal{P} + D^*JC)^*(D^*JD)^{-1}(B_{\mathrm{w}}^*\mathcal{P} + D^*JC) = A^*\mathcal{P} + \mathcal{P}A + C^*JC \tag{10.11}$$

has a solution $\mathcal{P} = \mathcal{P}^* \in \mathcal{B}(H, \mathrm{Dom}(B_{\mathrm{w}}^*))$ s.t. $s \mapsto K_{\mathrm{w}}(s-(A+BK_{\mathrm{w}}))^{-1}B$ is in $\mathrm{H}^\infty(\mathbf{C}^+;\mathcal{B}(U))$, where $K := -(D^*JD)^{-1}(B_{\mathrm{w}}^*\mathcal{P} + D^*JC)$.

(equivalently, s.t. K is stable and stabilizing).

Proof: (The labels of equivalent conditions follow roughly those of Theorem 9.9.10 and Corollary 10.2.13.)

The above equivalence is that of Corollary 10.2.13 with the additional requirement that $\mathbb{X} = I - \mathbb{F}$ is WR and $X \in \mathcal{GB}(U)$. (Note that we could again remove the J-coercivity assumption from the first condition if we required K to be uniformly minimizing.)

(a) This follows from the above and Corollary 10.2.13(a).

(b1) This follows from Theorem 10.6.3(f1)&(i)&(iv)&(iv').

(b2) This follows from (b1) and Theorem 10.6.3(i)&(iv)&(iv'). □

We extend one more classical result: if Σ is (approximately) observable and the cost function is somewhat standard, then $\mathcal{P} > 0$:

Lemma 10.2.16 ($\mathcal{P} > 0$) *If Σ is observable, $J > 0$, $\mathcal{J}(x_0, u) > 0$ for all $x_0 \in H$ and all nonzero $u \in \mathcal{U}_*^*(x_0)$, and there is a minimizing control for each $x_0 \in H$, then $\mathcal{P} > 0$, where $\mathcal{P}$ is the minimal cost operator.*

Proof: Obviously, $u = 0$ is strictly minimizing for $x_0 = 0$, hence the minimizing control is unique for each $x_0 \in H$, by Lemma 8.3.8. By Theorem 8.3.9(b1), we can define the minimal cost operator $\mathcal{P}$.

If $x_0 \in H$ and $0 \geq \langle x_0, \mathcal{P}x_0 \rangle = \mathcal{J}(x_0, u_{\text{crit}}(x_0))$, then $u_{\text{crit}}(x_0) = 0$, by the assumption. But then $\langle \mathbb{C}x_0, J\mathbb{C}x_0 \rangle = \mathcal{J}(x_0, 0) = 0$, hence then $\|J^{1/2}\mathbb{C}x_0\|_2 = 0$, i.e., $x_0 \in \operatorname{Ker}(\mathbb{C}) = \{0\}$, hence then $x_0 = 0$. We conclude that $\mathcal{P} > 0$. □

Naturally, all of the above theory can be applied to the dual of the LQR problem (which was formulated in [WR00]):

Remark 10.2.17 (Final state estimation problem) *Let* $\Sigma := \left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right] \in \text{WPLS}(U, H, Y)$. *In the* Optimal Final State Estimation Problem (OFSEP) *we wish to find an estimator* $\mathbb{H}_\circlearrowleft \in \mathcal{B}(\mathrm{L}^2(\mathbf{R}_+; U), H)$ *s.t.* $\mathbb{H}_\circlearrowleft \mathbb{D}$ *is an optimal estimate for* $\mathbb{C}$*, i.e.,* $\|\mathbb{B} - \mathbb{H}_\circlearrowleft \mathbb{D}\|$ *is minimal.*

The dual problem of this is the optimal open-loop L^2-stabilization problem, *where we wish to find a controller* $\mathbb{K}_\circlearrowleft \in \mathcal{B}(H, \mathrm{L}^2(\mathbf{R}_+; U))$ *s.t.* $\mathbb{D}\mathbb{K}_\circlearrowleft$ *is minimizes* $\mathbb{C}$*, i.e.,* $\|\mathbb{C} + \mathbb{D}\mathbb{K}_\circlearrowleft\|$ *is minimal (the solutions of this problem correspond one-to-one to those of the OFSEP for* Σ^{d} *through* $\mathbb{H}_\circlearrowleft = -\mathbb{K}_\circlearrowleft^{\mathrm{d}}$*); this obviously corresponds to a minimizing control in WPLS form (i.e., the corresponding state feedback need not be well-posed; cf. Theorem 8.3.9).*

If one requires $\mathbb{H}_\circlearrowleft$ *to be generated by some weakly regular* $H \in \mathcal{B}(H_{-1}, U)$ *that stabilizes* Σ*, equivalently, if one requires* $\mathbb{K}_\circlearrowleft$ *to be generated by some weakly regular* $K \in \mathcal{B}(U, H_1)$ *that stabilizes* Σ*, then one ends up with our LQR problem, hence one can use applicable theorems and corollaries from above. In particular, the OFSEP CARE becomes (here* $J = I$*)*

$$\begin{cases} HSH^* = A\mathcal{P} + \mathcal{P}A^* + BJB^*, \\ \qquad S = DJD^* + \operatorname*{w-lim}_{s\to+\infty} C_{\mathrm{w}}\mathcal{P}(s - A^*)^{-1}C_{\mathrm{w}}^*, \\ \qquad H = -S^{-1}(C_{\mathrm{w}}\mathcal{P} + DJB^*). \end{cases} \tag{10.12}$$

In any case, one can find a solution for the OFSEP by using the theory this section. If we minimize $\left\| \begin{bmatrix} \mathbb{B} - \mathbb{H}_\circlearrowleft \mathbb{D} \\ \mathbb{H}_\circlearrowleft \end{bmatrix} \right\|$ *as in [WR00], the dual cost function becomes that of Section 10.1; in particular, then (the duals of) Theorems 10.1.4 and 10.1.6 apply. (Cf. also Definitions 6.6.10 and 6.6.21.)* □

Section 5 of [WR00] explores the connection between the OFSEP and estimatability (in the open-loop form only, excluding closed-loop systems, factorizations and CAREs). The main idea of this formulation (OFSEP) of the dual problem is to guarantee the existence of a solution regardless whether the solution is of output injection form or not.

For WPLSs with bounded C (e.g., for finite-dimensional systems; see [IOW]), one often defines the system and the cost function without any reference to the output:

Remark 10.2.18 (Bounded C) *Let* $\left[\begin{smallmatrix}\mathbb{A} & \mathbb{B}\\ \mathbb{C} & \mathbb{D}\end{smallmatrix}\right] \in \mathrm{WPLS}(U,H,Y)$, *and let* C *be bounded. Set*

$$\widetilde{J} := \begin{bmatrix}C & D\end{bmatrix}^* J \begin{bmatrix}C & D\end{bmatrix} = \begin{bmatrix}C^*JC & C^*JD\\ D^*JC & D^*JD\end{bmatrix} \in \mathcal{B}(H \times U). \tag{10.13}$$

Then $\langle y, Jy\rangle = \langle \left[\begin{smallmatrix}x\\ u\end{smallmatrix}\right], \widetilde{J}\left[\begin{smallmatrix}x\\ u\end{smallmatrix}\right]\rangle$, *where* $x := \mathbb{A}x_0 + \mathbb{B}u$, $y := \mathbb{C}x + \mathbb{D}u = Cx + Du$.

The converse is trivial: given $\mathbb{A}$, $\mathbb{B}$ *and* $\widetilde{J}$, *one may just take* $C := \left[\begin{smallmatrix}I\\ 0\end{smallmatrix}\right]$, $D := \left[\begin{smallmatrix}0\\ I\end{smallmatrix}\right]$ *to get* $y = \left[\begin{smallmatrix}x\\ u\end{smallmatrix}\right]$, *and set* $J := \widetilde{J}$ *to obtain the same cost function as above.*

If we minimize over $\mathcal{U}_{\mathrm{exp}}$, *then we do not have to know* y; *knowledge on* x, u *and* $\mathcal{J}(x_0,u)$ *is enough, hence then it is not a problem that different* $\mathbb{C}$, $\mathbb{D}$ *and* J *may result in same* $\widetilde{J}$. □

The above remark can be used when translation our results to the language of several of the articles where a bounded output operator is assumed, and conversely. Note that the above remark can be generalized to arbitrary regular WPLSs.

Notes

The special case of a standard cost function (as in Section 10.1) was explained in the notes on p. 555. Implications “(Crit4SOS+)⇒(Crit1SOS+)–(Crit3SOS+)” of Corollary 10.2.13 and the SR case of “(Crit4stable+)⇒(Crit1stable+WR)–(Crit2stable+WR’)” are more or less implicitly contained in [S97b] and [WW]. These were extended for jointly stabilizable and detectable systems in [S98b] (cf. Theorems 10.2.11 and 10.2.14). See [WR00] for the first the paragraphs of Remark 10.2.17. All these results treat only the case $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{out}}$.

Corollary 10.2.9 generalizes Theorem 3.10 of [Keu], which replaces Hypothesis 9.2.1 by the stronger assumption that Σ is a Pritchard–Salamon system; most classical finite-dimensional results (including Theorem 16.3.3 of [LR], Section 14.3 of [ZDG], Section 5.2.2 of [GL] and Corollary 4.5.7 of [IOW]) are special cases of Corollary 10.2.9 or of Corollary 10.2.8, all these results assume positive J-coercivity or something stronger, as one can show by applying Proposition 10.3.2. All these results treat only the case $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{exp}}$.

In a sense, Corollary 10.2.9 uses the weakest possible coercivity assumption (positive J-coercivity; this can be relaxed if eCAREs ($D^*JD \not\gg 0$) are allowed). We do not know analogous results for more general systems (than Pritchard–Salamon systems), but there are some results with stronger coercivity or stabilizability assumptions for several subsets of WPLSs, as explained above and in the notes on p. 555.

10.3 Standard assumptions

Euclid taught me that without assumptions there is no proof. Therefore, in any argument, examine the assumptions.

— Eric Temple Bell (1883–1960)

In this section we study popular assumptions of LQR (minimization) problems; more exactly, we list sufficient (and necessary) conditions for positive J-coercivity over $\mathcal{U}_{\text{exp}}$ or $\mathcal{U}_{\text{out}}$.

In classical results (in particular, in most minimization results for finite-dimensional [LR] [GL] [ZDG] [IOW] or Pritchard–Salamon systems [Keu] [LW]), one usually assumes some of (i)–(iv) of Proposition 10.3.2 (see also Remark 10.3.3) or something stronger, and shows that the CARE has an exponentially stabilizing solution iff the system is exponentially stabilizable, and that in either case the solution leads to the unique minimizing state feedback over $\mathcal{U}_{\text{exp}}$. In those rare results where the assumptions are weaker than positive J-coercivity, the CARE does not need to have a solution, although the eCARE or something similar might have (as in Corollary 10.2.5).

Sometimes the minimization is done over $\mathcal{U}_{\text{out}}$ and the assumption is one of (ai)–(biv) of Proposition 10.3.1. The purpose of this section is to show that all these assumptions are equivalent to or stronger than positive J-coercivity. This also leads to a list of different ways to verify the positive J-coercivity of a given WPLS and thus make the minimization results of Section 10.2 more applicable.

Popular classical assumptions for LQR, H^2 and H^∞ problems include conditions "no invariant zeros on $i\mathbf{R}\cup\{\infty\}$" (see (iii) far below) and "no transmission zeros on $i\mathbf{R}\cup\{\infty\}$" ((aiv) below); for minimal finite-dimensional systems these are equivalent, as shown below. We start by showing that the latter condition is equivalent to I-coercivity over $\mathcal{U}_{\text{out}}$ (recall that $\mathcal{U}_{\text{out}}(0) := \{u \in \mathrm{L}^2(\mathbf{R}_+;U) \,\big|\, \mathbb{D}u \in \mathrm{L}^2\}$):

Proposition 10.3.1 ($\mathcal{U}_{\text{out}}$: $y \in \mathrm{L}^2 \Rightarrow u \in \mathrm{L}^2$) *Let* $\Sigma := \left[\begin{smallmatrix}\mathbb{A} & \mathbb{B}\\ \mathbb{C} & \mathbb{D}\end{smallmatrix}\right] \in \mathrm{WPLS}(U,H,Y)$ *and* $J = J^* \in \mathcal{B}(Y)$. *Consider the following conditions:*

(ai) $\mathcal{J}(0,u) \ge \varepsilon\left(\|u\|_2^2 + \|\mathbb{D}u\|_2^2\right)$ *for some* $\varepsilon > 0$ *and all* $u \in \mathcal{U}_{\text{out}}(0)$*;*
i.e., $\mathbb{D}$ *is positively* J*-coercive over* $\mathcal{U}_{\text{out}}$.

(aii) $\|\mathbb{D}u\|_2 \ge \varepsilon\|u\|_2$ $(u \in \mathrm{L}^2(\mathbf{R}_+;U))$ *for some* $\varepsilon > 0$.

(aiii) $\widehat{\mathbb{D}}(s)^*\widehat{\mathbb{D}}(s) \ge \varepsilon I$ *for all* $s \in i\mathbf{R}\cup\{\infty\}$ *and some* $\varepsilon > 0$.

(aiv) $\widehat{\mathbb{D}}(s)u_0 \ne 0$ *for all* $s \in i\mathbf{R}\cup\{\infty\}$ *and all nonzero* $u_0 \in U$.

(bii) $\langle \mathbb{D}u, J\mathbb{D}u\rangle \ge \varepsilon\|u\|_2^2$ $(u \in \mathcal{U}_{\text{out}}(0))$ *for some* $\varepsilon > 0$.

(biii) $\widehat{\mathbb{D}}(s)^*J\widehat{\mathbb{D}}(s) \ge \varepsilon I$ *for all* $s \in i\mathbf{R}\cup\{\infty\}$ *and some* $\varepsilon > 0$.

(biv) $\widehat{\mathbb{D}}(s)^*J\widehat{\mathbb{D}}(s) > 0$ *for all* $s \in i\mathbf{R}\cup\{\infty\}$.

(bv) $D^*JC = 0$, $D^*JD \gg 0$, $C^*JC \ge 0$.

We have the following implications:

(a) **($J \gg 0$)** *Assume that* $J \gg 0$. *Then (ai)⇔(aii)⇔(bii). If, in addition,* $\dim U \times H \times Y < \infty$*, then (ai)–(biv) are equivalent.*

(b) **(Rational $\widehat{\mathbb{D}}$)** *Assume that* $\dim U \times H \times Y < \infty$. *Then (ai)⇔(bii)⇔(biii)⇔(biv)⇐(bv).*

If, in addition, Σ is exponentially stabilizable and exponentially detectable (e.g., minimal), then also (i)–(vi) of Proposition 10.3.2 are equivalent to (ai) (see Proposition 10.3.2(d)).

(c) **($\mathbb{D} \in$ TIC)** *Assume that $\mathbb{D}$ is stable. Then (ai) $\Leftrightarrow$ (bii) $\Leftrightarrow$ $\mathbb{D}^* J \mathbb{D} \geq \varepsilon I$ $\Leftrightarrow$ $\langle \widehat{\mathbb{D}}(s) u_0, J \widehat{\mathbb{D}}(s) u_0 \rangle_Y \geq \varepsilon \|u_0\|_U^2$ a.e., for all $u_0 \in U$.*

If U is separable or $\widehat{\mathbb{D}}$ is piecewise continuous, then a fifth equivalent condition is that $(\widehat{\mathbb{D}})^* J \widehat{\mathbb{D}} \geq \varepsilon I$ a.e.

(d) Assume that $\widetilde{J} := \begin{bmatrix} C & D \end{bmatrix}^* J \begin{bmatrix} C & D \end{bmatrix} =: \left[\begin{smallmatrix} Q & N^* \\ N & R \end{smallmatrix}\right]$ satisfies $\widetilde{J} \geq 0$, $R \gg 0$, and $\dim \operatorname{Ker}(\widetilde{J}) = \dim \operatorname{Ker}(Q)$, *and* $\dim U \times H \times Y < \infty$.

Then there is $\varepsilon > 0$ s.t. $\langle \left[\begin{smallmatrix} x_0 \\ u_0 \end{smallmatrix}\right], \widetilde{J} \left[\begin{smallmatrix} x_0 \\ u_0 \end{smallmatrix}\right] \rangle \geq \varepsilon \|u_0\|_U^2$ for all $x_0 \in H$, $u_0 \in U$. Therefore, $\mathcal{J}(x_0, u) \geq \varepsilon \|u\|_2^2$ for all measurable $u : \mathbf{R}_+ \to U$ and all $x_0 \in H$. In particular, then (bii)–(biv) hold.

(Consequently, if we wish to optimize over all measurable controls, we may replace $\begin{bmatrix} C & D \end{bmatrix}$ by $\widetilde{J}^{1/2}$ and J by $I \in \mathcal{B}(H \times Y)$ to make $\mathcal{U}_{\text{out}}(x_0)$ equal to $\{u \mid \mathcal{J}(x_0, u) < \infty\}$; the new system is observable if $\left[\begin{smallmatrix} A \\ Q \end{smallmatrix}\right]$ is.)

Condition (aiv) is the standard assumption that $\widehat{\mathbb{D}}$ has a full column rank on $i\mathbf{R} \cup \{\infty\}$. Equivalently, one can say that $\widehat{\mathbb{D}}$ has no *transmission zeros* on $i\mathbf{R} \cup \{\infty\}$ (see, e.g., Lemma 3.27 of [ZDG]).

Note that, when $J \geq 0$, $\mathbb{D}$ is J-coercive iff $\mathbb{D}$ is positively J-coercive; for stable $\mathbb{D}$ this holds iff $\mathbb{D}^* J \mathbb{D} \gg 0$ (cf. (c)). Note also that (ai) is included in (a) and (c) only (since (bii)–(biv) do not require that $\langle \mathbb{D}u, J\mathbb{D}u \rangle \geq \varepsilon \|\mathbb{D}u\|_2^2$).

When $J \gg 0$, the condition in (bii) holds for all $u \in \mathcal{U}_{\text{out}}(0)$ iff it holds for all $u \in \mathrm{L}^2_{\text{loc}}$.

When $\dim U \times H \times Y < \infty$, the function $\widehat{\mathbb{D}}$ is rational, hence $\widehat{\mathbb{D}}_{|i\mathbf{R}}$ is then well defined also for unstable $\mathbb{D}$. In this case, we define $\langle \widehat{\mathbb{D}} u_0, J \widehat{\mathbb{D}} u_0 \rangle$ as the limit of itself at the poles of A, e.g., if $\widehat{\mathbb{D}}(\cdot) u_0$ has a pole at s_0, it is considered that $\|\widehat{\mathbb{D}}(s_0) u_0\| = \infty$ (in (aiii), (aiv), (biii) and (biv)). Thus, (aiii) holds iff $\|\widehat{\mathbb{D}}(s) u_0\| \geq \varepsilon \|u_0\|$ for $u_0 \in U$ and $s \in i\mathbf{R} \setminus \sigma(A)$. Naturally, $\widehat{\mathbb{D}}(\pm\infty) = D$ (when $\dim H < \infty$). Note also that, in this finite-dimensional case, a minimal system is exponentially stabilizable and detectable (see [LR, p. 91]).

Proof of Proposition 10.3.1: (a) By Lemma 8.4.11(d2), we have (ai)⇔(bii). Trivially, $\langle \mathbb{D}u, J\mathbb{D}u \rangle = \infty \geq \varepsilon \|u\|_2$ whenever $u \in \mathrm{L}^2(\mathbf{R}_+, U) \setminus \mathcal{U}_{\text{out}}(0)$, hence (aii)⇔(bii). The rest of (a) follows from (b).

(b) We assume that $\dim U \times H \times Y < \infty$ and divide the proof in parts.

1° *"If $u, \mathbb{D}u \in \mathrm{L}^2(\mathbf{R}_+; *)$, then $\mathcal{L}\mathbb{D}u = \widehat{\mathbb{D}}\widehat{u}$ a.e. on $i\mathbf{R}$":* (In fact, this holds also whenever, e.g., the set N below is at most countable (this happens whenever A is compact) or $\mathbb{D}$ is stable (then $\mathcal{L}\mathbb{D}u = \widehat{\mathbb{D}}\widehat{u}$ a.e. on $i\mathbf{R}$, by (3.36)).)

Because $\dim H < \infty$, the set $N := \sigma(A)$ is finite, and there is $\omega \in \mathbf{R}$ s.t. $\mathbb{D} \in \mathrm{TIC}_\omega$ and $\widehat{\mathbb{D}} \in \mathrm{H}(\mathbf{C} \setminus N; \mathcal{B}(U, Y))$.

Set $\Omega := \mathbf{C}^+ \setminus N$. The functions $\widehat{\mathbb{D}}\widehat{u} \in \mathrm{H}(\Omega;Y)$ and $G := \widehat{\mathbb{D}u} \in \mathrm{H}^2(\mathbf{C}^+;Y)$ coincide on $\mathbf{C}^+_\omega$, hence they are identical on Ω, by Lemma D.1.2(e). Consequently, $\widehat{\mathbb{D}}\widehat{u}|_{i\mathbf{R}} \in \mathrm{L}^2$ is the boundary function of $G \in \mathrm{H}^2$ a.e., by (a1)(1.) and (a2) of Theorem 3.3.1.)

(Note that if, e.g., $\widehat{\mathbb{D}}(s) = (s-1)^{-1}$, then the map $\widehat{u} \mapsto \widehat{\mathbb{D}}\widehat{u}$ on $i\mathbf{R}$ induces a TI_0 map that is different from $\mathbb{D}$ (which is in $\mathrm{TIC}_\infty \setminus \mathrm{TIC}$). Above we showed that these two maps coincide on the set $\{u \in \mathrm{L}^2 \,|\, \mathbb{D}u \in \mathrm{L}^2\}$. Another example in this direction is provided below Example 3.3.10.)

2° *(biii)⇒(biv):* This is trivial.

3° *(biii)⇒(bii):* Assume (biii). Let $u \in \mathcal{U}_{\mathrm{out}}(0)$. Then $\mathcal{L}\mathbb{D}u = \widehat{\mathbb{D}}\widehat{u}$ a.e. on $i\mathbf{R}$, by 1°, hence

$$\langle \mathbb{D}u, J\mathbb{D}u\rangle = (2\pi)^{-1/2}\langle \widehat{\mathbb{D}}\widehat{u}, J\widehat{\mathbb{D}}\widehat{u}\rangle \geq (2\pi)^{-1/2}\varepsilon\|\widehat{u}\|_2^2 = \varepsilon\|u\|_2^2, \tag{10.14}$$

by the Plancherel Theorem (see (D.36)). Thus, (bii) holds.

4° *(bii)⇒(biii):* Set $F := \widehat{\mathbb{D}}(-\bar{\cdot})^* J\widehat{\mathbb{D}}$, so that $F \in \mathrm{H}(N^c;\mathcal{B}(U))$, where $N \subset \mathbf{C}$ is finite, and $F = \widehat{\mathbb{D}}^* J\widehat{\mathbb{D}}$ on $i\mathbf{R} \cup \{\infty\}$.

Assume that (biii) is false (we aim to show that then also (bii) is false). Then there are $\varepsilon', \varepsilon'' < \varepsilon$, $ir_0 \in i\mathbf{R} \cup \{\infty\}$, $u_0 \in K := \{u_0 \in U \,|\, \|u_0\| = 1\}$ s.t. $G(ir_0) < \varepsilon' < \varepsilon''$, where $G(ir) := \langle u_0, F(ir)u_0\rangle$ $(r \in \mathbf{R})$. We may assume that $ir_0 \in i\mathbf{R} \setminus N$ (alter r_0 slightly if necessary).

Note that $G \in \mathrm{H}(N^c)$ and $G(ir) \in \mathbf{R}$ for all $r \in \mathbf{R}$. Let

$$\widehat{g}(s) := \prod_p (s-p)/(s+p+2), \tag{10.15}$$

where p runs over the poles of $\widehat{\mathbb{D}}u_0$ on $\mathbf{C}^+_{-1}$, counting multiplicities.

Then $\widehat{g}\widehat{\mathbb{D}}u_0 \in \mathrm{H}^\infty(\mathbf{C}^+_{-1};Y)$, $\widehat{g} \in \mathrm{H}^\infty(\mathbf{C}^+_{-1})$ and $|\widehat{g}| \leq 1$ on $\mathbf{C}^+_{-1}$. Set $a := |\widehat{g}(ir_0)|$, $M := \|J\|\|\widehat{\mathbb{D}}\widehat{g}u_0\|$, so that $|G||\widehat{g}|^2 \leq M$ on $\overline{\mathbf{C}^+} \cup \{\infty\}$, $M < \infty$ and $a > 0$. Given $\varepsilon_0 \in (0,a)$, there is $\delta > 0$ s.t. $a - \varepsilon_0 < |\widehat{g}(ir)| < a + \varepsilon_0$ on $J_\delta := i(r-\delta, r+\delta)$.

For each $t \in (0,\delta)$, we choose $f_t \in \mathrm{L}^2(\mathbf{R}_+)$ as in Lemma D.1.24, and set $\widehat{h}_t := \widehat{f}_t\widehat{g} \in \mathrm{H}^2(\mathbf{C}^+)$, so that $\widehat{\mathbb{D}}\widehat{h}_t u_0 \in \mathrm{H}^2$; in particular, $u_t := h_t u_0 \in \mathcal{U}_{\mathrm{out}}(0)$. Now

$$R := 2\pi\langle \mathbb{D}h_t u_0, J\mathbb{D}h_t u_0\rangle_{\mathrm{L}^2(\mathbf{R}_+;Y)} = \int_{i\mathbf{R}} \langle \widehat{h}_t u_0, F\widehat{h}_t u_0\rangle_U \, dm = \int_{i\mathbf{R}} G|\widehat{g}\widehat{f}_t|^2 \, dm \tag{10.16}$$

$$\leq \int_{J_\delta} \varepsilon'(a+\varepsilon_0)^2 |\widehat{f}_t|^2 \, dm + \int_{i\mathbf{R}\setminus J_\delta} M|\widehat{f}_t|^2 \, dm \leq 2\pi\varepsilon'(a+\varepsilon_0)^2 + M\varepsilon_t, \tag{10.17}$$

where $\varepsilon_t \to 0$ as $t \to 0+$. By choosing t small enough, we get $R < 2\pi\varepsilon''(a+\varepsilon_0)^2$ and $\|\widehat{h}_t\|_2^2 \geq 2\pi(a-\varepsilon_0)^2$. Because $\varepsilon_0 \in (0,a)$ was arbitrary, we have $\langle \mathbb{D}\widehat{h}_t u_0, J\mathbb{D}\widehat{h}_t u_0\rangle \leq \varepsilon''\|h_t u_0\|_2^2$ for ε_0 and t small enough. Therefore, (bii) does not hold.

5° *(ai)⇒(bii):* This follows from Lemma 8.4.11(d1).

6° *(biv)⇒(ai):* Set $F := \widehat{\mathbb{D}}(-\bar{\cdot})^* J\widehat{\mathbb{D}}$, so that $F = \widehat{\mathbb{D}}^* J\widehat{\mathbb{D}}$ on $i\mathbf{R} \cup \{\infty\}$ and $F \in \mathrm{H}(N^c;\mathcal{B}(U))$ (in fact, F is rational), where $N \subset \mathbf{C}$ is finite.

6.1° *"(biv)⇒(biii)" when $\mathbb{D}$ is exponentially stable:* Now $F(i\cdot) \in \mathcal{C}_b(i\mathbf{R} \cup \{\infty\}; \mathcal{B}(U))$. Assume (biv).

Obviously, $(r, u_0) \mapsto \langle u_0, F(ir)u_0 \rangle$ is continuous $i\mathbf{R} \cup \{\infty\} \times U \to (0, +\infty)$. Therefore, $\varepsilon := \min_{r \in i\mathbf{R} \cup \{\infty\}, u_0 \in K} \langle u_0, F(ir)u_0 \rangle$ exists, where $K := \{u_0 \in U \,\big|\, \|u_0\| = 1\}$. By (biv), we have $\varepsilon > 0$, hence (biii) holds (with this ε).

6.2° *(biv)⇒(ai):* Assume (biv). Being rational, $\widehat{\mathbb{D}}$ has an (exponential) r.c.f. $\widehat{\mathbb{D}} = \widehat{\mathbb{N}}\widehat{\mathbb{M}}^{-1}$ (by Lemma 6.5.10(b)). Since

$$0 < \langle \mathbb{D}\mathbb{M}u_\circlearrowleft, J\mathbb{D}\mathbb{M}u_\circlearrowleft \rangle = \langle \mathbb{N}u_\circlearrowleft, J\mathbb{N}u_\circlearrowleft \rangle \quad (u_\circlearrowleft \in \mathrm{L}^2(\mathbf{R}_+; U)), \tag{10.18}$$

we have $\widehat{\mathbb{N}}^* J \widehat{\mathbb{N}} \geq \varepsilon I$ for some $\varepsilon > 0$, by 5.1°. By Lemma 8.4.11(a2)&(b1), it follows that $\mathbb{D}$ (resp. $\mathbb{N}$) is positively J-coercive over $\mathcal{U}_{\text{out}}$ (resp. $\mathcal{U}_{\text{out}}^{\Sigma_\flat}$), hence (ai) holds.

7° *(bv)⇒(biv):* Assume (bv). Then $\widehat{\mathbb{D}}(s)^* J \widehat{\mathbb{D}}(s) = D^*JD + B^*(s - A)^{-*}C^*JC(s-A)^{-1}B \geq D^*JD \gg 0$ for all $s \in i\mathbf{R} \cup \{\infty\}$, hence then (biv) holds.

8° *(i)–(vi):* Proposition 10.3.2(d) provides the last equivalence.

(c) Now $\mathcal{U}_{\text{out}}(0) = \mathrm{L}^2(\mathbf{R}_+; U)$, hence $\mathbb{D}^* J \mathbb{D} \geq \varepsilon I$ is equivalent to (bii), by Lemma 6.4.6. Trivially, (ai)⇒(bii); the converse also holds since $\|\mathbb{D}u\|_2 \leq \|\mathbb{D}\|\|u\|_2$. By Theorem 3.1.3(e2), also the remaining two equivalences hold.

(d) *Finding $\varepsilon > 0$:* Assume (iv). We have $\widetilde{J} = P^* \begin{bmatrix} \widetilde{Q} & 0 \\ 0 & R \end{bmatrix} P = E^*E$, where $\widetilde{Q} := Q - NR^{-1}N^*$, $P := \begin{bmatrix} I & 0 \\ -R^{-1}N^* & I \end{bmatrix} \in \mathcal{GB}(H \times U)$ (hence $\widetilde{Q} \geq 0$), $E = \begin{bmatrix} \widetilde{Q}^{1/2} & 0 \\ 0 & R^{1/2} \end{bmatrix} P$. Thus, $0 \leq \langle z, \widetilde{Q}z \rangle \leq \langle z, Qz \rangle$ for all $z \in H \times U$, hence $\operatorname{Ker}(Q) \subset \operatorname{Ker}(\widetilde{Q})$. But $\dim \operatorname{Ker}(\widetilde{Q}) = \dim \operatorname{Ker}(Q) < \infty$, by (iv), hence $\operatorname{Ker}(Q) = \operatorname{Ker}(\widetilde{Q})$. Because $0 \leq \widetilde{Q}$, we must have

$$\operatorname{Ker}(Q) \subset \operatorname{Ker}(NR^{-1}N^*) = \operatorname{Ker}(N^*). \tag{10.19}$$

Therefore, $\operatorname{Ker}(\widetilde{Q}) \subset \operatorname{Ker}(N^*)$, hence $\operatorname{Ker}(\widetilde{Q}^{1/2}) \subset \operatorname{Ker}(N^*)$. By Lemma A.3.1(f), we have $L\widetilde{Q}^{1/2} = -R^{-1}N^*$ for some $L \in \mathcal{B}(U)$. Choose $\varepsilon_R > 0$ s.t. $R \geq \varepsilon_R^2 I$. Given $x_0 \in H$ and $u_0 \in U$, set $v_0 := u_0 - R^{-1}N^*x_0$, so that

$$\kappa(x_0, u_0) := \langle \begin{bmatrix} x_0 \\ u_0 \end{bmatrix}, \widetilde{J} \begin{bmatrix} x_0 \\ u_0 \end{bmatrix} \rangle = \langle v_0, Rv_0 \rangle + \langle x_0, \widetilde{Q}x_0 \rangle = \|E \begin{bmatrix} x_0 \\ u_0 \end{bmatrix}\|^2. \tag{10.20}$$

To obtain a contradiction, assume that $x_0 \in H$ and $u_0 \in U$ are s.t. $\|u_0\| = 1$ but $\|E \left[\begin{smallmatrix} x_0 \\ u_0 \end{smallmatrix}\right]\|^2 < \varepsilon := \min\{\varepsilon_R/2, 1/4\|L\|^2\} > 0$. Then $\|v_0\|_U < \varepsilon\varepsilon_R^{-1} \leq 1/2$, hence $\|R^{-1}N^*x_0\| \geq 1/2$, hence $\widetilde{Q}^{1/2}x_0 \geq 1/2\|L\|$, hence $\kappa(x_0, u_0) \geq 1/4\|L\|^2 \geq \varepsilon$, a contradiction, as required. Thus, $\widetilde{J} \geq \varepsilon I$.

2° *The other claims:* Since $y = Cx + Du$, where $x := \mathbb{A}x_0 + \mathbb{B}\tau u$, we have

$$\mathcal{J}(x_0, u) = \int_0^\infty \langle \begin{bmatrix} x \\ u \end{bmatrix}, \widetilde{J} \begin{bmatrix} x \\ u \end{bmatrix} \rangle dm \geq \varepsilon \|u\|_2^2 \quad (u \in \mathrm{L}^2(\mathbf{R}_+; U)). \tag{10.21}$$

Thus, the second claim holds. Condition (bii) follows from this, by (b) also (biii) holds (and (biv) if $J \geq 0$).

Let us now study the new system with $\begin{bmatrix} C' & D' \end{bmatrix} := \widetilde{J}^{1/2}$. As shown above, $\mathcal{J}(x_0, u) = \infty$ for all $u \notin \mathrm{L}^2$. With this new setting, $\mathcal{U}'_{\text{out}}(x_0)$ (for Σ') consists of all $u \in \mathrm{L}^2(\mathbf{R}_+;U)$ for which (10.21) is finite; in particular, we may have $\mathbb{C}x_0 + \mathbb{D}u \notin \mathrm{L}^2$ (but $\mathbb{C}'x_0 + \mathbb{D}'u \in \mathrm{L}^2$). Thus, $\mathcal{U}'_{\mathrm{out}}$ includes the original $\mathcal{U}_{\mathrm{out}}$.

Then $(C')^*C' = Q$, hence then the observability of $\left[\begin{smallmatrix} A \\ Q \end{smallmatrix}\right]$ implies that of $\left[\begin{smallmatrix} A \\ C' \end{smallmatrix}\right]$, by the Hautus test (see p. 7 of [IOW]; obviously, $\operatorname{rank} \left[\begin{smallmatrix} \lambda - A \\ (C')^*C' \end{smallmatrix}\right] \le \operatorname{rank} \left[\begin{smallmatrix} \lambda - A \\ C' \end{smallmatrix}\right]$). $\square$

Next we study J-coercivity over $\mathcal{U}_{\mathrm{exp}}(0) := \{u \in \mathrm{L}^2(\mathbf{R}_+;U) \,|\, \mathbb{B}\tau u \in \mathrm{L}^2\}$ (we shall have $C_{\mathrm{c}} := C_{\mathrm{w}}$, $D_{\mathrm{c}} := D$ for most applications):

Proposition 10.3.2 ($\mathcal{U}_{\mathrm{exp}}$: $y \in \mathrm{L}^2 \Rightarrow u, x \in \mathrm{L}^2$) *Let* $\Sigma := \left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right] \in \mathrm{WPLS}(U,H,Y)$ *have compatible generators* $(C_{\mathrm{c}}, D_{\mathrm{c}})$. *Let* $J = J^* \in \mathcal{B}(Y)$, *and set*

$$\kappa(x_0,u_0) := \langle (C_{\mathrm{c}}x_0 + D_{\mathrm{c}}u_0), J(C_{\mathrm{c}}x_0 + D_{\mathrm{c}}u_0)\rangle, \qquad \mathcal{J}(0,u) := \langle \mathbb{D}u, J\mathbb{D}u\rangle. \tag{10.22}$$

We have the following implications between the conditions (i)–(iii') given below:

(a1) Condition (i) is invariant under admissible state feedback (in the sense that if $\Sigma_\flat$ *is the corresponding closed-loop system, then* $\left[\begin{smallmatrix} \mathbb{A}_\flat & \mathbb{B}_\flat \\ \mathbb{C}_\flat & \mathbb{D}_\flat \end{smallmatrix}\right]$ *satisfies (i) iff* $\left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right]$ *satisfies (i)).*

(a2) Conditions (i)–(iii') are invariant under admissible state feedback by a compatible state feedback operator K_{c} *(in the above sense, with* $(C_\flat)_{\mathrm{c}} := C_{\mathrm{c}} + DK_{\mathrm{c}}$, $(D_\flat)_{\mathrm{c}} := D_{\mathrm{c}}$*).*

(b) If Σ *is estimatable, then* $\mathcal{U}_{\mathrm{out}} = \mathcal{U}_{\mathrm{exp}}$, *hence then (i) becomes equivalent to (ai) of Proposition 10.3.1.*

(c) (i)⇔(i")⇐(ii)⇐(iii)⇒(iii') (without further assumptions).

(d) **(dim$<\infty$)** *Assume that* $\dim U \times H \times Y < \infty$. *Then (iii)⇔(iii')⇐(vii).*

Assume, in addition, that Σ *is exponentially stabilizable. Then (i)–(vi) are equivalent to each other (and to (ai) and (bii)–(biv) of Proposition 10.3.1 if* Σ *is exponentially detectable). Moreover, in (ii), (ii'), (iii) and (iv), we may replace "*$r \in \mathbf{R}$*" by "*$r \in E$*", where* $E \subset \mathbf{R}$ *is dense.*

(e1) **(BwCARE)** *Assume that Hypothesis 9.2.1 holds for* $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{exp}}$, Σ *has a compatible exponentially stabilizing state feedback operator, and* $D_{\mathrm{c}} = D$.

Then (v)⇐(i')⇔(ii')⇔(vi)⇒(i)⇔(i")⇔(ii)⇔(iii)⇒(iii'). If, in addition, $D^*JD \gg 0$, *then (i)–(iii), (v) and (vi) are equivalent.*

(e2) **(Bounded or smoothing B)** *Assume that* $\pi_{[0,1)}\mathbb{A}B \in \mathrm{L}^1([0,1);\mathcal{B}(U,H))$, $\mathbb{D} \in \mathrm{ULR}$, Σ *is optimizable, and* $D_{\mathrm{c}} = D$.

Then (i)–(iii) are equivalent. If, in addition, Hypothesis 9.2.1 holds for $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{exp}}$, *then (i)–(iii), (v) and (vi) are equivalent.*

(f) If Σ *has an exponentially stabilizing compatible state feedback operator, then (i')⇔(ii'), and (i)⇔(i")⇔(ii)⇔(iii)⇒(iii').*

(g1) If $\mathbb{D}$ *is ULR and* $D_{\mathrm{c}} = D$, *then (i')⇒(i), and (ii')⇒(ii).*

(g2) If Σ has an exponentially stabilizing compatible state feedback operator s.t. $\mathbb{D}_\flat \in \mathrm{SHPR}$ (or $\mathbb{B}_\flat\tau \in \mathrm{SHPR}$ and $\mathbb{D} \in \mathrm{SLR}$), then (ii)⇒(ii'), and (i)⇒(i') (with $D_{\rm c} = D_\flat$).

(i) $\mathcal{J}(0,u) \geq \varepsilon\left(\|u\|_2^2 + \|\mathbb{B}\tau u\|_2^2\right)$ for some $\varepsilon > 0$ and all $u \in \mathcal{U}_{\exp}(0)$; i.e., $\mathbb{D}$ is positively J-coercive over $\mathcal{U}_{\exp}$.

(i') $\mathcal{J}(0,u) \geq \varepsilon\|\mathbb{B}\tau u\|_2^2$ for some $\varepsilon > 0$ and all $u \in \mathcal{U}_{\exp}(0)$, and $D_{\rm c}^* J D_{\rm c} \gg 0$.

(i") $\mathcal{J}(0,u) \geq \varepsilon\left(\|u\|_2^2 + \|\mathbb{B}\tau u\|_2^2 + \|\mathbb{D}u\|_2^2\right)$ for some $\varepsilon > 0$ and all $u \in \mathcal{U}_{\exp}(0)$.

(ii) There is $\varepsilon > 0$ s.t.

$$(ir-A)x_0 = Bu_0 \implies \kappa(x_0,u_0) \geq \varepsilon(\|x_0\|_H^2 + \|u_0\|_U^2) \quad (x_0 \in H,\ u_0 \in U,\ r \in \mathbf{R}). \tag{10.23}$$

(ii') $D_{\rm c}^* J D_{\rm c} \gg 0$ and there is $\varepsilon > 0$ s.t.

$$(ir-A)x_0 = Bu_0 \implies \kappa(x_0,u_0) \geq \varepsilon\|x_0\|_H^2 \quad (x_0 \in H,\ u_0 \in U,\ r \in \mathbf{R}). \tag{10.24}$$

(iii) There is $\varepsilon > 0$ s.t. $T_{ir}^* \left[\begin{smallmatrix} I_H & 0 \\ 0 & J \end{smallmatrix}\right] T_{ir} \geq \varepsilon I$ $(r \in \mathbf{R})$ on $H \times U$, where $T_{ir} := \left[\begin{smallmatrix} A-ir & B \\ C_{\rm c} & D_{\rm c} \end{smallmatrix}\right]$.

(iii') $T_{ir}^* \left[\begin{smallmatrix} I_H & 0 \\ 0 & J \end{smallmatrix}\right] T_{ir} > 0$ $(r \in \mathbf{R})$. Equivalently,

$$r \in \mathbf{R}\ \&\ \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right] \neq \left[\begin{smallmatrix} x_0 \\ u_0 \end{smallmatrix}\right] \in H \times U\ \&\ (ir-A)x_0 = Bu_0 \implies \kappa(x_0,u_0) > 0. \tag{10.25}$$

(iv) There is $\varepsilon > 0$ s.t. $\langle u_0, \widehat{\mathbb{D}}(s)^* J \widehat{\mathbb{D}}(s) u_0\rangle \geq \varepsilon\left(\|u_0\|_U^2 + \|(s-A)^{-1}Bu_0\|_H^2\right)$ for a.e. $s \in i\mathbf{R}$.

(v) There is a unique minimizing $u \in \mathcal{U}_{\exp}(x_0)$ for each $x_0 \in H$, and $D^*JD \gg 0$.

(vi) The $B_{\rm w}^*$-CARE has an exponentially stabilizing solution and $D^*JD \gg 0$.

(vii) (C,A) has no unobservable nodes on $i\mathbf{R}$, $J = I$, $D^*D > 0$ and $D^*C = 0$.

In classical LQR results, the assumptions are usually written in some of the following forms:

Remark 10.3.3 *Obviously, any $J \gg 0$ is equivalent to $J = I$ in all above conditions. Therefore, for $J \gg 0$, we get the following equivalent forms of above conditions:*
(i) $\|\mathbb{D}u\|_2 \geq \varepsilon(\|\mathbb{B}\tau u\|_2 + \|u\|_2)$;
(ii) $(ir-A)x_0 = Bu_0 \Rightarrow \|C_{\rm c}x_0 + D_{\rm c}u_0\|_Y \geq \varepsilon(\|x_0\|_H + \|u_0\|_U)$;
(ii') $D_{\rm c}^*D_{\rm c} \gg 0$, and $(ir-A)x_0 = Bu_0 \Rightarrow \|C_{\rm c}x_0 + D_{\rm c}u_0\|_Y \geq \varepsilon\|x_0\|_H$;
(iii') $T_{ir} := \left[\begin{smallmatrix} A-ir & B \\ C_{\rm c} & D_{\rm c} \end{smallmatrix}\right] : H \times U \to H \times Y$ has a full column rank (i.e., $T_{ir}^*T_{ir} \gg 0$) for all $r \in \mathbf{R}$;
(iv) $\|\widehat{\mathbb{D}}(s)u_0\|_Y \geq \varepsilon(\|u_0\|_U + \|(s-A)^{-1}Bu_0\|_H)$ for a.e. $s \in i\mathbf{R}$ and all $u_0 \in U$. □

Recall that we set $\|x\|_X = \infty$ for $x \notin X$, hence $\|T_s \left[\begin{smallmatrix} x_0 \\ u_0 \end{smallmatrix}\right]\|_{H\times Y} = \infty \geq \varepsilon\|\left[\begin{smallmatrix} x_0 \\ u_0 \end{smallmatrix}\right]\|_{H\times U}$ for all $x_0 \in H \setminus H_B$, $\varepsilon > 0$. In (iv) and (iv'), the values of $\|(s-A)^{-1}Bu_0\|_H$,

$\|\widehat{\mathbb{D}}(s)u_0\|_U$ and $\langle u_0, \widehat{\mathbb{D}}(s)^* J \widehat{\mathbb{D}}(s)u_0\rangle$ are defined by continuity at ∞ and at the poles of $(s-A)^{-1}$ (in particular, the value $+\infty$ is allowed).

Condition (iii') says that Σ has no *invariant zeros* on $i\mathbf{R}\cup\{\infty\}$ (see, e.g., Definition 3.16 of [ZDG]). If A is bounded (this is the case when $\dim H<\infty$), then $H=H_{-1}=H_1$, hence then necessarily $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ are bounded, $C_c = C$ and $D_c = D$.

When applying the above results, we often assume that $\mathbb{D}$ is WR, so that we can take $C_c = C_w$, $D_c = D$.

Example 9.13.5 shows that positive J-coercivity over $\mathcal{U}_{out}$ does not imply positive J-coercivity over $\mathcal{U}_{exp}$, not even when Σ is one-dimensional and exponentially stabilizable with $D^*JD \gg 0$.

Proof of Proposition 10.3.2: *Remarks:* The numbers $\varepsilon > 0$ in (i)–(iv) need not be equal, but they are always independent of x_0, u_0 and r. Since $\langle \left[\begin{smallmatrix} \tilde{x} \\ \tilde{y} \end{smallmatrix}\right], \left[\begin{smallmatrix} I_H & 0 \\ 0 & J \end{smallmatrix}\right] \left[\begin{smallmatrix} \tilde{x} \\ \tilde{y} \end{smallmatrix}\right]\rangle_{H\times U} := \|\tilde{x}\|_H + \langle \tilde{y}, J\tilde{y}\rangle_Y = \infty$ for $\tilde{x}\in H_{-1}\setminus H$, we have $\langle T_{ir} \left[\begin{smallmatrix} x_0 \\ u_0 \end{smallmatrix}\right], \left[\begin{smallmatrix} I_H & 0 \\ 0 & J \end{smallmatrix}\right] T_{ir} \left[\begin{smallmatrix} x_0 \\ u_0 \end{smallmatrix}\right]\rangle_{H\times U} = \infty$ when $(A-ir)x_0 + Bu_0 \notin H$; in particular, (iii) need to be checked for $x_0 \in H_B$ only.)

About the proof: By Lemma 6.7.8, (i) is equivalent to (i") and $\mathcal{U}_{exp}(0) = \{u \in \mathrm{L}^2(\mathbf{R}_+;U) \mid \mathbb{B}\tau u, \mathbb{D}u \in \mathrm{L}^2\}$. Thus, we may and will neglect (i") in the rest of the proof.

We shall use the notation of Definition 6.6.10 when referring to state feedback.

(a1) This is Theorem 8.4.5(d).

(a2) For (i) this follows from (a1); for (i') this follows from the formula $\mathcal{U}_{exp}(0) = \widetilde{\mathbb{M}}\,\mathcal{U}^{\Sigma_\flat}_{exp}(0)$ from Theorem 8.4.5(c1) (since $(D_\flat)_c := D_c$, by (6.144)), so we concentrate on (ii)–(iii'). Let K_c be an admissible compatible state feedback operator for Σ, and let $\Sigma_\flat$ be the corresponding closed-loop system. Then

$$\begin{bmatrix} A-s & B \\ C_c & D_c \end{bmatrix} \begin{bmatrix} I & 0 \\ K_c & I \end{bmatrix} = \begin{bmatrix} A+BK_c-s & B \\ C_c+D_cK_c & D_c \end{bmatrix} = \begin{bmatrix} A_\flat - s & B_\flat \\ (C_\flat)_c & (D_\flat)_c \end{bmatrix} =: T^\flat_s \tag{10.26}$$

for all $s\in\mathbf{C}$, by Proposition 6.6.18(d2).

In 1°–4°, we assume a claim for $\left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right]$ (as above), and prove the same claim for $\left[\begin{smallmatrix} \mathbb{A}_\flat & \mathbb{B}_\flat \\ \mathbb{C}_\flat & \mathbb{D}_\flat \end{smallmatrix}\right]$. The converse is always obtained by using $-K_c$ for $\left[\begin{smallmatrix} \mathbb{A}_\flat & \mathbb{B}_\flat \\ \mathbb{C}_\flat & \mathbb{D}_\flat \end{smallmatrix}\right]$ (see Proposition 6.6.18(d2)).

1° *Claim (ii) is K_c-invariant:* Assume (ii). Given $x_0\in H$, $v_0\in U$, set $u_0 := x_0 + K_c v_0$, $y_0 := C_c x_0 + D_c u_0$, so that $\left[\begin{smallmatrix} x_0 \\ u_0 \end{smallmatrix}\right] = \left[\begin{smallmatrix} I & 0 \\ K_c & I \end{smallmatrix}\right]\left[\begin{smallmatrix} x_0 \\ v_0 \end{smallmatrix}\right]$ and $y_0 = (C_\flat)_c x_0 + (D_\flat)_c v_0$, by (10.26).

Now $\kappa(x_0,u_0) = \langle y_0, Jy_0\rangle =: \kappa_\flat(x_0,v_0)$, so that obtain (ii) for $\Sigma_\flat$ too, because $\|x_0\|^2 + \|v_0\|^2 \le (M_0+2)^2(\|x_0\|^2 + \|u_0\|^2)$, where M_0 is as in Lemma 6.3.21 for $\left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{K} & \mathbb{F} \end{smallmatrix}\right]$ (note that $\|v_0\| \le \|x_0\| + M_0\|u_0\|$, since $v_0 = x_0 - K_c u_0$).

2° *Claim (ii') is K_c-invariant:* Drop the last part from the above proof.

3° *Claim (iii) is K_c-invariant:* Assume (iii). Given $x_0\in H$, $v_0\in U$, $r\in\mathbf{R}$, set $u_0 := x_0 + K_c v_0$ and

$$\begin{bmatrix} x_1 \\ y_0 \end{bmatrix} := T_{ir} \begin{bmatrix} x_0 \\ u_0 \end{bmatrix} = T^\flat_{ir} \begin{bmatrix} x_0 \\ v_0 \end{bmatrix}, \tag{10.27}$$

as in (10.26).

By (10.29) and (iii), the sum $\|x_1\|_H^2 + \langle y_0, Jy_0\rangle_Y$ is at least $\varepsilon(\|x_0\|_H^2 + \|u_0\|_U^2)$, hence at least $(\|x_0\|_H^2 + \|v_0\|_U^2)\varepsilon/(M_0+1)^2$, as in 1°. Thus, (iii) holds for $\left[\begin{array}{c|c} \mathbb{A}_\flat & \mathbb{B}_\flat \\ \hline \mathbb{C}_\flat & \mathbb{D}_\flat \end{array}\right]$.

4° *Claim (iii') is K_c-invariant:* This is analogous to 3°.

(b) This is given in Lemma 8.3.3.

(c) 1° *(iii)⇒(iii'):* This is trivial. (Moreover, the equivalence in (iii') is obvious from (10.29).)

2° *(ii)⇒(i):* Assume (ii). Let $u \in \mathcal{U}_{\exp}(0)$, so that $u, x, y \in L^2(\mathbf{R}_+; *)$, where $x := \mathbb{B}\tau u$ and $y := \mathbb{D}u$. By Lemma 6.3.20, we have $(\cdot - A)\widehat{x} = B\widehat{u}$ and $\widehat{y} = C_c\widehat{x} + D_c\widehat{u}$ a.e. on $i\mathbf{R}$. By (3.34), we have

$$\langle\widehat{y}, J\widehat{y}\rangle = \int_{\mathbf{R}} \kappa(\widehat{x}(ir), \widehat{u}(ir))\,dr \geq \varepsilon(\|\widehat{x}\|_2^2 + \|\widehat{u}\|_2^2) \tag{10.28}$$

hence $\mathcal{J}(0,u) = \langle y, Jy\rangle \geq \varepsilon(\|x\|_2^2 + \|u\|_2^2)$, by (3.34). Therefore, (i) holds.

3° *(iii)⇒(ii):* Now

$$\left\langle \begin{bmatrix} x_0 \\ u_0 \end{bmatrix}, T_{ir}^* \begin{bmatrix} I & 0 \\ 0 & J \end{bmatrix} T_{ir} \begin{bmatrix} x_0 \\ u_0 \end{bmatrix} \right\rangle = \|Bu_0 - (ir - A)x_0\|_H^2 + \kappa(x_0, u_0). \tag{10.29}$$

Thus, if $(A - ir)x_0 + Bu_0 = 0$, then (iii) implies that $\kappa(x_0, u_0) \geq \varepsilon(\|x_0\|_H^2 + \|u_0\|_U^2)$ for some $\varepsilon > 0$ independent of x_0 and u_0. Thus, then (ii) holds.

(f) By (c), we only have to show the implications "(i)⇒(ii)⇒(iii)", and "(i')⇔(ii')". This will be done below.

1.1° *(ii)⇒(iii) for exponentially stable Σ:* Let Σ be exponentially stable. Assume (ii) with $\varepsilon \in (0, 1/2)$. By Lemma A.4.4(g1), there is $M < \infty$ s.t. $\|(ir - A)^{-1}z_0\|_H \leq M\|z_0\|_H$ for all $r \in \mathbf{R}$ and $z_0 \in H$ (because $\omega_A < 0$).

Let $\|x_0\|_H^2 + \|u_0\|_U^2 = 1$ and $r \in \mathbf{R}$. Assume that

$$\|z_1\|_H < \delta := \varepsilon/3(1 + M + \|\widehat{\mathbb{C}}\|^2\|J\| + 2\|\widehat{\mathbb{C}}\|\|J\|\|\widehat{\mathbb{D}}\|), \tag{10.30}$$

where $z_1 := (ir - A)x_0 - Bu_0$. Set $x_1 := (ir - A)^{-1}z_1$, $x_2 := x_0 - x_1$, so that $(ir - A)x_2 = Bu_0$, i.e., $x_2 = (ir - A)^{-1}Bu_0$. It follows that

$$y_0 := C_c x_2 + D_c u_0 = \widehat{\mathbb{D}}(ir)u_0 \quad \text{and} \quad C_c x_0 = C_c x_2 + \widehat{\mathbb{C}}(ir)z_1, \tag{10.31}$$

hence $\kappa(x_2, u_0) = \langle y_0, Jy_0\rangle$ and $\kappa(x_0, u_0) = \langle y_0 + \widehat{\mathbb{C}}(ir)z_1, J(y_0 + \widehat{\mathbb{C}}(ir)z_1)\rangle$. Consequently,

$$\kappa(x_0, u_0) - \kappa(x_2, u_0) = \langle\widehat{\mathbb{C}}(ir)z_1, J\widehat{\mathbb{C}}(ir)z_1\rangle + 2\,\mathrm{Re}\langle\widehat{\mathbb{C}}(ir)z_1, Jy_0\rangle \tag{10.32}$$

$$\leq \|\widehat{\mathbb{C}}\|^2\|J\|\delta^2 + 2\|\widehat{\mathbb{C}}\|\|J\|\|\widehat{\mathbb{D}}\|\delta < \varepsilon/3 \tag{10.33}$$

(note that $\delta < \varepsilon/3 < 1$, hence $\delta^2 < \delta$). But $\|x_2\|^2 + \|u_0\|^2 \geq 1 + \|x_2\|^2 - \|x_0\|^2 \geq 1 - 2\|x_1\|\|x_0\| \geq 1 - 2M\|z_1\| \geq 1 - 2\varepsilon/3$, hence $\kappa(x_2, u_0) \geq \varepsilon(1 - 2\varepsilon/3) > 2\varepsilon/3$, by (ii). By (10.32), we obtain that $\kappa(x_0, u_0) > 2\varepsilon/3 - \varepsilon/3 = \varepsilon/3$.

Thus, (10.29)$\geq \min(\delta^2, \varepsilon/6) =: \varepsilon'$, hence (iii) holds with ε' in place of ε.

1.2° *(ii)⇒(iii) for Σ having an exponentially stabilizing K_c:* Apply 1° and twice (a2).

2.1° *(i)⇒(ii) for exponentially stable Σ:* Let Σ be exponentially stable and assume that (ii) does not hold for some $\gamma > 0$, so that there are $x_0 \in H$, $u_0 \in U$ and $r_0 \in \mathbf{R}$ s.t. $(ir_0 - A)x_0 = Bu_0$, $\kappa(x_0, u_0) < \gamma$ and $\|x_0\|^2 + \|u_0\|^2 = 1$. Note that $x_0 = (ir_0 - A)^{-1}Bu_0$.

Let $\varepsilon' \in (0,1)$ be arbitrary. Choose $\delta > 0$ s.t. $\kappa(x_r, u_0) < \gamma$ and $\|x_r\|^2 + \|u_0\|^2 > 1 - \varepsilon'$ for $|r - r_0| < \delta$, where $x_r := (ir - A)^{-1}Bu_0$. Set

$$M := \max\{1, \|\mathbb{D}\|^2 \|J\|\}. \tag{10.34}$$

By Lemma D.1.24, there is $f \in \mathrm{L}^2(\mathbf{R}_+)$ s.t. $\int_{|r-r_0|>\delta} |\widehat{f}(ir)|^2\, dr < \varepsilon'/M$ and $\|\widehat{f}\|_2 = 1$. Set $u := fu_0$, $\widehat{y} := \widehat{\mathbb{D}}\widehat{u}$, so that $\|\widehat{y}(ir)\| \le \|\mathbb{D}\| \|\widehat{f}(ir)\|$ $(r \in \mathbf{R})$, and hence

$$\int_{|r-r_0|>\delta} \langle \widehat{y}, J\widehat{y}\rangle\, dr \le \int_{|r-r_0|>\delta} M\|\widehat{f}\|^2\, dr < M\varepsilon'/M = \varepsilon', \tag{10.35}$$

$$\int_{|r-r_0|<\delta} \langle \widehat{y}, J\widehat{y}\rangle\, dr = \int_{|r-r_0|<\delta} \kappa(x_r, u_0)|\widehat{f}|^2\, dr < \gamma, \quad \text{and} \tag{10.36}$$

$$\int_{i\mathbf{R}} (\|\widehat{u}\|^2 + \|\widehat{x}\|^2)\, dr > (1-\varepsilon') \int_{|r-r_0|<\delta} \|\widehat{f}\|^2\, dm > (1-\varepsilon')^2 \tag{10.37}$$

In particular, $\mathcal{J}(0,u) < (\gamma + \varepsilon')/2\pi$ and $\|u\|^2 + \|x\|^2 > (1-\varepsilon')^2/2\pi$. Because $\varepsilon' > 0$ was arbitrary, we see that (i) cannot hold for any $\varepsilon > \gamma$.

Thus, if (i) holds for some $\varepsilon > 0$, we must also have (ii) for the same ε (otherwise (ii) would fail for some $\gamma < \varepsilon$ too, hence (i) would be false for ε, a contradiction).

2.2° *(i)⇒(ii) for Σ having an exponentially stabilizing K_{c}:* Apply 3.1° and twice (a2).

3° *(i')⇔(ii')*: Define Σ' and $\mathcal{J}'$ by setting $J' := \left[\begin{smallmatrix} J & 0 \\ 0 & I \end{smallmatrix}\right]$, $C'_{\mathrm{c}} := \left[\begin{smallmatrix} C_{\mathrm{c}} \\ 0 \end{smallmatrix}\right]$, $D'_{\mathrm{c}} := \left[\begin{smallmatrix} D_{\mathrm{c}} \\ I \end{smallmatrix}\right]$ (thus, $\mathcal{J}' = \mathcal{J} + \|u\|_2^2$). Then for $\mathcal{J}'$, condition (i) is satisfied, hence so is (ii), by 2°. But $\kappa'(x_0, u_0) = \kappa(x_0, u_0) + \|u_0\|_U^2$, hence (ii') is satisfied by $\mathcal{J}$. The converse is analogous, using "(ii)⇒(i)" (hence it does not even require the stabilizability assumption).

(g1) 1° *(ii')⇒(ii):* To derive a contradiction, assume that $\mathbb{D}$ is ULR, $D_{\mathrm{c}} = D$ and (ii') holds but (ii) does not hold. Choose $\varepsilon' > 0$ s.t. $D^*JD \gg \varepsilon' I$. Then there are $\{r_n\} \subset \mathbf{R}$, $\{x_n\} \subset H$, $\{u_n\} \subset U$ s.t. $\|u_n\|_U = 1$ $(n \in \mathbf{N})$, $(ir_n - A)x_n = Bu_n$ and $\kappa(x_n, u_n) \to 0$, as $n \to \infty$. Consequently, $x_n \to 0$ (by (ii')).

Given $\delta > 0$, we obtain from Lemma 6.3.22 that there is $M < \infty$ s.t. $\|C_{\mathrm{w}} x_n\|_Y \le M\|x_n\|_H + \delta\|u_n\|_U/2$ $(n \in \mathbf{N})$. Consequently, there is $N_\delta \in \mathbf{N}$ s.t. $\|C_{\mathrm{w}} x_n\|_Y < \delta$ for $n \ge N_\delta$. It follows that

$$\kappa(x_n, u_n) = \langle u_n, D^*JDu_n\rangle + 2\,\mathrm{Re}\,\langle C_{\mathrm{w}} x, JDu_n\rangle + \langle C_{\mathrm{w}} x, JC_{\mathrm{w}} x\rangle \tag{10.38}$$

$$\ge \varepsilon' - 2\|D\|\|J\|\delta - \|J\|\delta^2. \tag{10.39}$$

Because $\delta > 0$ was arbitrary, we have $\liminf_{n\to\infty} \kappa(x_n, u_n) \ge \varepsilon'$, a contradiction, as required.

2° *(i')⇒(i):* To derive a contradiction, assume that $\mathbb{D}$ is ULR, $D_{\mathrm{c}} = D$ and (i') holds but (i) does not hold. Choose $\varepsilon' > 0$ s.t. $D^*JD \gg \varepsilon' I$. Then there are $\{u_n\} \subset \mathcal{U}_{\exp}(0)$ s.t. $\|u_n\|_2 + \|x_n\|_2 = 1$ $(n \in \mathbf{N})$, and $\langle \mathbb{D}u, J\mathbb{D}u\rangle \to 0$, as $n \to \infty$, where $x_n := \mathbb{B}\tau u_n$. Consequently, $\|x_n\|_2 \to 0$ and $\|u_n\|_2 \to 1$ (by (i')), as $n \to \infty$.

Given $\delta > 0$, choose M as in 1°. Since $(ir-A)\widehat{x}_n(ir) = B\widehat{u}_n(ir)$ for a.e. $r \in \mathbf{R}$, by Lemma 6.3.20, we obtain that

$$\|C_{\mathrm{w}}\widehat{x}_n(ir)\|_Y \le M\|\widehat{x}_n(ir)\|_H + \frac{\delta}{2}\|\widehat{u}_n(ir)\|_U \quad \text{for a.e. } r \in \mathbf{R} \tag{10.40}$$

and for all $n \in \mathbf{N}$. Consequently, there is $N_\delta > 0$ s.t. $\|C_{\mathrm{w}}\widehat{x}_n(ir)\|_Y \le \delta\|\widehat{u}_n(ir)\|_U$ for a.e. $r \in \mathbf{R}$ and all $n \ge N_\delta$, hence $\|C_{\mathrm{w}}\widehat{x}_n\|^{\le}_{\mathrm{L}^2(i\mathbf{R};Y)} 2\pi\delta^2$ for all $n \ge N_\delta$ (because $\|\widehat{u}_n\|_2^2 = 2\pi\|u_n\|_2^2 \le 2\pi$). But

$$2\pi\mathcal{J}(0,u_n) \ge \langle D\widehat{u}_n, JD\widehat{u}_n\rangle_{\mathrm{L}^2} - \|J\|\|C_{\mathrm{w}}\widehat{x}_n\|_2^2 - 2\|J\|\|D\widehat{u}_n\|_2^2\|C_{\mathrm{w}}\widehat{x}_n\|_2^2, \tag{10.41}$$

by (6.90), hence $\mathcal{J}(0,u_n) \ge \varepsilon'\|u_n\|_2^2 - M'\delta^2 - M'\|u_n\|_2\delta$ for $n \ge N_\delta$, where $M' := \|J\|\max\{\|D\|,1\}$. Consequently, $\liminf \mathcal{J}(0,u_n) \ge \varepsilon' - M'\delta^2$. Because $\delta > 0$ was arbitrary, we have $\liminf \mathcal{J}(0,u_n) \ge \varepsilon'$, a contradiction, as required.

(g2) Let K_{c} be the exponentially stabilizing compatible state feedback operator. Note that $\mathbb{D}_\flat \in$ SHPR in either case, by the assumption or by Lemma 6.3.23.

1° *Case* $K_{\mathrm{c}} = 0$*:* Assume that (i) holds (by (c), this is the case if (ii) holds). By Lemma 6.3.6(c2), we have $D_\flat^* J D_\flat \gg 0$, i.e., $D^*JD \gg 0$. Thus, (i') holds (and (ii') if (ii) holds).

2° *(ii)⇒(ii'), and (i)⇒(i') assuming that* K_{c} *stabilizes* Σ *exponentially,* $\mathbb{B}_\flat\tau \in$ SHPR *and* $\mathbb{D}_\flat \in$ SLR*:* Assume that K_{c} is an exponentially stabilizing compatible state feedback operator for Σ. Assume that the corresponding closed-loop system $\Sigma_\flat$ is s.t. $\mathbb{D}_\flat \in$ SHPR. Assume that (ii) (resp. (i)) holds for $D_{\mathrm{c}} = D_\flat =: D$.

(By applying twice Proposition 6.6.18(d2) (see also Lemma 6.6.14), we see that $-K_{\mathrm{c}}$ is an admissible compatible state feedback operator for $\Sigma_\flat$, and $((C_\flat)_{\mathrm{w}} - D_\flat K_{\mathrm{c}}, D_\flat)$ is a compatible pair for Σ. Thus condition (ii) for $D_{\mathrm{c}} = D_\flat$ is well defined.)

Then (ii) (resp. (i)) holds for $\left[\begin{array}{c|c}\mathbb{A}_\flat & \mathbb{B}_\flat\\ \hline \mathbb{C}_\flat & \mathbb{D}_\flat\end{array}\right]$, by (a2), hence (ii') (resp. (i')) holds for $\left[\begin{array}{c|c}\mathbb{A}_\flat & \mathbb{B}_\flat\\ \hline \mathbb{C}_\flat & \mathbb{D}_\flat\end{array}\right]$, by 1°, hence (ii') (resp. (i')) holds for Σ, by (a2).

(e1) The first chain of implications follows from (f), (g1) and (g2) except for (v) and (vi).

"(ii')⇒(v)": This follows from Lemma 10.2.4 (since (ii') implies (i)). "(v)⇔(vi)⇒(i)": This follows from Theorem 9.2.16 "(vi)⇒(i')": If (vi) holds, then $D^*JD \gg 0$ and (i) holds, by the above, hence then (i') holds.

(e2) 1° *(i)⇒(i')–(iii')&(v):* Assume (i). By Lemma 9.2.17 and its proof, $D^*JD \gg 0$ (hence (i') holds) and there is an exponentially stabilizing ULR $K \in \mathcal{B}(H,U)$, hence (i)–(iii') and (v) hold, by (f).

2° *(i')/(ii)/(ii')/(iii)⇒(i):* This follows from (g1) and (f).

3° *(i)&(i')⇔(vi)⇔(v) under Hypothesis 9.2.1:* Assume Hypothesis 9.2.1 for $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{exp}}$. By Theorem 9.2.16, (i)&(i') implies (vi) and (v) implies (vi). Conversely, if (vi) holds, then we obtain from Theorems 9.2.16 and 9.9.1(k)&(a2) that (v) holds, $\mathbb{D}$ is J-coercive and $\mathcal{J}(0,\cdot) \ge 0$, hence $\mathbb{D}$ is positively J-coercive, i.e., (i) holds.

(d) Assume that $\dim U, \dim H, \dim Y < \infty$. (Note that (iv) is not well defined in general, but it is well defined in this finite-dimensional case.)

1° *(iii')⇒(iii):* Then A,B,C,D are bounded. Let $K := \{(x,u) \in H \times U \mid \|x\|_H^2 + \|u\|_U^2 = 1\}$. The function

$$F(r,x_0,u_0) := \langle \left[\begin{smallmatrix} x_0 \\ u_0 \end{smallmatrix}\right], T_{ir}^* \left[\begin{smallmatrix} I & 0 \\ 0 & J \end{smallmatrix}\right] T_{ir} \left[\begin{smallmatrix} x_0 \\ u_0 \end{smallmatrix}\right] \rangle = \|Ax_0 + Bu_0 - irx_0\|^2 + \kappa(x_0,u_0) \quad (10.42)$$

is continuous on $H \times U$ for a fixed $r \in \mathbf{R}$, hence $M := \max_{x,u \in K} F(0,x,u) < \infty$.

Obviously, there is $R < \infty$ s.t. $F(r,x,u) > M$ for all $|r| > R$ and $(x,u) \in K$. But F is obviously continuous on $[-R,R] \times K$, hence $\varepsilon := \min_{r \in \mathbf{R},\,(x,u) \in K} F(r,x,u)$ exists. By (iii'), $\varepsilon > 0$. Apparently, (iii) holds for this ε.

2° *(vii)⇒(iii'):* The first condition in (vii) means that $\mathrm{Ker}(\left[\begin{smallmatrix} ir-A \\ C \end{smallmatrix}\right]) = \{0\}$ for all $r \in \mathbf{R}$. If $u_0 \neq 0$, then $\kappa(x_0,u_0) \geq \langle u_0, D^*Du_0 \rangle > 0$. If $u_0 = 0$ and $(ir - A)x_0 = Bu_0$ for some $r \in \mathbf{R}$, then we have $Cx_0 \neq 0$, hence then $\kappa(x_0,u_0) = \langle Cx_0, Cx_0 \rangle > 0$.

3° *(ii)⇔(iv):* Let ε be as in (ii), and set $M = \|C\| + \|D\|$. If $ir \in i\mathbf{R} \setminus \sigma(A) =: R$ and $(ir - A)x_0 = Bu_0$, then $Cx_0 + Du_0 = \widehat{\mathbb{D}}(ir)u_0$, hence then $F := \widehat{\mathbb{D}}^* J \widehat{\mathbb{D}}$ satisfies $\langle u_0, F(ir)u_0 \rangle = \kappa(x_0,u_0)$, so that (ii) becomes equivalent to (iv) for $ir \in R$. Since R is dense in $i\mathbf{R} \cup \{\infty\}$, and both sides of the inequalities in (ii) and (iv) are continuous $i\mathbf{R} \cup \{\infty\} \to (0,+\infty]$, any $ir \in i\mathbf{R} \cup \{\infty\}$ will do. (Recall that we interpret the values as their limits as $\mathbf{R} \ni r \to r_0$ for $r_0 = \infty$ and for terms having a pole at ir_0.)

4° *The rest of (d) except E:* If Σ is exponentially stabilizable [and detectable], then we get the other implications from (e) [and (b) and Proposition 10.3.1(b)].

5° *The dense set $E \subset \mathbf{R}$:* In the proof of (ii)⇔(ii')⇔(iii) (including the invariance of (ii), (ii') and (iii) w.r.t. $K_c = K$ in (a2)), we can restrict r to any $E \subset \mathbf{R}$. But $T_\cdot \in \mathcal{C}(i\mathbf{R}; \mathcal{B}(U \times H))$, hence (iii) is invariant under the replacement of $\mathbf{R}$ by its dense subset. The same holds for (iv), because both sides of the inequality in (iv) are continuous $i\mathbf{R} \to [0,+\infty]$. □

In conditions (i)–(i") of Proposition 10.3.2, we posed requirements on $u \in \mathcal{U}_{\mathrm{exp}}(0)$ only; here we show that even in the finite-dimensional case, this is strictly weaker than requiring the same for all $u \in \mathrm{L}^2(\mathbf{R}_+;U)$ (or for all $u \in \mathcal{U}_{\mathrm{out}}(0)$), and that none of (i)–(iv') implies the same for all $u \in \mathcal{U}_{\mathrm{out}}(0)$ (naturally, this cannot be the case when Σ is exponentially detectable, by (b)):

Example 10.3.4 Let $A = 1 = B = D$, $C = 0$, $\mathbf{C} = U = H = Y$. Then $\mathbb{D}u = u$ for all u, so that $\mathcal{U}_{\mathrm{out}}(0) = \mathrm{L}^2(\mathbf{R}_+;U)$, but $(s - A)^{-1}B = (s-1)^{-1}$, hence $\mathcal{L}\mathbb{B}\tau u = (\cdot - 1)^{-1}\widehat{u}$, so that $\mathcal{U}_{\mathrm{exp}}(0) = \{u \in \mathrm{L}^2 \mid \widehat{u}(1) = 0\}$. Moreover, then $|\widehat{\mathbb{D}}| = 1$ and $|(s-A)^{-1}B| \leq 1$ on $i\mathbf{R}$, hence condition (ii') of Proposition 10.3.2 holds for $\varepsilon = 1$, hence (i)–(iv') hold, by (d) (because Σ is exponentially stabilizable).

Nevertheless, we have $\mathcal{J}(0,u) = \|u\|_2^2$ and $\|\mathbb{B}\tau u\|_2 = \infty$ for all $u \in \mathcal{U}_{\mathrm{out}}(0) \setminus \mathcal{U}_{\mathrm{exp}}(0)$, hence we cannot allow for arbitrary $u \in \mathrm{L}^2(\mathbf{R}_+;U)$, not even arbitrary $u \in \mathcal{U}_{\mathrm{out}}(0)$ in (i) (nor in (i') or in (i")). ◁

Notes

Many of the above results are known for finite-dimensional systems, part of them also more generally. The method of the proof of “(ii’)⇔(i’)” in Proposition 10.3.2 is partially from [Keu, Section 3], where these two conditions, (v) and (vi) are shown to be equivalent for Pritchard–Salamon systems.

The two propositions show that even in the finite-dimensional case, several classical results contain superfluous or redundant assumptions. See the notes on p. 555 for more on minimization problems and their assumptions.

Proposition 10.3.1 also holds in its discrete-time form, whereas Proposition 10.3.2 needs to be rewritten (since S takes the role of D^*JD); see Proposition 15.2.2.

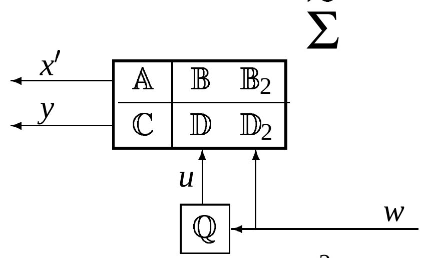

Figure 10.1: The H^2 problem

10.4 The H^2 problem

Grabel's Law:
2 is not equal to 3 – not even for large values of 2.

In this section, we shall show how the solution for the minimization problem of Section 10.2 (for Σ) provides also the solution of the H^2 full information and state feedback problems (for $\widetilde{\Sigma}$) as a corollary.

We first show that given an additional (bounded) input operator $B_2 \in \mathcal{B}(W,H)$, the norm $\|\mathbb{D}u + \mathbb{C}B_2 w_0\|_2$ is minimized by $u(t) = Kx(t)$ (Theorem 10.4.2), i.e., that K solves the H^2 problem. Then we allow for unbounded B_2 but require that K is bounded (Proposition 10.4.3). Finally, we give a corollary, whose regularity, coercivity and stabilizability assumptions guarantee that K necessarily exists and is bounded, so that it also solves the H^2 problem (Corollary 10.4.4).

Standing Hypothesis 10.4.1 ($J = I$, B_2) *Throughout this section, we assume the existence of an additional WR input operator $B_2 \in \mathcal{B}(W, H_{-1})$, such that Σ can be extended to*

$$\widetilde{\Sigma} := \left[\begin{array}{c|cc} \mathbb{A} & \mathbb{B} & \mathbb{B}_2 \\ \hline \mathbb{C} & \mathbb{D} & \mathbb{D}_2 \end{array}\right] \in \mathrm{WPLS}(U \times W, H, Y) \quad \textit{with generators} \quad \left[\begin{array}{c|cc} A & B & B_2 \\ \hline C & D & 0 \end{array}\right]. \tag{10.43}$$

We also assume that $J = I$.

(By B_2 being WR we mean that $\mathbb{D}_2$ is WR. The assumption $J = I$ implies that minimization refers to the cost function $\mathcal{J}(x_0,u) := \|\mathbb{C}x_0 + \mathbb{D}u\|_2^2$ (see Section 10.2). This could easily be relaxed to obtain a more general H^2 problem.)

In the H^2 *problem*, we wish to find a "controller" $\mathbb{Q}$ that minimizes the norm

$$\|\widehat{\mathbb{D}}\widehat{\mathbb{Q}} + \widehat{\mathbb{D}}_2\|_{\mathrm{H}^2_{\mathrm{strong}}(\mathbf{C}^+;\mathcal{B}(W,Y))} \tag{10.44}$$

where $\widehat{\mathbb{D}}_2(s) = C(s-A)^{-1}B_2$ (see Figure 10.1; this is a generalization of the traditional H^2 problem, by Remark 10.4.6). But $\|\widehat{\mathbb{E}}\|_{\mathrm{H}^2_{\mathrm{strong}}} = \sup_{\|w_0\|_W \le 1} \|\mathbb{E}w_0\delta_0\|_2$ for any $\widehat{\mathbb{E}} \in \mathrm{H}^\infty_\infty$; indeed, the Laplace transform is an isometric isomorphism of $\mathcal{B}(W, \mathrm{L}^2(\mathbf{R}_+;Y))$ onto $\mathrm{H}^2_{\mathrm{strong}}(\mathbf{C}^+;\mathcal{B}(W,Y))$, by Lemma F.3.4(d). (Here $\mathbb{E}w_0\delta_0 := \mathcal{L}^{-1}\widehat{\mathbb{E}}w_0$.) Thus, an equivalent definition for the H^2 problem as finding for an arbitrary $w_0 \in W$ a "stabilizing" control (see the proofs of Theorem

10.4.2 and Proposition 10.4.3 for explanations) s.t.

$$\mathcal{J}_{\mathrm{H}^2}(w_0,u) := \frac{1}{2\pi}\|\widehat{\mathbb{D}}\widehat{u}+\widehat{\mathbb{D}}_2 w_0\|^2_{\mathrm{H}^2(\mathbf{C}^+;Y)} \quad (=\|\mathbb{D}u+\mathbb{C}B_2w_0\|_2^2 \text{ if } B_2\in\mathcal{B}(W,H)) \tag{10.45}$$

is minimized. Thus, we are minimizing the energy of the impulse response (the output corresponding to "second input $w_0\delta_0$", where δ_0 is the Dirac delta function) of the system.

This implies that the assumption

$$\widehat{\mathbb{D}}_2 \in \mathrm{H}^2_{\mathrm{strong}}(\mathbf{C}^+_\omega;\mathcal{B}(W,Y)) \text{ for some } \omega>0 \tag{10.46}$$

is necessary (because $\widehat{\mathbb{D}}\widehat{u}\in\mathrm{H}^2_\omega$ for all $\widehat{u}\in\mathcal{L}[\mathrm{L}^2_\vartheta(\mathbf{R}_+;U)]$, where $\omega\ge\vartheta$ is s.t. $\widehat{\mathbb{D}}\in\mathrm{H}^\infty_\omega$ (and where ϑ is s.t. the controls are required to be ϑ-stable), it follows that we must have $\widehat{\mathbb{D}}_2w_0\in\mathrm{H}^2_\omega$, for all $w_0\in W$).

Therefore, $\begin{bmatrix}\mathbb{D} & \mathbb{D}_2\end{bmatrix}$ has a realization with a bounded B_2 (see (6.213), although one is usually more interested in the original system, where B_2 may be unbounded. Usually one also requires the control law $\mathbb{Q}: w_0\mapsto u$ to be of a specific form (e.g., a state feedback controller).

Due to the above, we start with the case $B_2\in\mathcal{B}(W,H)$ (which implies that (10.46) holds). As explained above, at least with such systems we end up with the minimization problem of Section 10.2 except that the initial states are restricted to $B_2[W]$; in particular:

Theorem 10.4.2 ($\mathcal{U}_*^*$: LQR $\Rightarrow$ H^2) *Assume that $B_2\in\mathcal{B}(W,H)$ and that there is a minimizing WR state feedback operator K over $\mathcal{U}_*^*$.*

Then K solves the H^2 problem (strictly if K is strictly minimizing), i.e., it leads to the minimization of the cost (10.45), for each $w_0\in W$; see Figure 10.2.

Thus, in this case, the state feedback and full information H^2 problems have a common solution, namely the minimizing K. See Section 10.2 for sufficient and necessary conditions for the existence of K.

Assume for a while that also B is bounded. Then a sufficient condition for the existence of K is that Σ is positively J-coercive (see Section 10.3) and "$\mathcal{U}_*^*$-stabilizable" (i.e., $\mathcal{U}_*^*(x_0)\neq\emptyset$ for all $x_0\in H$) if, e.g., $\mathcal{U}_*^*=\mathcal{U}_{\mathrm{out}}$ or $\mathcal{U}_*^*=\mathcal{U}_{\mathrm{exp}}$.

Moreover, then K is necessarily ULR and the optimal controller $\mathbb{Q}$ becomes the map $\mathbb{Q}=\mathbb{K}_\circlearrowleft B_2: W\to\mathrm{L}^2(\mathbf{R}_+;U)$ (since $\mathbb{C}_\circlearrowleft B_2w_0=\mathbb{C}B_2w_0+\mathbb{D}\mathbb{K}_\circlearrowleft B_2$; note that $\widehat{\mathbb{Q}}=$"$\widehat{\mathbb{F}_{\circlearrowleft 2}}$", where $\mathbb{F}_{\circlearrowleft 2}\in\mathrm{TIC}_\infty$ ($\in\mathrm{TIC}$ if $\mathcal{U}_*^*=\mathcal{U}_{\mathrm{exp}}$) is the inverse Laplace transform of $\widehat{\mathbb{Q}}$ as a H^∞_∞ function, whereas $\mathbb{Q}$ is the inverse Laplace transform of $\widehat{\mathbb{Q}}\in\mathcal{B}(W,\mathrm{L}^2(\mathbf{R}_+;U))$ (when, e.g., $\mathcal{U}_*^*=\mathcal{U}_{\mathrm{out}}$ or $\mathcal{U}_*^*=\mathcal{U}_{\mathrm{exp}}$)).

Proof of Theorem 10.4.2: By definition, the control $u:=\mathbb{K}_\circlearrowleft B_2w_0$ minimizes the cost $\|\mathbb{C}B_2w_0+\mathbb{D}u\|_2^2$ over $u\in\mathcal{U}_*^*(B_2w_0)$ (which in case $\mathcal{U}_*^*=\mathcal{U}_{\mathrm{exp}}$ means the controls $u\in\mathrm{L}^2(\mathbf{R}_+;U)$ that make $x:=\mathbb{A}B_2w_0+\mathbb{B}\tau u$ (and hence $y:=\mathbb{C}B_2w_0+\mathbb{D}u$) stable). (If this minimization is strict for all $x_0\in H$ in place of B_2w_0, then it is strict for a particular $w_0\in W$.)

N.B. If $\mathcal{U}_*^*=\mathcal{U}_{\mathrm{exp}}$, then K is an exponentially stabilizing state feedback operator (and it minimizes the H^2 norm over all such operators as well as over

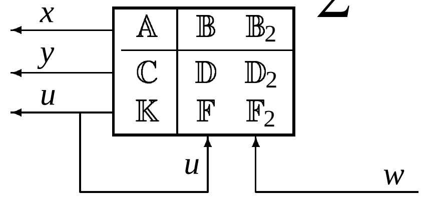

Figure 10.2: The H^2 state feedback problem

all $u \in \mathcal{U}_{\exp}(B_2 w_0)$, for each $w_0 \in W$). $\square$

In classical H^2 problems, one has $\mathcal{U}_*^* = \mathcal{U}_{\exp}$ (and $\dim U \times H \times Y < \infty$), and one assumes that the original system Σ is exponentially stabilizable and positively J-coercive, so that there is a unique (bounded) minimizing state feedback operator. Thus, then Theorem 10.4.2 becomes applicable.

Next we drop the assumption that B_2 is bounded and assume, instead, that the cost-minimizing state feedback operator K is bounded. We again show that K also solves the H^2 problem (assuming that $\mathcal{U}_*^* = \mathcal{U}_{\exp}$ and that the two necessary "$\mathrm{L}^2_{\text{strong}}$" assumptions hold):

Proposition 10.4.3 ($\mathcal{U}_{\mathbf{exp}}$: LQR $\Rightarrow$ H^2) *Assume that $\mathcal{U}_*^* = \mathcal{U}_{\exp}$. Assume that $\pi_{[0,1)}\mathbb{A}B_2 w_0 \in \mathrm{L}^2([0,1);H)$ and $\pi_{[0,1)}C_{\mathrm{w}}\mathbb{A}B_2 w_0 \in \mathrm{L}^2([0,1);Y)$ for all $w_0 \in W$ (this is necessary for the existence of admissible controls for each $w_0 \in W$).*

Assume also that there is a bounded minimizing state feedback operator K for Σ.

Then K solves the H^2 problem (strictly if K is strictly minimizing); i.e., the minimum of $\|\widehat{\mathbb{D}}\widehat{u} + \widehat{\mathbb{D}}_2 w_0\|_{\mathrm{H}^2(\mathbf{C}^+;Y)}$ over $\mathcal{U}_{\exp}$-stabilizing controls u is equal to

$$\|\widehat{\mathbb{D}}_{\circlearrowleft 2} w_0\|_{\mathrm{H}^2(\mathbf{C}^+;Y)} = \|\mathbb{C}_{\circlearrowleft} B_2 w_0\|_{\mathrm{L}^2(\mathbf{R}_+;Y)} = \|(C_{\circlearrowleft})_{\mathrm{L,s}}\mathbb{A}_{\circlearrowleft}^{\cdot} B_2 w_0\|_{\mathrm{L}^2(\mathbf{R}_+;Y)} \tag{10.47}$$

The above assumptions on B_2 hold iff $(\cdot - A)^{-1}B_2 \in \mathrm{H}^2_{\text{strong}}(\mathbf{C}^+_\omega;\mathcal{B}(W,H))$ and $\widehat{\mathbb{D}}_2 \in \mathrm{H}^2_{\text{strong}}(\mathbf{C}^+_\omega;\mathcal{B}(W,Y))$ for some $\omega \in \mathbf{R}$, by Lemma 6.8.1(a)&(d1).

Here $\mathcal{U}_{\exp}$-stabilizing controls mean such $u \in \mathrm{L}^2(\mathbf{R}_+;U)$ that the state of Σ with "initial state $B_2 w_0$" is in L^2. Since $\mathbb{B}\tau u \in \mathrm{L}^2_\infty$, the condition $\pi_{[0,1)}\mathbb{A}B_2 w_0 \in \mathrm{L}^2([0,1);H)$ is necessary; it was shown above that $\pi_{[0,1)}C_{\mathrm{w}}\mathbb{A}B_2 w_0 \in \mathrm{L}^2([0,1);Y)$ (and $D_2 = 0$) is also necessary. Thus, the assumptions at the beginning of the proposition do not reduce generality.

Note that we cannot require that $x \in \mathcal{C}$ unless $\mathbb{A}B_2 w_0 \in \mathcal{C}(\mathbf{R}_+;H)$ for all $w_0 \in W$ (e.g., B_2 is bounded), since $\mathbb{B}\tau u \in \mathcal{C}$ (for any $u \in \mathrm{L}^2_\infty(\mathbf{R}_+;U)$); see (10.50).

Proof of Proposition 10.4.3: 1° $\widetilde{\Sigma}_{\circlearrowleft}$*:* Since K is bounded, it extends $\widetilde{\Sigma}$ to

$$\widetilde{\Sigma}_{\text{ext}} := \left[\begin{array}{c|cc} \mathbb{A} & \mathbb{B} & \mathbb{B}_2 \\ \hline \mathbb{C} & \mathbb{D} & \mathbb{D}_2 \\ \mathbb{K} & \mathbb{F} & \mathbb{F}_2 \end{array}\right] \in \mathrm{WPLS}(U \times W, H, Y \times U) \quad \text{with generators} \quad \left[\begin{array}{c|cc} A & B & B_2 \\ \hline C & D & 0 \\ K & 0 & 0 \end{array}\right], \tag{10.48}$$

by Lemma 6.3.16(c). It follows from that $\left[\begin{smallmatrix} K \\ 0 \end{smallmatrix}\right]$ is a bounded ULR exponentially stabilizing state feedback operator for $\widetilde{\Sigma}$, by Lemma 6.6.11 (alternatively, we can use the state feedback operator $\begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix}$ for $\widetilde{\Sigma}_{\mathrm{ext}}$). The corresponding closed-loop system is given by

$$\widetilde{\Sigma}_{\circlearrowleft} := \left[\begin{array}{c|cc} \mathbb{A}_{\circlearrowleft} & \mathbb{B}_{\circlearrowleft} & \mathbb{B}_{\circlearrowleft 2} \\ \hline \mathbb{C}_{\circlearrowleft} & \mathbb{D}_{\circlearrowleft} & \mathbb{D}_{\circlearrowleft 2} \\ \mathbb{K}_{\circlearrowleft} & \mathbb{F}_{\circlearrowleft} & \mathbb{F}_{\circlearrowleft 2} \end{array}\right] = \left[\begin{array}{c|cc} \mathbb{A}+\mathbb{B}_{\circlearrowleft}\mathbb{K} & \mathbb{B}\mathbb{M} & \mathbb{B}_{\circlearrowleft}\mathbb{F}_2+\mathbb{B}_2 \\ \hline \mathbb{C}+\mathbb{D}_{\circlearrowleft}\mathbb{K} & \mathbb{D}\mathbb{M} & \mathbb{D}_{\circlearrowleft}\mathbb{F}_2+\mathbb{D}_2 \\ \mathbb{M}\mathbb{K} & \mathbb{M}-I & \mathbb{M}\mathbb{F}_2 \end{array}\right] \tag{10.49}$$

(use (6.133) with $\mathbb{M} = \begin{bmatrix} \mathbb{M} & \mathbb{M}\mathbb{F}_2 \\ 0 & I \end{bmatrix}$ or (6.125) with $I - L\mathbb{D} = \begin{bmatrix} \mathbb{X} & -\mathbb{F}_2 \\ 0 & I \end{bmatrix}$; two left columns of $\widetilde{\Sigma}_{\circlearrowleft}$ are exactly $\Sigma_{\circlearrowleft}$, the closed-loop system of Σ corresponding to K).

2° *We have $\mathbb{A}_{\circlearrowleft} B_2 w_0, (C_{\circlearrowleft})_{\mathrm{w}}\mathbb{A}_{\circlearrowleft} B_2 w_0 \in \mathrm{L}^2$:* Let $w_0 \in W$ be arbitrary. Since $\mathbb{K} = K\mathbb{A}$, we have

$$\mathbb{A}_{\circlearrowleft} B_2 w_0 = \mathbb{A} B_2 w_0 + \mathbb{B}_{\circlearrowleft}\tau K \mathbb{A} B_2 u_0 \in \mathrm{L}^2([0,1);H) \tag{10.50}$$

since $\mathbb{A}B_2 w_0 \in \mathrm{L}^2([0,1);H)$ and $K \in \mathcal{B}(H,U)$. By Lemma 6.8.1(a), it follows that $\mathbb{A}_{\circlearrowleft} B_2 w_0 \in \mathrm{L}^2(\mathbf{R}_+;U)$, (since $\omega_{\mathbb{A}_{\circlearrowleft}} < 0$). Analogously,

$$\widehat{\mathbb{D}}_{\circlearrowleft 2} w_0 = \widehat{\mathbb{D}_{\circlearrowleft}}\widehat{\mathbb{F}}_2 w_0 + \widehat{\mathbb{D}}_2 w_0 = \widehat{\mathbb{D}_{\circlearrowleft}} K(\cdot - A)^{-1} B_2 w_0 + \widehat{\mathbb{D}}_2 w_0 \in \mathrm{H}^2(\mathbf{C}^+_{\omega};Y) \tag{10.51}$$

for some $\omega > 0$, hence $\widehat{\mathbb{D}}_{\circlearrowleft 2} w_0 \in \mathrm{H}^2(\mathbf{C}^+;Y)$ (i.e., $(C_{\circlearrowleft})_{\mathrm{w}}\mathbb{A}_{\circlearrowleft} B_2 w_0 \in \mathrm{L}^2(\mathbf{R}_+;Y)$), by Lemma 6.8.1(d1).

3° *$\mathcal{U}_{\mathrm{exp}}$-stabilizing controls u:* Let $w_0 \in W$. Assume that $u \in \mathrm{L}^2(\mathbf{R}_+;U)$ is *$\mathcal{U}_{\mathrm{exp}}$-stabilizing*, by which we mean that $x \in \mathrm{L}^2$, where

$$x := \mathbb{A} B_2 w_0 + \mathbb{B}\tau u = \mathbb{A}_{\circlearrowleft} B_2 w_0 + \mathbb{B}_{\circlearrowleft}\tau(\mathbb{X}u - \mathbb{K} B_2 w_0) \tag{10.52}$$

(we used here the facts that $\mathbb{A}_{\circlearrowleft} = \mathbb{A} + \mathbb{B}_{\circlearrowleft}\tau\mathbb{K}$ and $\mathbb{B}_{\circlearrowleft} = \mathbb{B}\mathbb{X}^{-1}$). By assumption, $\mathbb{A}_{\circlearrowleft} B_2 w_0 \in \mathrm{L}^2$, hence $\mathbb{B}_{\circlearrowleft}\tau u_{\circlearrowleft} \in \mathrm{L}^2$, where $u_{\circlearrowleft} := \mathbb{X}u - \mathbb{K}B_2 w_0 = u - Kx \in \mathrm{L}^2$. Then (recall that $\widehat{\mathbb{F}}_2(s) = K(s-A)^{-1}B_2 = \widehat{\mathbb{K}}(s)B_2$)

$$\widehat{y} := \widehat{\mathbb{D}}_2 w_0 + \widehat{\mathbb{D}}\widehat{u} = \widehat{\mathbb{D}}_2 w_0 + \widehat{\mathbb{D}_{\circlearrowleft}}\widehat{\mathbb{K}} B_2 w_0 - \widehat{\mathbb{D}_{\circlearrowleft}}\widehat{\mathbb{K}} B_2 w_0 + \widehat{\mathbb{D}_{\circlearrowleft}}\widehat{\mathbb{X}}\widehat{u} = \widehat{\mathbb{D}}_{\circlearrowleft 2} w_0 + \widehat{\mathbb{D}_{\circlearrowleft}} u_{\circlearrowleft}. \tag{10.53}$$

4° *Minimum at $u_{\circlearrowleft} = 0$:* Let $w_0 \in W$ be arbitrary, and set $\widehat{z} := \widehat{\mathbb{D}}_{\circlearrowleft 2} w_0$, so that $z = (C_{\circlearrowleft})_{\mathrm{L,s}}\mathbb{A}_{\circlearrowleft}^{\cdot} B_2 w_0$, by Lemma 6.8.1(d1). For each $n \in \mathbf{N}$, choose $t_n \in (0,1/n)$ s.t. $x_n := \mathbb{A}_{\circlearrowleft}^{t_n} B w_0 \in H$. Then

$$z_n := (C_{\circlearrowleft})_{\mathrm{L,s}}\mathbb{A}_{\circlearrowleft}^{\cdot}\mathbb{A}_{\circlearrowleft}^{t_n} B_2 w_0 = \mathbb{C}_{\circlearrowleft} x_n \in \mathcal{X} \quad (n \in \mathbf{N}). \tag{10.54}$$

But $z_n = \tau^{-t_n}\pi_+\tau^{t_n} z$, hence $z_n \to z$ in $\mathrm{L}^2(\mathbf{R}_+;Y)$ as $n \to \infty$, by Corollary B.3.8. Consequently,

$$\langle z, \mathbb{D}_{\circlearrowleft} u_{\circlearrowleft}\rangle_{\mathrm{L}^2} = \lim_n \langle \mathbb{C}_{\circlearrowleft} x_n, \mathbb{D}_{\circlearrowleft} u_{\circlearrowleft}\rangle_{\mathrm{L}^2} = 0 \quad (u_{\circlearrowleft} \in \mathrm{L}^2). \tag{10.55}$$

Thus, if u and $u_{\circlearrowleft}$ are as in 3°, then $\langle \widehat{\mathbb{D}}_{\circlearrowleft 2} w_0, \widehat{\mathbb{D}_{\circlearrowleft}} u_{\circlearrowleft}\rangle_{\mathrm{H}^2} = 0$, so that

$$\|\widehat{\mathbb{D}}_2 w_0 + \widehat{\mathbb{D}}\widehat{u}\|^2_{\mathrm{H}^2} = \|\widehat{\mathbb{D}}_{\circlearrowleft 2} w_0\|^2_{\mathrm{H}^2} + \|\widehat{\mathbb{D}}_{\circlearrowleft} u_{\circlearrowleft}\|^2_{\mathrm{H}^2}. \tag{10.56}$$

Consequently, the minimum is achieved at $u_{\circlearrowleft} = 0$. This minimum is strict iff

$\mathbb{D}$ is injective (on $\mathcal{U}_{\rm exp}$-stabilizing controls u, e.g., on $\mathrm{L}^2(\mathbf{R}_+;U)$). $\square$

When Σ is very regular (cf. Theorem 9.2.3) and $D^*C=0$ (or C is bounded), the assumptions of the proposition are satisfied:

Corollary 10.4.4 ($\mathbf{H^2}$ problem when $\mathbb{A}Bu_0\in\mathrm{L}^2$) *Assume that $\mathcal{U}_*^*=\mathcal{U}_{\rm exp}$, $\mathbb{A}B_2w_0\in\mathrm{L}^2([0,1);H)$ and $C_{\rm w}\mathbb{A}B_2w_0\in\mathrm{L}^2([0,1);Y)$ for all $w_0\in W$. Assume also that Hypothesis 9.2.1 holds, $D^*C\in\mathcal{B}(H,U)$, and $D^*D\gg 0$.*

Then there is a minimizing control for each $x_0\in H$ iff Σ is optimizable and $\mathbb{D}$ is I-coercive (see Proposition 10.3.2). Assume that this is the case.

Then there is a bounded strictly minimizing state feedback operator K for Σ (corresponding to the unique exponentially stabilizing solution $(\mathcal{P},D^*D,K)$ of the $B_{\rm w}^*$-CARE), and K solves the H^2 problem strictly; i.e., the minimum of $\|\widehat{\mathbb{D}}\widehat{u}+\widehat{\mathbb{D}}_2w_0\|_{\mathrm{H}^2(\mathbf{C}^+;Y)}$ over $\mathcal{U}_{\rm exp}$-stabilizing controls u is strict and equal to $\|\widehat{\mathbb{D}}_{\circlearrowleft 2}w_0\|_{\mathrm{H}^2(\mathbf{C}^+;Y)}$.

In particular, the problem can be solved in the parabolic case (when Hypothesis 9.5.1 holds, $\gamma<1/4$, $\beta<-1/2$, $D^*D\gg 0$ and $D^*C\in\mathcal{B}(H,U)$), by Theorem 9.2.3.

Alternatively, when B and B_2 are bounded, $D^*C\in\mathcal{B}(H,U)$, $D^*D\gg 0$ and $\mathbb{D}$ is J-coercive, all assumptions of the corollary are satisfied. In either case, also the state becomes continuous. Hypothesis 9.2.1 is also satisfied in the case that $\mathbb{A}Bu_0\in\mathrm{L}^2([0,1);H)$ and $C_{\rm w}\mathbb{A}Bu_0\in\mathrm{L}^2([0,1);Y)$ for all $u_0\in U$, i.e., that also B satisfies the assumptions posed on B_2.

Proof of Corollary 10.4.4: By Corollary 10.2.9, there is a minimizing control for each $x_0\in H$ iff Σ is optimizable and $\mathbb{D}$ is (necessarily positively, since $I\ge 0$) I-coercive, and such a control is necessarily unique, with $(\mathcal{P},D^*D,K)$ being the unique exponentially stabilizing solution of the $B_{\rm w}^*$-CARE. Since $D^*C\in\mathcal{B}(H,U)$, the operator K is bounded. The rest follows from Proposition 10.4.3. $\square$

Remark 10.4.5 ($x_0\neq 0$) *As one observes from the proofs, the (state feedback) controller $\begin{bmatrix}\mathbb{Q}_0 & \mathbb{Q}\end{bmatrix}:(x_0,w)\mapsto u:=\mathbb{K}_\circlearrowleft x_0+\mathbb{K}_\circlearrowleft B_2w_0$ of Theorem 10.4.2, Proposition 10.4.3 and Corollary 10.4.4 actually minimizes the norm*

$$\|\widehat{\mathbb{C}}x_0+\widehat{\mathbb{D}}\widehat{u}+\widehat{\mathbb{D}}_2w_0\|_{\mathrm{H}^2(\mathbf{C}^+;Y)} \tag{10.57}$$

for any $x_0\in H$ (not merely for $x_0=0$ as in the H^2 problem). Indeed, the "cost"

$$\|\widehat{\mathbb{C}_\circlearrowleft}x_0+\widehat{\mathbb{D}_\circlearrowleft}\widehat{u_\circlearrowleft}+\widehat{\mathbb{D}_{\circlearrowleft 2}}w_0\|_{\mathrm{H}^2}^2=\|\widehat{\mathbb{C}_\circlearrowleft}x_0+\widehat{\mathbb{D}_{\circlearrowleft 2}}w_0\|_{\mathrm{H}^2}^2+\|\widehat{\mathbb{D}_\circlearrowleft}\widehat{u_\circlearrowleft}\|_{\mathrm{H}^2}^2 \tag{10.58}$$

induced by K corresponds to the control law $\begin{bmatrix}\mathbb{Q}_0 & \mathbb{Q}\end{bmatrix}$, where $\widehat{u_\circlearrowleft}$ is the external input. For $u_\circlearrowleft\in\mathrm{L}^2_{\rm c}(\mathbf{R}_+;U)$ we have $\|\widehat{\mathbb{D}_\circlearrowleft}\widehat{u_\circlearrowleft}\|_{\mathrm{H}^2}^2=\langle\widehat{u_\circlearrowleft},S\widehat{u_\circlearrowleft}\rangle$ (where $S=D^*D$ if Σ is regular enough; this is the case in Corollary 10.4.4), by (9.162); if $\mathbb{D}_\circlearrowleft$ is stable (e.g., $\mathcal{U}_*^*=\mathcal{U}_{\rm exp}$), then this is the case for all $u_\circlearrowleft\in\mathrm{L}^2(\mathbf{R}_+;U)$.

Thus, not only is there a unique $\mathcal{J}_{\mathrm{H}^2}$-minimizing u for each w_0 (and $x_0=0$), there is a unique $\mathcal{J}_{\mathrm{H}^2}$-minimizing u for each w_0 and x_0, and this u is of state *feedback form, namely generated by the ($\mathcal{J}$-)minimizing K (in particular, the (optimal) function from state to u is static).*

As noted below Proposition 10.4.3, conditions $\mathbb{A}B_2 w_0, C_{\mathrm{w}}\mathbb{A}B_2 w_0 \in \mathrm{L}^2_{\mathrm{loc}}$ ($w_0 \in W$) are necessary for the existence of a solution over $\mathcal{U}_{\mathrm{exp}}$. $\square$

Remark 10.4.6 (Traditional H^2 problem) *Traditionally, in the (finite-dimensional) H^2 problem one minimizes the (squared) norm*

$$\|\widehat{\mathbb{D}}_{\circlearrowleft 2}\|^2 := \sum_{q\in Q} \|\widehat{\mathbb{D}}_{\circlearrowleft 2} w_q\|^2_{\mathrm{H}^2}, \tag{10.59}$$

where $\{w_q\}_{q\in Q}$ is an orthonormal base for W (this form is defined as the H^2 norm of the trace of $\widehat{\mathbb{D}}^*_{\circlearrowleft 2}\widehat{\mathbb{D}}_{\circlearrowleft 2}$, see p. 265 of [IOW] for the equivalence of the two definitions).

Since we minimize $\|\widehat{\mathbb{D}}_{\circlearrowleft 2} w_0\|^2_{\mathrm{H}^2}$ for each $w_0 \in W$, our solution a fortiori *solves the traditional H^2 problem provided that it is solvable, i.e., that the norm can be made finite (e.g., when $\dim W < \infty$ and the conditions of of Theorem 10.4.2, Proposition 10.4.3 or Corollary 10.4.4 are satisfied). Thus, when $\dim W = \infty$, we have to strengthen our optimizability assumption the include the condition that*

$$\sum_{q\in Q} \|\widehat{\mathbb{D}}_2 w_q + \widehat{\mathbb{D}}\widehat{u}_q\|^2_{\mathrm{H}^2} < \infty \tag{10.60}$$

for some functions $u_q \in \mathcal{U}^*_*(B_2 w_q)$ ($q \in Q$) to guarantee a solution also for the traditional H^2 problem (if B_2 is unbounded, we must extend $\mathcal{U}^*_*$ to $B_2[\{w_q\}]$ as in 3° of the proof of Proposition 10.4.3). $\square$

We have solved above the state feedback and full information H^2 problems (they have a common solution). In classical literature, one usually also solves other special H^2 problems and then the general H^2 (dynamic partial feedback) control problem for a given (even more extended) system

$$\left[\begin{array}{c|cc} \mathbb{A} & \mathbb{B} & \mathbb{B}_2 \\ \hline \mathbb{C} & \mathbb{D} & \mathbb{D}_2 \\ \mathbb{C}_2 & \mathbb{D}_{21} & \mathbb{D}_{22} \end{array}\right] \in \mathrm{WPLS}(U\times W, H, Y\times Y_2) \quad \text{with generators} \quad \left[\begin{array}{c|cc} A & B & B_2 \\ \hline C & D & 0 \\ C_2 & 0 & D_{22} \end{array}\right]. \tag{10.61}$$

We end this section by a brief overview of the general H^2 problem. One usually poses for $\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B}_2 \\ \hline \mathbb{C}_2 & \mathbb{D}_{22} \end{array}\right]^{\mathrm{d}}$ the same assumptions as on Σ (thus guaranteeing the solution for the LQR problem for that system).

It seems that it is straightforward to extend the classical results for infinite-dimensional systems by using the separation result of Proposition 10.4.3 (and its proof). (An alternative approach would be to formulate the problems in I/O map form and solve them as in Section 12.3.)

Indeed, since $\widehat{\mathbb{D}_{\circlearrowleft}}$ is (I, D^*D)-inner (and it can be made inner by replacing K by $\left[\begin{array}{c|c} K' & F' \end{array}\right]$ with $K' := XK$, $F' := I - X$, $X := (D^*D)^{1/2}$), the cost for a (dynamic partial feedback) controller $\mathbb{Q}_{\circlearrowleft} : w_0 \mapsto u_{\circlearrowleft}$ is equal to $\mathcal{J}_{\mathrm{H}^2} = \|\mathbb{D}_{\circlearrowleft} w_0\|_2^2 + \|\mathbb{Q}_{\circlearrowleft} w_0\|_2^2$. Thus, the solution of the general H^2 problem is obtained by finding the optimal

$\mathbb{Q}_{\circlearrowleft}$, i.e., by solving the H^2 output estimation problem (OEP) for

$$\left[\begin{array}{c|cc} \mathbb{A} & \mathbb{B} & \mathbb{B}_2 \\ \hline -\mathbb{K} & \mathbb{X} & -\mathbb{F}_2 \\ \mathbb{C}_2 & \mathbb{D}_{21} & \mathbb{D}_{22} \end{array}\right] \in \mathrm{WPLS}(U\times W, H, U\times Y_2) \quad \text{with generators} \quad \left[\begin{array}{c|cc} A & B & B_2 \\ \hline -K & I & 0 \\ C_2 & 0 & D_{22} \end{array}\right]. \tag{10.62}$$

Since we cannot use the theory of Section 10.2 directly for the H^2 OEP (unlike for the H^2 full information problem (FIP)), we leave it to the interested reader complete the details under suitable regularity assumptions, such as those of Lemma 6.8.5 for $p = 2 = q$ or those of Hypothesis 9.5.7(3.) (no special assumptions seem to be needed in the discrete-time case). See, e.g., pp. 316 and 395–396 of [ZDG], Chapter 9 of [IOW] or Chapter VIII of [DGKF] for details for finite-dimensional systems.

Notes

See, e.g., [AM90], [AM79], [KS] or [GL, p. 207] for the motivation and the history of the H^2 problem and [IOW] or [ZDG] for complete solutions in the finite-dimensional case. The first state-space solution seems to be given in [DGKF] and a very general one in [IOW].

The classical assumptions for the state feedback and full information H^2 problems are positive J-coercivity, exponential stabilizability and $D^*D \gg 0$, so that corresponding results are contained Corollary 10.4.4 (since B and C are bounded for finite-dimensional systems).

Naturally, by taking causal adjoints we obtain a solution of the dual problems, the H^2 output injection and full control problems (see [GL]). Their stochastic counterpart is called the Kalman filter problem (see, e.g., [GL] or [LR]).

Also the general H^2 problem has a stochastic counterpart, the so called *Linear Quadratic Gaussian (LQG)* problem; it has been studied also for infinite-dimensional systems, see [CP78].

According to [Helton85], p. 17, G. Zames indicated in the late 1970s how the H^∞ problem is usually physically better motivated than the H^2 problem. Since that time the former problem (whose solutions are more complicated) has become much more popular than the latter one.

10.5 Real lemmas

Rocky's Lemma of Innovation Prevention:
Unless the results are known in advance, funding agencies will reject the proposal.

In this section, we present the Bounded Real Lemma (in two forms) and the Strict Positive Real Lemma, which allow one to use the Riccati equation to verify whether $\|\mathbb{D}\|_{\mathrm{TIC}} \le \gamma$ or $\mathrm{Re}\langle \mathbb{D}\cdot, \cdot\rangle \gg 0$, respectively.

We give our results for "L^1-type" systems; see Theorem 10.6.5(e) for alternative regularity assumptions. For general regular systems only the sufficiency part holds (unless we accept IAREs in place of CAREs). See Theorems 15.4.1 and 15.4.3 and Proposition 15.4.2 for the discrete-time counterparts of these results (and to get an overview of this section without any regularity considerations).

Our theorems do not need any verification of stability/stabilizability; this is based on the fact that we may apply the theory of the next section in case "$C^*JC \le 0$", as one observes from the proofs.

Thus, under sufficient regularity, a uniform Riccati inequality has a solution iff Σ is exponentially stable and $\|\mathbb{D}\| < \gamma$:

Theorem 10.5.1 (Generalized Strict Bounded Real Lemma) *Assume that* $\gamma > 0$.

(a) If (1.) or (2.) or (5.) of Hypothesis 9.2.2 holds, or if C is bounded and at least one of the following conditions holds:

1. $\dim Y < \infty$*;*
2. $\pi_{[0,1)}\mathbb{A}B \in \mathrm{L}^1([0,1);\mathcal{B}(U,H))$*;*
3. $\pi_{[0,1)}\mathbb{A}Bu_0 \in \mathrm{L}^1([0,1);H)$ *for all* $u_0 \in U$ *and* $D^*C = 0$*;*
4. $D^*C = 0$ *and* $(\widehat{\mathbb{D}} - D)u_0 \in \mathrm{H}^2_{\mathrm{strong}}(\mathbf{C}^+;\mathcal{B}(U,Y))$ *for all* $u_0 \in U$*;*

then the following are equivalent:

(i) Σ *is exponentially stable and* $\|\mathbb{D}\| < \gamma$*;*

(ii) There is $\mathcal{P} \le 0$ *s.t.* $\mathcal{P}[H] \subset \mathrm{Dom}(B_{\mathrm{w}}^*)$ *and*

$$\begin{bmatrix} A^*\mathcal{P} + \mathcal{P}A - C^*C & (B_{\mathrm{w}}^*\mathcal{P} - D^*C)^* \\ B_{\mathrm{w}}^*\mathcal{P} - D^*C & \gamma^2 I - D^*D \end{bmatrix} \gg 0 \quad \text{on} \quad \mathrm{Dom}(A) \times U. \tag{10.63}$$

(iii) There is $\mathcal{P} \le 0$ *s.t.* $S := \gamma^2 I - D^*D + \text{s-lim}_{s\to+\infty} B_{\mathrm{w}}^*\mathcal{P}(s-A)^{-1}B$ *exists and*

$$\begin{bmatrix} A^*\mathcal{P} + \mathcal{P}A - C^*C & (B_{\mathrm{w}}^*\mathcal{P} - D^*C)^* \\ B_{\mathrm{w}}^*\mathcal{P} - D^*C & S \end{bmatrix} \gg 0 \quad \text{on} \quad \mathrm{Dom}(A) \times U. \tag{10.64}$$

(b) If $\pi_{[0,t)}\mathbb{A}B \in \mathrm{L}^1([0,t);\mathcal{B}(U,H))$, $\pi_{[0,t)}C_{\mathrm{w}}\mathbb{A} \in \mathrm{L}^1([0,t);\mathcal{B}(H,Y))$, *and* $\pi_{[0,t)}C_{\mathrm{w}}\mathbb{A}B \in \mathrm{L}^1([0,t);\mathcal{B}(U,Y))$ *for some* $t > 0$*, then (i)⇔(iii)⇐(ii).*

(c1) If $\mathbb{D}$ is ULR, then we have (i)⇐(iii)⇐(ii).

(c2) Any solution of (ii), (iii), (ii') or (iii') is strictly negative ($\mathcal{P} < 0$). Under the assumptions of (a), there is an exponentially stabilizing solution (if (i) holds).

(d) If $\mathbb{D}$ is SR, then we have (ii)⇔(ii')⇒(iii)⇔(iii'), where

(ii') $\|D\| < \gamma$, and there is $\mathcal{P} \le 0$ s.t. $\mathcal{P}[H] \subset \mathrm{Dom}(B_{\mathrm{w}}^*)$ and

$$(B_{\mathrm{w}}^*\mathcal{P} - D^*C)^*(\gamma^2 I - D^*D)^{-1}(B_{\mathrm{w}}^*\mathcal{P} - D^*C) \ll A^*\mathcal{P} + \mathcal{P}A - C^*C. \tag{10.65}$$

(iii') There is $\mathcal{P} \le 0$ s.t.

$$S := (\gamma^2 - D^*D) + \operatorname*{s-lim}_{\alpha\to+\infty} B_{\mathrm{w}}^*\mathcal{P}(\alpha - A)^{-1}B \gg 0, \quad \text{and} \tag{10.66}$$

$$(B_{\mathrm{w}}^*\mathcal{P} - D^*C)^*S^{-1}(B_{\mathrm{w}}^*\mathcal{P} - D^*C) \ll A^*\mathcal{P} + \mathcal{P}A - C^*C. \tag{10.67}$$

We always require that $\mathcal{P} \in \mathcal{B}(H)$. As in Definition 9.1.5, conditions (iii) and (iii') include the requirements that the terms are defined (in particular, that the limits converge strongly; as in Remark 9.1.6, it follows that $\mathcal{P}[H_B] \subset \mathrm{Dom}(B_{\mathrm{w}}^*)$). If B is bounded, then $B_{\mathrm{w}}^* = B^*$ and $\mathrm{Dom}(B_{\mathrm{w}}^*) = H$, so that then ((a) applies and) (ii) becomes essentially simpler.

Recall that "$C^*JC \le 0$" means that $\langle x_0, C^*JCx_0\rangle_{\langle H_1,(H_1)^*\rangle} \le 0$ for all $x_0 \in H_1 := \mathrm{Dom}(A)$. The phrase "on $\mathrm{Dom}(A) \times U$" refers to the inner product $\langle \left[\begin{smallmatrix} x_1 \\ u_1 \end{smallmatrix}\right], \left[\begin{smallmatrix} x_2 \\ u_2 \end{smallmatrix}\right]\rangle := \langle x_1, x_2\rangle_{H_1,H_{-1}^*} + \langle u_1, u_2\rangle_U$, since H is the pivot space and $H_1 := \mathrm{Dom}(A)$.

Note that (10.65) (resp. (10.66)&(10.67)) is an "inequality form" of the B_{w}^*-CARE (resp. of the (strongly regular) CARE). Hypothesis 9.5.1 is stronger than the assumption of (b), and $\mathbb{D}$ is ULR whenever the assumptions of (a) or (b) hold.

See Theorem 10.6.5(e) (with Σ_{aug} and J_{aug} in place of Σ and J, i.e., with substitutions $\widetilde{C} = \begin{bmatrix} C \\ 0 \\ I \end{bmatrix}$, $\widetilde{D} = \begin{bmatrix} D \\ I \\ 0 \end{bmatrix}$, $\widetilde{J} = \mathrm{diag}(-I, \gamma^2, \varepsilon)$), for alternative regularity assumptions.

Note that we could also have the real lemmas of this section based on IAREs instead of B_{w}^*-CAREs or CAREs, to make them look like their discrete-time counterparts.

If one replaces J by $-J$ in the proof, the operators $\mathcal{P} \le 0$ and $S \gg 0$ are replaced by $-\mathcal{P} \ge 0$ and $S \ll 0$, so that the condition (ii) becomes "$\|D\| < \gamma$, and there is $\mathcal{P} \ge 0$ s.t. $\mathcal{P}[H] \subset \mathrm{Dom}(B_{\mathrm{w}}^*)$ and

$$-(B_{\mathrm{w}}^*\mathcal{P} + D^*C)^*(\gamma^2 I - D^*D)^{-1}(B_{\mathrm{w}}^*\mathcal{P} + D^*C) \ll A^*\mathcal{P} + \mathcal{P}A + C^*C\text{"}; \tag{10.68}$$

(special cases of) this condition is common in the literature, but we have made the choice that leads to a positive Popov operator and to the setting of Section 10.6. Obviously, analogous changes can be made to the other conditions (multiply the inequalities by -1 and replace $\mathcal{P}$ by $-\mathcal{P}$).

Proof of Theorem 10.5.1: 1° Define $J_{\mathrm{aug}} := \begin{bmatrix} -I & 0 \\ 0 & \gamma^2 I \end{bmatrix}$, $\mathbb{C}_{\mathrm{aug}} := \begin{bmatrix} \mathbb{C} \\ 0 \end{bmatrix}$, $\mathbb{D}_{\mathrm{aug}} := \begin{bmatrix} \mathbb{D} \\ I \end{bmatrix}$ and $\Sigma_{\mathrm{aug}} := \left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline \mathbb{C}_{\mathrm{aug}} & \mathbb{D}_{\mathrm{aug}} \end{array}\right]$.

Then $\|\mathbb{D}\|_{\rm TIC} < \gamma$ iff $\mathbb{D}_{\rm aug}^* J_{\rm aug} \mathbb{D}_{\rm aug} = \gamma^2 I - \mathbb{D}^*\mathbb{D} \gg 0$. Moreover, $\Sigma_{\rm aug}$ is exponentially stable iff Σ is, by Lemma 6.1.10(a1), and $C_{\rm aug}^* J_{\rm aug} C_{\rm aug} = -C^*C \leq 0$ (and $D_{\rm aug}^* J_{\rm aug} C_{\rm aug} = -D^*C$ and $D_{\rm aug}^* J_{\rm aug} D_{\rm aug} = \gamma^2 I - D^*D$ whenever $\mathbb{D}$ is regular). Consequently, the theorem follows directly from Theorem 10.6.5, by 2°.

2° *The assumptions of (a) imply those of (a) or (e) of Theorem 10.6.5 for* $\Sigma_{\rm aug}$*:* Obviously, (1.) or (2.) or (5.) of Hypothesis 9.2.2 for Σ implies that for $\Sigma_{\rm aug}$.

Assume then that C is bounded. Case 1. was given in Theorem 10.6.5(e). Cases 2., 3., and 4. (with (i)) imply (4.), (3.) and (6.) of Hypothesis 9.2.2, respectively, because (i) implies that $\|D\| < \gamma$ (hence that $\gamma^2 - D^*D \gg 0$), by Lemma 6.3.2(e) (thus, we need Theorem 10.6.5(e) in case 4.). □

Note that the spectral factorization $\gamma^2 I - \mathbb{D}^*\mathbb{D} = \mathbb{X}^* S \mathbb{X}$ is equivalent to the normalized factorization $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$, $\gamma^2 \mathbb{M}^*\mathbb{M} - \mathbb{N} * \mathbb{N} = S$, $\mathcal{B}(U) \ni S \gg 0$, $\mathbb{M} := \mathbb{X}^{-1} \in \mathcal{G}\mathrm{TIC}(U)$, $\mathbb{N} := \mathbb{D}\mathbb{M} \in \mathrm{TIC}(U,Y)$.

When one wishes to find an estimate for $\mathbb{D}$ without requiring Σ to be exponentially stable, one should use the proposition below instead of the above theorem (and one should know, a priori, that $\mathbb{C}$ is stable).

Proposition 10.5.2 (Nonexp. $\|\mathbb{D}\|_{\rm TIC} < \gamma$) *Assume that $\mathbb{D}$ is SR and $\gamma > 0$.*

If (ii) or (iii) holds, then $\mathbb{D} \in \mathrm{TIC}$ and $\|\mathbb{D}\| \leq \gamma$.

Conversely, if $\Sigma \in \mathrm{SOS}$, $\|\mathbb{D}\| < \gamma$, and $\Sigma_{\rm aug}$, $J_{\rm aug}$ satisfy (2.) (resp. (6.)) of Hypothesis 10.6.1, then (iii) (resp. (iii) and (ii)) holds (also with "$=$" in place of "$\geq$").

Here we have referred to the following conditions:

(ii) $\|D\| < \gamma$*, and there is $\mathcal{P} \leq 0$ s.t. $\mathcal{P}[H] \subset \mathrm{Dom}(B_{\rm w}^*)$ and*

$$\begin{bmatrix} A^*\mathcal{P} + \mathcal{P}A - C^*C & (B_{\rm w}^*\mathcal{P} - D^*C)^* \\ B_{\rm w}^*\mathcal{P} - D^*C & \gamma^2 I - D^*D \end{bmatrix} \geq 0 \quad on \quad \mathrm{Dom}(A) \times U. \tag{10.69}$$

(iii) There is $\mathcal{P} \leq 0$ s.t. $S := \gamma^2 I - D^*D + \text{s-lim}_{s\to+\infty} B_{\rm w}^*\mathcal{P}(s-A)^{-1}B \gg 0$, and

$$\begin{bmatrix} A^*\mathcal{P} + \mathcal{P}A - C^*C & (B_{\rm w}^*\mathcal{P} - D^*C)^* \\ B_{\rm w}^*\mathcal{P} - D^*C & S \end{bmatrix} \geq 0 \quad on \quad \mathrm{Dom}(A) \times U. \tag{10.70}$$

Moreover, we have (ii)⇔(ii')⇒(iii)⇔(iii'), where (ii') and (iii') are from Theorem 10.5.1(d) with "$\leq$" in place of "$\ll$".

If $\mathbb{B}$ is strongly stable, then we can replace "$\mathcal{P} \leq 0$" by $\mathcal{P} = \mathcal{P}^*$ everywhere in this proposition.

(See the proof of Theorem 10.5.1 for $\Sigma_{\rm aug}$ and $J_{\rm aug}$. Note that here we may have $\mathcal{P} = 0$ (take $\Sigma = 0$) whereas $\mathcal{P} < 0$ in the theorems of this section.)

Thus, if $\mathbb{C}$ is stable, $\mathbb{D}$ is SR, and any of (1.)–(10.) (resp. (1.)–(8.)) of Lemma 10.6.2(c) (with $D^*JC \mapsto D^*C$ and $D^*JD \mapsto \gamma^2 - D^*D$) holds, then Hypothesis 10.6.1(2.) (resp. (6.)) holds, and we can estimate $\|\mathbb{D}\|$ as follows:

Take some $\gamma > 0$, and then verify, whether the Riccati inequality condition (iii) (resp. (ii)) has any solutions. If so, then $\|\mathbb{D}\| \leq \gamma$, otherwise $\|\mathbb{D}\| \geq \gamma$. Then vary

γ and find an estimate for $\|\mathbb{D}\|$ by, e.g., a binary search. (Also the corresponding Riccati equation (that is, "$=$" in place of "$\geq$") can be used.)

Proof of Proposition 10.5.2: The proof of Theorem 10.5.1 applies here too, with Proposition 10.6.4 in place of Theorem 10.6.5.

(If $\mathbb{B}^t$ is strongly stable, then, in the proof of Proposition 10.6.4(d), we have $\gamma^2 I - \mathbb{D}^{t\,*}\mathbb{D}^t \geq -\mathbb{B}^{t\,*}\mathcal{P}\mathbb{B}^t \to 0$, which implies that $\mathbb{D} \in \mathrm{TIC}$ and $\|\mathbb{D}\| \leq \gamma$. Therefore, we do not have to assume $\mathbb{D}$ to be stable for the last claim unlike in Proposition 10.6.4(d).)

Remark: As noted in the proof of Proposition 10.6.4(b), in the converse claim we can choose $\mathcal{P} \leq 0$ so that it is P-SOS-r.c.-stabilizing (also with "$=$" in place of "$\geq$" in (iii) (resp. in (iii) and (ii)). □

In classical literature, an operator $\mathbb{D} \in \mathrm{TIC}(U)$ is called *strictly positive* if $\mathrm{Re}\langle \mathbb{D}\cdot,\cdot\rangle \gg 0$, i.e., if $\mathbb{D}+\mathbb{D}^* \gg 0$ (one can show that $\sigma(\mathbb{D}) \subset \mathbf{C}^+$ is a necessary condition; it is sufficient for normal $\mathbb{D}$). An equivalent condition is that $\widehat{\mathbb{D}}+\widehat{\mathbb{D}}^* \geq \varepsilon I$ in $\mathrm{L}^\infty_{\mathrm{strong}}(i\mathbf{R};\mathcal{B}(U))$ for some $\varepsilon > 0$. We use this definition in the following generalized extension of the classical Strictly Positive Real Lemma:

Theorem 10.5.3 (Generalized Strictly Positive (Real) Lemma)

(a) If C is bounded and $\dim Y < \infty$, or if (1.) or (2.) or (5.) of Hypothesis 9.2.2 holds, then the following are equivalent:

(i) Σ is exponentially stable and $\mathbb{D}$ is strictly positive;

(ii) There is $\mathcal{P} \leq 0$ s.t. $\mathcal{P}[H] \subset \mathrm{Dom}(B_{\mathrm{w}}^*)$ and

$$\begin{bmatrix} A^*\mathcal{P}+\mathcal{P}A & (B_{\mathrm{w}}^*\mathcal{P}+C)^* \\ B_{\mathrm{w}}^*\mathcal{P}+C & D+D^* \end{bmatrix} \gg 0 \quad on \quad \mathrm{Dom}(A)\times U. \tag{10.71}$$

(iii) There is $\mathcal{P} \leq 0$ s.t. $S := D+D^* + \text{s-lim}_{s\to+\infty} B_{\mathrm{w}}^*\mathcal{P}(s-A)^{-1}B$ exists and

$$\begin{bmatrix} A^*\mathcal{P}+\mathcal{P}A & (B_{\mathrm{w}}^*\mathcal{P}+C)^* \\ B_{\mathrm{w}}^*\mathcal{P}+C & S \end{bmatrix} \gg 0 \quad on \quad \mathrm{Dom}(A)\times U. \tag{10.72}$$

(b) If $\pi_{[0,t)}\mathbb{A}B \in \mathrm{L}^1([0,t);\mathcal{B}(U,H))$, $\pi_{[0,t)}C_{\mathrm{w}}\mathbb{A} \in \mathrm{L}^1([0,t);\mathcal{B}(H,Y))$, and $\pi_{[0,t)}C_{\mathrm{w}}\mathbb{A}B \in \mathrm{L}^1([0,t);\mathcal{B}(U,Y))$ for some $t > 0$, then (i)$\Leftrightarrow$(iii)$\Leftarrow$(ii).

(c1) If $\mathbb{D}$ is ULR, then we have (i)$\Leftarrow$(iii)$\Leftarrow$(ii).

(c2) Any solution of (ii), (iii), (ii') or (iii') is strictly negative ($\mathcal{P} < 0$). Under the assumptions of (a), there is an exponentially stabilizing solution (if (i) holds).

(d) If $\mathbb{D}$ is SR, then we have (ii)$\Leftrightarrow$(ii')$\Rightarrow$(iii)$\Leftrightarrow$(iii'), where

(ii') $D+D^* \gg 0$, and there is $\mathcal{P} \leq 0$ s.t. $\mathcal{P}[H] \subset \mathrm{Dom}(B_{\mathrm{w}}^*)$ and

$$(B_{\mathrm{w}}^*\mathcal{P}+C)^*(D+D^*)^{-1}(B_{\mathrm{w}}^*\mathcal{P}+C) \ll A^*\mathcal{P}+\mathcal{P}A \tag{10.73}$$

(iii') There is $\mathcal{P} \le 0$ s.t.

$$S := D + D^* + \operatorname*{s-lim}_{\alpha\to+\infty} B_{\mathrm{w}}^* \mathcal{P}(\alpha - A)^{-1} B \gg 0, \quad \textit{and} \tag{10.74}$$

$$(B_{\mathrm{w}}^* \mathcal{P} + C)^* S^{-1} (B_{\mathrm{w}}^* \mathcal{P} + C) \ll A^* \mathcal{P} + \mathcal{P} A. \tag{10.75}$$

□

(The proof of Theorem 10.5.1 applies mutatis mutandis, with $J_{\mathrm{aug}} := \left[\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right]$ (so that $\mathbb{D}_{\mathrm{aug}}^* J_{\mathrm{aug}} \mathbb{D}_{\mathrm{aug}} = \mathbb{D} + \mathbb{D}^*$). Also the comments below Theorem 10.5.1 apply, mutatis mutandis.)

We may also use a binary search for finding an estimate for supremal $\gamma > 0$ s.t. $\mathrm{Re}\langle \mathbb{D}\cdot, \cdot\rangle \ge \gamma^2 I$; then a positive real -variant of Proposition 10.5.2 applies (such a result holds with the same proof, mutatis mutandis).

We can again rewrite the conditions for $\mathcal{P} \mapsto -\mathcal{P} \ge 0$, $S \mapsto -S \ll 0$ and $J \mapsto -J$; e.g., (ii') becomes: $D + D^* \gg 0$, and there is $\mathcal{P} \ge 0$ s.t. $\mathcal{P}[H] \subset \mathrm{Dom}(B_{\mathrm{w}}^*)$ and

$$-(B_{\mathrm{w}}^* \mathcal{P} - C)^* (D + D^*)^{-1} (B_{\mathrm{w}}^* \mathcal{P} - C) \ll A^* \mathcal{P} + \mathcal{P} A. \tag{10.76}$$

Notes for Sections 10.5 and 10.6

In Section 8 of [S98b], it was shown that if "(i)" holds and the corresponding spectral factor is sufficiently regular, then "(iii)" holds; this applies to both "real lemmas".

The results of this section generalize most analogous results, including Theorem 3.7.1 and Problem 3.25 of [GL] (finite-dimensional case), Section 4.5 of [Oostveen] (the strongly stabe case with bounded B and C), and Remark 3.14 of [Keu] (exponentially stable Pritchard–Salamon systems); all these require one to assume, a priori, that Σ is strongly stable, and to check whether the a solution is stabilizing. One obtains such lemmas by using Theorem 10.6.3 (whose conditions (iv) and (v) are popular in the literature) instead of Theorem 10.6.5 in our proofs.

An exception to this is the Strict Bounded Real Lemma given in Section 7.1 of [IOW], which equals Theorem 10.5.1(a) restricted to finite-dimensional systems. Analogously, Theorem 10.6.5 is the generalization of Theorems 4.6.1–4.6.2 of [IOW]. In both cases, the proofs of [IOW] cannot be extended, because $\mathcal{P} < 0$ does not imply that $\mathcal{P} \ll 0$ when $\dim H = \infty$. The strength of our results reflect the power of the integral notation ($\left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right]$ and the IARE instead of $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ and the CARE), which allows one to observe connections not visible from the generator notation. Similar remarks apply to corresponding discrete-time results (Section 15.4).

Propositions 10.5.2 and 10.6.4 might be new even for finite-dimensional systems. See Chapter 5 for notes on (i)–(ii") of Theorem 10.6.3(a); also the necessity of the existence of a solution to the CARE is well known [WW] [S97b].

10.6 Positive Popov operators ($0 \ll \mathbb{D}^* J\mathbb{D} = \mathbb{X}^*\mathbb{X}$)

Positive, *adj.:*
Mistaken at the top of one's voice.
— Ambrose Bierce (1842–1914), "The Devil's Dictionary"

In this section, we present necessary and sufficient conditions for the uniform positivity of the Popov operator (i.e., for $\mathbb{D}^* J\mathbb{D} \geq \varepsilon I$ for some $\varepsilon > 0$; equivalently, $\widehat{\mathbb{D}}^* J\widehat{\mathbb{D}} \geq \varepsilon I$ in $\mathrm{L}^\infty_{\text{strong}}(i\mathbf{R}; \mathcal{B}(U))$), in terms of [regular] spectral factorizations and Riccati equations or Riccati inequalities. In the finite-dimensional setting, the connection between these concepts is rather simple; the same holds in the infinite-dimensional setting with bounded input and output operators. Therefore, to have an nice overview of the contents of this section, the reader might first wish to read our corresponding discrete-time results, namely Lemma 15.3.1, Proposition 15.3.2 and Theorem 15.3.3.

When working with stable finite-dimensional or Pritchard–Salamon systems, or with stable WPLSs having $\mathbb{D} \in$ MTIC, the uniform positivity of the Popov operator ($\mathbb{D}^* J\mathbb{D} \gg 0$) implies that it has a ULR spectral factorization (by Lemma 10.6.2(b)). For general (or even for ULR) stable WPLSs, this is not the case, by Proposition 9.13.1(c1), and therefore we must sometimes replace the CARE theory by the IARE theory.

To overcome this difficulty, we often assume that $\mathbb{D} \in$ MTIC or that Σ is otherwise sufficiently regular; that is, we assume some of the six alternative hypotheses below:

Hypothesis 10.6.1 ($\mathbb{D}$ admits positive regular SpF) *We have* $\mathbb{D} \in \mathrm{WR} \cap \mathrm{TIC}$, *and if some* $\mathbb{X} \in \mathcal{G}\mathrm{TIC}(U)$ *satisfies* $\mathbb{X}^*\mathbb{X} = \mathbb{D}^* J\mathbb{D}$*, then*

(1.) $\mathbb{X} \in \mathrm{WR}$ *and* $X \in \mathcal{G}\mathcal{B}(U)$.

(2.) $\mathbb{X} \in \mathrm{SR}$ *and* $X \in \mathcal{G}\mathcal{B}(U)$.

(3.) $\mathbb{X} \in \mathrm{UR}$.

(4.) $\mathbb{X} \in \mathrm{ULR}$.

(5.) $\mathbb{X} \in \mathrm{ULR}$ *and* $X^*X = D^*JD$.

(6.) the B_{w}^**-CARE has a stable P-I/O-stabilizing solution. Moreover,* $\Sigma \in \mathrm{SOS}$ *and* $\mathbb{D} \in \mathrm{ULR}$.

Note that for any $\mathbb{D} \in \mathrm{TIC}(U,Y)$, we have $\mathbb{X}^*\mathbb{X} = \mathbb{D}^* J\mathbb{D}$ for some $\mathbb{X} \in \mathcal{G}\mathrm{TIC}(U)$ iff $\mathbb{D}^* J\mathbb{D} \gg 0$, by Lemma 6.4.7(a). Conditions (1.)–(5.) of the above hypothesis just require that $\mathbb{D}$ and $\mathbb{X}$ (if any) are regular (by Lemma 6.4.5(a), all possible $\mathbb{X}$'s differ by an unitary constant, hence they are equally regular).

By Theorem 10.6.3(f1)&(i)&(iii)&(d), condition (1.) implies for any $\Sigma \in \mathrm{SOS}$ that the CARE has a unique stable P-I/O-stabilizing solution $\mathcal{P} \geq 0$; condition (6.) just says that this solution must also solve the B_{w}^*-CARE, i.e., that $\mathcal{P}[H] \subset \mathrm{Dom}(B_{\mathrm{w}}^*)$.

Next we list sufficient conditions for the above hypotheses (see Standing Hypothesis 10.6.6 for $\widetilde{\mathcal{A}}_+$):

Lemma 10.6.2

(a1) We have (6.)⇒(5.)⇒(4.)⇒(3.)⇒(2.)⇒(1.) in Hypothesis 10.6.1.

(a2) If $\dim U < \infty$*, then (1.)–(3.) of Hypothesis 10.6.1 are equivalent.*

(b) If $\mathbb{D} \in \widetilde{\mathcal{A}}_+$*, then* $\mathbb{D}$ *satisfies (1.)–(4.) of Hypothesis 10.6.1.*

(c) Assume that $\Sigma \in \mathrm{SOS}$ *and at least one of conditions (1.)–(10.) below holds. Then* $\mathbb{D}$ *satisfies Hypothesis 10.6.1(1.)–(5.).*

(1.) B is bounded (i.e., $B \in \mathcal{B}(U,H)$*);*
(2.) **(Analytic** $\mathbb{A}$**)** *Hypotheses 9.5.1 and 9.5.7 hold.*
(3.) $C = \begin{bmatrix} C_1 \\ 0 \end{bmatrix} \in \mathcal{B}(H, Y_1 \times Y_2)$, $\dim Y_1 < \infty$*, and* $\mathbb{B}$ *is stable;*
(4.) $\mathbb{A}B \in \mathrm{L}^1(\mathbf{R}_+; \mathcal{B}(U,H))$ *and* $C \in \mathcal{B}(H,Y)$*;*
(5.) $\mathbb{A}B \in \mathrm{L}^1([0,1]; \mathcal{B}(U,H)), C \in \mathcal{B}(H,Y)$ *and* $\mathbb{A}$ *is exponentially stable;*
(6.) C is bounded, $D^*JC = 0$*, and* $\mathbb{D} \in \mathcal{B}(U,Y) + \mathcal{B}(U, \mathrm{L}^1(\mathbf{R}_+;Y))*$*;*
(7.) $\mathbb{D} \in \mathcal{B}(U,Y) + \mathcal{B}(U, \mathrm{L}^2(\mathbf{R}_+;Y))*$*;*
(8.) Hypothesis 9.2.1 (or 9.2.2) is satisfied and $D^*JD \gg 0$*;*
(9.) $\mathbb{D} \in \mathrm{MTIC}^{\mathrm{L}^1}$*;*
(10.) **(Analytic** $\mathbb{A}$**)** *Hypothesis 9.5.1 holds and* $\mathbb{A}$ *is exponentially stable.*

(d) Assume that Σ *is stable and that at least one of conditions (1.)–(8.) above holds. Then* Σ *satisfies Hypothesis 10.6.1(1.)–(6.).*

Assumptions (1.)–(7.) of (c) are roughly the stable versions of (1.)–(7.) of Hypothesis 9.2.2.

Proof: (a1) Trivially, (5.)⇒(4.)⇒(3.), and (2.)⇒(1.) By Theorem 10.6.3(b)(ii)&(iii), (6.) implies (5.). By Proposition 6.3.1(b1), (3.) implies (2.).

(a2) Use (a1) and Lemma 6.3.2(a1)&(a2).

(b) This follows from Standing Hypothesis 10.6.6.

(c) 1° *Cases (9.) and (10.):* We have $\mathbb{D} \in \mathrm{MTIC}^{\mathrm{L}^1}$, by Lemma 9.5.2. Therefore, this follows from Theorem 8.4.9 (and $\mathbb{X} \in \mathrm{MTIC}^{\mathrm{L}^1} \subset \mathrm{UHPR}$).

2° *Cases (1.)–(8.) & (d):* (N.B. We do not know whether Hypothesis 9.2.2 is sufficient without the assumption $D^*JD \gg 0$. As shown below, $\mathbb{D}^*J\mathbb{D} \gg 0 \Rightarrow D^*JD \gg 0$ under any of (1.)–(8.).)

By Proposition 9.2.4, (3.) implies (7.), i.e., that $\widehat{\mathbb{D}} \in \mathrm{H}^2_{\mathrm{strong}}(\mathbf{C}^+; \mathcal{B}(U,Y))$. By Lemma 6.8.3(a), (5.) implies (4.). Therefore, any of (1.)–(8.) implies that Hypothesis 9.2.1 holds (see Hypothesis 9.2.2 and Theorem 9.2.3).

By Theorem 6.9.1(d2), we have $\widehat{\mathbb{D}} \in \mathrm{H}^2_{\mathrm{strong}} \cap \mathrm{H}^\infty(\mathbf{C}^+; \mathcal{B}(U,Y))$ in (1.); the same holds in (7.), hence in (3.) too. In case (2.), we have $\mathbb{D} \in \mathrm{UHPR} \subset \mathrm{SHPR}$, by Lemma 9.6.3. In cases (4.), (5.) and (6.), we have $\mathbb{D} \in \mathrm{SMTIC}^{\mathrm{L}^1} \subset \mathrm{SHPR}$, by Theorem 2.6.4(h1).

Assume that $\mathbb{D}^*J\mathbb{D} \gg 0$, i.e., that $\mathbb{D}^*J\mathbb{D} \geq \varepsilon I$ for some $\varepsilon > 0$. Then $D^*JD \geq \varepsilon I$, by Lemma 6.3.5 (cases (1.), (3.) and (7.)) or by Lemma 6.3.6(b) (cases (2.), (4.), (5.) and (6.)) or by assumption (case (8.)). By Lemma 6.4.7(a), we have $\mathbb{D}^*J\mathbb{D} = \mathbb{X}^*\mathbb{X}$ for some $\mathbb{X} \in \mathcal{G}\mathrm{TIC}(U)$, Therefore, there is a J-critical, strictly minimizing, stable, SOS-stabilizing state feedback pair $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ for Σ over $\mathcal{U}_{\rm out}$, by Corollary 9.9.11(Crit1SOS)&(Crit4SOS), and $\mathbb{F} = I - \mathbb{X}$, by (9.140).

Thus, we can apply Theorem 9.2.9(v)&(iii) to obtain that the $B_{\rm w}^*$-CARE has a solution $(\mathcal{P}, S, \left[\begin{array}{c|c} \widetilde{\mathbb{K}} & \widetilde{\mathbb{F}} \end{array}\right])$ with $S = D^* JD \gg 0$, $\widetilde{\mathbb{X}} := I - \widetilde{\mathbb{F}} \in$ ULR and $\widetilde{X} = I$. By Theorem 9.9.1(a1)&(f2), we have $\widetilde{\mathbb{X}} = E\mathbb{X}$ for some $E \in \mathcal{GB}(U)$ (hence for $E = X^{-1}$). Consequently, $\widetilde{\mathbb{X}} \in \mathcal{G}\mathrm{TIC}(U)$.

It follows that $\widetilde{\mathbb{N}} := \mathbb{D}\widetilde{\mathbb{X}}^{-1}$ is stable, hence $\widetilde{\mathbb{N}}^* J \widetilde{\mathbb{N}} = S$, by Theorem 9.9.1(g2), so that $\mathbb{D}^* J\mathbb{D} = \widetilde{\mathbb{X}}^* S \widetilde{\mathbb{X}}$. By Lemma 6.4.5(a), it follows that $S = (E^*)^{-1} I E^{-1} = X^* X$. Thus, $D^* JD = S = X^* X$, as required.

3° *Remarks:* The assumption that $\mathbb{C}$ is stable is superfluous (for (c)) in (4.), (5.), (9.) and (10.) and the assumption that $\mathbb{D}$ is stable is redundant in (4.), (5.), (6.), (9.) and (10.).

We note that $\mathbb{X} \in \mathcal{G}\mathrm{MTIC}^{\mathrm{L}^1}$ in cases (4.), (5.), (9.) and (10.) (by 1°), and $\mathbb{X}, \mathbb{X}^{-1} \in \mathcal{B} + \mathrm{H}^\infty \cap \mathrm{H}^2_{\rm strong}(\mathbf{C}^+; \mathcal{B}(U))$ in cases (1.), (3.) and (7.). The latter claim follows from Theorem 4.1.6(j), and from the fact that $\mathbb{D}$ has a stable realization with a bounded B, by Theorem 6.9.1(a)&(d2) and Corollary 6.9.7, so that we can obtain $\mathbb{X}$ for that realization instead of Σ (note that (1.)–(5.) of Hypothesis 10.6.1 depend on $\mathbb{D}$ and J only).

(d) Part "if" from the last claim follows from 2° above; and "only if" from Proposition 9.8.11. □

In fact, the solution of the $B_{\rm w}^*$-CARE in Hypothesis 10.6.1(6.) is actually stable and P-SOS-r.c.-stabilizing, and we have the classical equivalence between positive J-coercivity, I-spectral factorization and Riccati equations in this generality too:

Theorem 10.6.3 ($\mathbb{D}^* J\mathbb{D} \gg 0 \Leftrightarrow$ SpF $\Leftrightarrow$ CARE) *Assume that Σ is SOS and ULR.*

(a) If Hypothesis 10.6.1(6.) holds and Σ is strongly stable, then (i)–(iv') are equivalent (and (v) if Σ is exponentially stable).

- *(i)* $\mathbb{D}^* J\mathbb{D} \gg 0$;
- *(i')* $\mathbb{D}^* J\mathbb{D} \gg 0$ *and* $D^* JD \in \mathcal{GB}(U)$;
- *(ii)* $\mathbb{D}^* J\mathbb{D} = \mathbb{X}^* S\mathbb{X}$ *for some* $\mathbb{X} \in \mathcal{G}\mathrm{TIC}(U)$ *and* $S \gg 0$;
- *(ii')* $\mathbb{D}^* J\mathbb{D} = \mathbb{X}^* \mathbb{X}$ *for some* $\mathbb{X} \in \mathcal{G}\mathrm{TIC}(U)$;
- *(ii")* $\mathbb{D}^* J\mathbb{D} = \mathbb{X}^* \mathbb{X}$ *for some* $\mathbb{X} \in \mathcal{G}\mathrm{TIC}(U)$*, and* $\mathbb{D}, \mathbb{X} \in$ ULR *and* $D^* JD = X^* X \gg 0$;
- *(iii) the $B_{\rm w}^*$-CARE (or CARE or IARE) has a stable P-I/O-stabilizing solution with* $S \gg 0$;
- *(iii') the $B_{\rm w}^*$-CARE has a stable P-SOS-r.c.-stabilizing with* $S \gg 0$;
- *(iv) the $B_{\rm w}^*$-CARE (or CARE or IARE) has a solution with* $S \gg 0$ *and* $\mathbb{M} \in$ TIC;
- *(iv') the $B_{\rm w}^*$-CARE has a strongly stabilizing solution with* $S \gg 0$;
- *(iv") the $B_{\rm w}^*$-CARE has a stable strongly r.c.-stabilizing solution with* $S \gg 0$;

(v) the B_{w}^*-CARE has an exponentially stabilizing solution with $S \gg 0$.

(b) If Hypothesis 10.6.1(6.) holds, then (i)–(iii') are equivalent.

(c1) We have (iii')⇒(iii)⇒(ii')⇔(ii)⇔(i), and (iv)⇐(iv')⇐(iv")⇒(iii')⇒(ii")⇒(i')⇒(i).

(c2) If Σ is strongly stable, then (iii')⇔(iv')⇔(iv"), and (iii)⇔(iv).

(d) The solutions of the B_{w}^*-CARE mentioned in (a)–(b) are unique and equal, and they solve (ii) (and (ii') and (ii")) if we replace $\mathbb{X}$ by $S^{1/2}\mathbb{X}$).

(e) If Σ is strongly stable and the eIARE has a solution with $S \geq 0$, then $\mathbb{D}^* J\mathbb{D} \geq 0$.

(f1) **(CARE)** *Replace "(6.)" by "(1.)", and remove (i') and (ii") and "ULR", and replace "B_{w}^*-CARE" by "CARE", everywhere in this theorem. Then (a)–(e) still hold.*

(f2) **(General WPLSs)** *Remove (i'), (ii"), "ULR" and Hypothesis 10.6.1(6.), and replace "B_{w}^*-CARE" by "IARE", everywhere in this theorem. Then (a)–(e) still hold.*

The proposition provides us necessary and sufficient conditions for (i) (i.e., for the positive J-coercivity over $\mathcal{U}_{\mathrm{out}}$) under different stability and regularity assumptions. Such conditions are needed for positive and bounded real lemmas and for minimization problems; the reader can find here additional equivalent conditions for those results under same or different assumptions.

Recall that $S := D^*JD$ for the B_{w}^*-CARE (but not necessarily for the CARE or IARE), and that any solution of the B_{w}^*-CARE (and any WR solution of the eCARE) is a solution of the eIARE.

In the unstable case, we have three alternatives for minimization: 1. If Σ is regular enough, we may use the B_{w}^*-CARE results of Section 9.2. 2. If Σ is stabilizable with closed-loop system $\Sigma_\flat$ as in Hypothesis 10.6.1(6.) (or (1.)), we may combine the above result and Proposition 9.12.4 (cf. Theorem 10.2.14(b1)&(b2)). 3. In the general case, we have to be satisfied with results such as Theorem 10.2.11 and Corollaries 10.2.5(a), 10.2.6 and 10.2.12.

Proof of Theorem 10.6.3: (a) Trivially, (v) implies (iv'); the converse (for exponentially stable Σ) follows from Corollary 6.6.9 (in fact, $\mathcal{P}$ is then exponentially stable and exponentially r.c.-stabilizing). The rest follows from (b) and (c2).

(b) 1° *(i)⇒(ii')&(iii'):* Assume (i), so that $X^*X \gg 0$ and the B_{w}^*-CARE has a P-I/O-stabilizing solution $(\mathcal{P}, S, K)$ s.t. $\mathbb{X}_K = (X^*X)^{-1/2}\mathbb{X} \in \mathcal{G}\mathrm{TIC}(U)$, where $\widehat{\mathbb{X}}_K := I - K_{\mathrm{w}}(\cdot - A)^{-1}B$, $\mathbb{X} \in \mathcal{G}\mathrm{TIC}(U) \cap \mathrm{ULR}$, $X^*X \gg 0$ and $\mathbb{X}^*\mathbb{X} = \mathbb{D}^*J\mathbb{D}$, by the hypothesis. Consequently, (iii') holds, by Proposition 9.8.11.

Since $\widehat{\mathbb{X}}_K \in \mathrm{ULR}$, we have $\mathbb{X} \in \mathrm{ULR}$ and hence $X \in \mathcal{GB}(U)$. From (9.162), we obtain that $\langle (X^*X)^{1/2}u, (X^*X)^{1/2}v\rangle = \langle u, Sv\rangle$ for all $u, v \in \mathrm{L}^2_{\mathrm{c}}$ (since $\mathbb{D}_{\circlearrowleft} = \mathbb{D}\mathbb{X}_K^{-1} = \mathbb{D}\mathbb{X}^{-1}(X^*X)^{1/2}$), hence $X^*X = S$; in particular, $S \gg 0$. But $S = D^*JD$, hence $D^*JD = X^*X \gg 0$, so that (ii') holds.

2° The rest of the equivalence follows from (c1).

(c1) Implications "(iii')⇐(iv")⇒(iv')⇒(iv)", "(iii')⇒(iii)", "(ii')⇒(ii)", and "(i')⇒(i)" are trivial, and "(ii')⇒(i')" is obvious. Equivalence "(i)⇔(ii)" follows from Lemma 6.4.7(a), We obtain "(iii)⇒(ii)" from Proposition 9.8.11 (in particular, a stable P-$\mathbb{M}$-stabilizing solution suffices) (and Proposition 9.2.7(b)).

Finally, we obtain "(iii')⇒(ii')" from Proposition 9.2.7(b) and Proposition 9.8.11(c), by replacing $\mathbb{X}$ by $S^{1/2}\mathbb{X}$.

(c2) This follows from Proposition 9.8.11(b).

(d) This follows from the above proofs (especially of that of (c1)).

(e) By (9.160), we have $\langle \mathbb{D}^t u, J\mathbb{D}^t u\rangle \geq -\langle \mathbb{B}^t u, \mathcal{P}\mathbb{B}^t u\rangle \to 0$, as $t \to +\infty$, because $\mathbb{B}$ is strongly stable (by Lemma 6.1.13), hence $\mathbb{D}^* J\mathbb{D} \geq 0$. (Note that whenever $S \geq 0$, $\mathcal{P} \leq 0$, we have $\langle \mathbb{D}^t u, J\mathbb{D}^t u\rangle \geq 0$ for all t, so that $\mathbb{D}^* J\mathbb{D} \geq 0$ if $\mathbb{D}$ is stable.)

(f1) This follows as above (or from (f2)) (note from Proposition 10.7.2 that any solution of the CARE with $S \gg 0$ is a WR solution of the IARE and from Proposition 10.7.1 that such a solution is stable if Σ is).

(f2) Also this follows as above. □

For a SOS-stable system $\left[\begin{array}{c|c}\mathbb{A} & \mathbb{B}\\ \hline \mathbb{C} & \mathbb{D}\end{array}\right]$ with sufficient regularity, condition (ii) below implies that $\mathbb{D}^* J\mathbb{D} \geq 0$, and, conversely, $\mathbb{D}^* J\mathbb{D} \gg 0$ implies that (ii) holds (by (b) below). For several applications, such as the "strict bounded real lemma" of Proposition 10.5.2, this "almost equivalence" is in practice as good as an equivalence.

Proposition 10.6.4 ($\mathbb{D}^* J\mathbb{D} \geq 0$ ⇔ CARI) *Assume that $\mathbb{D}$ is SR and $C^*JC \leq 0$. Then we have (ii)⇒(iii)⇒(i)⇐(iv) for the following conditions:*

(i) $\mathbb{D}^{t\,*} J\mathbb{D}^t \geq 0$ *for all* $t \geq 0$.

(ii) $D^*JD \gg 0$, *and there is* $\mathcal{P} \leq 0$ *s.t.* $\mathcal{P}[H] \subset \mathrm{Dom}(B_{\mathrm{w}}^*)$ *and*

$$\begin{bmatrix} A^*\mathcal{P} + \mathcal{P}A + C^*JC & (B_{\mathrm{w}}^*\mathcal{P} + D^*JC)^* \\ B_{\mathrm{w}}^*\mathcal{P} + D^*JC & D^*JD \end{bmatrix} \geq 0 \quad \text{on} \quad \mathrm{Dom}(A) \times U. \tag{10.77}$$

(iii) There is $\mathcal{P} \leq 0$ *s.t.* $S := D^*JD + \text{s-lim}_{s\to+\infty} B_{\mathrm{w}}^*\mathcal{P}(s-A)^{-1}B \gg 0$, *and*

$$\begin{bmatrix} A^*\mathcal{P} + \mathcal{P}A + C^*JC & (B_{\mathrm{w}}^*\mathcal{P} + D^*JC)^* \\ B_{\mathrm{w}}^*\mathcal{P} + D^*JC & S \end{bmatrix} \geq 0 \quad \text{on} \quad \mathrm{Dom}(A) \times U. \tag{10.78}$$

(iv) $\mathbb{D} \in \mathrm{TIC}$ *and* $\mathbb{D}^* J\mathbb{D} \geq 0$.

Moreover, the following hold:

(a) If $\mathbb{D} \in \mathrm{TIC}$, *then we have (i)⇔(iv).*

(b) If $\Sigma \in \mathrm{SOS}$, $\mathbb{D}^* J\mathbb{D} \gg 0$, *and (2.) (resp. (6.)) of Hypothesis 10.6.1 holds, then (i), (iii) and (iv) (resp. (i)–(iv)) hold; in fact, we can have equality in (10.81) (resp. in (10.79)).*

(c) We have (ii)⇔(ii')⇒(iii)⇔(iii') (even for a fixed $\mathcal{P}$*), where*

(ii') $D^*JD \gg 0$, *and there is* $\mathcal{P} \leq 0$ *s.t.* $\mathcal{P}[H] \subset \mathrm{Dom}(B_{\mathrm{w}}^*)$ *and*

$$(B_{\mathrm{w}}^*\mathcal{P} + D^*JC)^*(D^*JD)^{-1}(B_{\mathrm{w}}^*\mathcal{P} + D^*JC) \leq A^*\mathcal{P} + \mathcal{P}A + C^*JC. \tag{10.79}$$

(iii') There is $\mathcal{P} \le 0$ s.t.

$$S := D^*JD + \operatorname*{s-lim}_{\alpha\to+\infty} B_{\mathrm{w}}^*\mathcal{P}(\alpha - A)^{-1}B \gg 0, \quad \textit{and} \tag{10.80}$$

$$(B_{\mathrm{w}}^*\mathcal{P} + D^*JC)^*S^{-1}(B_{\mathrm{w}}^*\mathcal{P} + D^*JC) \le A^*\mathcal{P} + \mathcal{P}A + C^*JC. \tag{10.81}$$

(d) If $\mathbb{D} \in \mathrm{TIC}$ and $\mathbb{B}$ is strongly stable, then we can replace "$\mathcal{P} \le 0$" by $\mathcal{P} = \mathcal{P}^*$ everywhere in this proposition.

(The comments below Theorem 10.5.1 apply here too, mutatis mutandis. See Theorem 10.6.3 for more on (b).)

Proof: (Naturally, condition "$C^*JC \le 0$" means that $\langle x_0, C^*JCx_0\rangle \le 0$ for all $x_0 \in \mathrm{Dom}(A)$. In the proof below, the proofs of (c) and (a) come logically first.)

We get "(ii)$\Rightarrow$(iii)" as in the proof of Proposition 9.2.7(a). Implication "(iv)$\Rightarrow$(i)" follows from (a).

To complete the first claim, assume (iii) (i.e., (iii')). By Proposition 9.11.9(e)&(c), there is $\mathbb{X} \in \mathrm{TIC}_\infty(U)$ s.t. $\mathbb{X}^{t\,*}S\mathbb{X}^t \le \mathbb{D}^{t\,*}J\mathbb{D}^t + \mathbb{B}^{t\,*}\mathcal{P}\mathbb{B}^t$ for all $t \ge 0$, hence $\mathbb{D}^{t\,*}J\mathbb{D}^t \ge 0$. Thus, (iii) implies (i).

(a) Since $C^*JC \le 0$, we have $C_{\mathrm{s}}^*JC_{\mathrm{s}} \le 0$ on $\mathrm{Dom}(C_{\mathrm{s}}) \supset H_B$ (because $\langle C_{\mathrm{s}}x_0, JC_{\mathrm{s}}x_0\rangle_Y = \lim_{s\to+\infty}\langle Cs(s-A)^{-1}x_0, JCs(s-A)^{-1}x_0\rangle_Y \le 0$ for all $x_0 \in \mathrm{Dom}(C_{\mathrm{s}})$). Therefore, for any $t \ge 0$ and $u \in \mathrm{L}^2([0,t);U)$ we have $\int_0^\infty = \int_0^t + \int_t^\infty$, i.e.,

$$\langle \mathbb{D}u, J\mathbb{D}u\rangle_{\mathrm{L}^2} = \langle u, \mathbb{D}^{t\,*}J\mathbb{D}^t u\rangle_{\mathrm{L}^2} + \int_t^\infty \langle C_{\mathrm{s}}\mathbb{B}\tau^r u, JC_{\mathrm{s}}\mathbb{B}\tau^r u\rangle\, dr \le \langle u, \mathbb{D}^{t\,*}J\mathbb{D}^t u\rangle_{\mathrm{L}^2}, \tag{10.82}$$

by Theorem 6.2.13(a2). Consequently, $\mathbb{D}^*J\mathbb{D} \ge 0$ implies that (i) holds. The converse is obvious (use Corollary B.3.8).

(b) Obviously, (iv) holds, hence so does (i), by (a). By (b)&(f1)&(i)&(iii') of Theorem 10.6.3, condition (iii') (resp. (ii')) above holds (with equality in (10.81) (resp. in (10.79))) if we can show that $\mathcal{P} \le 0$. By (c), then the rest of (b) holds too.

Condition $C^*JC \le 0$ implies that $\langle \mathbb{C}x_0, J\mathbb{C}x_0\rangle_{\mathrm{L}^2} = \int_0^\infty \langle \mathbb{A}^t x_0, C^*JC\mathbb{A}^t x_0\rangle\, dt \le 0$ for all $x_0 \in \mathrm{Dom}(A)$, hence $\mathbb{C}^*J\mathbb{C} \le 0$, by density. Since $S \gg 0$, it follows that $\mathbb{C}^*J\mathbb{C} - \mathbb{K}^*S\mathbb{K} \le 0$, hence $\mathcal{P} \le 0$, by (9.142).

(Note that the $\mathcal{P}$ above is P-SOS-r.c.-stabilizing. We could use (1.) instead of (2.) of Hypothesis 10.6.1 if we would have "w-lim" in (10.80).)

(c) Implication "(ii)$\Rightarrow$(iii)" was shown above. If $\mathrm{Dom}(A) = H$ (i.e., if A is bounded), then the equivalences follow from Lemma A.3.1(p2) (with columns and rows interchanged); even general A, the proof of Lemma A.3.1(p2) applies.

(d) The above proofs still apply except that the proof of "(iii)$\Rightarrow$(i)" must be altered as follows: Assume (iii). When $\mathbb{B}^t$ is strongly stable, the inequality $\mathbb{X}^{t\,*}S\mathbb{X}^t \le \mathbb{D}^{t\,*}J\mathbb{D}^t + \mathbb{B}^{t\,*}\mathcal{P}\mathbb{B}^t$ $(t \ge 0)$ implies that $\mathbb{D}^{t\,*}J\mathbb{D}^t \ge -\mathbb{B}^{t\,*}\mathcal{P}\mathbb{B}^t \to 0$, as $t \to +\infty$, i.e., that $\mathbb{D}^*J\mathbb{D} \ge 0$. By (a), this is equivalent to (i). $\square$

Next we show that, under sufficient regularity, the uniform Riccati inequality has a solution iff Σ is exponentially stable and the Popov operator is uniformly positive:

Theorem 10.6.5 ($\mathbb{D}^*J\mathbb{D} \gg 0 \Leftrightarrow$ CARI) *Assume that $C^*JC \leq 0$.*

(a) If any of Hypothesis 9.2.2(1.)–(6.) holds (the references to Theorem 8.3.9 may be ignored), then the following are equivalent:

(i) Σ is exponentially stable and $\mathbb{D}^*J\mathbb{D} \gg 0$.

(ii) There is $\mathcal{P} \leq 0$ s.t. $\mathcal{P}[H] \subset \mathrm{Dom}(B_{\mathrm{w}}^*)$ and

$$\begin{bmatrix} A^*\mathcal{P}+\mathcal{P}A+C^*JC & (B_{\mathrm{w}}^*\mathcal{P}+D^*JC)^* \\ B_{\mathrm{w}}^*\mathcal{P}+D^*JC & D^*JD \end{bmatrix} \gg 0 \quad \text{on} \quad \mathrm{Dom}(A)\times U. \tag{10.83}$$

(iii) There is $\mathcal{P} \leq 0$ s.t. $S := D^*JD + \text{s-lim}_{s\to+\infty} B_{\mathrm{w}}^*\mathcal{P}(s-A)^{-1}B$ exists, and

$$\begin{bmatrix} A^*\mathcal{P}+\mathcal{P}A+C^*JC & (B_{\mathrm{w}}^*\mathcal{P}+D^*JC)^* \\ B_{\mathrm{w}}^*\mathcal{P}+D^*JC & S \end{bmatrix} \gg 0 \quad \text{on} \quad \mathrm{Dom}(A)\times U. \tag{10.84}$$

(b) If $\pi_{[0,t)}\mathbb{A}B \in \mathrm{L}^1([0,t);\mathcal{B}(U,H))$, $\pi_{[0,t)}C_{\mathrm{w}}\mathbb{A} \in \mathrm{L}^1([0,t);\mathcal{B}(H,Y))$, and $\pi_{[0,t)}C_{\mathrm{w}}\mathbb{A}B \in \mathrm{L}^1([0,t);\mathcal{B}(U,Y))$ for some $t>0$, then (i)$\Leftrightarrow$(iii)$\Leftarrow$(ii).

(c1) If $\mathbb{D}$ is ULR, then we have (i)$\Leftarrow$(iii)$\Leftarrow$(ii).

(c2) Any solution of (ii), (iii), (ii') or (iii') is strictly negative ($\mathcal{P} < 0$). Under the assumptions of (a), there is an exponentially stabilizing solution (if (i) holds).

(d) If $\mathbb{D}$ is SR, then we have (ii)$\Leftrightarrow$(ii')$\Rightarrow$(iii)$\Leftrightarrow$(iii') (even for a fixed $\mathcal{P}$), where

(ii') $D^*JD \gg 0$, and there is $\mathcal{P} \leq 0$ s.t. $\mathcal{P}[H] \subset \mathrm{Dom}(B_{\mathrm{w}}^*)$ and

$$(B_{\mathrm{w}}^*\mathcal{P}+D^*JC)^*(D^*JD)^{-1}(B_{\mathrm{w}}^*\mathcal{P}+D^*JC) \ll A^*\mathcal{P}+\mathcal{P}A+C^*JC. \tag{10.85}$$

(iii') There is $\mathcal{P} \leq 0$ s.t.

$$\begin{cases} K^*SK \ll A^*\mathcal{P}+\mathcal{P}A+C^*JC & \text{on } \mathrm{Dom}(A)\ , \\ S = D^*JD + \underset{s\to+\infty}{\text{s-lim}}\, B_{\mathrm{w}}^*\mathcal{P}(s-A)^{-1}B & \text{on } U\ , \\ K = -S^{-1}(B_{\mathrm{w}}^*\mathcal{P}+D^*JC) & \text{on } \mathrm{Dom}(A)\ . \end{cases} \tag{10.86}$$

and $S \gg 0$ (for some S and K).

(e) We can replace the assumptions of (a) (resp. of (b)) by any assumption that together with (i) leads to Hypothesis 10.6.1(6.) (resp. (2.)) (or to $D^*JD \gg 0$ and Hypothesis 9.2.1) for $\widetilde{\Sigma}$ and $\widetilde{J}$ (see the proof), for all sufficiently small $\varepsilon > 0$.

One such assumption is that $C \in \mathcal{B}(H,Y)$ and $\dim Y < \infty$.

See also the comments below Theorem 10.5.1. The case where "$\gg$" is replaced by "$\geq$" is covered by Proposition 10.6.4.

Proof: (Naturally, condition "$C^*JC \leq 0$" means that $\langle x_0, C^*JCx_0\rangle \leq 0$ for all $x_0 \in \mathrm{Dom}(A)$.) Let $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{exp}}$.

We shall use (d) tacitly throughout the proof.

1° *"(i)⇒(ii)&(iii)" under Hypothesis 9.2.2:* Assume (i) and Hypothesis 9.2.2. Then $\mathbb{L} := \mathbb{B}\tau \in \mathrm{TIC}(U,H)$, by Lemma 6.1.10, so that $\mathbb{D}^* J\mathbb{D} - \varepsilon\mathbb{L}^*\mathbb{L} \gg 0$ for some $\varepsilon > 0$. Define $\widetilde{\Sigma} := \left(\begin{smallmatrix} A & B \\ \widetilde{C} & \widetilde{D} \end{smallmatrix}\right) \in \mathrm{WPLS}(U,H,Y\times U)$ by $\widetilde{C} := \left[\begin{smallmatrix} C \\ I \end{smallmatrix}\right]$, $\widetilde{D} := \left[\begin{smallmatrix} D \\ 0 \end{smallmatrix}\right]$, and set $\widetilde{J} := \left[\begin{smallmatrix} J & 0 \\ 0 & -\varepsilon I \end{smallmatrix}\right]$.

Then also $\widetilde{\Sigma}$ satisfies Hypothesis 9.2.2 (including the requirements of Theorem 8.3.9(b2)). Since $\widetilde{\mathbb{D}}^* \widetilde{J}\widetilde{\mathbb{D}} = \mathbb{D}^* J\mathbb{D} - \varepsilon\mathbb{L}^*\mathbb{L} \gg 0$, the B_{w}^*-CARE, the IARE and the CARE for $\widetilde{\Sigma}$ and $\widetilde{J}$ have an exponentially stabilizing solution $(\mathcal{P},S,K)$ with $S = D^*JD \in \mathcal{GB}(U)$, by Theorem 9.2.9(i) and Proposition 8.3.10.

But the CARE for $\widetilde{\Sigma}$ and $\widetilde{J}$ equals (10.86) with "$= -\varepsilon I +$" in place of "$\ll$". Therefore, we have established (ii') and (iii') once we show that $\mathcal{P} \le 0$ and $S \ge 0$.

By Theorem 9.9.1(a2), $S \ge 0$, hence $S \gg 0$. Condition $C^*JC \le 0$ implies that $\langle \mathbb{C}x_0, J\mathbb{C}x_0\rangle_{\mathrm{L}^2} = \int_0^\infty \langle \mathbb{A}^t x_0, C^*JC\mathbb{A}^t x_0\rangle\, dt \le 0$ for all $x_0 \in \mathrm{Dom}(A)$, hence $\mathbb{C}^* J\mathbb{C} \le 0$, by density. Since $\widetilde{\mathbb{C}}^* \widetilde{J}\widetilde{\mathbb{C}} = \mathbb{C}^* J\mathbb{C} - \varepsilon\mathbb{J}^*\mathbb{J} < 0$, where $(\mathbb{J}x_0)(t) := \mathbb{A}^t x_0$ $(t \ge 0)$, we obtain from (9.142) that $\mathcal{P} = \widetilde{\mathbb{C}}^* \widetilde{J}\widetilde{\mathbb{C}} - \mathbb{K}^* S\mathbb{K} < 0$, as required (recall that $S \gg 0$).

Remark: $\mathcal{P} < 0$ and $\mathcal{P}$ is exponentially stabilizing for Σ when (i) holds (since $\mathcal{P}$ is exponentially stabilizing for $\widetilde{\Sigma}$, hence for (A,B) too). (When the assumptions of (a) are satisfied and (i) holds, there is one such solution; however, the inequalities (ii) and (iii) might have non-stabilizing solutions too.)

2° *"(ii)⇒(iii)" (whenever $\mathbb{D}$ is SR):* This follows from the proof of Proposition 9.2.7(a).

3° *"(iii)⇒(i)" (whenever $\mathbb{D} \in$ ULR):* By Proposition 9.11.9(e), $\left[\begin{smallmatrix} A & B \\ K & 0 \end{smallmatrix}\right]$ generate a SR WPLS. Since $A^*(-\mathcal{P}) + (-\mathcal{P})A \ll 0$ and $-\mathcal{P} \ge 0$, the semigroup $\mathbb{A}$ is exponentially stable (and $-\mathcal{P} > 0$), by Lemma 9.12.2(d).

By Proposition 9.11.9(d), we have $\mathbb{X}^{t\,*} S\mathbb{X}^t + \varepsilon\mathbb{L}^{t\,*}\mathbb{L}^t \le \mathbb{D}^{t\,*} J\mathbb{D}^t + \mathbb{B}^{t\,*}\mathcal{P}\mathbb{B}^t$ for some $\varepsilon > 0$, where $\mathbb{L} := \mathbb{B}\tau \in \mathrm{TIC}_\infty(H,U)$. Since $S \gg 0$ and $\mathcal{P} \le 0$, we have $\varepsilon\mathbb{L}^{t\,*}\mathbb{L}^t \le \mathbb{D}^{t\,*} J\mathbb{D}^t$, hence $\mathbb{D}^* J\mathbb{D} \ge \varepsilon\mathbb{L}^*\mathbb{L}$, hence $\mathbb{D}$ is positively J-coercive, by Proposition 10.3.2(g1) (with $D_{\mathrm{c}} = D$; here we need the assumption that $\mathbb{D}$ is ULR); in particular $\mathbb{D}^* J\mathbb{D} \ge \varepsilon I$.

4° *Remarks:* From 1° we observe that the implication "(i)⇒(ii)" (resp. "(i)⇒(iii)") also holds under a suitable variant of (6.) (resp. of (2.)) or Hypothesis 10.6.1.

(a) Since Hypothesis 9.2.2 implies that $\mathbb{D}$ is ULR, the equivalence follows from 1°–3°.

(b) By Lemma 6.8.5, $\mathbb{D}$ is ULR, hence we have (ii)⇒(iii)⇒(i). If (i) holds, then $\mathbb{D},\mathbb{L} \in \mathrm{MTIC}^{\mathrm{L}^1}$ (hence $\widetilde{\mathbb{D}} \in \mathrm{MTIC}^{\mathrm{L}^1}$), by Lemma 6.8.5(a), hence then we obtain "(i)⇒(iii)" from 1° with Theorem 9.2.14(c3) in place of Theorem 9.2.9(i).

(c1) This follows from 2° and 3°.

(c2) This was observed in 3°.

(d) Use the proof of Lemma A.3.1(p4) (with rows and columns interchanged) for "(ii)⇔(ii')" and for "(iii)⇔(iii')", and 2° for "(ii)⇒(iii)".

(e) (Note that $\mathbb{D} \in \widetilde{\mathcal{A}}_+$ is not sufficient, since $\widetilde{\mathbb{D}}$ also contains $\mathbb{B}\tau$.)

1° If Σ is exponentially stable, $C \in \mathcal{B}(B,Y)$ and $\dim Y < \infty$, then $\widetilde{\Sigma}$ and $\widetilde{J}$ satisfy Hypothesis 10.6.1(6.) (and hence (2.)) for all $\varepsilon > 0$, by Lemma 10.6.2(d)(6.).

2° If $D^*JD \gg 0$, (i) holds, and Hypothesis 9.2.1 holds for $\widetilde{\Sigma}$ and $\widetilde{J}$, then Hypothesis 10.6.1(6.) is satisfied by $\widetilde{\Sigma}$ and $\widetilde{J}$ for $\varepsilon > 0$ s.t. $D^*JD - \varepsilon I \gg 0$, by Lemma 10.6.2(d)(8.).

3° Under Hypothesis 10.6.1(6.) (resp. (2.)), assumption (i) still implies that the B_{w}^*-CARE (resp. the CARE) for $\widetilde{\Sigma}$ and $\widetilde{J}$ has an exponentially stabilizing solution (cf. 1° (resp. the proof of (b))). Therefore, 1° (resp. the proof of (b)) still applies; the rest follows from (c1). □

Throughout this chapter, $\widetilde{\mathcal{A}}_+$ is assumed to be MTIC or something almost as regular:

Standing Hypothesis 10.6.6 (Class $\widetilde{\mathcal{A}}_+$ is ULR and admits positive SpF)

(1.) $\mathcal{B} \subset \widetilde{\mathcal{A}}_+ \underset{a}{\subsetneq} \mathrm{TIC} \cap \mathrm{ULR}$ *(see Definition 6.2.4), and*

(2.) if $\mathbb{D} \in \widetilde{\mathcal{A}}_+(U,Y)$, $J = J^* \in \mathcal{B}(Y)$, *and* $\mathbb{D}^* J \mathbb{D} \gg 0$, *then* $\mathbb{D}^* J \mathbb{D} = \mathbb{X}^* \mathbb{X}$ *for some* $\mathbb{X} \in \mathcal{G}\widetilde{\mathcal{A}}_+(U)$.

Thus, we have weakened Hypothesis 8.4.7 to only cover the positive case; consequently, all MTIC classes are now applicable without dimensionality districtions (see (a1) below):

Lemma 10.6.7 ($\widetilde{\mathcal{A}}_+$)

(a1) All classes listed in Theorem 8.4.9 and their exponentially stable versions (see "$\mathcal{A}_{\mathrm{exp}}$" in the Theorem) satisfy Standing Hypothesis 10.6.6.

(a2) Hypothesis 8.4.7 is stronger than Standing Hypothesis 10.6.6.

(a3) The class TIC *satisfies (2.) of Standing Hypothesis 10.6.6.*

(b) Let $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$ *be a q.r.c.f. with* $\mathbb{N},\mathbb{M} \in \widetilde{\mathcal{A}}_+$. *Then the following are equivalent*

(i) $\mathbb{N}^* J \mathbb{N} \gg 0$*;*

(ii) $\mathbb{N}^* J \mathbb{N} = \mathbb{X}^* \mathbb{X}$ *for some* $\mathbb{X} \in \mathcal{G}\widetilde{\mathcal{A}}_+$*;*

(iii) $\mathbb{D}$ *has a* (J,I)*-inner q.r.c.f.* $\mathbb{D} = \mathbb{N}'\mathbb{M}'^{-1}$.

Moreover, we have $\mathbb{N}',\mathbb{M}' \in \widetilde{\mathcal{A}}_+$ *and* $M' \in \mathcal{G}\mathcal{B}(U)$ *for any* $(J,*)$*-inner q.r.c.f.* $\mathbb{D} = \mathbb{N}'\mathbb{M}'^{-1}$ *of* $\mathbb{D}$.

Proof: (a1) By Theorem 5.2.8, we may now allow for $\dim U = \infty$ in (β) too; the rest follows from (a2) and Theorem 8.4.9.

(a2) Assume that $\mathbb{D}^* J \mathbb{D} \gg 0$, i.e., that $\pi_+ \mathbb{D}^* J \mathbb{D} \pi_+ \gg 0$ (see Lemma 6.4.6), so that $\mathbb{D}^* J \mathbb{D} = \mathbb{X}^* S \mathbb{X}$ for some $S \in \mathcal{G}\mathcal{B}(U)$. Then $S \geq 0$, hence $S \gg 0$. Replace $\mathbb{X}$ by $S^{1/2}\mathbb{X}$ to satisfy Standing Hypothesis 10.6.6.

(a3) This is Lemma 6.4.7(a).

(b) This follows as in the proof of Corollary 8.4.14 (see also Lemma 8.4.11). □

(See the notes on p. 595.)

10.7 Positive Riccati equations ($\mathcal{J}(0,\cdot) \geq 0$)

I made it a rule to forbear all direct contradictions to the sentiments of others, and all positive assertion of my own. I even forbade myself the use of every word or expression in the language that imported a fixed opinion, such as "certainly", "undoubtedly", etc. I adopted instead of them "I conceive", "I apprehend", or "I imagine" a thing to be so or so; or "so it appears to me at present".

— Autobiography of Benjamin Franklin (1706–1790)

In this section, we shall give additional auxiliary results for Riccati equations in the positive (minimization) case, where $\mathcal{J}(0,\cdot) \geq 0$ (e.g., $J \geq 0$). Recall from Lemma 10.2.2, that $\mathcal{J}(0,\cdot) \geq 0$ implies that a control is minimizing iff it is J-critical.

Such a solution is of state feedback form iff it corresponds to a $\mathcal{U}_*^*$-stabilizing solution of the eIARE. Since such a solution is necessarily nonnegative, we are only interested in nonnegative solutions of the eIARE (or of the eCARE) in this section.

In Proposition 10.7.2 (resp. 10.7.1) we show that a solution of the CARE (resp. the IARE for a stable system) with $S \gg 0$ is WR (resp. well-posed and stable). Analogous results for Riccati inequalities were given in Proposition 9.11.9.

In the two latter propositions we show that for standard LQR cost functions, solutions of the IARE or CARE are well-posed, admissible and stabilizing provided that certain additional assumptions hold.

Proposition 10.7.1 ($S \gg 0$) *Let Σ be [strongly] stable. Assume that the IARE for Σ and J has a solution $(\mathcal{P}, S, \begin{bmatrix} \mathbb{K} \mid \mathbb{F} \end{bmatrix})$ s.t. $S \gg 0$.*

Then $\mathbb{K}$ and $\mathbb{F}$ are stable, and $\mathbb{X}^* S \mathbb{X} = \mathbb{D}^* J \mathbb{D} + \text{s-lim}_{t\to+\infty} \tau(-t)\mathbb{B}^* \mathcal{P} \mathbb{B} \tau(t)$ [$\mathbb{X}^* S \mathbb{X} = \mathbb{D}^* J \mathbb{D}$].

Proof: By Lemma 9.10.1(b4), the eIARE implies that (9.153)–(9.161) hold. The stability of $\mathbb{X}$ follows from (9.160) as that of $\mathbb{N}$ in Proposition 10.7.3(a); similarly, we obtain the stability of $\mathbb{K}$ from (9.159). The claim on $\mathbb{X}^* S \mathbb{X}$ follows from Proposition 9.12.7(a). □

We recall from Proposition 9.11.8 that a solution of the CARE with $S \gg 0$ is WR:

Proposition 10.7.2 ($S \gg 0$) *Let Σ be WR. If the CARE has a solution $\mathcal{P}$ with $S \gg 0$, then $\mathcal{P}$ is a WR solution of the IARE.* □

For $J \gg 0$, any admissible nonnegative solution is at least $\begin{bmatrix} \mathbb{C} \mid \mathbb{D} \end{bmatrix}$-stabilizing; for the standard LQR cost function with $C^*C \gg 0$, such a solution is exponentially stabilizing:

Proposition 10.7.3 ($J \gg 0$) *Assume that the eIARE for Σ and $J \gg 0$ has an admissible solution $\mathcal{P} \geq 0$. Then (a)–(c3) hold:*

(a) The maps $\mathbb{C}_\circlearrowleft$ and $\mathbb{D}_\circlearrowleft$ are stable and $S \geq 0$.

(b) Conversely, any minimizing solution is nonnegative.

(c1) If $\mathbb{C} = \begin{bmatrix} \mathbb{C}_1 \\ 0 \end{bmatrix}$, $\mathbb{D} = \begin{bmatrix} \mathbb{D}_1 \\ I \end{bmatrix}$, and $J = \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix} \gg 0$, then (10.87) is satisfied.

(c2) If $\mathbb{D} \in \mathrm{SR}$ and $\begin{bmatrix} C & D \end{bmatrix}^* J \begin{bmatrix} C & D \end{bmatrix} \geq \varepsilon \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}$ on $H_1 \times U$ for some $\varepsilon > 0$, then (10.87) is satisfied.

(c3) If $C \in \mathcal{B}(H,Y)$, $C^*C \gg 0$ and $\begin{bmatrix} C & D \end{bmatrix}^* J \begin{bmatrix} C & D \end{bmatrix} \geq \varepsilon \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}$ on $H_1 \times U$ for some $\varepsilon > 0$, then (10.87) is satisfied and Σ is estimatable, hence then (d3) applies, and $\mathcal{P}$ is the unique nonnegative admissible solution of the eIARE and minimizing over $\mathcal{U}_{\mathrm{exp}}$ (and $\mathcal{U}_{\mathrm{out}}$, $\mathcal{U}_{\mathrm{sta}}$ and $\mathcal{U}_{\mathrm{str}}$) and exponentially r.c.-stabilizing.

Assume, in addition, that there is $\varepsilon > 0$ s.t.

$$\langle \mathbb{C}x_0 + \mathbb{D}u, J(\mathbb{C}x_0 + \mathbb{D}u)\rangle_{\mathrm{L}^2(\mathbf{R}_+;Y)} \geq \varepsilon \|u\|_2^2 \quad (x_0 \in H,\ u \in \mathrm{L}^2_{+,\infty}). \tag{10.87}$$

(d) Then $\mathbb{D}$ is positively J-coercive over $\mathcal{U}_{\mathrm{out}}$, $S \gg 0$, $\mathcal{P}$ is SOS-stabilizing, and there is a unique minimizing control over $\mathcal{U}_{\mathrm{out}}$ for each $x_0 \in H$. Moreover, (d1)–(d6) hold:

(d1) If the minimizing control over $\mathcal{U}_{\mathrm{out}}$ is given by some state feedback pair $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$, then $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ is the pair (modulo $E \in \mathcal{GB}(U)$) determined by the smallest nonnegative admissible solution of the eIARE.

(d2) Assume that Σ is [strongly [exponentially]] stable.

Then so are Σ_{ext} and $\Sigma_\circlearrowleft$. In particular, then $\mathcal{P}$ is stable, [strongly [exponentially]] r.c.-stabilizing [and strictly minimizing over $\mathcal{U}_{\mathrm{out}}$, $\mathcal{U}_{\mathrm{sta}}$ and $\mathcal{U}_{\mathrm{str}}$ [and $\mathcal{U}_{\mathrm{exp}}$], and $\mathcal{P}$ is the unique nonnegative admissible solution].

(d3) Assume that Σ is estimatable or exponentially q.r.c.-stabilizable. Then $\mathcal{P}$ is the unique nonnegative admissible solution, and it is exponentially q.r.c.-stabilizing and strictly minimizing over $\mathcal{U}_{\mathrm{out}}$, $\mathcal{U}_{\mathrm{sta}}$, $\mathcal{U}_{\mathrm{str}}$ and $\mathcal{U}_{\mathrm{exp}}$.

Thus, in the case described in (c3), we only have to find one nonnegative solution and check whether it is exponentially stabilizing. If not, then there is no minimizing state feedback pair over any of $\mathcal{U}_{\mathrm{out}}$–$\mathcal{U}_{\mathrm{exp}}$.

The assumption in (c1) is equivalent to having the cost function $\mathcal{J}(x_0,u) = \langle u, Ru\rangle + \langle y_1, Qy_1\rangle$, where $y_1 = \mathbb{C}_1 x_0 + \mathbb{D}_1 u$, $R \gg 0$, $Q \gg 0$ (note that here $y = \begin{bmatrix} y_1 \\ u \end{bmatrix}$). See also Theorem 9.2.10 and Corollary 15.1.6.

Although the assumptions in (d) or (c) (and the existence of $\mathcal{P}$) imply that there is a unique minimizing control over $\mathcal{U}_{\mathrm{out}}$, we do not know in general whether such a control can be given in the feedback form (not even whether there is a minimal control among such controls), neither whether a smallest solution would be cost-minimizing. In discrete time we have no such problems, see, e.g., Corollary 15.1.6 (neither in continuous time when, e.g., B is bounded; see Theorems 9.9.6, 9.2.10 and 10.1.4(b4)&(b6)).

Proof of Proposition 10.7.3: (a) 1° $\mathbb{N} := \mathbb{D}_{\circlearrowleft}$ *is stable and* $S \geq 0$*:* For all $u \in \pi_+ \mathrm{L}^2$, $t > 0$, we have

$$0 \leq \langle \mathbb{N}^t u, J\mathbb{N}^t u\rangle \leq \langle u, Su\rangle \leq \|S\|\|u\|_2^2, \tag{10.88}$$

by (9.157), hence $\|J^{1/2}\mathbb{N}^t u\|_2^2 \leq \|S\|\|u\|_2^2$. But $\|\mathbb{N}^t u\|_2^2 \leq M^2\|J^{1/2}\mathbb{N}^t u\|_2^2$ for some $M < \infty$, hence $\|\mathbb{N}\| \leq M\|S\|^{1/2} < \infty$, i.e., $\mathbb{N} = \mathbb{D}_{\circlearrowleft}$ is stable, by Lemma 6.1.12. From (10.88) we also observe that $S \geq 0$ (take $u = u_0\chi_{[0,1)}$).

2° $\mathbb{C}_{\circlearrowleft}$ *is stable:* By (9.155) and the Monotone Convergence Theorem, $\|J^{1/2}\mathbb{C}_{\circlearrowleft} x_0\|_2 \leq \langle x_0, \mathcal{P}x_0\rangle$ ($x_0 \in H$), hence $\mathbb{C}_{\circlearrowleft}$ is stable.

(b) This follows from equation $\mathcal{P} = \mathbb{C}_{\circlearrowleft}^* J\mathbb{C}_{\circlearrowleft}$ (see Theorem 9.9.1(a2)&(g2)). (In fact, any $\mathbb{C}$-P-stabilizing solution is nonnegative, by Lemma 9.10.1(d1).)

(c1) Now $\mathcal{J}(x_0,u) \geq \langle u, Ru\rangle \geq \varepsilon^2\|u\|_2^2$ for some $\varepsilon^2 > 0$.

(c2) Now $y := \mathbb{C}x_0 + \mathbb{D}u = C_{\mathrm{s}}x + Du$ a.e., hence $\langle y, Jy\rangle \geq \varepsilon\|u\|_2^2$, where $x := \mathbb{A}x_0 + \mathbb{B}\tau u$ (note that $\langle \begin{bmatrix} C_{\mathrm{s}} & D\end{bmatrix}\begin{bmatrix} x_0 \\ u_0\end{bmatrix}, J\begin{bmatrix} C_{\mathrm{s}} & D\end{bmatrix}\begin{bmatrix} x_0 \\ u_0\end{bmatrix}\rangle \geq \varepsilon\|u_0\|_U^2$ can be extended to $\mathrm{Dom}(C_{\mathrm{s}}) \times U$, by first replacing x_0 by $r(r-A)^{-1}x_0$, and then letting $r \to +\infty$).

(c3) Because $C^*C \gg 0$, we have $(C^*C)^{-1}C^*C = I$, hence Σ is estimatable by a bounded $\mathtt{H}$, by Lemma 6.6.25. Consequently, (d3) applies (by (c2)). By Lemma 6.6.26, $\begin{bmatrix} \mathbb{K} \mid \mathbb{F} \end{bmatrix}$ is exponentially r.c.-stabilizing (jointly with $\mathtt{H}$).

(d) (Naturally, we allow the value $+\infty$ for the norms.)

1° $\mathcal{P}$ *is SOS-stabilizing:* Let $u_{\circlearrowleft} \in \mathrm{L}^2(\mathbf{R}_+;U)$, $x_0 \in H$. Set

$$u := \mathbb{K}_{\circlearrowleft} x_0 + \mathbb{M}u_{\circlearrowleft}, \;\; y := \mathbb{C}x_0 + \mathbb{D}u = \mathbb{C}_{\circlearrowleft} x_0 + \mathbb{D}_{\circlearrowleft} u_{\circlearrowleft}. \tag{10.89}$$

By (a), y is stable, hence so is u. Consequently, $\mathbb{K}_{\circlearrowleft}$ and $\mathbb{M}$ are stable, by Lemma 6.1.12. Thus, $\Sigma_{\circlearrowleft} \in \mathrm{SOS}$.

2° $S \gg 0$ *and* $\mathbb{D}$ *is positively J-coercive over* $\mathcal{U}_{\mathrm{out}}$*:* These follow from Proposition 10.3.1(a) and Lemma 9.10.3.

3° *Unique minimizing control over* $\mathcal{U}_{\mathrm{out}}$*:* Because $\mathcal{P}$ is SOS-stabilizing, we have $\mathbb{K}_{\circlearrowleft} x_0 \in \mathcal{U}_{\mathrm{out}}(x_0)$ consequently, $\mathcal{U}_{\mathrm{out}}(x_0) \neq \emptyset$, for each $x_0 \in H$. Thus, by Theorem 8.4.3, there is a unique minimizing control for each x_0.

(d1) This follows from Theorem 9.9.1(a2), since any admissible nonnegative solution is output-stabilizing, by (d).

(d2) If Σ is [strongly [exponentially]] stable, then so is Σ_{ext}, by Proposition 10.7.1, and $\Sigma_{\circlearrowleft}$, by Corollary 6.6.9 (here "$I - L\mathbb{D}$"$= \mathbb{X} \in \mathcal{G}$TIC) and Lemma 6.6.8(c). Now $\mathbb{D}, \mathbb{X}, \mathbb{X}^{-1}$ are [[exponentially]] stable, hence $\begin{bmatrix} \mathbb{K} \mid \mathbb{F} \end{bmatrix}$ is [strongly [exponentially]] r.c.-stabilizing.

[By Theorem 9.9.10(e2)&(c1)&(b), $\mathcal{P}$ is minimizing over $\mathcal{U}_{\mathrm{out}}$, $\mathcal{U}_{\mathrm{sta}}$ and $\mathcal{U}_{\mathrm{str}}$ [and $\mathcal{U}_{\mathrm{exp}}$], hence unique.]

(d3) A nonnegative admissible solution $(\mathcal{P}', S', \begin{bmatrix} \mathbb{K}' \mid \mathbb{F}' \end{bmatrix})$ is SOS-stabilizing and has $S' \gg 0$, by (d), hence it is exponentially q.r.c.-stabilizing, by Theorem 6.7.15(c2)&(b1), hence minimizing over $\mathcal{U}_{\mathrm{out}}$, $\mathcal{U}_{\mathrm{sta}}$, $\mathcal{U}_{\mathrm{str}}$ and $\mathcal{U}_{\mathrm{exp}}$ and unique, by Theorem 9.9.10(e2)&(c1)&(b).

□

In usual quadratic minimization problems, one need not check whether $\mathcal{P}$ is admissible (see Section 10.1 for applications):

Proposition 10.7.4 ($\mathcal{J} = \langle y, Qy\rangle + \langle u, Ru\rangle$) *Let* $\Sigma = \left[\begin{array}{c|c}\mathbb{A} & \mathbb{B}\\ \hline \mathbb{C} & \mathbb{D}\end{array}\right] \in \mathrm{WPLS}(U,H,Y)$ *be WR. Let* $Y = \mathcal{Y}\times U$, $\mathbb{C} = \begin{bmatrix}\mathbb{C}_1\\ 0\end{bmatrix}$, $\mathbb{D} = \begin{bmatrix}\mathbb{D}_1\\ I\end{bmatrix}$, $J = \begin{bmatrix}Q & 0\\ 0 & R\end{bmatrix}$, $Q \geq 0$, $R \gg 0$. *Assume that* $\mathbb{D} \in \mathrm{UR}$ *or* $\dim U < \infty$.

Let the CARE

$$\begin{cases} K^*SK = A^*\mathcal{P} + \mathcal{P}A + C^*JC, \\ S = D^*JD + \lim_{s\to+\infty} B_{\mathrm{w}}^*\mathcal{P}(s-A)^{-1}B, \\ K = -S^{-1}(B_{\mathrm{w}}^*\mathcal{P} + D^*JC) \end{cases} \tag{10.90}$$

have a solution $\mathcal{P} \in \mathcal{B}(H)$, $\mathcal{P} \geq 0$ *s.t.* $\lim_{s\to+\infty} B_{\mathrm{w}}^*\mathcal{P}(s-A)^{-1}B \geq 0$ *or* $S \gg 0$.

Then $\mathcal{P}$ *is UR and admissible.*

If $Q \gg 0$, *then* $\mathcal{P}$ *is SOS-stabilizing and Proposition 10.7.3 applies; in particular, if* Σ *is estimatable, then* $\mathcal{P}$ *is the unique nonnegative solution, strictly minimizing over* $\mathcal{U}_{\mathrm{out}}$ *and exponentially q.r.c-stabilizing.*

Note that for $\dim U < \infty$ (resp. $\mathbb{D} \in \mathrm{UR}$), the limit in CARE converges uniformly (as required above) iff the limit in CARE converges weakly (resp. iff $\mathbb{F} \in \mathrm{UR}$, by Lemma 9.11.5(e)).

Proof: Choose some $\omega > \max(0, \omega_A)$. If $\lim_{s\to+\infty} B_{\mathrm{w}}^*\mathcal{P}(s-A)^{-1}B \geq 0$, then $S \geq D^*JD = D_1^*QD_1 + R \geq R \gg 0$. Therefore, $S \gg 0$ under either assumption.

By Proposition 10.7.2, $\mathcal{P}$ is a WR solution of the CARE. The map $\mathbb{X}$ is UR, by Lemma 6.3.2(a1)&(a2) or Lemma 9.11.5(e).

Substitute $z = s$ into (9.188) to observe that

$$\widehat{\mathbb{X}}(s)^*S\widehat{\mathbb{X}}(s) \geq \widehat{\mathbb{D}}(s)^*J\widehat{\mathbb{D}}(s) \quad (s \in \mathbf{C}^+_\omega) \tag{10.91}$$

(because $\mathcal{P} \geq 0$). But $\widehat{\mathbb{D}}(s)^*J\widehat{\mathbb{D}}(s) = \widehat{\mathbb{D}}_1(s)^*Q\widehat{\mathbb{D}}_1(s) + R \geq R \gg 0$ $(s \in \mathbf{C}^+_\omega)$, hence $\widehat{\mathbb{X}}(s)^*S\widehat{\mathbb{X}}(s) \geq R \gg 0$.

By Proposition 2.2.5, we have $\mathbb{X} \in \mathcal{G}\mathrm{TIC}_\infty(U)$ (this is why we wanted $\mathbb{X}$ to be UR). Thus, $\mathcal{P}$ is also admissible, and $S \gg 0$. The rest follows now from Proposition 10.7.3. □

If, e.g., Hypothesis 9.5.1 holds, then we have $\|sB^*(s-A)^{-*}\mathcal{P}(s-A)^{-1}B\|_{\mathcal{B}(U)} \to 0$, as $s \in \mathbf{C}^+_\omega$, $|s| \to \infty$, by Lemma 9.4.2(k). Therefore, in that case, self-adjoint solutions are UR and admissible even without the nonnegativity assumption (since then we have, instead of (10.91), that $\widehat{\mathbb{X}}(s)^*S\widehat{\mathbb{X}}(s) \geq \widehat{\mathbb{D}}(s)^*J\widehat{\mathbb{D}}(s) - \varepsilon I \geq R - \varepsilon I$ for some $\varepsilon > 0$, when $\operatorname{Re} s$ is big enough, by (9.188)), hence also then $\widehat{\mathbb{X}} \in \mathcal{G}\mathrm{H}^\infty_\infty$).

Notes

Part of Proposition 10.7.3 is well known for certain subclasses; see, e.g., Theorem 3.3 of [PS87] or Section 6.2 of [CZ]. In the finite-dimensional case, [LR] is a comprehensive reference for both general and positive Riccati equations and inequalities.