Chapter 1

Introduction

From the wreck of the past, which hath perish'd,
Thus much I at least may recall,
It hath taught me that what I most cherish'd
Deserved to be dearest of all.

— Lord Byron (1788–1824), "Stanzas to Augusta"

In Section 1.1, we summarize the main contributions of this monograph, avoiding any technicalities. Readers wishing to get a somewhat more accurate picture on the actual results should consult Section 1.2, where we give a glance at each chapter by explaining its contents but yet avoiding most technical details and generality.

Some conventions on notation, proofs and hypotheses are explained in Section 1.3. See the end of the book for symbols, concepts, abbreviations, references and index.

1.1 On the contributions of this book

Our ultimate goal has been to develop the H^∞ Four-Block Problem theory in Chapter 12. This has required us to first develop several other parts of the theory that are of independent interest, such as the Riccati equation theory, the cost minimization theory, the dynamic feedback theory, the WPLS theory or the discrete-time theory, all of which are mainly generalizations of existing theory for finite-dimensional or smooth infinite-dimensional systems.

Our main results include the following:

1. On (generalized) Optimal Control and Riccati equations for WPLSs, we have

   (a) established the relations between different classical coercivity assumptions (Section 10.3), generalized them to WPLSs and applied them to solve the general control problem (Section 8.4).

   (b) formulated Integral Algebraic Riccati Equations to establish the corresponding equivalence in continuous time. This also allowed us to reduce several problems to discrete time, where input and output operators are bounded.

   (c) established the corresponding equivalence for (classical-type) Continuous-time Algebraic Riccati Equations (under weak regularity) (Chapter 9).

      i. The implication from the existence of a solution of the control problem to the existence of a solution of the Riccati Equation was already established by G. Weiss, M. Weiss and O. Staffans under stronger regularity and very strong stabilizability and detectability assumptions.

      ii. We have also shown the existence of a smoother solution under several different additional regularity assumptions (e.g., Section 9.2).

   (d) established the Continuous-time Riccati equations on the domain of the closed-loop semigroup generator for general (possibly irregular) WPLSs (Section 9.7; extension of [FLT]).

   (e) treated all the above for both the exponentially stabilizing controls and for the recently-popular strongly or output-stabilizing controls and others, thus providing new results even for finite-dimensional systems.

2. On specific control problems for WPLSs, we have extended the finite-dimensional results by, in addition to the above, solving

   (a) the H^∞ full-information control problem in terms of the Riccati equation (Chapter 11).

i. In the stable case, the existence of a solution was already shown by O. Staffans, assuming the existence of corresponding spectral factorization; a similar statement applies to the LQR problem below.

(b) the general (measurement feedback, or four-block) H^∞ control problem in terms of two Riccati equations and a spectral radius condition (Chapter 12). We have shown that the recent theory of controllers with an internal loop (cf. [CWW96]) is shown to be intimately connected to a general solution of this problem, and all such solutions are also covered.

(c) the cost minimization (LQR) problem, showing the existence of a solution equivalent to the existence of any solution of the corresponding Riccati equation (the solution need not be stabilizing or even admissible a priori). We have also derived similar generalizations of Strict Bounded and Strictly Positive (Real) Lemmas (Chapter 10).

3. On WPLS system theory, we have

   (a) introduced compatibility, which allows one to write any WPLS in a differential form regardless of regularity (Section 6.3).

   (b) introduced an infinite-dimensional weakly coprime factorization concept (Sections 6.4 and 6.5) and applied it to establish the stability and uniqueness of a solution of certain Riccati equations and control problems (this is particularly useful when the solution is not required to be exponentially stabilizing). This concept and compatibility have already become the subjects of leading researchers' articles.

   (c) characterized the transfer functions (equivalently, impulse responses) having a Pritchard–Salamon realization (thus correcting the errors in [KMR] to which we also provide a counter-example). Similarly, we have characterized transfer functions realizable with bounded input or output operators. (Section 6.9)

   (d) generalized the equivalence between exponential dynamic stabilizability and exponential stabilizability and detectability (Theorem 7.2.4).

4. The infinite-dimensional control theory has been limited by several open problems in harmonic and functional analysis and function theory. This has lead us to solve those most intimately connected to our work, e.g., we have

   (a) generalized the L^2 Fourier multiplier theorem to the case of functions with values in Hilbert spaces (the separable case was already known) and beyond (Theorem 1.2.2).

   (b) generalized similarly the existence result of the boundary function of a H^∞ function (Theorem 1.2.3).

(c) developed a theory of strongly measurable operator valued functions, including the the completeness of L^∞_{strong} (and incompleteness of L^p_{strong}) and its applications including the two above results (Appendix F).

(d) shown the existence of a spectral factorization for convolutions with (Hilbert space) operator-valued measures having a discrete part plus an L^1 part (assuming the invertibility of the Toeplitz operator; see Theorems 1.2.4 and 1.2.5).

(e) extended to the infinite-dimensional case the classical [ClaGoh] H^2 spectral factorization for any Popov function having an invertible Toeplitz operator (Theorem 9.14.6).

Finally, of all the above we also present corresponding infinite-dimensional discrete-time results, which become rather elegant since, in this case, the input and output operators are naturally bounded.

For a control theorist, the generalization of Riccati equation theory to the regular WPLS setting (particularly 1b and 1c above) and the general H^∞ and minimization problems (2.) may rise above the rest.

To observe in detail the other new results in this monograph, the reader should read the "Notes" at the end of each section. There we discuss earlier research in same direction, including any known similar results under less general systems, settings or assumptions.

The size of this book requires some explanations. For the first, the chapters of this monograph are so intimately connected to each other that it would have been impossible to remove a single chapter without destroying, e.g., the proofs in Chapter 12.

If we had limited ourselves only to very smooth systems or to discrete-time systems, the size of this book would have probably fallen by more than half but its contribution even by much more. Indeed, most problems but also most value in our work is in its generality. Certainly, we might have presented our solutions only in terms of factorizations (which are given as an intermediary stage in our proofs), but the Riccati equations are really the form of the classical solutions and something that provides a practical way to solve the problems.

Sometimes the Riccati equations become very complicated for general regular WPLSs, hence we have presented more beautiful corollaries for important special cases, such as for the case where the I/O map is the convolution with a measure. Moreover, the realization of the optimal control in the form of a state feedback or dynamic feedback controller requires the existence of certain factorizations that need not exist in the general case (see Example 11.3.7).

One of the objectives of this book has been to state and prove results of a technical nature that are too long to be published in ordinary research articles but that are necessary building blocks for the final results.

1.2 A summary of this book

We now start a rather self-contained summary, aiming to give the reader a motivation for and a picture of the theory treated in each chapter, by starting with a non-technical description and then presenting some results. We strongly recommend for the reader to read the summaries in this section before diving into the technicalities of the actual chapters.

The results mentioned below are just examples from the theory; here we have usually favored simple, important examples to more general but more complex ones. See the chapters themselves for further definitions, results, details, explanations and references.

Outside the appendices, the letters H, U, W, Y and Z will denote complex Hilbert spaces of arbitrary dimensions unless something else is indicated.

Part I: TI Operator Theory

The appendices and Part I of the book contain results in harmonic and functional analysis (vector-valued functions, shift-invariant operators, transfer functions and boundary functions, the Corona Theorem and spectral factorization among others) that are needed in the control theory of Parts II–IV. Many of the results are also of independent interest. A fast track to WPLSs is to first have a glance at subsections 2.1.1–2.1.7 and then go directly to Part II.

Chapter 2: TI and MTI Operators (MTI $\subset$ TI)

In Chapter 2, we study the theory TI_ω, the space of bounded, shift-invariant operators $\mathrm{L}^2 \to \mathrm{L}^2$, where the L^2 space may have a weight and the functions have their values in a Hilbert space. We also present certain smooth subclasses of TI, particularly MTI, the convolutions with a (vector-valued) measure with no singular continuous part.

Our contributions include the theory of the intersection $\mathrm{TI}_\omega \cap \mathrm{TI}_{\omega'}$ and its causal part for two weights $\omega, \omega' \in \mathbf{R}$ (see 2.1.9–2.1.11 and 3.1.6), necessary and sufficient conditions for losslessness and certain results on static operators and signature operators.

Technically, $\mathrm{TI}_\omega(U,Y)$ is the space of bounded time-invariant linear operators $\mathrm{L}^2_\omega(\mathbf{R};U) \to \mathrm{L}^2_\omega(\mathbf{R};Y)$, where U and Y are Hilbert spaces of arbitrary dimensions, $\omega \in \mathbf{R}$, and

$$\|u\|_{\mathrm{L}^2_\omega} := \Big(\int_{\mathbf{R}} \mathrm{e}^{-2\omega t}\|u(t)\|_U^2\,dt\Big)^{1/2} \tag{1.1}$$

for Bochner-measurable $u : \mathbf{R} \to U$ (thus, $\mathrm{L}^2_0 = \mathrm{L}^2$, $\mathrm{L}^2_\omega := \{\mathrm{e}^{\omega\cdot}u(\cdot) \,\big|\, u \in \mathrm{L}^2\}$). The *time-invariance* of $\mathbb{D} \in \mathrm{TI}_\omega$ means that $\mathbb{D}\tau(t) = \tau(t)\mathbb{D}$ for all $t \in \mathbf{R}$, where $\tau(t)u := u(\cdot + t)$.

The maps in $\mathrm{TIC}_\omega(U,Y) := \{\mathbb{D} \in \mathrm{TI}_\omega(U,Y) \,\big|\, \pi_-\mathbb{D}\pi_+ = 0\}$ are called *causal* (or sometimes Toeplitz operators); here $\pi_+u := \chi_{\mathbf{R}_+}u$ and $\pi_-u := \chi_{\mathbf{R}_-}u$ for all functions u, and χ_E is the characteristic function of a set E. The following is well known:

Theorem 1.2.1 *For each* $\mathbb{D} \in \mathrm{TIC}_\omega(U,Y)$, *there is a unique function* $\widehat{\mathbb{D}} \in \mathrm{H}^\infty(\mathbf{C}^+_\omega; \mathcal{B}(U,Y))$, *called the* transfer function *(or symbol) of* $\mathbb{D}$, *s.t.* $\widehat{\mathbb{D}u} = \widehat{\mathbb{D}}\hat{u}$ *on* $\mathbf{C}^+_\omega$ *for all* $u \in \mathrm{L}^2_\omega(\mathbf{R}_+;U)$. *The mapping* $\mathbb{D} \mapsto \widehat{\mathbb{D}}$ *is an isometric isomorphism onto.* □

Here $\mathcal{B}(U,Y)$ denotes the space of bounded linear operators $U \to Y$, $\mathrm{H}^\infty(\mathbf{C}^+_\omega; \mathcal{B}(U,Y))$ denotes the Banach space of bounded holomorphic functions $\mathbf{C}^+_\omega \to \mathcal{B}(U,Y)$, and $\hat{u}$ denotes the Laplace transform

$$\hat{u}(s) := \int_{\mathbf{R}} \mathrm{e}^{-st} u(t)\, dt \qquad (s \in \mathbf{C}^+_\omega := \{s \in \mathbf{C} \mid \operatorname{Re} s > \omega\}) \tag{1.2}$$

of u. Thus, the elements of $\mathrm{TIC}_\infty(U,Y) := \cup_{\omega\in\mathbf{R}} \mathrm{TIC}_\omega(U,Y)$ correspond one-to-one to the holomorphic $\mathcal{B}(U,Y)$-valued functions that are bounded on some right half-plane; such functions are generally called "proper" or "well-posed". The set of the I/O maps of WPLSs is exactly TIC_∞ (see Section 6.1). Transfer functions are studied also in Chapter 3.

In Section 2.2, we study the invertibility of TIC_ω (and TI_ω) operators. In Section 2.3, we develop sufficient conditions for a TIC operator to be static, that is, the multiplication operator induced by an element of $\mathcal{B}(U,Y)$. We also give certain results that will be used in connection with the *signature operators* of optimization problems, Riccati equations and spectral factorizations.

Also Section 2.4 treats signature operators. A main result of this section is that for any $S \in \mathcal{B}(U \times Y)$, the following are equivalent:

(i) $S = \mathbb{E}^* \begin{bmatrix} I_U & 0 \\ 0 & -I_Y \end{bmatrix} \mathbb{E}$ for some $\mathbb{E} \in \mathcal{G}\mathrm{TIC}(U \times Y)$;

(ii) $S = E^* \begin{bmatrix} I_U & 0 \\ 0 & -I_Y \end{bmatrix} E$ for some $E \in \mathcal{G}\mathcal{B}(U \times Y)$.

(Recall that $\mathcal{G}$ denotes the subset of invertible operators.)

Section 2.5 treats the concept "(J,S)-losslessness" (close to "(J,S)-dissipativity"), which is often studied in connection with H^∞ problems and indefinite inner products (losslessness is roughly equivalent to the nonnegativity of the corresponding Riccati operator). There are two widely-used definitions of losslessness whose exact connection has been unknown. We develop necessary and/or sufficient conditions for both concepts and show that they coincide when the input spaces are finite-dimensional.

In Section 2.6, we define the subclass $\mathrm{MTI}(U,Y)$ ("M" for "measures") as the operators $\mathbb{D} \in \mathrm{TI}(U,Y)$ that are of the form

$$(\mathbb{E}u)(t) = \sum_{k=0}^{\infty} T_k u(t-t_k) + \int_{-\infty}^{\infty} f(t-r)u(r)\, dr, \tag{1.3}$$

i.e., of the form $\mathbb{E}u = \mu * u$, where the measure μ consists of a function $f \in \mathrm{L}^1(\mathbf{R}; \mathcal{B}(U,Y))$ plus a discrete part with $T_k \in \mathcal{B}(U,Y)$ and $t_k \in \mathbf{R}$ for all $k \in \mathbf{N}$, s.t.

$$\|\mathbb{E}\|_{\mathrm{MTI}} := \|f\|_{\mathrm{L}^1} + \sum_{k\in\mathbf{N}} \|T_k\|_{\mathcal{B}(U,Y)} < \infty. \tag{1.4}$$

The *Wiener class* $\mathrm{MTI}^{\mathrm{L}^1}$ refers to the elements of MTI of form $u \mapsto Tu_0 + f * u$ (i.e., no delays). The class $\mathrm{MTIC} := \mathrm{MTI} \cap \mathrm{TIC}$ (resp. $\mathrm{MTIC}^{\mathrm{L}^1} := \mathrm{MTI}^{\mathrm{L}^1} \cap \mathrm{TIC}$) consists of those elements of MTI (resp. $\mathrm{MTIC}^{\mathrm{L}^1}$) that correspond to measures supported on $\mathbf{R}_+$. In [CD80] and [CZ] among others, the class MTIC (or "$\mathcal{A}(0)$") has been studied for finite-dimensional U and Y.

The basic properties of these classes are listed in Section 2.6. They share most properties of maps with rational transfer functions; in particular, they have the same spectral factorization properties (see Section 5.2). These properties allow us to show (in Part III) that classical conditions for the solvability of standard control problems are necessary and sufficient also for systems whose I/O maps belong to MTIC (such conditions are sufficient but not necessary for general WPLSs); some of this has already been established for less general systems (see, e.g., [CD80] or [CW99]).

Chapter 3: Transfer Functions ($\widehat{\mathrm{TI}} = \mathrm{L}^\infty_{\mathrm{strong}}$, $\widehat{\mathrm{TIC}} = \mathrm{H}^\infty$)

We study the *Laplace and Fourier transforms* (or *transfer functions* or *symbols*) of TI and TIC maps, that is, (causal and general) time-invariant maps $\mathrm{L}^2 \to \mathrm{L}^2$.

Our main results are two generalizations to unseparable Hilbert spaces, first one of the Fourier multiplier theorem ("$\widehat{\mathrm{TI}}(U,Y) = \mathrm{L}^\infty_{\mathrm{strong}}(i\mathbf{R};\mathcal{B}(U,Y))$") and then of the fact that an operator-valued H^∞ function over the right half-plane has a boundary function in strong L^∞ on the imaginary axis as its "strong pointwise limit", in a very natural sense.

We first show that "$\widehat{\mathrm{TI}}(U,Y) = \mathrm{L}^\infty_{\mathrm{strong}}(i\mathbf{R};\mathcal{B}(U,Y))$" (Theorem 3.1.3(a1)):

Theorem 1.2.2 *For each* $\mathbb{E} \in \mathrm{TI}(U,Y)$*, there is a unique (*symbol*)* $\widehat{\mathbb{E}} \in \mathrm{L}^\infty_{\mathrm{strong}}(i\mathbf{R};\mathcal{B}(U,Y))$ *s.t.* $\widehat{\mathbb{E}}\widehat{u} = \widehat{\mathbb{E}u}$ *a.e. for all* $u \in \mathrm{L}^2(\mathbf{R};U)$*. This mapping* $\mathbb{E} \mapsto \widehat{\mathbb{E}}$ *is an isometric isomorphism of* $\mathrm{TI}(U,Y)$ *onto* $\mathrm{L}^\infty_{\mathrm{strong}}(i\mathbf{R};\mathcal{B}(U,Y))$. □

(The separable case of this claim is well-known. Here $i\mathbf{R}$ is the imaginary axis, and $\widehat{\mathbb{E}} \in \mathrm{L}^\infty_{\mathrm{strong}}(i\mathbf{R};\mathcal{B}(U,Y))$ means that $\widehat{\mathbb{E}} : i\mathbf{R} \to \mathcal{B}(U,Y)$ is s.t. $\widehat{\mathbb{E}}u_0 \in \mathrm{L}^\infty(i\mathbf{R};Y)$ for all $u_0 \in U$. It follows that $\|\widehat{\mathbb{E}}\|_{\mathrm{L}^\infty_{\mathrm{strong}}} := \sup_{\|u_0\|_U \le 1} \|\widehat{\mathbb{E}}u_0\|_\infty < \infty$, by Lemma F.1.6.)

Then we go on to show that this *Fourier transform* restricts to an isometric isomorphism of $\mathrm{TI}_a(U,Y) \cap \mathrm{TI}_b(U,Y)$ onto $\mathrm{H}^\infty(\mathbf{C}_{a,b};\mathcal{B}(U,Y))$, where $\mathrm{H}^\infty(\mathbf{C}_{a,b};\mathcal{B}(U,Y))$ refers to bounded holomorphic functions $\mathbf{C}_{a,b} \to \mathcal{B}(U,Y)$ and $\mathbf{C}_{a,b} := \{s \in \mathbf{C} \mid a < \mathrm{Re}\, s < b\}$, and that $\widehat{\mathbb{E}}\widehat{u} = \widehat{\mathbb{E}u}$ on $\mathbf{C}_{a,b}$ (both sides of the equation being holomorphic) for all $u \in \mathrm{L}^2_a(\mathbf{R};U) \cap \mathrm{L}^2_b(\mathbf{R};U)$.

In Sections 3.1 and 3.2, we also give further results on the Fourier transform and weaker forms of the two results mentioned above for arbitrary Banach spaces U and Y and L^p in place of L^2 (and "TI^p_ω" in place of TI_ω). These can be considered as extensions of the so called *Fourier multiplier theory*.

In Section 3.3, we establish several results on the boundary functions of holomorphic functions, the most important of which is the following (Theorem 3.3.1(c1)):

Theorem 1.2.3 *For each* $f \in \mathrm{H}^\infty(\mathbf{C}_0^+;\mathcal{B}(U,Y))$*, there is a* boundary function $f_0 \in \mathrm{L}^\infty_{\text{strong}}(i\mathbf{R};\mathcal{B}(U,Y))$ *s.t.* f_0u_0 *is the nontangential limit of* fu_0 *a.e. on* $i\mathbf{R}$ *for all* $u_0 \in U$. □

(The separable case of this theorem was given in [Thomas].) As the observant reader already may have guessed, f_0 is the Fourier transform of $\mathbb{D}$, where $\mathbb{D} \in \mathrm{TI}(U,Y)$ is s.t. $\widehat{\mathbb{D}} = f$. This justifies the use of "$\widehat{\mathbb{D}}$" to denote both the Fourier transform $\widehat{\mathbb{D}} \in \mathrm{L}^\infty_{\text{strong}}(\omega + i\mathbf{R};\mathcal{B}(U,Y))$ and the transfer function (Laplace transform) $\widehat{\mathbb{D}} \in \mathrm{H}^\infty(\mathbf{C}_\omega^+;\mathcal{B}(U,Y))$ of a map $\mathbb{D} \in \mathrm{TIC}_\omega(U,Y)$.

Some counter-examples are given to show that Theorem 1.2.3 is not true for general Banach spaces nor with H^2 in place of H^∞.

We also give further results on transfer functions; these results will be needed for the WPLS theory of Parts II and III.

Chapter 4: Corona Theorems and Inverses

In this chapter, we first show that any causal inverses of I/O maps preserve smoothness and then we do the same for causal left inverses (most of this consists of combinations of known results). The latter only holds for finite-dimensional input spaces, but we present partial results on the infinite-dimensional case, on which we shall later build our quasi-coprime factorization theory for WPLSs.

In Theorem 4.1.1, we list the following equivalent conditions for the invertibility of any $\mathbb{D} \in \widetilde{\mathcal{A}}(U,Y)$, where $\widetilde{\mathcal{A}}$ stands for TIC, MTIC, CTIC or for some of their subclasses mentioned above:

(i) $\mathbb{D} \in \mathcal{G}\widetilde{\mathcal{A}}$;

(ii) $\mathbb{D} \in \mathcal{G}\mathrm{TIC}$;

(iii) $\pi_+\mathbb{D}\pi_+ \in \mathcal{G}\mathcal{B}(\pi_+\mathrm{L}^2)$;

(iv) $\widehat{\mathbb{D}} \in \mathcal{G}\mathrm{H}^\infty$, i.e., $\widehat{\mathbb{D}}^{-1}$ exists and is bounded on $\mathbf{C}^+$.

In particular, $\widetilde{\mathcal{A}}$ is inverse-closed in TIC. The same holds for the set of maps that are "exponentially $\widetilde{\mathcal{A}}$". For the case $\dim U = \dim Y < \infty$, there are several other equivalent conditions, such as (v) $\inf_{\mathbf{C}^+} |\det(\widehat{\mathbb{D}})| > 0$; (vi) $\mathbb{D}$ is left-invertible in TIC (see the Corona equivalence below for more).

We also give analogous results on TI, MTI, CTI and their (noncausal) subclasses (e.g., $\mathbb{E} \in \mathrm{MTI}$ is invertible in MTI iff $\widehat{\mathbb{E}}$ is boundedly invertible on $i\mathbf{R}$) and further invertibility results.

Then we study the Corona Theorem and its consequences following the methods of M. Vidyasagar. In case $\mathbb{D} \in \widetilde{\mathcal{A}}(U,Y)$, $\dim U < \infty$, we list the following equivalent conditions for the left-invertibility of $\mathbb{D}$:

(i) $\mathbb{V}\mathbb{D} = I$ for some $\mathbb{V} \in \widetilde{\mathcal{A}}(Y,U)$;

(ii) $\mathbb{V}\mathbb{D} = I$ for some $\mathbb{V} \in \mathrm{TIC}(Y,U)$;

(iii) $\widehat{\mathbb{D}}(s)^*\widehat{\mathbb{D}}(s) \geq \varepsilon I$ for all $s \in \mathbf{C}^+$ and some $\varepsilon > 0$;

(iv) $\|\mathbb{D}u\|_{\mathrm{L}^2_\omega} \geq \varepsilon\|u\|_{\mathrm{L}^2_\omega}$ for all $u \in \mathrm{L}^2_\omega(\mathbf{R};U)$, $\omega > 0$ and some $\varepsilon > 0$;

(v) $\mathbb{D}^*\pi_-\mathbb{D} \geq \varepsilon\pi_-$ on L^2 for some $\varepsilon > 0$;

(vi) $\mathbb{D}^{t\,*}\mathbb{D}^t \geq \varepsilon\pi_{[0,t)}$ for all $t > 0$ and some $\varepsilon > 0$.

(Here $\mathbb{D}^t := \pi_{[0,t)}\mathbb{D}\pi_{[0,t)}$.) It follows that $\mathbb{N} \in \widetilde{\mathcal{A}}(U,Y)$ and $\mathbb{M} \in \widetilde{\mathcal{A}}(U)$ are right coprime over $\widetilde{\mathcal{A}}$, i.e., $\widetilde{\mathbb{X}}\mathbb{M} - \widetilde{\mathbb{Y}}\mathbb{N} = I$ for some $\widetilde{\mathbb{X}}, \widetilde{\mathbb{Y}} \in \widetilde{\mathcal{A}}$, iff $\widehat{\mathbb{N}}(s)^*\widehat{\mathbb{N}}(s) + \widehat{\mathbb{M}}(s)^*\widehat{\mathbb{M}}(s) \geq \varepsilon I$ for all $s \in \mathbf{C}^+$ and some $\varepsilon > 0$. Moreover, for most of these classes an equivalent condition is that $\mathbb{D}$ can be complemented to an invertible map $\begin{bmatrix}\mathbb{D} & \mathbb{F}\end{bmatrix}$ over $\widetilde{\mathcal{A}}$. Therefore, for these classes the existence of right or left coprime factors in $\widetilde{\mathcal{A}}$ implies the existence of a doubly-coprime factorization over $\widetilde{\mathcal{A}}$.

The Corona Theorem does not extend to infinite-dimensional U, but we give several partial results for the infinite-dimensional case.

Chapter 5: Spectral Factorization ($\mathbb{E} = \mathbb{Y}^*\mathbb{X}$, $\mathbb{D}^*J\mathbb{D} = \mathbb{X}^*S\mathbb{X}$)

We study *spectral factorization* ("canonical factorization") in the sense of Israel Gohberg et al. This means factoring the given time-invariant map as the product of a non-causal and a causal invertible time-invariant map (with the inverses having the same properties).

In the frequency domain, spectral factorization equals writing a given operator-valued essentially bounded measurable function on the unit circle as the product $\widehat{\mathbb{Y}}^*\widehat{\mathbb{X}}$, where $\widehat{\mathbb{Y}}$ and $\widehat{\mathbb{X}}$ are (the nontangential limits at the circle of) operator-valued bounded, boundedly invertible holomorphic functions on the unit disc; that is, given $\mathbb{E} \in \mathrm{L}^\infty(\partial\mathbf{D};\mathbf{C}^{n\times n})$, finding $\widehat{\mathbb{Y}}, \widehat{\mathbb{Y}}^{-1}, \widehat{\mathbb{X}}, \widehat{\mathbb{X}}^{-1} \in \mathrm{H}^\infty(\mathbf{D};\mathbf{C}^{n\times n})$ such that $\mathbb{E} = \widehat{\mathbb{Y}}^*\widehat{\mathbb{X}}$ a.e. on $\partial\mathbf{D}$, (in case of unseparable Hilbert spaces in place of $\mathbf{C}^n$, this product must not be interpreted pointwise).

This factorization is an extremely important tool in solving stable control problems, and even the unstable case can often be reduced to the stable one.

For rational transfer functions (equivalently, for finite-dimensional systems), the existence of such a factorization for a bounded time-invariant map $\mathrm{L}^2 \to \mathrm{L}^2$ is equivalent to the invertibility of the Toeplitz operator of this map (The map to be factorized is typically the cost function (or Popov operator) of a control problem.)

Since this necessary Toeplitz invertibility condition is not sufficient for general (non-rational) indefinite maps, the classical conditions for the existence of a solution to a control problem cannot be generalized to general WPLSs, not even if we were not be interested on the regularity of the controller. This makes these factorization results essential for much of the theory, as well as the fact that the regularity implied by these results makes it possible to write down the Riccati equations for the problems and to obtain smooth controllers. Thus, in our most general results in later sections and in some other special cases, we have to use different methods to obtain results, often with fewer equivalent conditions or more complicated formulae.

We also mention that though the spectral factorization need not exist, there are yet "H^2 spectral factors", as shown by Gohberg et al. [ClaGoh] for finite-dimensional Hilbert spaces. We extend this result to the general case in Theorem 9.14.6.

Section 5.1 consists of rather straight-forward derivation of required results from the literature. In Section 5.2, we treat the convolutions with measures consisting of a discrete part plus an (uniformly measurable) L^1 part. Our main contribution is Lemma 5.2.3, by which we can reduce the factorization of such convolutions to the separate factorizations of the discrete and absolutely continuous parts of the measure, which already have been gradually solved during the last three decades.

The positive case of the lemma has already been proved by J. Winkin [Winkin] (for finite-dimensional input and output spaces). Though the existence of a spectral factorization is always guaranteed in the positive case (assuming the invertibility of the corresponding Toeplitz operator), it is important to know the smoothness of the factor, as explained above.

The main corollaries of our lemma are that such convolutions maps have spectral factors, and that these are of the same form as the original maps. This allows one to formulate the solutions to WPLS control problems as in the classical case, though with several technical complications due to the unboundedness of input and output operators (see Section 9.1). These corollaries can be written in the form of the following two theorems:

Theorem 1.2.4 (Positive MTI spectral factorization) *Let U be a Hilbert space, and let $\mathcal{A}$ be one of the classes* TI, MTI, $\mathrm{MTI}^{\mathrm{L}^1}$. *Let $\mathbb{E} \in \mathcal{A}(U)$, and set $\widetilde{\mathcal{A}} := \mathcal{A} \cap \mathrm{TIC}$.*

Then $\mathbb{E} \gg 0$ iff $\mathbb{E}$ has a factorization

$$\mathbb{E} = \mathbb{X}^*\mathbb{X}, \text{ where } \mathbb{X} \in \mathcal{G}\widetilde{\mathcal{A}}(U). \tag{1.5}$$

Moreover, if $\mathbb{E} \in \mathcal{A}_{\mathrm{exp}}$, then $\mathbb{X}^{\pm 1} \in \widetilde{\mathcal{A}}_{\mathrm{exp}}$. □

(The class $\mathcal{A}_{\mathrm{exp}}$ (resp. $\widetilde{\mathcal{A}}_{\mathrm{exp}}$) consists of "exponentially stable $\mathcal{A}$ (resp. $\mathcal{A}_{\mathrm{exp}}$) maps". By "$\mathbb{E} \gg 0$" (or "$0 \ll \mathbb{E}$") we mean that $\mathbb{E} \geq \varepsilon I$ for some $\varepsilon > 0$.)

If $\mathbb{E} \in \mathrm{MTI} = \mathcal{A}$, then $\widehat{\mathbb{E}}$ and $\widehat{\mathbb{X}}$ are continuous in $i\mathbf{R}$, hence then (1.5) is equivalent to "$\widehat{\mathbb{E}}(it) = \widehat{\mathbb{X}}(it)^*\widehat{\mathbb{X}}(it)$ for all $t \in \mathbf{R}$, $\mathbb{X}, \mathbb{X}^{-1} \in \mathrm{MTIC}(U)$".

The general (indefinite) case is analogous except that for some classes $\mathcal{A}$, our result requires U to be finite-dimensional:

Theorem 1.2.5 (MTI spectral factorization) *Let $\mathbb{E} \in \mathcal{A}(U)$, where $\mathcal{A}$ and $\widetilde{\mathcal{A}}$ are as in Theorem 5.2.7. Then the* Toeplitz operator *(or Wiener–Hopf operator) $\pi_+\mathbb{E}\pi_+ \in \mathcal{B}(\mathrm{L}^2(\mathbf{R}_+;U))$ is invertible iff $\mathbb{E}$ has a spectral factorization*

$$\mathbb{E} = \mathbb{Y}^*\mathbb{X}, \text{ where } \mathbb{X}, \mathbb{Y} \in \mathcal{G}\widetilde{\mathcal{A}}(U). \tag{1.6}$$

Moreover, if $\mathbb{E} \in \mathcal{A}_{\mathrm{exp}}$, then $\mathbb{X}^{\pm 1}, \mathbb{Y}^{\pm 1} \in \widetilde{\mathcal{A}}_{\mathrm{exp}}$. □

(Note that $\pi_+\mathbb{E}\pi_+ \in \mathcal{B}(\mathrm{L}^2(\mathbf{R}_+;U))$ iff $\mathbb{E}\pi_+ + \pi_- \in \mathcal{B}(\mathrm{L}^2(\mathbf{R};U))$.)

In fact, in the two theorems above, also several other subclasses of MTI can take the place of $\mathcal{A}$ (see Theorems 5.2.8 and 5.2.7). We also state a few other results concerning the spectral factorization of TI maps and some results on other subclasses.

If the assumption "$\mathbb{E} \in \mathcal{A}(U)$" is replaced by "$\mathbb{E} \in \mathrm{TI}(U)$", then the "generalized canonical factors" $\mathbb{X}$ and $\mathbb{Y}$ of $\mathbb{E}$ need no longer be stable in the indefinite case (but their Cayley transforms are invertible in H^2 over the unit disc). For $\dim U < \infty$, this can be found in [CG81] or in [LS] (with the Cayley transforms of $\widehat{\mathbb{X}}^{\pm 1}$ and $\widehat{\mathbb{Y}}^{\pm 1}$ being invertible in H^2 over the unit disc). We show that this theory has an extension for the case where U is an arbitrary Hilbert space (see p. 148 and Theorem 9.14.6).

To emphasize the importance of spectral factorization, we note that one of the main themes of this monograph is the equivalence of the following four conditions for several control problems for an exponentially stable WPLS:

(I) the problem has a (nonsingular) solution;

(II) the Popov Toeplitz operator of the problem is invertible;

(III) the Popov operator of the problem has a spectral factorization;

(IV) the Riccati equation of the problem has a stabilizing solution.

For the case where the WPLS is merely stable, we get almost the same results and the unstable case is somewhat analogous (it can often be reduced to the [exponentially] stable case).

For systems with a I/O map in MTIC (and hence the Popov operator in MTI), the equivalence "(II)⇔(III)" follows from either of the two theorems above (the former one covers more classes of I/O maps but is only applicable in minimization problems).

The equivalence "(I)⇔(II)" will be established in Chapter 8 and in the sections corresponding to the particular control problems; equivalence "(III)⇔(IV)" will be established in Section 9.1 (assuming sufficient regularity of the I/O map and the spectral factor; MTI maps are sufficiently regular for our purposes; hence, for such systems, we have a complete equivalence of (I)–(IV)).

The I/O map of a finite-dimensional system is rational, hence in MTI (if stable). Therefore, in the standard finite-dimensional theory we always have the equivalence of (I)–(IV).

Theorem 1.2.5 is not true for $\mathcal{A} = \mathrm{TI}$, not even when $U = \mathbf{C}^2$ (by Example 8.4.13), and the equivalence "(III)⇔(IV)" does not even hold for all regular systems (by Proposition 9.13.1(c1)). For these reasons, some of our results in Chapters 9–12 pose additional regularity assumptions on the system; most of them are satisfied by systems having a MTIC I/O map (cf. Theorem 8.4.9).

Part II: Continuous-Time Control Theory

This part contains the theory of *well-posed linear systems (WPLSs)*: system theory, regularity, spectral and coprime factorization and stabilization (by static feedback, state feedback, output injection or dynamic feedback).

Chapter 6: Well-Posed Linear Systems (WPLS)

Chapters 6 and 7 present an extensive theory on *Well-Posed Linear Systems (WPLSs)*: state-space and frequency-domain theory, stability, regularity, factorization, state feedback, output injection, static and dynamic output feedback and relations to Pritchard–Salamon systems and other special cases.

Some of the results in these chapters are rather straight-forward extensions of existing theory or generalizations of classical results, though yet useful for control problems. The main new contributions of Chapter 6 include the following (in the order of appearance):

1. the relations between the stabilities of different parts of a WPLS (from Lemma 6.1.10 to Example 6.1.14);

2. several, often very technical regularity results needed in the Riccati equation theory;

3. compatibility theory (to write also irregular WPLSs in a differential form as in (1.7));

4. infinite-dimensional quasi- and pseudo-coprime factorization theory and corresponding stabilization theory (Sections 6.4–6.7). This theory serves almost as well as the classical coprime factorization theory for the stabilizability and uniqueness analysis of the solutions of Riccati equations, but these strictly weaker coprimess properties are sometimes more easily verified, and quasi-coprimeness is preserved under discretization in both directions, thus allowing one to reduce several proofs to discrete time.

5. new results on the generators of closed-loop systems (part of Proposition 6.6.18);

6. equivalent conditions for different stability and stabilizability properties (particularly parts of Theorems 6.7.10 and 6.7.15);

7. theory of systems with a smoothing semigroup (Section 6.8, particularly Lemma 6.8.5);

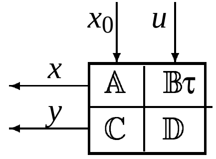
8. the characterization of those transfer functions (equivalently, of I/O maps) that have realizations having a bounded input or output operator or a Pritchard–Salamon realization (Theorems 6.9.1 and 6.9.6);

Also almost all of our results in Chapters 6–12 will be given in a WPLS setting, therefore we motivate these systems briefly below.

Linear time-invariant control systems are usually governed by the equations

$$x'(t) = Ax(t) + Bu(t), \quad y(t) = Cx + Du, \quad x(0) = x_0 \quad (t \geq 0), \tag{1.7}$$

where the *generators* $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \in \mathcal{B}(H \times U, H \times Y)$ of the system are matrices, or more generally, linear operators in Hilbert spaces of arbitrary dimensions, and $u : \mathbf{R}_+ \to U$ is the input, $x : \mathbf{R}_+ \to H$ is the state and $y : \mathbf{R}_+ \to Y$ is the output of the system. If the generators are bounded, then the solution of (1.7) is obviously



$$x = \mathbb{A}x_0 + \mathbb{B}\tau u$$
$$y = \mathbb{C}x_0 + \mathbb{D}u$$

Figure 1.1: Input/state/output diagram of a WPLS $\left[\begin{smallmatrix}\mathbb{A} & \mathbb{B}\\ \mathbb{C} & \mathbb{D}\end{smallmatrix}\right]$

given by the system

$$\begin{cases} x(t) &= \mathbb{A}(t)x_0 + \mathbb{B}\tau(t)u \\ y &= \mathbb{C}x_0 + \mathbb{D}u, \end{cases} \quad \text{where} \tag{1.8}$$

$$\begin{aligned} \mathbb{A}(t) &= e^{At}, & \mathbb{B}\tau(t)u &= \int_0^t \mathbb{A}(t-s)Bu(s)\,ds, \\ \mathbb{C}x_0 &= C\mathbb{A}(\cdot)x_0, & \mathbb{D}u &= C\mathbb{B}\tau(t)u + Du. \end{aligned} \tag{1.9}$$

The formulae (1.8)–(1.9) are actually valid for rather unbounded generators. Therefore, WPLSs are defined by requiring $\mathbb{A}$ to be a strongly continuous semigroup, $\mathbb{D}$ to be time-invariant and causal, $\mathbb{B}$ and $\mathbb{C}$ to be compatible with $\mathbb{A}$ and $\mathbb{D}$, and $\left[\begin{smallmatrix}\mathbb{A}(t) & \mathbb{B}\tau(t)\\ \mathbb{C} & \mathbb{D}\end{smallmatrix}\right]$ being linear and continuous $H \times \mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}_+;U) \to H \times \mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}_+;Y)$ for each $t \geq 0$, equivalently, that

$$\|x(t)\|_H^2 + \int_0^t \|y(s)\|_Y^2\,ds \leq K_t\big(\|x_0\|_H^2 + \int_0^t \|u(s)\|_U^2\,ds\big) \tag{1.10}$$

for some (equivalently, all) $t > 0$, where K_t depends on t only. An equivalent formulation is given in Definition 6.1.1, where we use the unique natural extensions of $\mathbb{B}$ and $\mathbb{D}$ that allow the inputs to be defined on the whole real line, thus simplifying several formulae.

Abstract linear system theory has been gradually developed since Rudolf Kalman's work in [KFA], by William Helton [Helton76a], Paul Fuhrmann and others until Dietmar Salamon and Anthony Pritchard [PS85] [PS87] formulated the *Pritchard–Salamon systems*, which are formally close to WPLSs. These systems have been extensively studied in eighties and early nineties, but they do not cover all interesting examples. This motivated Salamon to define WPLSs in [Sal87].

The *Lax–Phillips scattering theory* [LP] and the operator-based model theory of Béla Sz.-Nagy and Ciprian Foiaş [SF] gave a remarkable impact to the research already on the seventies, and these theories have been shown to be equivalent to WPLSs (see Chapter 11 of [Sbook]). Thus, also the system theory based on the Lax–Phillips model and extensively developed in Soviet Union by D.Z. Arov and others (independently from WPLSs; see [AN] and its reference list) has exactly the WPLS framework.

Until then, research had been divided by different ways to represent a system, for example:

(1.) in terms of partial differential equations or differential delay equations [Lions] [FLT],

(2.) in terms of the generators $\left[\begin{smallmatrix}A & B\\ C & D\end{smallmatrix}\right]$ [Helton76a] [Fuhrmann81], (3.) as a frequency domain relationship between inputs and outputs [CG97],

(4.) as a dynamical system (e.g., WPLS) in the sense of Kalman [KFA],

(5.) by fractional representations [Vid] [CD78]

as noted by Ruth Curtain [Curtain97], who emphasized the need for a theory covering both state-space and frequency-domain aspect and unifying all the above representations; the work of Salamon and George Weiss in the late eighties showed that WPLSs satisfy this need. Thereafter WPLSs have become an increasingly popular subject in some parts of control theory, being the most general widely-used class of infinite-dimensional linear systems.

The more specialized approaches still have their advantages in the study of special cases. One of the most important examples of this is the work of Irena Lasiecka, Roberto Triggiani and others (see [LT00a], [LT00b] and references therein), who have solved state feedback problems corresponding to several important PDEs and rather coercive cost functions, by using a more ad hoc approach (of type "(1.)"). At its best, the abstract WPLS approach can complement the others by providing a different insight and an abundance of results including those common for rather general systems and cost functions, thus removing the need to "reinvent the wheel" over and over again.

We study the basic properties, stability, realization theory, dual systems and generators of WPLSs in Section 6.1. For any WPLS, there are generators $B \in \mathcal{B}(U, \mathrm{Dom}(A^*)^*)$ and $C \in \mathcal{B}(\mathrm{Dom}(A), Y)$ satisfying (1.9) in a strong sense (e.g., $\int_0^t \mathbb{A}(s) Bu(-s)\, ds$ converges in $\mathrm{Dom}(A^*)^*$ but its value belongs to the smaller space H and equals $\mathbb{B}u$; also the formula $x' = Ax + Bu$ holds in $\mathrm{Dom}(A^*)^*$ a.e.), as shown by Salamon [Sal89] and Weiss [W89a] [W89b]. Salamon also observed that any TIC_∞ map (or proper transfer function) can be realized as a WPLS.

A WPLS need not have a well-defined feedthrough operator ("D"), but all systems of practical interest seem to have one; such WPLSs are called *regular*. Regularity is treated in Sections 6.2 and 6.3. An equivalent definition of [weak] regularity is that the transfer function has a [weak] limit (necessarily the same $D \in \mathcal{B}(U, Y)$) at infinity along the positive real axis. All weakly regular systems satisfy (1.9) in a weak sense, and the classical formulae such as $\widehat{\mathbb{D}}(s) = D + C(s - A)^{-1}B$ hold if we replace C by its *weak Weiss extension* C_w.

Regularity is an extremely important property, because feedthrough operators are of fundamental importance for much of the control theory. For example, optimal control problems are most often solved through Riccati equations that are written in terms of the generators of the system, including the (feedthrough) operator D.

For general WPLSs, equations (1.9) and the classical formulae such as $\widehat{\mathbb{D}}(s) = D + C_\mathrm{ext}(s - A)^{-1}B$ still hold in a very weak sense for certain *compatible* pairs (C_ext, D); their theory is developed in Section 6.3, which also contains additional results on different forms of regularity, on H^p transfer functions, on the relations between a WPLS and its generators and on reachability and observability.

In Sections 6.4 and 6.5, we define and study coprime, spectral and lossless factorizations. The importance of these factorizations is due to the equivalence on p. 21, with coprime factorization taking the place of spectral factorization in

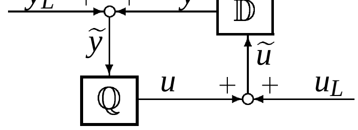


Figure 1.2: Dynamic output feedback controller $\mathbb{Q}$ for $\mathbb{D} \in \mathrm{TIC}_\infty(U,Y)$

"(III)" in the unstable case, and due to the strong connection between coprime factorization and dynamic stabilization. We also present two weak forms of coprimeness, which are useful in the infinite-dimensional settings, the weaker of them being invariant under (inverse) discretization and hence allowing us to reduce several results to the simpler discrete-time theory.

Thus, the connection between presentations (2.)–(5.) of p. 23 is established in Sections 6.1–6.5. Connection to (1.) is beyond the scope of this book. Instead, we study WPLS theory, with emphasis on Riccati equations and optimal control.

Sections 6.6 and 6.7 treat state feedback, output injection and static output feedback. Since our interest is not limited to exponential stabilization, but we often only require that the controller makes the closed-loop system stable or strongly stable (this has become increasingly popular lately), we meet certain additional difficulties.

In Section 6.8, we study systems whose semigroup is smoothing (e.g., $\mathbb{A}Bu_0 \in H$ a.e. on $\mathbf{R}_+$ for each $u_0 \in U$). In Section 6.9, we show that a transfer function $\widehat{\mathbb{D}}$ has a realization with bounded B iff $\widehat{\mathbb{D}} - \widehat{\mathbb{D}}(+\infty) \in \mathrm{H}^2_{\mathrm{strong}}$ over some right half-plane. We also establish analogous results for realizations with bounded C and for Pritchard–Salamon realizations.

Chapter 7: Dynamic Stabilization

In this chapter, we treat different forms of dynamic stabilization. In *dynamic output feedback* (Section 7.1), the output is fed back to the input through a *Dynamic Feedback Controller*, in order to stabilize and control the plant, as in Figure 1.2.

As one can verify from Figure 1.2, the map from the original input to the output of the plant $\mathbb{D} : u \mapsto y$ becomes $\mathbb{D}(I - \mathbb{Q}\mathbb{D})^{-1} : u_L \mapsto y$.

We have above treated only the I/O maps of the plant and of the controller. We shall also study the problem where the plant and the controller have to be stabilized internally too (see Figure 7.2), but most such results are obtained as corollaries of the I/O theory, since a controller stabilizes a system exponentially iff it I/O-stabilizes the system and both the system and the controller are optimizable and estimatable (this is an extension of the classical concept "exponentially stabilizable and exponentially detectable"), as shown in [WR00], cf. Theorem 7.2.3(c1).

The main new contributions of this section include the relations between external and internal stability of the controlled system (Theorems 7.2.3 and 7.2.4), particularly the extension of the equivalence (1.11); certain results of the internal loop theory required by the H^∞ 4BP theory, including the corollaries on dynamic partial feedback; and the relation between the stabilization of the controlled part and the stabilization of the whole plant in partial feedback (Lemmas 7.3.5 and 7.3.6 and Theorem 7.3.11).

In Chapter 7, we extend most classical results (such as the connection to coprime factorization and Youla parametrization of all stabilizing controllers) to the infinite-dimensional case and present some new results. For example, we extend (see Theorem 7.2.4(c)) the classical equivalence

$$\text{exponentially DF-stabilizable} \quad \Longleftrightarrow \quad \text{exponentially stabilizable and detectable} \tag{1.11}$$

to a large subclass on WPLS (including the parabolic systems of Section 9.5).

In Section 7.2, we study the more general *controllers with internal loop*, where $\mathbb{Q}$ need not be well-posed (i.e., proper), as long as the closed-loop system is still well-posed; classical "fractional $\mathrm{H}^\infty/\mathrm{H}^\infty$" controllers fall into this category. For example, if $\mathbb{D} \in \mathrm{TIC}_\infty(U,Y)$ has the *doubly coprime factorization (d.c.f.)* $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1} = \widetilde{\mathbb{M}}^{-1}\widetilde{\mathbb{N}}$, where $\mathbb{M},\mathbb{N},\widetilde{\mathbb{M}},\widetilde{\mathbb{N}} \in \mathrm{TIC}$, $\mathbb{M},\widetilde{\mathbb{M}} \in \mathcal{G}\mathrm{TIC}_\infty$, and

$$\begin{bmatrix} \mathbb{M} & \mathbb{Y} \\ \mathbb{N} & \mathbb{X} \end{bmatrix} \begin{bmatrix} \widetilde{\mathbb{X}} & -\widetilde{\mathbb{Y}} \\ -\widetilde{\mathbb{N}} & \widetilde{\mathbb{M}} \end{bmatrix} = \begin{bmatrix} I_U & 0 \\ 0 & I_Y \end{bmatrix} = \begin{bmatrix} \widetilde{\mathbb{X}} & -\widetilde{\mathbb{Y}} \\ -\widetilde{\mathbb{N}} & \widetilde{\mathbb{M}} \end{bmatrix} \begin{bmatrix} \mathbb{M} & \mathbb{Y} \\ \mathbb{N} & \mathbb{X} \end{bmatrix} \tag{1.12}$$

for some $\mathbb{X},\mathbb{Y},\widetilde{\mathbb{X}},\widetilde{\mathbb{Y}} \in \mathrm{TIC}$, then all stabilizing DF-controllers with internal loop for $\mathbb{D}$ are given by the *Youla parametrization*

$$(\mathbb{T}+\mathbb{M}\mathbb{U})(\mathbb{S}+\mathbb{N}\mathbb{U})^{-1} = (\widetilde{\mathbb{S}}+\widetilde{\mathbb{N}}\widetilde{\mathbb{U}})^{-1}(\widetilde{\mathbb{T}}+\widetilde{\mathbb{M}}\widetilde{\mathbb{U}}), \tag{1.13}$$

where the parameter $\mathbb{U}$ ranges over $\mathrm{TIC}(U)$ (Theorem 7.2.14). The controller (1.13) is well-posed iff $\mathbb{S}+\mathbb{N}\mathbb{U}$ (equivalently, $\widetilde{\mathbb{S}}+\widetilde{\mathbb{N}}\widetilde{\mathbb{U}}$) is invertible in TIC_∞. By shifting stability, we obtain an analogous result on exponential stabilization. We also give a series of results that do not require the plant to have a d.c.f.

Part of the results of Chapter 7 have been established earlier in the works of R. Curtain, R. Rebarber, G. Weiss, M. Weiss and others.

In Section 7.3 we study *dynamic partial output feedback (DPF)*, where the controller can access only a part of the output ("the measurement") and it can affect only part of the input, as in Figure 1.4. (see Figure 7.8 for the I/O part). Consequently, the map $\mathbb{D}_{12} : w \mapsto z$ from the external input w to the actual output z becomes

$$\mathcal{F}_\ell(\mathbb{D},\mathbb{Q}) := \mathbb{D}_{12} + \mathbb{D}_{11}\mathbb{Q}(I-\mathbb{D}_{21}\mathbb{Q})^{-1}\mathbb{D}_{22} \, : \, w \mapsto z \tag{1.14}$$

when the controller is applied to the system. All stabilizing DPF-controllers for $\mathbb{D} \in \mathrm{TIC}_\infty(U \times W, Z \times Y)$ are given by the Youla formula (1.13) applied to $\mathbb{D}_{21}$ in place of $\mathbb{D}$ if $\mathbb{D}$ satisfies standard "stabilizability and detectability" assumptions (by Lemma 7.3.6(b2)).

We list the corollaries of DF-stabilization theory for DPF-stabilization, as above, and present DPF-specific results (with and without internal loop, both I/O theory and state-space theory).

The above results and the further theory developed in Chapter 7 are used in Chapter 12 for the H^∞ *Four-Block Problem (*H^∞ *4BP)*, where one tries to find a stabilizing dynamic partial feedback controller that minimizes the norm of $w \mapsto z$

(or makes it less than a given constant $\gamma > 0$).

Part III: Riccati equations and Optimal control

This part contains a theory on optimal control (both in an abstract setting, and as an application to WPLSs) and Riccati equations, with applications to minimization (LQR and H^2) problems and to the H^∞ full-information and four-block problems.

Chapter 8: Optimal Control ($\frac{d}{du}\mathcal{J} = 0$)

We present an abstract theory on optimization and optimal control in state feedback form (Sections 8.1 and 8.2) and the application of this theory to WPLSs (Sections 8.3 and 8.4) with guidelines to problems finite time interval (Section 8.5) and to systems where the input operator (B) is allowed to be more unbounded than that of WPLSs (Section 8.6). We solve the generalized control problem, whose (possibly indefinite) cost function covers most standard control problems.

Our main contributions include the generalization of the classical coercivity assumption to general WPLSs and cost functions, and the fact that this assumption leads to a solution of the generalized control problem (see Theorems 8.4.3 and 8.3.9); this was already extended to stable WPLSs by O. Staffans. An important part of our theory are also the methods to treate simultaneously all forms of stabilization (i.e., whether one requires the "optimal control" to be, e.g., exponentially, strongly or merely output-stabilizing). These results will then be applied in the derivation of the Riccati equation, LQR and H^∞ theories in the chapters to follow.

We study the critical points of a given cost function and the case where such control corresponds to a stabilizing state feedback pair. Such an "optimal" state feedback pair corresponds to a "stabilizing" solution of the Riccati equation, as shown in Chapter 9. The corresponding special control problems are solved in Chapters 10–12.

Given a WPLS $\left[\begin{smallmatrix}\mathbb{A} & \mathbb{B}\\ \mathbb{C} & \mathbb{D}\end{smallmatrix}\right]$ and a cost operator $J = J^* \in \mathcal{B}(Y)$, we consider the *cost function*

$$\mathcal{J}(x_0,u) := \int_0^\infty \langle y(t), Jy(t)\rangle_Y \, dt, \quad \text{where} \quad y := \mathbb{C}x_0 + \mathbb{D}u \quad (x_0 \in H,\ u : \mathbf{R}_+ \to U) \tag{1.15}$$

and u is required to be exponentially stabilizing, strongly stabilizing, stabilizing or something similar, depending on how stable one wishes the closed-loop system to be.

This covers all quadratic (definite or indefinite) costs on the input, state and output (extend $\mathbb{C}$ and $\mathbb{D}$ suitably if necessary, e.g., replace $\mathbb{C}$ by $\left[\begin{smallmatrix}\mathbb{C}\\ 0\end{smallmatrix}\right]$ and $\mathbb{D}$ by $\left[\begin{smallmatrix}\mathbb{D}\\ I\end{smallmatrix}\right]$ to cover cross terms of u and y). In particular, minimization, H^∞ and similar control problems are covered. The solutions of such problems correspond to the controls that are critical points of $\mathcal{J}$, i.e., for which the Fréchet derivative of $\mathcal{J}(x_0,\cdot)$ is zero; we call such controls *J-critical*.

In Section 8.4, we define and study *J-coercivity*, which is a generalization of the standard nonsingularity assumptions of several control problems (including the "*J*-coercivity" assumptions defined in [S97b]–[S98d], the "Popov Toeplitz invertibility" condition in the stable case and the "no transmission zeros" and "no invariant zeros" conditions in the positive case). We show that any "stabilizable" *J*-coercive WPLS has a unique *J*-critical ("optimal") control for each initial state, and that this *J*-critical control can be presented in WPLS form (this generalizes the corresponding result in [FLT]).

However, the corresponding feedback need not be well-posed without additional assumptions on the system, as illustrated in Examples 8.4.13 and 11.3.7. This leads to some additional difficulties in the Riccati equation theory (the situation is the same even in the case studied in [FLT]). Sections 8.3 and 8.4 also contains a series a further results on *J*-critical controls and *J*-coercivity and on the connection of the latter to spectral and coprime factorizations.

The control problems for unstable systems are traditionally reduced to the stable case by preliminary stabilization, when the optimal control is required to be exponentially stabilizing. We show that this is possible for WPLSs too, give a counter-example for other forms of stabilization and develop more complicated tricks to overcome this problem (Theorem 8.4.5).

In the last two section of Chapter 8, we give guidelines on how to extend our optimization and Riccati equation results for problems on finite time interval (Section 8.5) and for more general systems than WPLSs (Section 8.6). These results are not used elsewhere in this monograph.

Chapter 9: Riccati Equations and *J*-Critical Control

It was shown independently in [WW] and [S97b]–[S98d] that, in the (stable) regular case, the optimal cost operator of certain control problems satisfies a generalized (operator) Riccati equation. We established the converse implication from a stabilizing solution of the Riccati equation to the existence of an optimal control in [Mik97b]. In Chapter 9, we extend both results to the general optimization context of Chapter 8, thus covering also general unstable systems and more singular problems (under weaker regularity assumptions).

We also simplify the equation and the assumptions in several special cases, present a priori sufficient assumptions for the required regularity, and provide weaker results for less regular settings. Moreover, the connection to spectral or coprime factorization and further aspects (such as uniqueness, Riccati inequalities and certain pathologies) are addressed. Possibly ill-posed or irregular optimal controls and corresponding generalized Riccati equations are covered in Section 9.7 (for bounded output operators, a special case of this was solved in [FLT]). We describe below the main results of this chapter.

The existence of a unique regular optimal state feedback operator for a regular WPLS is equivalent to the existence of a (necessarily unique) $\mathcal{U}_*^*$-stabilizing solution of the *Continuous-time Algebraic Riccati Equation (CARE)* and from one the other can be computed (see Theorem 9.9.1; read "optimal" as "*J*-critical"). This extends most similar results in the literature.

When we optimize over exponentially stabilizing controls or state feedback operators, the term "$\mathcal{U}_*^*$-stabilizing" is equivalent to "exponentially stabilizing" (a WPLS is exponentially stable iff its semigroup $\mathbb{A}$ satisfies $\|\mathbb{A}(t)\|_H \leq Me^{\omega t}$ $(t \geq 0)$ for some $\omega < 0$, $M < \infty$). To make things easier, we illustrate this under rather strong assumptions:

Theorem 1.2.6 ($\mathcal{U}_{\mathbf{exp}}$: Unique minimum $\Leftrightarrow$ B_{w}^*-CARE$\Leftrightarrow$ J-coercive) *Assume that the WPLS* $\left[\begin{smallmatrix}\mathbb{A} & \mathbb{B}\\ \mathbb{C} & \mathbb{D}\end{smallmatrix}\right]$ *and the cost operator* $J = J^* \in \mathcal{B}(Y)$ *are s.t.* $\pi_{[0,1)}\mathbb{A}B \in \mathrm{L}^1([0,1);\mathcal{B}(U,H))$, $C \in \mathcal{B}(H,Y)$ *and* $D^*JD \gg 0$. *Then the following are equivalent:*

(i) There is a unique minimizing exponentially stabilizing state feedback operator.

(ii) There is a unique minimizing control over $\mathcal{U}_{\mathrm{exp}}(x_0) := \{u \in \mathrm{L}^2(\mathbf{R}_+;U) \,\big|\, x \in \mathrm{L}^2(\mathbf{R}_+;H)\}$ *for each initial state* $x_0 \in H$.

(iii) The Riccati equation

$$(B_{\mathrm{w}}^*\mathcal{P} + D^*JC)^*(D^*JD)^{-1}(B_{\mathrm{w}}^*\mathcal{P} + D^*JC) = A^*\mathcal{P} + \mathcal{P}A + C^*JC \qquad (1.16)$$

has a solution $\mathcal{P} = \mathcal{P}^* \in \mathcal{B}(H)$ *s.t.* $\mathcal{P}[H] \subset \mathrm{Dom}(B_{\mathrm{w}}^*)$ *and the semigroup generated by* $A - BK$ *is exponentially stable, where* $K := -(D^*JD)^{-1}(B_{\mathrm{w}}^*\mathcal{P} + D^*JC)$.

(iv) Σ *is optimizable and* $\mathbb{D}$ *is J-coercive over* $\mathcal{U}_{\mathrm{exp}}$.

(v) Σ *is exponentially stabilizable and there is* $\varepsilon > 0$ *satisfying*

$$(ir - A)x_0 = Bu_0 \Rightarrow \langle Cx_0 + Du, J(Cx_0 + Du)\rangle \geq \varepsilon\|x_0\|_H^2 \quad (x_0 \in H,\ u_0 \in U,\ r \in \mathbf{R}).$$

If (iii) holds, then K is bounded ($K \in \mathcal{B}(H,U)$) *and it is the unique minimizing exponentially stabilizing state feedback operator. The minimal cost equals* $\langle x_0, \mathcal{P}x_0\rangle$ *for each* $x_0 \in H$. □

(This is a special case of Corollary 10.2.9 combined with Theorem 9.2.3.)

Thus, the optimal control corresponds to the state feedback $u(t) = Kx(t)$ $(t \geq 0)$, where K is as above. Here B_{w}^* denotes the Weiss extension of $B^* \in \mathcal{B}(\mathrm{Dom}(A^*), U)$. The Riccati equation (1.16) is given on $\mathrm{Dom}(A)$ (see (9.14)). See (1.17) for the more complicated general CARE.

When $\mathcal{J}(x_0,u) = \|Cx\|_2^2 + \|u\|_2^2$, i.e., $C = \left[\begin{smallmatrix} C_1 \\ 0\end{smallmatrix}\right]$, $D = \left[\begin{smallmatrix} 0 \\ I\end{smallmatrix}\right]$, $J = I$, then (1.16) becomes $(B_{\mathrm{w}}^*\mathcal{P})^*B_{\mathrm{w}}^*\mathcal{P} = A^*\mathcal{P} + \mathcal{P}A + C^*C$, the minimizing feedback is given by $u(t) = -B_{\mathrm{w}}^*\mathcal{P}x(t)$ $(t \geq 0)$, and the closed-loop semigroup is generated by $A + BK = A - BB_{\mathrm{w}}^*\mathcal{P}$.

As explained on p. 27, we can have cross terms of u and y in the cost, e.g., replace C by $\left[\begin{smallmatrix} C \\ 0\end{smallmatrix}\right]$ and D by $\left[\begin{smallmatrix} D \\ I\end{smallmatrix}\right]$ to obtain another WPLS and, correspondingly, a "more general" (actually, less general) "standard" form of the Riccati equation, as in, e.g., Remark 9.1.14.

However, the theory of Section 8.3 also allows optimization over various other sets ("$\mathcal{U}_*^*$") of controls than $\mathcal{U}_{\mathrm{exp}}$, e.g., for those which make the state and output strongly stable for each initial state ("$\mathcal{U}_{\mathrm{str}}$"). Correspondingly, the regular optimal state feedback operator (if any) over $\mathcal{U}_{\text{str}}$ corresponds to the unique solution of the CARE that is $\mathcal{U}_{\text{str}}$-stabilizing, i.e., that stabilizes the state and output strongly for each initial state.

In the literature of infinite-dimensional systems, it has become popular to only require that the output is stable for each initial state and possibly also for each stable external input to the feedback loop. In this case the condition "$\mathcal{U}_*^*$-stabilizing" becomes rather complicated (Definition 9.8.1).

If the system is exponentially detectable, then all the cases mentioned above (and certain others) coincide with exponential stabilization, but this assumption is sometimes too strong. If the system is "coprime stabilizable" (in a suitable, rather weak sense; this assumption always holds when the system is output stable (resp. stable, strongly stable)), then optimization over output-stabilizable (resp. stabilizable, strongly stabilizable) controls corresponds to the "coprime stabilizing" solution of the CARE, and the equivalence of (I)–(IV) on p. 21 holds, see Section 9.1 for details. However, this solution need not be exponentially stabilizing, and the same CARE may also have an exponentially stabilizing solution (see Example 9.13.2; naturally, in a minimization problem the optimal cost becomes higher for stronger stabilizability requirements). Part of these results seem to be new even for finite-dimensional systems.

Very regular systems, such as those of Theorem 1.2.6, are studied in Section 9.2. For them the CARE becomes rather elegant and similar to its finite-dimensional counterparts, as part (iii) of the theorem shows. Such systems cover analytic systems (hence most parabolic-type problems) having rather unbounded input and output operators, as shown in Section 9.5.

In the general case, the optimal control need not correspond to a (well-posed) state feedback operator, as explained in Chapter 8. Nevertheless, such control corresponds to a generalized Riccati equation, as illustrated in Section 9.7 (for WPLSs with a bounded output operator ("C") and a rather coercive cost function, this was shown in [FLT] by F. Flandoli, I. Lasiecka and R. Triggiani). However, since these equations are given on the (unknown) domain of the closed-loop semigroup generator rather than on $\text{Dom}(A)$, it becomes very difficult to solve the Riccati equation and thus obtain the (possibly non-well-posed) feedback operator.

As mentioned above, the existence of a (well-posed) regular state feedback operator for a regular WPLS is equivalent to the CARE having a solution, but in this general case the CARE becomes rather complex: we have to find $\mathcal{P} = \mathcal{P}^* \in \mathcal{B}(H)$ satisfying

$$\begin{cases} K^*SK = A^*\mathcal{P} + \mathcal{P}A + C^*JC & \in \mathcal{B}(\text{Dom}(A), \text{Dom}(A)^*) \\ S = D^*JD + \operatorname*{w-lim}_{s\to+\infty} B_\text{w}^*\mathcal{P}(s-A)^{-1}B & \in \mathcal{B}(U) \\ SK = -(B_\text{w}^*\mathcal{P} + D^*JC) & \in \mathcal{B}(\text{Dom}(A), U). \end{cases} \tag{1.17}$$

Obviously, S and K are uniquely determined by $\mathcal{P}$ if S is one-to-one, which corresponds to a unique optimal control. The optimal state feedback is given by $u(t) = K_\text{w}x(t)$ for a.e. $t \geq 0$. See Definition 9.1.5 for details (and Definition 9.8.1 for noninvertible signature operators).

Note that whereas the special case (1.16) is close the finite-dimensional

CARE, this general form looks almost like the discrete-time Riccati equation; in particular, the *signature operator* S may differ from D^*JD, as observed in [S97b] and [WW]. In the notes to Section 9.8 we explain how the signature properties of the problem are determined by S, not by D^*JD, even when the I/O map is a simple delay. Thus, the situation is analogous to the (finite-dimensional) discrete-time setting, where the signature operator $S := D^*JD + B^*\mathcal{P}B$ takes the role of D^*JD.

We also list several cases in which the CARE can be simplified and cases in which an optimal control is always given by a well-posed regular state feedback pair (and hence corresponds to a CARE; see, e.g., Remark 9.9.14).

The optimal control is given by a well-posed state feedback iff the *Integral Algebraic Riccati Equation (IARE)* has an $\mathcal{U}_*^*$-stabilizing solution, regardless of regularity. While IAREs are not particularly apt for engineering purposes, they provide a link to discrete-time Riccati equations, and this allows us the prove several results whose continuous-time proofs would seem intractable due to the unboundedness of input and output operators. The IAREs also allow us to treat the connection between optimal control and Riccati equations separately from regularity considerations. Naturally, for regular WPLSs, the solutions of the CARE are exactly the solutions of the IARE corresponding to regular feedback. Also these questions are addressed in Section 9.8. Several further properties of Riccati equations are treated in the rest of the chapter. Much of our theory concerning for $\mathcal{U}_*^* \neq \mathcal{U}_{\text{exp}}$ is new even for finite-dimensional systems.

In Section 9.14, we give an extension of the generalized canonical factorization theory to the case of infinite-dimensional input and output spaces (see also p. 148).

Chapter 10: Quadratic Minimization (LQR)

For control problems with a positive Popov operator, one traditionally shows that under certain conditions any solution of the Riccati equation is unique, admissible and exponentially stabilizing. One of our main contributions in this and preceding chapter is the extension of the above fact to WPLSs and partially also to the non-exponentially stabilizing case; this is technically very challenging due to the unbounded input and output operators, which, e.g., make it hard to show when the "optimal feedback" is well posed.

As corollaries, we get several results that formally look like the classical ones. These corollaries include Theorem 1.2.7 below, (b4)&(c1)&(c2) of Theorem 10.1.4, the Strict Bounded and Strictly Positive (Real) Lemmas, and the equivalence between optimizability and exponential stabilizability for systems with a smoothing semigroup (Theorem 9.2.12). We also solve several minimization problems with more general stabilizability or regularity assumptions.

Important new contributions of the chapter also include the connection between different classical coercivity assumptions and their generalizations to WPLSs, including J-coercivity (Section 10.3).

In Section 10.2, we study minimization problems, by which we refer to the minimization of the cost function (1.15). Theorem 1.2.6 is a corollary of that section.

In Section 10.1 we study the special case of the cost function $\|y\|_2^2 + \|u\|_2^2$ and its variants. Under a mild detectability condition, there is at most one nonnegative solution of the CARE, hence we do not have to verify whether a solution is "$\mathcal{U}_*^*$-stabilizing":

Theorem 1.2.7 (LQR: $\min_u \int_0^\infty (\|x\|_H^2 + \|u\|_U^2)$) *Assume that the WPLS $\Sigma = \left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right]$ is uniformly regular (UR) and estimatable (e.g., that C is bounded and $C^*C \gg 0$). Consider, for some $R \in \mathcal{B}(U)$, $Q \in \mathcal{B}(Y)$ s.t. $R, Q \gg 0$, the cost function*

$$\mathcal{J}(x_0, u) := \int_0^\infty \big(\langle y(t), Qy(t)\rangle_Y + \langle u(t), Ru(t)\rangle_U\big)\, dt \quad (x_0 \in H,\ u \in \mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}_+; U)). \tag{1.18}$$

There is a UR minimizing state feedback operator for Σ iff there is a nonnegative solution $\mathcal{P} \in \mathcal{B}(H)$ satisfying the CARE

$$\begin{cases} K^*SK = A^*\mathcal{P} + \mathcal{P}A + C^*QC, \\ \quad S = R + D^*QD + \lim_{s\to+\infty} B_{\mathrm{w}}^*\mathcal{P}(s-A)^{-1}B, \\ \quad K = -S^{-1}(B_{\mathrm{w}}^*\mathcal{P} + D^*QC), \end{cases} \tag{1.19}$$

for some $K \in \mathcal{B}(H_1, U)$, $S \in \mathcal{B}(U)$, $S \gg 0$.

If such a solution exists, then it is the unique nonnegative solution of (1.19), K is a UR exponentially stabilizing state feedback operator for Σ, and K is the unique minimizing state feedback operator over all $u \in \mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}_+; U)$ (and over $\mathcal{U}_{\mathrm{exp}}$ and over $\mathcal{U}_{\mathrm{out}}$). □

(This follows from Theorem 10.1.4 and Remark 10.1.5.) For each CARE result in this monograph, including the one above, there is also a "B_{w}^*-CARE" variant that allows us to remove the limit term and simplify the formulation under any of the regularity assumptions of Hypothesis 9.2.2, as illustrated in Theorem 1.2.6.

Without the detectability (estimatability) condition, we observe that a minimizing state feedback operator over $\mathcal{U}_{\mathrm{exp}}$ corresponds to the maximal nonnegative solution of the CARE and a minimizing state feedback operator over $\mathcal{U}_{\mathrm{out}}$ corresponds to the minimal nonnegative solution of the CARE (Theorem 10.1.4). We also derive further results on such and more general minimization problems.

In Section 10.4 we show that the solution of the minimization problem leads to the solution of the H^2 full information and state feedback problems, where one wishes to find a controller ($\mathbb{Q}$; possibly induced by state feedback or dynamic output feedback) that minimizes the norm

$$\|\widehat{\mathbb{D}}\widehat{\mathbb{Q}} + \widehat{\mathbb{D}}_2\|_{\mathrm{H}^2_{\mathrm{strong}}(\mathbf{C}^+; \mathcal{B}(W,Y))}, \tag{1.20}$$

where $\widehat{\mathbb{Q}} : \widehat{w} \to \widehat{u}$ is the frequency-domain control law (determined by $\mathbb{Q}$) to an external (disturbance) input to the control input for the WPLS

$$\left[\begin{array}{c|cc} \mathbb{A} & \mathbb{B} & \mathbb{B}_2 \\ \hline \mathbb{C} & \mathbb{D} & \mathbb{D}_2 \end{array}\right] \quad \text{with generators} \quad \left[\begin{array}{c|cc} A & B & B_2 \\ \hline C & D & 0 \end{array}\right], \tag{1.21}$$

as in Figure 10.1. The above WPLS is obtained by adding a second input to the WPLS Σ, We assume that both $\mathbb{D}$ and $\mathbb{D}_2$ are WR. A stronger problem is to find, for each $w_0 \in W$, a "stabilizing" control u s.t. $\|\widehat{\mathbb{D}}\widehat{u} + \widehat{\mathbb{D}}_2 w_0\|_{\mathrm{H}^2(\mathbf{C}^+;Y)}$ is minimized, see Section 10.4 for details. We show that under minimal assumptions, a minimizing state feedback operator for the original system also solves the H^2 problem and the stronger problem formulated above.

In Section 10.3 we treat most standard assumptions for classical minimization problems and show that they are stronger than or equivalent to positive J-coercivity (over $\mathcal{U}_{\mathrm{exp}}$ or over $\mathcal{U}_{\mathrm{out}}$).

In Section 10.5, we present generalized versions of the Bounded Real Lemma, including the following:

Theorem 1.2.8 (Generalized Strict Bounded Real Lemma) *Assume that* $\gamma > 0$.

If C *is bounded and* $\dim Y < \infty$*, or if* B *is bounded, then the following are equivalent:*

(i) Σ *is exponentially stable and* $\|\mathbb{D}\| < \gamma$*;*

(ii) There is $\mathcal{P} \leq 0$ *s.t.* $\mathcal{P}[H] \subset \mathrm{Dom}(B_{\mathrm{w}}^*)$ *and*

$$\begin{bmatrix} A^*\mathcal{P} + \mathcal{P}A - C^*C & (B_{\mathrm{w}}^*\mathcal{P} - D^*C)^* \\ B_{\mathrm{w}}^*\mathcal{P} - D^*C & \gamma^2 I - D^*D \end{bmatrix} \gg 0 \quad on \quad \mathrm{Dom}(A) \times U. \tag{1.22}$$

Moreover, any solution of (ii) satisfies $\mathcal{P} < 0$.

In the Strict Positive Real Lemma, we present analogous conditions for the I/O map to satisfy $\mathbb{D} \in \mathrm{TIC}$ and $\mathrm{Re}\langle \mathbb{D}\cdot, \cdot\rangle \gg 0$ (i.e., $\widehat{\mathbb{D}} + \widehat{\mathbb{D}}^* \geq \varepsilon I$ in $\mathrm{L}^\infty_{\mathrm{strong}}(i\mathbf{R}; \mathcal{B}(U,Y))$ for some $\varepsilon > 0$). Naturally, there are also analogous results for unbounded B and C.

In Section 10.6, we present necessary and sufficient conditions for the uniform positivity of the Popov operator ($\mathbb{D}^* J \mathbb{D} \gg 0$), in terms of spectral factorizations and Riccati equations or inequalities. Section 10.7 present additional results for positive Riccati equations (say, with positive signature operator, $S \gg 0$).

Chapter 11: The H^∞ Full-Information Control Problem (FICP)

The H^∞ control problems refer to the minimization of the output of a plant in the presence of a disturbance input. The name "H^∞" comes from the minimization of the (controlled closed-loop system) $\mathrm{L}^2 \to \mathrm{L}^2$ norm from disturbance to output, which equals the H^∞ norm of the corresponding transfer function, by Theorem 1.2.1.

In the FICP, studied in this chapter, we can produce the control signal knowing exactly the state and disturbance of the system, whereas in the Four-Block Problem of Chapter 12 the controllers only input is a separate measurement output.

Our main results state that, given $\gamma > 0$, there is a controller achieving a norm less than γ iff the Riccati equation (1.24) has a nonnegative stabilizing solution. Moreover, if this is the case, then there is such a controller that consists of pure

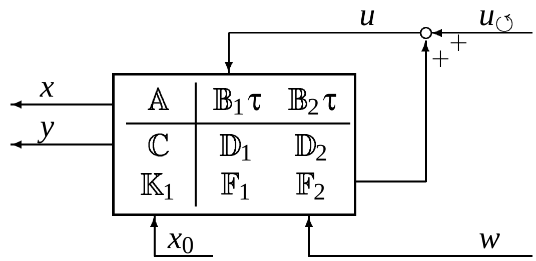


Figure 1.3: The H^∞ FICP

state feedback (with no measurement of the disturbance). This generalizes the classical results to this problem. For WPLSs, O. Staffans had already proved the necessity part of the implication for stable systems having an L^1 impulse response. We also formulate the solution in terms of J-lossless factorizations and solve the corresponding discrete-time problem.

Technically, we study a system of form $\Sigma = \left[\begin{array}{c|cc} \mathbb{A} & \mathbb{B}_1 & \mathbb{B}_2 \\ \hline \mathbb{C} & \mathbb{D}_1 & \mathbb{D}_2 \end{array}\right]$, with input space $U \times W$ instead of U. If $\mathbb{D} = \begin{bmatrix} \mathbb{D}_1 & \mathbb{D}_2 \end{bmatrix} \in \mathrm{TIC}_\infty(U \times W, Y)$ is regular, we can write this as

$$\begin{cases} x' = Ax + B_1 u + B_2 w, \\ y = Cx + D_1 u + D_2 w, \end{cases} \tag{1.23}$$

(if $B = \begin{bmatrix} B_1 & B_2 \end{bmatrix} \in \mathcal{B}(U \times W, H_{-1})$ and $C \in \mathcal{B}(\mathrm{Dom}(A), Y)$ are unbounded, then the dynamics (1.23) are satisfied only in the sense described in Theorem 6.2.13).

We have divided the input space in two to model a setting where only part of the input (called the *control*), $u : \mathbf{R}_+ \to U$, is accessible by the controller, whereas the other part represents the *disturbance* (or uncertainties, sensor noise, modeling error) $w : \mathbf{R}_+ \to W$ to the system. The signal y is the *objective* or *error signal* whose norm is to be minimized.

In the optimal H^∞ *State-Feedback Control Problem (SFCP)*, one wishes to find a (pure) state feedback controller of form "$u(t) = Kx(t)$" (with e.g., $K \in \mathcal{B}(H, U)$) such that this feedback stabilizes the system exponentially and minimizes the norm $\|w \mapsto y\|_{\mathcal{B}(L^2, L^2)}$. In the optimal H^∞ *Full-Information Control Problem (FICP)*, the controller is allowed to be of form "$u(t) = K(t)x + F_2 w(t)$" (state feedback plus feedforward), as in Figure 1.3. (Here $\left[\begin{array}{c|cc} \mathbb{K} & \mathbb{F}_1 & \mathbb{F}_2 \end{array}\right]$ is the state feedback pair generated by K or $\left[\begin{array}{c|cc} K & 0 & F_2 \end{array}\right]$, and the signal $u_{\circlearrowleft}$ represents the external disturbances (or external inputs) in the feedback loop. The words "full information" refer to the fact that the controller has access to both the state and the disturbance.)

There is no direct method available (even in the finite-dimensional case) to find the exact optimum. Therefore, instead of the optimal problem, the corresponding *suboptimal* H^∞ *problem* is usually treated in the literature. In the suboptimal H^∞ problem, we search for an exponentially stabilizing controller such that $\|w \mapsto y\|_{\mathcal{B}(L^2, L^2)} < \gamma$, where $\gamma > 0$ is a given constant; such a controller is called *(γ-)suboptimal*. We extend the classical results by showing that there is a suboptimal state feedback controller iff the Riccati equation condition (iii) below is satisfied. By varying γ we can then find an estimate of the infimal γ and a corresponding (almost optimal) controller (e.g., by a binary search over γ's).

As mentioned above, under standard coercivity assumptions and certain regularity and normalization conditions (see, e.g., Theorem 11.1.4), the following are equivalent:

(i) there is a suboptimal control law $w \mapsto u$, and (A, B_1) is exponentially stabilizable;

(ii) there is a suboptimal state-feedback (plus feedforward) controller $u = K_w x + F_2 x$;

(ii') there is suboptimal pure state-feedback controller $u = K_w x$;

(iii) the Riccati equation

$$\mathcal{P}(B_1 B_1^* - \gamma^{-2} B_2 B_2^*)\mathcal{P} = A^*\mathcal{P} + \mathcal{P}A + C^*C, \tag{1.24}$$

(on $\mathrm{Dom}(A)$) has a nonnegative solution $\mathcal{P} \in \mathcal{B}(H)$ such that $A - (B_1 B_1^* - \gamma^{-2} B_2 B_2^*)\mathcal{P}$ generates an exponentially stable C_0-semigroup.

Moreover, if (iii) holds, then $K := -B_1^*\mathcal{P} \in \mathcal{B}(H, U)$ determines a suboptimal (pure) state-feedback controller for Σ (through $u(t) := Kx(t)$ $(t \geq 0)$). A solution $\mathcal{P}$ of (iii) is unique.

Here we have assumed that B is bounded, $D_2 = 0$ and $D_1^* \begin{bmatrix} C & D_1 \end{bmatrix} = \begin{bmatrix} 0 & I \end{bmatrix}$; see, e.g., (11.24) and (11.17) for the unsimplified forms of (iii) and K. (Also without the above simplifying assumptions, the suboptimal state feedback operator K is exponentially stabilizing (and uniformly regular, though not necessarily bounded), but we must add a signature condition to (iii); moreover, condition (ii') becomes strictly stronger than the other conditions (which remain equivalent to each other) unless a stronger signature condition is satisfied.)

We present analogous results under different regularity assumptions, and variants for $\mathcal{U}_{\mathrm{out}}$, $\mathcal{U}_{\mathrm{sta}}$ and $\mathcal{U}_{\mathrm{str}}$, i.e., where the suboptimal controller needs to be, e.g., merely strongly stabilizing instead of exponentially stabilizing. We also establish the sufficiency of the Riccati equation condition for arbitrary regular WPLSs (see Lemma 11.2.13). In Example 11.3.7(c), we show that, however, this condition is not necessary for general regular WPLSs.

In (i), we have allowed for an arbitrary control law $\mathrm{L}^2(\mathbf{R}_+; W) \mapsto \mathrm{L}^2(\mathbf{R}_+; U)$. If such a control law $\mathbb{Q} : w \mapsto u$ has a transfer function (e.g., $\mathbb{Q} \in \mathrm{TIC}_\infty(W, U)$), then the norm $\|w \mapsto y\|$ equals $\|\mathbb{D}_1 \mathbb{Q} + \mathbb{D}_2\|_{\mathrm{TIC}(W,Y)}$, or $\|\widehat{\mathbb{D}}_1 \widehat{\mathbb{Q}} + \widehat{\mathbb{D}}_2\|_{\mathrm{H}^\infty(\mathbf{C}^+; \mathcal{B}(W,Y))}$. By the above equivalence, this problem, the FICP and the SFCP are all equivalent (under simplifying assumptions and suitable regularity). Thus, if there is any suboptimal control law (and (A, B_1) is exponentially stabilizable), then there is actually a causal, linear, stable, time-invariant control law that can be implemented as an exponentially stabilizing state feedback controller (so that $\mathbb{Q} = (I - \mathbb{F}_1)^{-1}\mathbb{F}_2$). Condition (i) can also be formulated as a minimax problem, as explained in Section 11.1 (particularly on pp. 613 and 626).

In Section 11.2, we give proofs and additional variants for the above results, and we extend the (frequency-domain) J-lossless factorization results for the H^∞ FICP given in [Green] and [CG97] to MTIC and similar classes (Theorem 11.2.7).

The *Discrete-Time* H^∞ *FICP* is treated in Section 11.5, and the abstract H^∞ FICP in Section 11.7.

The H^∞ FICP is interesting both for its own merits and for the fact that it can be used to obtain a solution to the H^∞ 4BP presented below.

The methods used for the stable H^∞ FICP also apply to the (one-block) Nehari problem, where one wishes to estimate $d(\mathbb{D}, \mathrm{TIC}^*)$ or the Hankel norm $\|\pi_+\mathbb{D}\pi_-\|$ of some $\mathbb{D} \in \mathrm{TIC}$. Therefore, we take a brief look at this problem in Section 11.8, this includes the following:

Theorem 1.2.9 (Nehari) *Let* $\mathbb{D} \in \mathrm{MTIC}(W,U)$ *and* $\gamma > 0$*. If* $\dim U \times W < \infty$ *or* $\mathbb{D} \in \mathrm{MTIC}_{T\mathbf{Z}}$*, then the following are equivalent:*

(i) There is $\mathbb{U} \in \mathrm{TIC}(U,W)$ *s.t.* $\|\mathbb{D} + \mathbb{U}^*\|_{\mathcal{B}(\mathrm{L}^2)} < \gamma$ *(i.e.,* $d(\mathbb{D}, \mathrm{TIC}^*) < \gamma$*).*

(ii) The Hankel norm $\|\pi_+\mathbb{D}\pi_-\|$ *of* $\mathbb{D}$ *is less than* γ*.*

(iii) There is $\mathbb{X} \in \mathcal{G}\mathrm{TIC}(U \times W)$ *s.t.* $\mathbb{X}_{11} \in \mathcal{G}\mathrm{TIC}(U)$ *and* $\begin{bmatrix} I & \mathbb{D} \\ 0 & I \end{bmatrix}^* \begin{bmatrix} I & 0 \\ 0 & -\gamma^2 I \end{bmatrix} \begin{bmatrix} I & \mathbb{D} \\ 0 & I \end{bmatrix} = \mathbb{X} \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \mathbb{X}^*$*.*

(Recall that $\mathbb{D} \in \mathrm{MTIC}_{T\mathbf{Z}}$ means that $\mathbb{D}$ has an L^1 impulse response plus delays of form $\sum_{k=0}^\infty D_k \tau^{-kT}$ for some period $T > 0$.)

The factorization in (iii) is often called a co-spectral factorization. The norm $\|\pi_+\mathbb{D}\pi_-\|$ equals $\rho(\mathbb{B}\mathbb{B}^*\mathbb{C}^*\mathbb{C})^{1/2}$, where $\mathbb{B}\mathbb{B}^*$ and $\mathbb{C}^*\mathbb{C}$ are the reachability and observability Gramians, respectively, of any realization of $\mathbb{D}$ having stable input and output maps.

We do not treat the Nehari Riccati equations, since their theory would require lengthy additions to Chapter 9 due to the noncausality of the corresponding “closed-loop systems”.

Chapter 12: H^∞ Four-Block Problem ($\|\mathcal{F}_\ell(\mathbb{D}, \mathbb{Q})\| < \gamma$)

In the H^∞ *Four-Block Problem (*H^∞ *4BP)* (aka. “the standard H^∞ problem” or “the general regulator problem”), one tries to find a DPF-controller that makes the norm $w \mapsto z$ less than a given constant $\gamma > 0$ (see (1.14)), i.e., γ-*suboptimal*.

Consequently, as explained above, the difference to the H^∞ FICP is that now the controller does not have access to the disturbance, only to a part of the output (“the measurement”), as in Figure 1.4 (or in Figure 7.8; see Figures 7.10 and 7.11 for DPF-controllers with internal loop).

Thus, the goal of the engineer is again to minimize the norm from the external disturbance input to the objective output of the system. As in the previous chapter, we again generalize the classical result (previously generalized to Pritchard–Salamon systems by B. van Keulen [Keu]) that there is a γ-suboptimal exponentially stabilizing (measurement feedback) controller iff certain two independent Riccati equations have exponentially stabilizing nonnegative solutions and these (necessarily unique) solutions satisfy the standard spectral radius condition. We formulate the result also in terms of two nested J-lossless factorizations and solve the H^∞ discrete-time Four-Block Problem; in fact these two generalizations of classical results serve as parts of our lengthy proof.

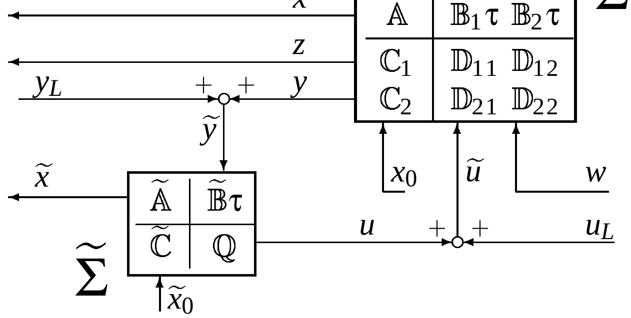


Figure 1.4: DPF-controller $\widetilde{\Sigma}$ for $\Sigma \in \mathrm{WPLS}(U \times W, H, Z \times Y)$

As in Section 7.3, the output "y" is now divided in two, namely "y"$= \left[\begin{smallmatrix} z \\ y \end{smallmatrix}\right]$, where z is the objective output to be minimized and y is a measurement that is fed into the controller. This corresponds to the dynamics

$$\begin{cases} x' = Ax + B_1 u + B_2 w, \\ z = C_1 x + D_{11} u + D_{12} w, \\ y = C_2 x + D_{21} u + D_{22} w, \end{cases} \tag{1.25}$$

with initial state $x_0 \in H$, disturbance input $w \in \mathrm{L}^2(\mathbf{R}_+;W)$, control input $u \in \mathrm{L}^2(\mathbf{R}_+;U)$, objective output $z \in \mathrm{L}^2(\mathbf{R}_+;Z)$ and measurement output $y \in \mathrm{L}^2(\mathbf{R}_+;Y)$ (the controller input). In the case of a general weakly regular system, equations (1.25) hold in the strong sense, see Theorem 6.2.13 for details.

We are then to find a controller $\mathbb{Q} : y \mapsto u$ s.t. the norm $w \mapsto z$ becomes small enough and that the closed-loop connection becomes exponentially stable (that is the main case; we only treat the case where the closed-loop system is merely required to be stable or strongly stable).

(We remind that the order of the subindices corresponding to u and w is often reversed in the literature; this also affects the formulae below.)

In Section 12.1, we present several versions of the standard result that the H^∞ 4BP has a solution iff the two H^∞ Riccati equations have nonnegative exponentially stabilizing solutions satisfying the coupling condition. Since we do not use any simplifying assumptions, our formulae become rather complicated. Therefore, we show here the simplified forms of those formulae (by making additional assumptions):

Theorem 1.2.10 ($\mathbf{H}^\infty$ 4BP) *Let $\gamma > 0$. Make the regularity and nonsingularity assumptions (A1)&(A2) of Theorem 12.1.4.*

Then there is an exponentially stabilizing DPF-controller for Σ (possibly with internal loop) satisfying $\|w \mapsto z\| < \gamma$ iff (1.)–(3.) of Theorem 12.1.4 hold. Under the normalizing conditions

$$D_{12} = 0 = D_{21}, \qquad D_{11}^* \begin{bmatrix} C_1 & D_{11} \end{bmatrix} = \begin{bmatrix} 0 & I \end{bmatrix} = D_{22} \begin{bmatrix} B_2^* \\ D_{22}^* \end{bmatrix}, \tag{1.26}$$

conditions (1.)–(3.) can be written as follows:

(1.) **($\mathcal{P}_\mathbf{X}$-CARE)** *There is $\mathcal{P}_\mathrm{X} \in \mathcal{B}(H, \mathrm{Dom}(B_\mathrm{w}^*))$ s.t. $\mathcal{P}_\mathrm{X} \geq 0$ on H, $A +$*

$(\gamma^{-2}B_2(B_2^*)_{\mathrm{w}} - B_1(B_1^*)_{\mathrm{w}})\mathcal{P}_{\mathrm{X}}$ *is exponentially stable, and*

$$((B_1^*)_{\mathrm{w}}\mathcal{P}_{\mathrm{X}})^*(B_1^*)_{\mathrm{w}}\mathcal{P}_{\mathrm{X}} - \gamma^{-2}((B_2^*)_{\mathrm{w}}\mathcal{P}_{\mathrm{X}})^*(B_2^*)_{\mathrm{w}}\mathcal{P}_{\mathrm{X}} = A^*\mathcal{P}_{\mathrm{X}} + \mathcal{P}_{\mathrm{X}}A + C_1^*C_1. \tag{1.27}$$

(2.) **($\mathcal{P}_{\mathrm{Y}}$-CARE)** *There is* $\mathcal{P}_{\mathrm{Y}} \in \mathcal{B}(H, \mathrm{Dom}(\begin{bmatrix} C_2 \\ C_1 \end{bmatrix}_{\mathrm{w}}))$ *s.t.* $\mathcal{P}_{\mathrm{Y}} \geq 0$ *on* H, $A^* + (\gamma^{-2}C_1^*(C_1)_{\mathrm{w}} - C_2^*(C_2^*)_{\mathrm{w}})\mathcal{P}_{\mathrm{Y}}$ *is exponentially stable, and*

$$((C_2)_{\mathrm{w}}\mathcal{P}_{\mathrm{X}})^*(C_2)_{\mathrm{w}}\mathcal{P}_{\mathrm{X}} - \gamma^{-2}((C_1)_{\mathrm{w}}\mathcal{P}_{\mathrm{X}})^*(C_1)_{\mathrm{w}}\mathcal{P}_{\mathrm{X}} = A\mathcal{P}_{\mathrm{X}} + \mathcal{P}_{\mathrm{X}}A^* + B_2B_2^*. \tag{1.28}$$

(3.) **(Coupling condition)** $\rho(\mathcal{P}_{\mathrm{X}}\mathcal{P}_{\mathrm{Y}}) < \gamma^2$.

Any solutions of (1.) or (2.) are unique. If (1.)–(3.) are satisfied, then all exponentially stabilizing DPF-controllers for Σ *satisfying* $\|w \mapsto z\| < \gamma$ *are the ones parametrized in Theorem 12.1.8, and the regularity claims of Theorem 12.1.4(a)&(b) apply.*

In (3.), ρ denotes the spectral radius. One of the alternative regularity assumptions in (A1) is that B is bounded and $\pi_{[0,1)}C_{\mathrm{w}}\mathbb{A} \in \mathrm{L}^1([0,1);\mathcal{B}(H, Z \times Y))$. For bounded B, the Riccati equation (1.27) takes the classical form

$$\mathcal{P}_{\mathrm{X}}(B_1B_1^* - \gamma^{-2}B_2B_2^*)\mathcal{P}_{\mathrm{X}} = A^*\mathcal{P}_{\mathrm{X}} + \mathcal{P}_{\mathrm{X}}A + C_1^*C_1. \tag{1.29}$$

See p. 618 for further simplification and remarks. Analogous remarks apply to (2.); e.g., for bounded C, the Riccati equation (1.28) becomes

$$\mathcal{P}_{\mathrm{X}}(C_2^*C_2 - \gamma^{-2}C_1^*C_1)\mathcal{P}_{\mathrm{X}} = A\mathcal{P}_{\mathrm{X}} + \mathcal{P}_{\mathrm{X}}A^* + B_2B_2^*. \tag{1.30}$$

Thus, the classical results become special cases of ours. We also give several results under weaker regularity assumptions (e.g., for the case where $\mathbb{A}B, C_{\mathrm{w}}\mathbb{A}, C_{\mathrm{w}}\mathbb{A}B \in \mathrm{L}^1_{\mathrm{loc}}$; this allows roughly twice as much unboundedness as the assumptions of a Pritchard–Salamon system).

In general we allow for DPF-controllers with internal loop, but we show that such a loop is not needed if $D_{21} = 0$ (i.e., one can use a well-posed controller in that case).

In Section 12.2, we give discrete forms of the results of Chapter 12. For them we need no regularity assumptions (since B and C are always bounded for "discrete-time WPLSs").

In Section 12.3, we study the frequency-domain H^∞ 4BP, where one is only given an I/O map $\mathbb{D} \in \mathrm{TIC}_\infty(U \times W, Z \times Y)$, and one wishes to find a controller (I/O map) $\mathbb{Q} \in \mathrm{TIC}_\infty(Y, U)$ s.t. the closed-loop connection becomes (I/O-)stable and satisfies $\|w \mapsto z\| < \gamma$ (see (1.14) for $\mathcal{F}_\ell(\mathbb{D}, \mathbb{Q}) : w \mapsto z$; we also treat the case where $\mathbb{Q}$ is allowed to have an internal loop). In particular, no state-space or internal stability considerations are required.

Michael Green showed in [Green] (Theorem 4.4) that the frequency-domain 4BP has a solution iff certain two nested spectral factorizations exist (in the rational finite-dimensional case). In Section 12.3 we extend this result and its earlier extensions to maps having a d.c.f. in MTIC (Theorem 12.3.6); we also provide partial results for more general settings. Our proof of Theorem 1.2.10 is based on both the frequency-domain 4BP and the H^∞ FICP. The rest of the chapter consists of proofs and minor results.

Part IV: Discrete-Time Control Theory

Part IV presents the discrete-time counterpart of the theory of Parts I–III. Primarily we list the continuous-time results that hold also for the *discrete-time well-posed linear systems (wpls's)* (cf. Theorem 13.3.13; our notation is much the same in both settings). Most proofs apply mutatis mutandis; we give explicit proofs when this is not the case. Several proofs in Parts I–III are actually reduced to the discrete time. Our main contributions in this part are mainly the same as those in continuous-time (Parts I–III), such as the solutions of the H^∞ problems (Sections 11.5 and 12.2).

Chapter 13: Discrete-Time Maps and Systems (ti & wpls)

In Chapter 13, we present briefly some facts on the discrete counterparts of WPLSs, which we call *discrete-time well-posed linear systems (wpls's)*. They are the systems governed by the difference equations

$$\begin{cases} x_{j+1} = Ax_j + Bu_j, \\ \quad y_j = Cx_j + Du_j, \end{cases} \qquad j \in \mathbf{Z}, \tag{1.31}$$

for some $\begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathcal{B}(H \times U, H \times Y)$. We show that almost all our continuous-time results have discrete-time analogies (see Theorem 13.3.13), and also many further results hold due to the boundedness of the generating operators (A, B, C, D).

In Section 13.1, we study bounded linear time-invariant maps $\ell^2_r(\mathbf{Z};U) \to \ell^2_r(\mathbf{Z};Y)$ ("$\mathrm{ti}_r(U,Y)$", where $\|u\|^2_{\ell^2_r(\mathbf{Z};U)} := \sum_k \|r^{-k}u_k\|^2_U$), for $r > 0$, and corresponding transfer functions. The Cayley transform is treated in Section 13.2. (These two sections correspond to Chapters 2 and 3; in particular, we extend the discrete-time Fourier multiplier and H^∞ boundary function theorems for I/O maps over unseparable Hilbert spaces).

In Section 13.3, we study wpls's (this corresponds to Chapter 6; also Chapters 4, 7 and 8 (and partially the rest of this monograph) are treated in Theorem 13.3.13).

In Section 13.4, we show how to obtain wpls's from WPLSs by discretization. This allows us to reduce several WPLS problems to wpls problems, which are often substantially simpler due to the bounded input and output operators.

Discrete-time Riccati equations (DAREs) and spectral and coprime factorization are treated in Chapter 14, minimization problems in Chapter 15, and H^∞ (and Nehari) problems in Sections 11.5 and 12.2.

Chapter 14: Riccati Equations (DARE)

In Chapter 14, we shall present the results of Chapters 9 and 5 (see the above summaries) in their discrete-time forms and supplement this by further results. In particular, we define and study infinite-dimensional *Discrete-time Algebraic Riccati Equations (DAREs)*.

We show that for the general cost function, the existence of an optimal control is equivalent for the DARE (1.32) to have a stabilizing solution. Moreover, the optimal controller can be computed from such a solution. We also show that a third equivalent condition is the generalization of the standard coercivity condition combined to exponential stability (Theorem 14.2.7).

Notice that the discrete-time H^∞ control problems are solved in Sections 11.5 and 12.2. The solutions are already known for finite-dimensional problems (see [IOW]).

Given an initial state $x_0 \in H$, we say that "$u \in \mathcal{U}_{\mathrm{exp}}(x_0)$" iff $u \in \ell^2(\mathbf{N};U)$ is such that $x \in \ell^2(H,U)$ (where x is determined by (1.31) with $x_0 = 0$); such controls (u) are sometimes called "exponentially stabilizing" (or "power stabilizing"). (It obviously follows that $y \in \ell^2(\mathbf{N},Y)$.)

One often wants to minimize or, more generally, optimize a cost function (i.e., to find a J-critical control) under the restriction $u \in \mathcal{U}_{\mathrm{exp}}(x_0)$. This problem has a unique solution iff the extended DARE has a solution:

Theorem 1.2.11 *There is a unique J-critical control for each $x_0 \in H$ iff the extended Discrete-time Algebraic Riccati Equation (eDARE)*

$$\begin{cases} K^*SK = A^*\mathcal{P}A - \mathcal{P} + C^*JC, \\ \quad S = D^*JD + B^*\mathcal{P}B, \\ \quad SK = -(D^*JC + B^*\mathcal{P}A), \end{cases} \tag{1.32}$$

has solution $(\mathcal{P},S,K)$ such that $\mathcal{P} = \mathcal{P}^* \in \mathcal{B}(H)$, S is one-to-one, $K \in \mathcal{B}(H,U)$ and $\sigma(A+BK) \subset \mathbf{D}$. Moreover, any such solution is unique.

If such a solution exists, then the J-critical control is determined by the state feedback $u_j = Kx_j$ and corresponding J-critical cost is given by $\langle x_0, \mathcal{P}x_0\rangle$, where x_0 is the initial state. □

(Here the cost function is of form $\mathcal{J}(x_0,u) := \sum_{k=0}^\infty \langle y, Jy\rangle_Y$ for some $J = J^* \in \mathcal{B}(Y)$. See Theorem 14.1.6 for the proof.)

The above theorem corresponds to $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{exp}}$ (in the discrete-time sense); analogous results hold for other $\mathcal{U}_*^*$'s (cf. Chapter 8). We also present some results in the singular case (where S is not one-to-one and K is not unique) and sufficient conditions for the existence of a unique J-critical control.

Chapter 15: Quadratic Minimization

This chapter is mostly the discrete-time counterpart of Chapter 10; see the above summary for Chapter 10 for corresponding problems and results, such as LQR and H^2 problems, extended minimization, coercivity, real lemmas and maximal solutions of Riccati inequalities/equations. Naturally, several additional discrete-time results are given; the following one solves the extended LQR (minimization) problem:

Corollary 1.2.12 (LQR: $\min \sum_{j=0}^{\infty}(\|y_j\|_Y^2 + \|u_j\|_U^2)$**)** *Let* $R, Q \gg 0$. *Then the following are equivalent:*

(i) there is a $\langle y, Qy\rangle_{\ell^2} + \langle u, Ru\rangle_{\ell^2}$*-minimizing control over all* $u : \mathbf{N} \to U$ *for each* $x_0 \in H$*;*

(ii) for each $x_0 \in H$ *there is* $u \in \ell^2(\mathbf{N}; U)$ *s.t.* $y \in \ell^2$*;*

(iii) the DARE

$$\begin{cases} \mathcal{P} = A^*\mathcal{P}A + C^*QC - K^*SK, \\ S = R + D^*QD + B^*\mathcal{P}B, \\ K = -S^{-1}(D^*QC + B^*\mathcal{P}A), \end{cases} \tag{1.33}$$

has a nonnegative solution $\mathcal{P}$.

If (iii) holds, then the smallest nonnegative solution is minimizing over all $u : \mathbf{N} \to U$.

There is a minimizing control over $\mathcal{U}_{\text{exp}}$ *iff the DARE has an exponentially stabilizing solution* $\mathcal{P}_+$*; such a solution is strictly minimizing over* $\mathcal{U}_{\text{exp}}$ *and the greatest nonnegative solution of the DARE.*

If Σ *is exponentially detectable (e.g.,* $C^*C \gg 0$*), then the DARE has at most one nonnegative solution, and such a solution is necessarily strictly minimizing over* $\mathcal{U}_{\text{exp}}$. □

In Section 15.5, we show that any strongly stabilizing solution of a positive DARE (or of the corresponding Riccati inequality) is the maximal one. We also study Riccati inequalities in the indefinite case.

Appendices A–F

In the appendices, we present mathematical knowledge that is necessary for a complete understanding of the proofs in the main part of this monograph. The readers unfamiliar with the theory of vector-valued functions might wish to have a glance at the beginnings of Appendices A, B and D before starting to read the main text, but most readers will probably visit the appendices only when in need to clarify some parts of the proofs in the main text.

Most of the appendices consists of vector-valued analogies of "well-known" scalar results, some of which are difficult to find in the literature even in the scalar case, whereas some of our results seem to be new even in the scalar case. Hopefully, the appendices can also serve as a reference for several results that have been commonly used in infinite-dimensional control theory without known references.

In the main text of this monograph, the vector spaces are assumed to be complex ($\mathbf{K} = \mathbf{C}$), but in the appendices, the scalar field $\mathbf{K}$ can be taken to be either $\mathbf{C}$ of $\mathbf{R}$ (in Appendix D and Sections A.4 and F.3, we assume that $\mathbf{K} = \mathbf{C}$, as explicitly stated there; in the other sections in the appendices we always state explicitly any such exceptions).

In Appendix A, we present standard definitions and several facts on algebra, topology and functional analysis, including several useful formulae for the inverses of operators between product spaces.

In Section B.1, we briefly present (Lebesgue) integration, differentiation, measurability and L^p and $\mathcal{C}$ function spaces. In the rest of Appendix B, we extend such concepts for functions with values in Banach spaces (we call such functions *vector-valued*). Our results include the density of finite-dimensional, smooth, compactly carried functions in vector-valued (Lebesgue) L^p spaces (even simultaneously for different p's and weight functions; see Theorem B.3.11), several integral inequalities and equalities (e.g., Theorems B.4.12 and B.4.16), certain product measurability results, differentiation formulae for integrals (Section B.5) and the basic theory of vector-valued Sobolev spaces (Section B.7).

In Appendix C, we briefly introduce vector-valued almost periodic functions.

In Appendix D, we study holomorphic vector-valued functions. This includes (Hardy) H^p spaces, Laplace and Fourier transforms and Poisson integral formulae. We also present some results on convolutions and on vector-valued measures.

In Appendix E, we present the Riesz–Thorin Interpolation Theorem, the Hausdorff–Young Theorem and similar results for vector-valued functions, with applications to control theory.

In Appendix F, we define spaces of *strongly measurable* functions ($f : Q \to \mathcal{B}(B, B_2)$, where $fx : Q \to B$ is (Bochner-)measurable for each $x \in B$) and *weakly measurable* functions ($\Lambda f x$ is measurable for each $x \in B$ and $\Lambda \in B_2^*$). In particular, we define and study $\mathrm{L}^p_{\text{strong}}$ and $\mathrm{L}^p_{\text{weak}}$ spaces (the main applications are contained in the above summary on Chapter 3) and $\mathrm{H}^p_{\text{strong}}$ and $\mathrm{H}^p_{\text{weak}}$ spaces (with applications in system theory). We also develop integration, convolution and Laplace transform theory for strongly or weakly measurable functions.

The completeness of $\mathrm{L}^\infty_{\text{strong}}$ (whereas $\mathrm{L}^p_{\text{strong}}$ is incomplete for $p < \infty$) and some $\mathrm{H}^p \cap \mathrm{H}^p$ type results at the end of Appendix D may be the deepest new results in the appendices, whereas many of the other results are more or less straight-forward generalizations and/or extensions of known facts.

1.3 Conventions

If we spoke a different language, we would perceive a somewhat different world.

— Ludwig Wittgenstein (1889–1951)

Most of the notation is explained at the point where it is used for the first time, and there is an extensive list of references, symbols, terms, abbreviations and acronyms at the end of this book (p. F.3). The correspondence of diagrams of systems to corresponding equations can be observed from Figure 6.1 (p. 155); in particular, inputs correspond to columns and outputs to rows, as in a matrix (and in [Sbook]).

Following the standard convention, in definitions we write *if* instead of *iff* (which means "if and only if"). An asterisk ("$*$") often denotes for something omitted (see the symbol list, p. 1038). By brackets ("$[\dots]$") we denote references (p. 1024) or optional parts; see p. 1037.

For clarity, we have chosen the "Blackboardbold" style to indicate the "integral" operators, e.g., a WPLS is of the form $\left[\begin{smallmatrix}\mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D}\end{smallmatrix}\right]$. As a result, we have to use ordinary bold letters ($\mathbf{C}$, $\mathbf{R}$, $\mathbf{Z}$, $\mathbf{N} = \{0,1,2,\dots\}$) for standard fields of (complex, real, integer, natural) numbers.

The generators of $\left[\begin{smallmatrix}\mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D}\end{smallmatrix}\right]$ are denoted by $\left[\begin{smallmatrix}A & B \\ C & \end{smallmatrix}\right]$ or $\left[\begin{smallmatrix}A & B \\ C & D\end{smallmatrix}\right]$, as in Section 1.2 (alternatively, see Definition 6.1.1, Lemma 6.1.16 and Definition 6.2.3). Similarly, the generators (feedthrough operators) of any other integral maps (always Blackboardbold) will usually be denoted by corresponding ordinary (capital) letters. Note also the bars separating the different parts of the system; this is helpful when the parts consist of larger expressions.

The order of proofs

The "integral" notation (1.8) of a system allows us to treat continuous-time and discrete-time problems in a unified way. This allows us to transfer continuous-time results to discrete time with a minimal effort: it suffices to just list which parts are valid also in discrete time, with same proofs (see Theorem 13.3.13). In particular, within the discrete-time theory (Part IV), any references to continuous-time results refer to corresponding discrete-time variants (ones having undergone the substitutions (13.63)).

However, the proofs of certain results rest on the boundedness of "differential" or difference operators, hence they are given first for discrete time and then extended to continuous time by discretization (because discrete-time systems always have bounded generators). Such results include the uniqueness of the solution of the Riccati equation, the two-Riccati formula of the H^∞ Four-Block Problem and several results on stabilization.

Because of this, to verify the proofs of the whole monograph, one might wish to first read and verify the results in their discrete-time form, and only then in their continuous-time form (see Theorems 13.3.13, 14.1.3, 15.1.1, 11.5.2 and 12.2.2 and their proofs for details and other possible orders). Nevertheless, all results that are valid in both continuous and discrete time are first stated in continuous time, and we give the proofs in their continuous-time forms whenever reasonably possible; in such cases the discrete-time analogies are just references to the continuous-time results and proofs, as in Theorem 13.3.13.

Most readers may read the book "as is", but the reader wishing to have a deeper insight (or to understand all proofs) has to study also the discrete-time part in order to completely absorb the continuous-time part. Conversely, readers interested in discrete-time results only may skip things such as generators and regularity of continuous-time systems, as well as related complicated technical methods.

Trying to balance between the properties needed from a reference manual and those needed for a "chronological" order of proofs, we have grouped some clearly related results together, thus placing some results before those needed in their proofs; we have tried to clarify the order of proofs in those cases.

Proofs

The proofs often contain extra information: remarks, clarifications of ambiguous statements in the theorems, weaker or alternative assumptions, or "counter-examples" showing that our assumptions are not superfluous, etc.

We place a square ("□") at the end of each proof, and at the end of each lemma, proposition, theorem, corollary or remark whose proof is only sketched or replaced by a reference to some other result.

There are several algebraic basic results (e.g., the Schur decomposition of an (operator) matrix) that are often used in control theory without a further mention. We have compiled them to the Operator Matrix Lemma A.1.1, which has helped us make many proofs dramatically shorter, simpler and easier and the results more elegant than in the early versions of this book (you do not want to know...).

Notes

At the end of most sections, there is a "Notes" subsection containing further remarks and external references, including any earlier forms of similar results in the literature (known to us). However, we often refer to a "more up-to-date reference" instead of the first author.

When reading the notes to discrete-time sections, one should also consult the notes to corresponding continuous-time sections. Note also the historical remarks of Section 1.2.

Hypotheses

At the beginning of each chapter, we list any standing hypotheses and assumptions of the chapter or of its parts.

Outside the appendices, any Banach and Hilbert spaces are complex and of arbitrary dimensions unless otherwise stated. In the appendices, the scalar field $\mathbf{K}$ may be either of $\mathbf{R}$ or $\mathbf{C}$ except that in Appendix D and Sections A.4 and F.3 we assume that $\mathbf{K} = \mathbf{C}$, as explicitly stated there.