Notation

Thou art a symbol and a sign
To Mortals of their fate and force;
Like thee, Man is in part divine,
A troubled stream from a pure source;
And Man in portions can foresee
His own funereal destiny.
— Lord Byron (1788–1824), "Prometheus"

We present here the notation used in this monograph. For readability, we use some redundancy. See also the standing hypotheses (if any) mentioned in the beginning of each chapter. Any alternative meanings of these symbols are specified explicitly.

For symbols we only give their continuous-time explanations, see Theorem 13.3.13 for corresponding discrete-time interpretation.

For the correspondence with the notation used by G. Weiss and M. Weiss (among others), see p. 166.

Brackets etc.

$\{x \in X \mid P(x)\}$: The set of all elements x of X for which $P(x)$ is true.

$\{x_n\}$: This refers to a sequence, usually on $\mathbf{N}$ or $\mathbf{Z}$, i.e., to $(x_n)_{n=0}^{\infty}$ or $(x_n)_{n=-\infty}^{\infty}$.

[S97b, Section 7]: A reference to Section 7 of [S97b] (see p. 1024 for the bibliography)). [Rud73, p. 131] is a reference to page 131 of [Rud73], and (1.16) is a reference to equation (1.16) of this text (see p. 29).

"Σ is [strongly/exponentially] stable iff $\widetilde{\Sigma}$ is [strongly/exponentially] stable": the conditions in brackets are optional and correspond to each other, i.e., this says that "Σ is stable iff $\widetilde{\Sigma}$ is stable", "Σ is strongly stable iff $\widetilde{\Sigma}$ is strongly stable", and "Σ is exponentially stable iff $\widetilde{\Sigma}$ is exponentially stable", as in Corollary 6.6.9. Analogously, "$a \in A$ iff $a \in B$ [and $a \in C$]" implies that both "$a \in A$ iff $a \in B$" and "$a \in A$ iff $a \in B$ and $a \in C$" hold.

$f[X], fX$: $f[X]$ is the set $\{f(x) \mid x \in X\}$, when f is a function defined (at least) on the set X; for linear f we may write fX. Special cases of this are the imaginary axis $i\mathbf{R}$ and the set $\mathrm{L}^1* := \{f* \mid f \in \mathrm{L}^1\}$; the latter means the convolution operators defined by a L^1 function, cf. Definition 2.6.3; analogously, $\mathbf{N}+1 = \{1,2,3,\ldots\}$.

$\left[\begin{array}{c|c}\mathbb{A} & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D}\end{array}\right]$: A WPLS (or a wpls), see Definition 6.1.1 (or 13.3.1).

$\left[\begin{array}{c|c}A & B \\ \hline C & D\end{array}\right]$: The generators of a WPLS (or a wpls).

$\left(\begin{array}{c|c}A & B \\ \hline C & D\end{array}\right)$: The WPLS (or wpls) generated by $\left[\begin{array}{c|c}A & B \\ \hline C & D\end{array}\right]$; see Lemmas 6.1.16 and 6.2.9(a) and Definition 6.2.3 (or Lemma 13.3.3).

$[a,b],\ [a,b),\ (a,b],\ (a,b)$: The intervals $\{r \in \mathbf{R} \,|\, a \le r \le b\}$, $\{r \in \mathbf{R} \,|\, a \le r < b\}$, $\{r \in \mathbf{R} \,|\, a < r \le b\}$, $\{r \in \mathbf{R} \,|\, a < r < b\}$, respectively.

$f(t,z) > 0$ $(t \in \mathbf{R},\ z \in \mathbf{C})$: This means that $f(t,z) > 0$ for all $t \in \mathbf{R}$ and all $z \in \mathbf{C}$.

$\langle x,y \rangle_H$: The (sesquilinear) inner product of $x,y \in H$ in the Hilbert space H.

$\langle x,\Lambda \rangle_{X,X^*}, \langle x,\Lambda \rangle_{X,X^{\mathrm{d}}}, \langle x,\Lambda \rangle_{X,X^{\mathrm{B}}}$: Each of these denotes the scalar $\Lambda x := \Lambda(x)$ (here $x \in X,\ \Lambda \in X^*$). Thus, by $\langle \cdot,\cdot \rangle_{X,X^*}$ we denote the pairing $X \times X^* \to \mathbf{K}$. The subindex d or B on H specifies explicitly how the multiplication in X^* is defined, see "Superscripts", p. F.3.

More non-letter symbols

$\cdot$: The place for the argument of a function; e.g., $\mathrm{e}^{\omega\cdot}$ denotes the function $t \mapsto \mathrm{e}^{\omega t}$.

$*$: In the place of an operator the asterisk $(*)$ refers to unspecified (irrelevant) entry, e.g., $\left[\begin{smallmatrix}1 & 2 \\ 3 & *\end{smallmatrix}\right]$ may have any right bottom entry; $(J,*)$-inner means (J,S)-inner for any S. Sometimes $*$ refers to anything (possibly void) that is omitted..

$\mu * f$: The convolution of μ and f, see the index. By $\mu*$ we mean the operator $f \mapsto \mu * f$.

$f \circ g$: The composite function $t \mapsto f(g(t))$.

$A \setminus B$: The set $\{x \in A \,|\, x \notin B\}$ (or $A \cap B^{\mathrm{c}}$).

$A := B$: A is defined to be equal to B. Equivalent to "$B =: A$".

$s \to +\infty$: s is real and s goes to $+\infty$; we often write ∞ for $+\infty$.

$x_n \rightharpoonup x$: "$x_n \to x$" weakly, i.e., $\langle x_n, y \rangle \to \langle x, y \rangle$ for all y (see Lemma A.3.1(h)–(j)).

$f : \Omega \to X$: f is a function of Ω into X; we may write, e.g., $f : \Omega \ni s \mapsto x_0 + x_3 s^3 \in X$ to specify a rule determining $f(s)$ for $s \in \Omega$.

$\cup, \cap$: Union, intersection, respectively; see Lemma A.3.17 for the norm of $X \cap Y$.

$A \subset B,\ B \supset A$: Either means that A is a subset of B ($x \in A \Rightarrow x \in B$; possibly $A = B$).

$A \underset{c}{\subset} B$: This means that $A \subset B$ and that this inclusion is continuous (see p. 890).

$A \underset{a}{\subset} B$: This means that A is an (algebraic) subclass of B (Definition 6.2.4).

$A + B$: The set $\{a + b \,|\, a \in A,\ b \in B\}$ (this is a special case of $f[X]$ above); see Lemma A.3.17 for corresponding norm.

$A \oplus B$: When A and B are closed subspaces of a Banach space, and $A \cap B = \{0\}$, we denote $A + B$ by $A \oplus B$ (cf. [Rud73]).

$X \times Y$: The Cartesian product $\{(x,y) \,|\, x \in X,\ y \in Y\}$. If X and Y are normed spaces, we use the norm $\|(x,y)\|_{X \times Y} := (\|x\|_X^2 + \|y\|_Y^2)^{1/2}$;

if X and Y are inner product spaces, we use the inner product $\langle (x,y),(x',y')\rangle_{X\times Y} := \langle x,x'\rangle_X + \langle y,y'\rangle_Y$; use induction for $\prod_{k=1}^n X_k := X_1 \times X_2 \times \cdots \times X_n$. Finally, $X^n := \prod_{k=1}^n X$. See also p. 870.

$\int_\Omega f\,dm$: The Bochner (i.e., Lebesgue, if f is $\mathbf{C}$- or $\mathbf{C}^n$-valued) integral (see p. 927) of f over Ω w.r.t. the Lebesgue measure m.

$\mathcal{F}, \mathcal{F}^{-1}$: See Theorem F.2.1, p. 1011.

$\exists, \&$: "$\exists$" means "exists", "&" means "and".

$\exists A^{-1}$: "There exists a bounded inverse of the operator A" (in particular, A is one-to-one and onto).

$(s-A)$: $(s-A) := sI - A$, when $s \in \mathbf{C}$ (see Lemma A.3.3 and Section A.4).

$\partial\Omega$: $\partial\Omega := \overline{\Omega} \cap \overline{\Omega^c}$ is the boundary of the set Ω.

$f|_A$: The restriction of the function f on the set A (we write just f when there is no risk of confusion).

$x \perp y$: This means that $\langle x,y\rangle = 0$; if S is a set, then $x \perp S$ means that $\langle x,y\rangle = 0$ for all $y \in S$; similarly for $S \perp x$ and $S \perp S'$.

$\|\cdot\|_X$: The norm on space X (see "Function and operator spaces" on p. 1045 for most X's).

$\|A\|_{\mathcal{B}(U,Y)}$: The standard operator norm $\|A\|_{\mathcal{B}(U,Y)} := \sup_{\|u_0\|_U \le 1} \|Au\|_Y$; note that $\|A\| = \sqrt{\max \sigma(A^*A)}$, by Theorem 11.28c of [Rud73], if U and Y are Hilbert spaces.

$\|\mathbb{D}\|$: If $\mathbb{D} \in \mathrm{TI}(U,Y)$ (see below), this refers always to the $\mathrm{TI}(U,Y)$ norm $\|\mathbb{D}\|_{\mathrm{TI}} := \|\mathbb{D}\|_{\mathcal{B}(\mathrm{L}^2(\mathbf{R};U),\mathrm{L}^2(\mathbf{R};Y))}$, not to $\|\mathbb{D}\|_{\mathcal{A}}$, even if $\mathbb{D} \in \mathcal{A}$, where $\mathcal{A}$ is a subclass of TI.

$\|\cdot\|_p$: $\|\cdot\|_p := \|\cdot\|_{\mathrm{L}^p}$; in particular, $\|x\|_p := \|x\|_{\ell^p}$ when x is a sequence.

$f \ge g$: This means that $f(q) \ge g(q)$ for all q in the (common) domain of f and g, when f and g are scalar functions.

$A \ge B$: iff $B \le A$ iff $A = A^*$, $B = B^*$ and $\langle x,(A-B)x\rangle \ge 0$ for all x (if A and B are linear operators); see also p. 872.

$A > B$: iff $B < A$ iff $A = A^*$, $B = B^*$ and $\langle x,(A-B)x\rangle > 0$ for all $x \ne 0$.

$A \gg B$: iff $B \ll A$ iff for some $\varepsilon > 0$ we have $\langle (A-B)x,x\rangle \ge \varepsilon\|x\|^2$ for all x; see Lemma A.3.1(b1) for more.

$\diamondsuit, \heartsuit$: Cayley transforms (Definition 13.2.2).

Bold letters

$\mathbf{R}$, $\mathbf{R}_+$, $\mathbf{R}_-$: $\mathbf{R} := (-\infty,\infty)$, $\mathbf{R}_+ := [0,\infty)$, and $\mathbf{R}_- := (-\infty,0]$.

$i\mathbf{R}$: $i\mathbf{R} := \{it \mid t \in \mathbf{R}\}$; we consider the differentiation, the Lebesgue measure m etc. for $f : i\mathbf{R} \to U$ as those for $f(i\cdot) : \mathbf{R} \to U$.

$\mathbf{C}$, $\mathbf{Q}$, $\mathbf{Z}$, $\mathbf{N}$: The complex/rational/integer/natural numbers, respectively; $\mathbf{N} := \{0,1,2,\dots\}$, $\mathbf{Z}_- := \{-1,-2,-3,\dots\}$.

$\mathbf{K}$: "The field of scalars"; outside the appendices we have $\mathbf{K} = \mathbf{C}$. In general, the results in the appendices also hold for $\mathbf{K} = \mathbf{R}$ (see the beginning of each appendix or section of an appendix for exceptions).

D: The *unit disc* $\mathbf{D} := \{z \in \mathbf{C} \mid |z| < 1\}$, hence $\partial\mathbf{D} = \{z \in \mathbf{C} \mid |z| = 1\}$ (the *unit cirle*) and $r\mathbf{D} = \{rz \mid z \in \mathbf{D}\} = \{z \in \mathbf{C} \mid |z| < r\}$ for any $r > 0$.

$\mathbf{C}^+_\omega$, $\mathbf{C}^+$: $\mathbf{C}^+_\omega := \{s \in \mathbf{C} \mid \operatorname{Re} s > \omega\}$, $\mathbf{C}^+ := \mathbf{C}^+_0$.

$\mathbf{C}^-_\omega$, $\mathbf{C}^-$: $\mathbf{C}^-_\omega := \{s \in \mathbf{C} \mid \operatorname{Re} s < \omega\}$, $\mathbf{C}^- := \mathbf{C}^-_0$.

$\mathbf{C}_J, \mathbf{C}_{a,b}$: $\mathbf{C}_J := \{s \in \mathbf{C} \mid \operatorname{Re} s \in J\}$, $\mathbf{C}_{a,b} := \mathbf{C}_{(a,b)} = \{s \in \mathbf{C} \mid a < \operatorname{Re} s < b\}$.

$\overline{\mathbf{C}^+} \cup \{\infty\}, i\mathbf{R} \cup \{\infty\}$: See the footnote on p. 72 for corresponding topologies.

$\mathbf{Я}$: Reflection around the origin: $(\mathbf{Я}u)(t) := u(-t)$, p. 782.

$\mathbf{Я}_{-1}$: The shifted (discrete-time) reflection: $(\mathbf{Я}_{-1}x)_i := x_{-1-i}$, p. 782.

Superscripts

B^A: The set of functions $A \to B$.

f': The derivative of f; see pp. 918 and 961.

$\overline{\alpha}$: the complex conjugate of α when $\alpha \in \mathbf{C}$.

$\overline{\Omega}$: $\overline{\Omega}$ is the closure of the set Ω.

$S^\perp$: The set $\{x \mid x \perp S\}$. By Section 4.9 of [Rud86], this is a closed subspace of the underlining Hilbert space.

$\widehat{u}$: The *Laplace transform* of u, i.e., $\widehat{u}(s) := \int_{\mathbf{R}} e^{-st} u(t)\, dt$; see Definition D.1.6. If u is considered as an element of $L^1_\omega(\mathbf{R}; X)$, then $\widehat{u}$ is often considered as a function $\omega + i\mathbf{R} \to X$; this function (restriction of $\widehat{u}$ to $\omega + i\mathbf{R}$) is called the *Fourier transform* of u. In discrete time, $\widehat{\ }$ stands for the Z-transform, p. 782, for signals; see "$\widehat{\mathbb{D}}$" below for operators.

$\widehat{\mu}$: The Laplace (or Fourier) transform of the (possibly vector-valued) measure μ; see Lemma D.1.12.

$\widehat{\mathbb{D}}$: The transfer function, (symbol, either "Laplace" or "Fourier" transform) of $\mathbb{D}$, when $\mathbb{D} \in \mathrm{TI}_\infty$ (see Theorems 2.1.2 and 3.1.3). In discrete time, see Lemmas 13.1.5 and 13.1.6 instead.

$\widehat{\mathbb{A}}, \widehat{\mathbb{B}}, \widehat{\mathbb{C}}$: See Theorem 6.2.11 (or Lemma 13.3.6 in discrete time).

$\widehat{\mathcal{V}}$: The set $\{\widehat{f} \mid f \in \mathcal{V}\}$ if $\mathcal{V}$ is a set.

$\Sigma^{\curvearrowleft}_{\mathrm{ext}}, \mathbb{A}^{\curvearrowleft}, \mathbb{B}^{\curvearrowleft}, \mathbb{C}^{\curvearrowleft}, \mathbb{D}^{\curvearrowleft}, \mathbb{K}^{\curvearrowleft}, \mathbb{F}^{\curvearrowleft}$: The partially-closed-loop system (state feedback through first input only), pp. 614–616.

$A^{1/2} = \sqrt{A}$: The (nonnegative) square root of A (Lemma A.3.1(b4)).

A^{-1}: The inverse of A.

$\mathbb{Y}\mathbb{X}^{-1}$, $\widetilde{\mathbb{X}}^{-1}\widetilde{\mathbb{Y}}$: Sometimes a map with coprime internal loop, Definition 7.2.11.

A^{-*}, A^{-d}: $A^{-*} := (A^*)^{-1} = (A^{-1})^*$, $A^{-d} := (A^d)^{-1} = (A^{-1})^d$.

X^*, X^{d}, X^{B}, X^{H}: Adjoint, sesquilinear adjoint (often w.r.t. a pivot space; n L^2_ω contexts we use L^2 as the pivot space; in state contexts we use the state space (usually H) as the pivot space), (bilinear) Banach adjoint and (sesquilinear) Hilbert adjoint of X, respectively, when X is an operator, and the corresponding dual space, when X is a normed space. The meaning of $()^*$ depends on the context; in pivot space contexts (this is usually the case outside the appendices) it stands for the Hilbert adjoint w.r.t. the pivot space; thus, we follow the standard convention in infinite-dimensional control theory. If X is a set of operators, then $X^* = (\cdot)^*[X]$, i.e., $X^* = \{T^* \mid T \in X\}$, etc. See pp. 896–899 for details and Definition 6.1.17 for a main application. Moreover, $(\cdot)^d$ also has an alternative meaning (“causal dual”) for systems and their components, see below.

$\mathbb{B}^*, \mathbb{C}^*, \mathbb{D}^*, \mathbb{E}^*$: This is explained above (Hilbert adjoint w.r.t. the pivot space L^2 (or w.r.t. ℓ^2 in discrete time)). See (6.2) and Definition 2.1.4 for details.

$\Sigma^{\mathrm{d}}, \mathbb{A}^{\mathrm{d}}, \mathbb{B}^{\mathrm{d}}, \mathbb{C}^{\mathrm{d}}, \mathbb{D}^{\mathrm{d}}$: (Causal) dual system or map. See Lemmas 6.1.4 and 3.3.8 for continuous time, and Proposition 13.3.5 and Lemma 13.1.8 for discrete time.

$\widehat{\mathbb{D}}^{\mathrm{d}}$: $\widehat{\mathbb{D}}^{\mathrm{d}}(s) := \widehat{\mathbb{D}^d}(s) = \widehat{\mathbb{D}}(\overline{s})^*$; see Lemmas 3.3.8 and 13.1.8.

$X^{n \times m}$: The set of matrices having n rows and m columns and elements from X; cf. (A.1). We set $X^n := X^{n \times 1}$.

E^c: The set of elements that do not belong to E (*complement*).

E^o: The interior of E (p. 867).

$\Sigma^o, \Sigma_I^o, \mathbb{D}^o, \mathbb{D}_I^o, \ldots$: The open and closed loop dynamic feedback systems or maps. See Definitions 7.1.1, 7.2.1 and 7.3.1..

$\begin{bmatrix} A & B \end{bmatrix}^{\mathrm{T}}$: The *transpose* $\begin{bmatrix} A^{\mathrm{T}} \\ B^{\mathrm{T}} \end{bmatrix}$ of $\begin{bmatrix} A & B \end{bmatrix}$.

$\mathbb{A}^t, \mathbb{B}^t, \mathbb{C}^t, \mathbb{D}^t$: $\mathbb{A}^t := \mathbb{A}(t)$, $\mathbb{B}^t := \mathbb{B}\tau\pi_{[0,t)}$, $\mathbb{C}^t := \pi_{[0,t)}\mathbb{C}$, $\mathbb{D}^t := \pi_{[0,t)}\mathbb{D}\pi_{[0,t)}$; see (6.5).

Further uses of superscripts are presented in “Function and operator spaces”, p. 1045.

Subscripts

$\Sigma_\sharp$: An output injection closed-loop system of Σ; see Definition 6.6.21.

$\Sigma_\flat = \left[\begin{array}{c|c} \mathbb{A}_\flat & \mathbb{B}_\flat \\ \hline \mathbb{C}_\flat & \mathbb{D}_\flat \\ \mathbb{K}_\flat & \mathbb{F}_\flat \end{array}\right]$: A (state feedback) closed-loop system of Σ; see Definition 6.6.10.

$\Sigma_\circlearrowleft = \left[\begin{array}{c|c} \mathbb{A}_\circlearrowleft & \mathbb{B}_\circlearrowleft \\ \hline \mathbb{C}_\circlearrowleft & \mathbb{D}_\circlearrowleft \\ \mathbb{K}_\circlearrowleft & \mathbb{F}_\circlearrowleft \end{array}\right]$: The closed-loop system corresponding to a solution $(\mathcal{P}, S, \left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right])$ of the Riccati equation.

$\Sigma_{\mathrm{crit}} = \left[\begin{array}{c} \mathbb{A}_{\mathrm{crit}} \\ \hline \mathbb{C}_{\mathrm{crit}} \\ \mathbb{K}_{\mathrm{crit}} \end{array}\right]$: A J-critical control in WPLS form; see Theorem 8.3.9.

$\Sigma_L, \mathbb{A}_L, \mathbb{B}_L, \ldots$: Static feedback closed-loop system and maps (Propositions 6.6.2 and 6.6.18).

$H_1, H_{-1}, H_1^*, H_{-1}^*, H_B, H_C^*, H_{C,K}^*$: See Lemma 6.1.16 and Definition 6.1.17.

$C_{\mathrm{c}}, D_{\mathrm{c}}$: A compatible pair; see Definition 6.3.8.

$\mathbb{D}_{\mathrm{d}}$: $\mathbb{D}_{\mathrm{d}} := \mathbb{D}^{\mathrm{d}}$ when $\mathbb{D}$ is a one-block operator; $\begin{bmatrix} \mathbb{D}_{11} & \mathbb{D}_{12} \\ \mathbb{D}_{21} & \mathbb{D}_{22} \end{bmatrix}^{\mathrm{d}} := \begin{bmatrix} \mathbb{D}_{22}^{\mathrm{d}} & \mathbb{D}_{12}^{\mathrm{d}} \\ \mathbb{D}_{21}^{\mathrm{d}} & \mathbb{D}_{11}^{\mathrm{d}} \end{bmatrix}$ (see also p. 740).

$\Sigma_{\mathrm{d}}, \mathbb{A}_{\mathrm{d}}, \mathbb{B}_{\mathrm{d}}, \mathbb{C}_{\mathrm{d}}$: See p. 740.

$X_{\mathbf{R}}$: The complex vector space X as a real vector space (Lemma A.3.21).

$C_{\mathrm{s}}, C_{\mathrm{w}}, C_{\mathrm{L,s}}, C_{\mathrm{L,w}}$: The strong Yosida, weak Yosida, strong Lebesgue and weak Lebesgue extensions of C; e.g., $C_{\mathrm{w}} x_0 := \text{w-lim}_{s \to +\infty} s(s-A)x_0$; see Proposition 6.2.8 for details.

$\mathcal{U}_{\mathrm{u}}$: The set of admissible controls, see p. 614 (or p. 681).

$\mathcal{U}_*^*, \mathcal{U}_{\mathrm{out}}, \mathcal{U}_{\mathrm{sta}}, \mathcal{U}_{\mathrm{str}}, \mathcal{U}_{\mathrm{exp}}$: Various sets of admissible controls (see Definition 8.3.2 and Hypothesis 9.0.1).

$\left[\ \mathbb{K}_{\mathrm{u}} \ \middle|\ \mathbb{F}_{\mathrm{u}}\ \right], \left[\begin{smallmatrix} \mathbb{H}_{\mathrm{y}} \\ \mathbb{G}_{\mathrm{y}} \end{smallmatrix}\right]$: Preliminarily stabilizing state feedback and output injection pairs, see p. 737.

$\mathbb{N}_{\mathrm{u}}, \mathbb{M}_{\mathrm{u}}, \widetilde{\mathbb{N}_{\mathrm{y}}}, \widetilde{\mathbb{M}_{\mathrm{y}}}$: Parts of a preliminary d.c.f., see p. 713 or p. 737.

$\Sigma_{\mathrm{X}}, \Sigma_{\mathrm{Y}}, \Sigma_{\mathrm{Z}}, \Sigma_{\mathbb{E}^{\mathrm{d}}}, \Sigma_{\mathbb{T}}$: See p. 744, 744, 747 (or 762), 747 or 753, respectively.

Further uses of subscripts are presented in “Function and operator spaces”, p. 1045.

Miscellaneous letters

Capital letters $(A, B, C, D, \ldots)$ often denote the generators (generating operators) of the corresponding maps $(\mathbb{A}, \mathbb{B}, \mathbb{C}, \mathbb{D}, \ldots)$, as in the third and fourth explanation below.

$\widetilde{\mathcal{A}}, \widetilde{\mathcal{A}}_+$: In Chapter 10 (resp. Chapters 11 and 12), the symbol $\widetilde{\mathcal{A}}_+$ (resp. $\widetilde{\mathcal{A}}$) stands for MTIC or something similar; see the standing hypotheses mentioned at the beginnings of these chapters (see Theorem 8.4.9 for suitable classes).

$\mathbb{A}Bu_0 \in \mathrm{L}^1([0,1); H)$: See Section 6.8.

A, B, C; $H_1, H_{-1}, H_1^*, H_{-1}^*$; H_B, H_C^*; B^*, C^*: See Lemma 6.1.16 and Definition 6.1.17.

D: The feedthrough operator of $\mathbb{D}$. See Definition 6.2.3.

D_j: The jth partial derivative; see Definition B.3.3, p. 918.

D^α: See p. 950.

$D(x,r)$: The disc $\{y \mid d(x,y) < r\}$.

$d(x,y)$: The distance $d(x,y) := \|x-y\|$ when x and y belong to a normed space.

$d(x,A)$: The distance $\inf_{a \in A} d(x,a)$ when d is a metric (see Section A.2 for metrics).

e^x: $\mathrm{e}^x := \sum_{k=0}^\infty x^k/k!$ when x is an element of a Banach algebra (e.g., $x \in \mathbf{C}$).

e_k: The vector $e_k := \chi_{\{k\}}$; thus, $\{e_k\}_{k \in \mathbf{Z}}$ is the natural orthonormal basis of $\ell^2(\mathbf{Z})$.

$\mathcal{F}$: The Fourier transform (not always).

$\mathcal{F}_\ell(\mathbb{D}, \mathbb{Q})$: The “lower linear fractions transformation” of $\mathbb{D}$ and $\mathbb{Q}$, i.e., the map $w \mapsto z$ of Figure 7.8 or of Figure 7.10; see (7.64) or (7.98, respectively (or Definition 7.3.1 or Lemma 12.3.2).

$\mathcal{G}$: The set (group) of invertible elements (e.g., $\mathcal{GB}(X,Y)$).

$H_{\mathbb{B}}, H_{\mathbb{C}}$: The reachability and observability subspace (Definition 6.3.25).

H, U, W, Y, Z: Often complex Hilbert spaces of arbitrary dimensions.

x,u,w,y,z: Often u (resp. $\left[\begin{smallmatrix} u \\ w \end{smallmatrix}\right]$) denotes the input, x the state, x_0 the initial state and y (resp. $\left[\begin{smallmatrix} z \\ y \end{smallmatrix}\right]$) the output of a system; see Figure 6.1 and Definition 6.1.5 (resp. Figure 7.9 and Definition 7.3.1).

$\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right], \Sigma_{\mathrm{ext}}, \Sigma_\flat$: See Definition 6.6.10 (state feedback).

$\left[\begin{smallmatrix} \mathbb{H} \\ \hline \mathbb{G} \end{smallmatrix}\right], \Sigma_\sharp, \Sigma_{\mathrm{Total}}$: See Definition 6.6.21 (output injection).

I: The identity operator (I_X denotes the identity on X).

i: The imaginary unit ($i = \sqrt{-1}$).

J_γ: $J_\gamma := \begin{bmatrix} I & 0 \\ 0 & -\gamma^2 I \end{bmatrix}$.

$\mathcal{J}, J$: The cost function and the cost operator, respectively. See Definitions 9.1.3, 8.3.2 and 8.1.3.

$\mathcal{L}, \mathcal{L}_{\mathrm{s}}, \mathcal{L}_{\mathrm{w}}$: The Laplace transform (weak, strong); see pp. 969 and F.3.1.

$P-, PB-, (P), (PB)$: See Definitions 9.1.5, 9.8.1, 9.8.4 and 14.1.1.

$\mathcal{P}$: Often a solution of the Riccati equation. Sometimes the J-critical cost operator $\mathcal{P} := \mathbb{C}_{\mathrm{crit}}^* J \mathbb{C}_{\mathrm{crit}}$ (Theorem 8.3.9(b1)).

$(\mathcal{P}, S, K)$, $(\mathcal{P}, S, \left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right])$: A solution of the Riccati equation. See Definitions 9.1.5, 9.8.1, 9.8.4 and 14.1.1.

$\mathbb{Q}, \mathbb{R}$: Definition 8.3.2 and Hypothesis 9.0.1 (and Hypothesis 14.0.1).

$\mathcal{T}_*$: The stability shift (Remarks 2.1.6, 6.1.9 and 13.3.9).

$\mathcal{U}_*^*$: The set of admissible controls (see Definition 8.3.2 and Hypothesis 9.0.1).

$\mathcal{U}, \mathcal{Y}$: The sets of admissible inputs and outputs, respectively; see Definitions 8.3.2 and 8.1.3.

Y^{s}, Y, Z: See Hypothesis 8.1.1 (Sections 8.1–8.2 only).

$Z^{\mathrm{s}}, Z^{\mathrm{u}}$: The sets of stable and unstable states (see Definition 8.3.2, Hypothesis 9.0.1 and Hypothesis 14.0.1).

$\mathcal{Z}$: The Z-transform: $(\mathcal{Z}u)(z) := \sum_{j \in \mathbf{Z}} z^j u_j$ (p. 782).

Greek letters

β, γ, ζ: See Hypothesis 9.5.1 (Sections 9.5 and 9.6 only).

γ: A fixed positive number in Chapters 11 and 12.

δ_t: The point mass at $t \in \mathbf{R}$; $\delta_t * f := f(\cdot - t) = \tau(-t) f$, when f is a function.

$\Delta^{\ell^2}, \Delta^{\mathrm{S}}$: Discretization operators Section 13.4.

ε_+: See Hypothesis 11.2.1 (Chapter 11 only).

ϑ: See Definition 8.3.2.

κ: See Sections 10.3 and 15.2.

π_J: $(\pi_J u)(s) := u(s)$ if $s \in J$ and $(\pi_J u)(s) := 0$ if $s \notin J$, i.e., $(\pi_J u)(\cdot) := \chi_J(\cdot) u(\cdot)$. Here J is a subset of $\mathbf{R}$. This operator is used both $\mathrm{L}^2(\mathbf{R}; U) \to \mathrm{L}^2(\mathbf{R}; U)$ and $\mathrm{L}^2(\mathbf{R}; U) \to \mathrm{L}^2(J; U)$.

π_+, π_-: $\pi_+ := \pi_{\mathbf{R}_+}$ and $\pi_- := \pi_{\mathbf{R}_-}$.

π^+, π^-: $\pi^+ := \pi_{\mathbf{N}}$, $\pi^- := I - \pi^+$ (Section 13.1).

$\widehat{\pi_\pm}, \widehat{\pm^\pm}$: $\widehat{\pi_\pm} \widehat{f} := \mathcal{L} \pi_\pm \mathcal{L}^{-1} \widehat{f}$, $\widehat{\pi^\pm} \widehat{f} := \mathcal{Z} \pi^\pm \mathcal{Z}^{-1} \widehat{f}$.

$\Pi*$: See Definition 2.6.3.

$\prod$: A product (of numbers or sets; cf. "$X \times Y$" above).

$\rho(A)$: The spectral radius $\rho(A) := \sup\{|z| \,\big|\, z \in \sigma(A)\} = \lim_{k\to\infty} \|A^k\|^{1/k} = \inf_{k\to\infty} \|A^k\|^{1/k} \le \|A\|$ of A (see Lemma A.3.3(r1)).

$\sigma(A)$: The spectrum $\mathbf{C} \setminus \{\lambda \in \mathbf{C} \,\big|\, \exists(\lambda - A)^{-1}\}$ (see pp. 871, 882 and 901).

$\Sigma = \left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D} \end{array}\right]$: A WPLS (or a wpls). See Definition 6.1.1 (or Definition 13.3.1).

$\sum$: A sum.

$\tau(t)$: The bilateral time-shift operator $\tau(t)u(s) = u(t+s)$ (this is a left-shift when $t > 0$ and a right-shift when $t < 0$).

$\phi_{\text{Cayley}}, \phi_{\text{Cayley}}^{-1}$: The Cayley function $\phi_{\text{Cayley}}(s) = \frac{1-s}{1+s} = \phi_{\text{Cayley}}^{-1}(s)$ (Lemma 13.2.1).

χ_J: The characteristic function of the set J, i.e., $\chi_J(s) := s$ if $s \in J$ and $\chi_J(s) := 0$ if $s \notin J$.

ω_A: The growth rate $\omega_A := \inf_{t>0}[t^{-1} \log \|\mathbb{A}(t)\|]$, when A is the infinitesimal generator of a semigroup $\mathbb{A}$. Note that $\omega_A \ge \sup \operatorname{Re} \sigma(A)$.

Function and operator spaces: generic notation

When $\mathcal{V}$ equals $\mathcal{B}$, $\mathcal{BC}$, $*\mathrm{TI}_*^*$ or ti_*^*, we use some or all of the following conventions:

$\mathcal{V}(X)$, $\mathcal{V}$: We set $\mathcal{V}(X) := \mathcal{V}(X,X)$. We write just $\mathcal{V}$ when we do not wish to specify X and Y; e.g., "$S, T \in \mathcal{B}$" means that S and T are linear and bounded (but they need not have same domain and range spaces).

$\mathcal{GV}$: This stands for the set of invertible (in $\mathcal{V}$) elements of $\mathcal{V}$, e.g., $\mathcal{GV}(X,Y) = \{T \in \mathcal{V}(X,Y) \,\big|\, ST = I_X \;\;\&\;\; TS = I_Y$ for some $S \in \mathcal{V}(Y,X)\}$.

$\mathcal{V}_\infty$: This stands for the union of $\mathcal{V}_\omega$ for all ω.

$\mathcal{V}_{\text{exp}}$: The set of "exponentially stable functions of type $\mathcal{V}$". We set $\mathcal{V}_{\text{exp}} := \cup_{\omega<0} \mathcal{V}_\omega$ (resp. $\mathcal{V}_{\text{exp}} := \cup_{\omega<1} \mathcal{V}_\omega$) if $\mathcal{V}_\omega$ is defined for all $\omega \in \mathbf{R}$ (resp. for $\omega > 0$ only).

When $\mathcal{V}$ equals $\mathcal{C}_*$, L_*^*, ℓ_*^*, H_*^* or W_*^*, we use some or all of the following conventions:

$\mathcal{V}(J;Y)$: Functions $J \to Y$ of type $\mathcal{V}$; we set $\mathcal{V}(J) := \mathcal{V}(J;\mathbf{K})$. We write just $\mathcal{V}$ when we do not wish to specify J and Y.

$\mathcal{GV}$: This stands for the set of invertible (in $\mathcal{V}$) elements of $\mathcal{V}$, e.g., $\mathcal{GV}(J;Y) = \{f \in \mathcal{V}(J;Y) \,\big|\, gf \equiv I \;\;\&\;\; fg \equiv I$ for some $g \in \mathcal{V}(J;*)\}$.

$\mathcal{V}_\infty$: This stands for the union of $\mathcal{V}_\omega$ for all ω.

$\mathcal{V}_{\text{strong}}$: $\mathcal{V}_{\text{strong}}(J;\mathcal{B}(X,Y)) := \{F \,\big|\, Fx \in \mathcal{V}(J;Y)$ for all $x \in X\}$, $\|F\|_{\mathcal{V}_{\text{strong}}} := \sup_{\|x\|_X \le 1} \|Fx\|_{\mathcal{V}}$.

$\mathcal{V}_{\text{weak}}$: $\mathcal{V}_{\text{weak}}(J;\mathcal{B}(X,Y)) := \{F \,\big|\, \Lambda Fx \in \mathcal{V}(J)$ for all $x \in X$, $\Lambda \in Y^*\}$, $\|F\|_{\mathcal{V}_{\text{weak}}} := \sup_{\|x\|_X \le 1,\ \|\Lambda\|_{Y^*} \le 1} \|\Lambda Fx\|_{\mathcal{V}}$.

(The strong and weak norms above become bounded in our applications due to the Uniform Boundedness Principle, Lemma A.3.4(O1).)

We often omit the subindex (if any) corresponding to the weight function if the weight function is the constant function 1 (e.g., $\ell^p = \ell^p_1$, $\mathrm{L}^p = \mathrm{L}^p_0$, $\mathrm{TI} = \mathrm{TI}_0$; note that this does not apply "WPLS" or "wpls", which equal WPLS_∞ and wpls_∞, respectively).

If $f \in \mathcal{V}_\omega$, $g \in \mathcal{V}_{\omega'}$ and $f = g$ on the intersection of their domains, then we identify f and g. (Then f and g are the unique elements of $\mathcal{V}_\omega$ and $\mathcal{V}_{\omega'}$ having that restriction on the intersection. For $*\mathrm{TI}^*_*$ (and ti^*_*, see Theorem 13.3.13) this fact is shown in Remark 2.1.9. See Lemma D.1.2(e) for the H^*_* identifications; for L^*_*, ℓ^*_* and W^*_* this is trivial.) We used this identification in the definition of $\mathcal{V}_\infty$ and $\mathcal{V}_{\exp}$.

Function and operator spaces

$\mathrm{AP}(\mathbf{R};X)$: The set of almost-periodic functions $\mathbf{R} \to X$ (see p. 957).

$\mathcal{B}(X,Y)$: The set of bounded linear operators $X \to Y$.

$\mathcal{BC}(X,Y)$: The set of compact linear operators from X into Y (p. 871).

$\mathcal{GB} + \mathcal{BC}$: See Lemma A.3.4.

$\mathcal{C}(J;X)$: The set of continuous functions $J \to X$; see p. 918 for more on $\mathcal{C}^*_*$.

$\mathcal{C}^k(J;X)$: The set of k times continuously differentiable functions $f \in \mathcal{C}(J;X)$; $\mathcal{C}^\infty := \cap_{k\in\mathbf{N}} \mathcal{C}^k$; we make similar definitions for the subspaces of $\mathcal{C}$ defined below.

$\mathcal{C}_\mathrm{b}(J;X)$: The set of bounded continuous functions $f \in \mathcal{C}(J;X)$; if J is an open or closed subset of $\mathbf{R}^n$ or an interval, then $\mathcal{C}_\mathrm{b}(J;X)$ is a Banach space with supremum norm, and $\mathcal{C}_\mathrm{bu}$ and $\mathcal{C}_0$ are closed subspaces of $\mathcal{C}_\mathrm{b}$.

$\mathcal{C}_\mathrm{bu}(J;X)$: The set of bounded and uniformly continuous functions $f \in \mathcal{C}(J;X)$.

$\mathcal{C}_0(J;X)$: The set of functions $f \in \mathcal{C}(J;X)$ vanishing at infinity (i.e., for all $\varepsilon > 0$ there is a compact $K \subset J$ s.t. $\|f(t)\|_X < \varepsilon$ for $t \in J \setminus K$); see Lemma B.3.4.

$\mathcal{C}_\mathrm{c}(J;X)$: The set of compactly supported functions $f \in \mathcal{C}(J;X)$ (note that $\mathcal{C}_\mathrm{c}(J;X) \subset \mathcal{C}_0(J;X)$).

$\mathcal{C}^\infty_\mathrm{c}(J;X)$: The set of compactly supported functions having continuous derivatives of all orders; see also Theorem B.3.11.

$\mathrm{c_c}, \mathrm{c_0}$: finite and vanishing sequences, respectively. See p. 919.

$\mathrm{H}(\Omega;X)$: Holomorphic functions $\Omega \to X$ (Appendix D); note that such functions are identified with their holomorphic extensions.

$\mathrm{H}^\infty(\Omega;X)$: Bounded holomorphic functions $\Omega \to X$ with supremum norm.

$\mathrm{H}^p, \mathrm{H}^p_\omega$: See Definition D.1.3, p. 964; for discrete time (on $r\mathbf{D}$), see Lemma D.1.15, p. 977.

$\mathrm{H}^p_\mathrm{strong}, \mathrm{H}^p_{\mathrm{strong},\omega}$: See Definition F.3.1, p. 1017.

ℓ^p_r, ℓ^p: By ℓ^p we denote L^p w.r.t. the counting measure. When u is a sequence, we set $\|u\|_p := \|u\|_{\ell^p}$; thus, then $\|u\|_p^p = \sum_k \|u_k\|^p$ $(1 \le p < \infty)$, $\|u\|_\infty = \sup_k \|u_k\|$. Finally, $\|u\|_{\ell^p_r} := \|r^{-\cdot} u\|_p = \|(r^{-k} u_k)\|_p$ $(r > 0)$.

$\ell^1_\pm, \ell^1_{\mathcal{BC}}, \ell^1_{\mathcal{BC},\pm}, \mathrm{c_c}$: See Section 13.1, p. 781.

$\mathrm{L}(J;X)$: The set of (equivalence classes of) (uniformly) Bochner measurable functions $J \to X$ (p. 911).

$\mathrm{L}_{\mathrm{strong}}(J;X)$; $\mathrm{L}_{\mathrm{weak}}(J;X)$: The set of (equivalence classes of) strongly; weakly measurable functions $J \to X$ (p. 998).

$\mathrm{L}^p(J;X)$: The Banach space of (equivalence classes of) X-valued L^p-functions on J. Thus, $\|f\|_{\mathrm{L}^p} := \|f\|_p$, where $\|f\|_\infty := \operatorname{ess\,sup}\|f\|_X$ and $\|f\|_p := \|f\|_{\mathrm{L}^p} = (\int_J \|f\|_X^p \, d\mu)^{1/p}$ when $1 \le p < \infty$; see also Definition B.3.1.

$\mathrm{L}^p(r\partial\mathbf{D};X)$: Here we identify $r\partial\mathbf{D}$ with $[0,2\pi)$ through $re^{i\cdot}$; cf. Lemma D.1.15.

$\mathrm{L}^p_\omega(J;X)$: The Banach space of (equivalence classes of) measurable functions $f : J \to X$ s.t. $\|f\|_{\mathrm{L}^p_\omega} := \|e^{\omega\cdot}\|_p < \infty$ (we must have $J \subset \mathbf{R}$). Thus, $\mathrm{L}^p = \mathrm{L}^p_0$, and $e^{\omega\cdot}$ becomes an isometric isomorphism $\mathrm{L}^p \to \mathrm{L}^p_\omega$. See Definition D.1.3.

$\mathrm{L}^p_{\mathrm{loc}}(J;X)$: The set of (equivalence classes of) functions $f : J \to X$ s.t. $f \in \mathrm{L}^p(K;X)$ whenever $K \subset J$ is compact; here $1 \le p \le \infty$. Note that $\mathrm{L}^p \subset \mathrm{L}^p_{\mathrm{loc}} \subset \mathrm{L}^1_{\mathrm{loc}}$ (and $\mathrm{L}^p_\omega \subset \mathrm{L}^p_{\mathrm{loc}}$ whenever $J \subset \mathbf{R}$). See also Definition B.3.1.

$\mathrm{L}^2_\infty(J;X)$; $\mathrm{L}^2_{+,\infty}$: The set $\cup_{\omega\in\mathbf{R}}\mathrm{L}^2_\omega(J;X)$; $\mathrm{L}^2_{+,\infty} := \mathrm{L}^2_\infty(\mathbf{R}_+;*)$.

$\mathrm{L}^p_c(J;X)$: The set $\{u \in \mathrm{L}^p(J;X) \,|\, u = 0 \text{ (a.e.) outside } [-T,T] \text{ for some } T > 0\}$ (i.e., the $\mathrm{L}^p(J;X)$ functions with a compact essential support).

$\mathrm{L}^p_{\mathrm{strong}}, \mathrm{L}^p_{\mathrm{weak}}$: See Definition F.1.4, p. 1002.

$\mathcal{S}(\mathbf{R};X)$: The space of rapidly decreasing functions, p. 978.

$\mathrm{W}^{k,p}_\omega(J;X), \mathrm{W}^{k,p}_{0,\omega}, \ldots$: The Sobolev spaces of k times weakly differentiably $\mathrm{L}^p_\omega(J,X)$ functions. See Section B.7.

If f is a X-valued function defined on a subinterval J of $\mathbf{R}$, we often identify f with its extension $\pi_J f$ having the value zero outside J. We use the same symbol π_J both for the embedding operator from $J \to X$ to $\mathbf{R} \to X$ and for the corresponding projection operator from $\mathbf{R} \to X$ to $J \to X$. With this interpretation, $\pi_+\mathrm{L}^2(\mathbf{R};X) = \mathrm{L}^2(\mathbf{R}_+;X) \subset \mathrm{L}^2(\mathbf{R};X)$ and $\pi_-\mathrm{L}^2(\mathbf{R};X) = \mathrm{L}^2(\mathbf{R}_-;X) \subset \mathrm{L}^2(\mathbf{R};X)$, etc.

Classes of time-invariant maps

See also Chapter 2 for TI$*$, Section 13.1 for ti$*$ and Section 2.6.3 for the other classes.

$\widetilde{\mathcal{A}}, \widetilde{\mathcal{A}}_+$: In Chapter 10 (resp. Chapters 11 and 12), the symbol $\widetilde{\mathcal{A}}_+$ (resp. $\widetilde{\mathcal{A}}$) stands for MTIC or something similar; see the standing hypotheses mentioned at the beginnings of these chapters (see Theorem 8.4.9 for suitable classes).

$\mathrm{TI}_\omega(U,Y)$: The (closed) subspace of operators $\mathbb{D} \in \mathcal{B}(\mathrm{L}^2_\omega(\mathbf{R};U);\mathrm{L}^2_\omega(\mathbf{R};Y))$ that are time-invariant (i.e., $\tau(t)\mathbb{D} = \mathbb{D}\tau(t)$ for all $t \in \mathbf{R}$). $\mathrm{TI} := \mathrm{TI}_0$, $\mathrm{TI}_\infty := \cup_{\omega\in\mathbf{R}}\mathrm{TI}_\omega$.

$\mathrm{TI}^p, \mathrm{TI}^p_*, \mathrm{TI}^{p,q}_\omega$: See Theorem 3.1.5.

$\mathrm{TI}^{\mathcal{C}_0}_\omega$, $\mathcal{C}_{0,\omega}$: See p. 92.

$\mathrm{TIC}_\omega(U,Y)$: The (closed) subspace of operators $\mathbb{D} \in \mathrm{TI}_\omega(U,Y)$ that are causal (i.e., $\pi_-\mathbb{D}\pi_+ = 0$). $\mathrm{TIC} := \mathrm{TIC}_0$, $\mathrm{TIC}_\infty := \cup_{\omega\in\mathbf{R}}\mathrm{TIC}_\omega$, $\mathrm{TIC}_{\exp} := \cup_{\omega<0}\mathrm{TIC}_\omega$.

$\mathrm{CTI}, \mathrm{CTIC}, \mathrm{CTI}^{\mathcal{BC}}, \mathrm{CTIC}^{\mathcal{BC}}$: Classes of TIC operators having continuous transforms (Definition 2.6.1).

$\mathrm{MTI}^*_*, \mathrm{MTIC}^*_*, \mathrm{SMTI}^*_*, \mathrm{SMTIC}^*_*$: Classes of TIC_∞ operators that are convolutions with certain kinds of measures (Definition 2.6.3).

$\mathrm{MTI}_X, \mathrm{MTIC}_X$: Certain kinds of measures with values in X, see Lemma D.1.12.

$\mathrm{ti}^*_*, \mathrm{tic}^*_*$: Discrete-time classes; see Definition 13.1.1, p. 783.

Abbreviated symbols

card A: The *cardinality* of A (see, e.g., Definition 151, p. 275 of [Kelley]). It suffices to know that $\operatorname{card} A \leq \operatorname{card} B$ iff there is a one-to-one function of A to B (equivalently, a function of B onto A). Consequently, $\operatorname{card} A = \operatorname{card} B$ iff $\operatorname{card} A \leq \operatorname{card} B$ and $\operatorname{card} B \leq \operatorname{card} A$ (equivalently, there is a one-to-one map of A onto B); see, e.g., [Kelley] for details. See also Lemma B.2.2.

det A: The determinant of the matrix A.

$\operatorname{diag}(A,B,C)$: the *diagonal* matrix $\left[\begin{smallmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{smallmatrix}\right]$ with diagonal elements (or blocks) A, B and C.

$\dim(H)$: the *dimension* of the Hilbert space H = the cardinality of an arbitrary orthonormal basis of H (Lemma A.3.1(a1)).

$\mathrm{Dom}(T)$: The *domain* (of definition) of the operator T.

ess range: Essential range, Lemma B.2.7.

ess sup, ess inf: Essential supremum, essential infimum. See p. 909.

$\mathrm{Ker}(T)$: The *kernel* $\mathrm{Ker}(T) := T^{-1}[\{0\}] := \{u \in U \mid Tu = 0\}$, when $T : U \to Y$.

$\mathrm{Leb}(f)$: The set of Lebesgue points of f (p. 942).

log: Logarithm with base exp.

$\mathrm{Ran}(T)$: The range $T[U] := \{T(u) \mid u \in U\}$, when $T : U \to Y$.

Re, Im: Real part, imaginary part ($\mathrm{Re}(x+iy) = x$, $\mathrm{Im}(x+iy) = y$ for $x, y \in \mathbf{R}$).

span: $\operatorname{span} E := \{\sum_{j=0}^n \alpha_j x_j \mid n \in \mathbf{N},\ \alpha_j \in \mathbf{K},\ x_j \in E\ (j = 0, \ldots, n)\}$.

sup, inf, max, min: Supremum, infimum, maximum, minimum, respectively. Recall from [Rud86] that $\inf \emptyset := +\infty$, $\sup \emptyset = -\infty$.

supp f: The *support* of f = the closure of $\{t \mid f(t) \neq 0\}$.

$\mathrm{supp}_\mathrm{d}\,\mathbb{E}$: The nonzero atoms of $\mathbb{E}$ (Definition 2.6.3).

w-lim, s-lim, lim: "lim" means the *limit* in the standard topology, which for operators is the uniform (i.e., norm) topology. "s-lim" and "w-lim" refer to strong and weak limits, respectively. E.g., if $F : \mathbf{R} \to \mathcal{B}(H)$, where H is a Hilbert space, then $\text{w-lim}_{s\to+\infty} F(s) = A$ means that $F(s)x \to Ax$ weakly for all $x \in H$, as $s \to +\infty$; cf. Lemma A.3.1(h)–(j).

Glossary

We should have a great many fewer disputes in the world if only words were taken for what they are, the signs of our ideas only, and not for things themselves.

— John Locke (1632–1704)

If $T \in \mathcal{B}(H_1, H_2)$, where H_1 and H_2 are Hilbert spaces, then we use the following terms for T (partially valid also for more general functions):

one-to-one: = *injective*, i.e., $Tx = 0 \Leftrightarrow x = 0$ (for all $x \in H_1$). Equivalent to "coercive" if $\dim H_1 < \infty$.

coercive: $\|Tx\| \geq \varepsilon\|x\|$ for all $x \in H_1$, i.e., $T^*T \gg 0$ = *left-invertible* = T^* is onto = full column rank (if matrix).

onto: $T[H_1] = H_2$, i.e., $TT^* \gg 0$ = *surjective* = *right-invertible* = T^* coercive = full row rank (if matrix).

invertible: = boundedly invertible = *bijection* = one-to-one and onto.

countably infinite: A set is countably infinite if it has the same cardinality as the set **N**, i.e., if there is a one-to-one function of **N** onto this set.

countable: A set is countable if it is finite or countably infinite; otherwise it is *uncountable*.

finite-dimensional vector space: A vector space spanned by a finite number of vectors.

finite-dimensional function: A vector-valued function whose values lie in a finite-dimensional subspace of the range space.

finite-dimensional system: A system whose input, state and output spaces are finite-dimensional. Recall that a transfer function has a finite-dimensional realization iff it is rational.

finite-dimensional theory: This refers to the theory of finite-dimensional systems.

classical theory: This refers usually to finite-dimensional theory (equivalently, to the theory of rational transfer functions).

time domain: This refers to **R** or $\mathbf{R}_+$ as the time horizon. The input, output and state signals in control systems are functions of time, with domain **R** or $\mathbf{R}_+$. Thus, their Laplace (or Fourier) transforms are defined on a subset of **C**; by *frequency domain* we refer to such subsets or to (a part of) **C** as the domain of the argument of these transformed functions.

state-space: State-space theory refers to theory on systems (where one can also speak of the state, not just on input and output) in contrast to *I/O-theory* or *frequency-domain theory*, which ignores the internal structure of systems and treats I/O maps instead of systems (compare this to the two definitions of admissible controllers in Definition 7.1.1). Thus (due to historical reasons), "frequency-domain" has two meanings: the first refers to working with Laplace transforms (as opposed to time-domain), and the second to working with I/O maps or transfer functions (as opposed to state-space).

discrete time: This refers to time domain $\mathbf{Z}$ or $\mathbf{N}$ in place of $\mathbf{R}$ or $\mathbf{R}_+$ as opposed to *continuous time* treated above, see Part IV.

discrete part: The discrete (atomic) part of a measure is explained in Section 2.6.

map: A map means a function. However, for most of the time, we reserve the word map for the "integral operators", such as $\mathbb{A}$, $\mathbb{B}$, $\mathbb{C}$ and $\mathbb{D}$ in Definition 6.1.1. See Definitions 7.2.1 and 7.2.11 and Lemma 7.2.7 for maps with internal loop.

well-posed: An I/O map is well posed (or *proper*) if it is in TIC_∞. A transfer function is well posed (or proper) if it is in H^∞_∞, i.e., if it is bounded on some right half-plane. (By Theorem 2.1.2, an I/O map is well posed iff its transfer function is well posed.) Lemma 7.2.7 shows when a map with internal loop is well posed.

stable: A function (signal) $u \in \mathrm{L}^2_{\mathrm{loc}}$ is called stable iff $u \in \mathrm{L}^2$. A map from vectors or signals to vectors or signals is called stable iff it is bounded w.r.t. to the standard norm for vectors and the L^2 norm for signals; see Definition 6.1.3 for details. In discrete-time, a function (signal; actually sequence) u is called stable iff $u \in \ell^2$; see Definition 13.3.1 for the stability of discrete-time maps

singular: A control problem is often called *singular* if the map from the control to the output is not coercive (or I-coercive over $\mathcal{U}^*_*$); otherwise it is *nonsingular*. Most control problems in the literature are nonsingular.

superfluous: An assumption is said to be superfluous if the claims are true even without the assumption ("and $n < 7$" is superfluous in "if $n > 1$ and $n < 7$, then n is positive").

redundant: As assumption is said to be redundant if it is implied by the other assumptions ("and $t = |t|$" is redundant in "if $t > 1$ and $t = |t|$, then $t^2 > t$").

greatest: If $\mathcal{A} \subset \mathcal{B}(H)$, then $A \in \mathcal{A}$ is the greatest element of $\mathcal{A}$ iff $A \geq A'$ for all $A' \in \mathcal{A}$ (iff $-A$ is the *smallest* element of $\mathcal{A}$). Recall that $A \in \mathcal{A}$ is *maximal* iff $A \leq A' \in \mathcal{A} \Rightarrow A = A'$. Obviously, a greatest element must be unique and maximal, whereas a maximal element need not be unique in general.

Abbreviations

a.e.: almost everywhere (or "almost every")

iff: if and only if

I/O: input/output ("from input to output")

p., pp.: page, pages

r.c., l.c., d.c., p.r.c., p.l.c., q.r.c., q.l.c.: See Definition 6.4.1.

r.c.f., l.c.f., d.c.f., p.r.c.f., p.l.c.f., q.r.c.f., q.l.c.f.: See Definition 6.4.4.

s.t.: such that

w.r.t.: with respect to

w.l.o.g.: without loss of generality

Acronyms

B_{w}^{*}-CARE: Certain simplified Riccati equation, see Definition 9.2.6

[e]CARE; [e]DARE; [e]IARE: Riccati equations. See Definition 9.8.1; 14.1.1; 9.8.4.

[e]CARI; [e]DARI; [e]IARI: Riccati inequalities; see the index.

FICP: Full-Information Control Problem; see Chapter 11 (or Section 10.4).

LQR: Linear Quadratic Regulator, see Chapter 10.

SF: State Feedback, see Definition 6.6.10 (sometimes this refers to pure state feedback, where the feedthrough term (the "F" of $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$) is zero, see Definition 11.1.2).

SpF: See Definition 8.4.6, p. 384

$*\mathrm{TI}_{*}^{*}$, $*\mathrm{TIC}_{*}^{*}$, ti_{*}^{*}, tic_{*}^{*}: See "Classes of time-invariant maps", p. 1046

TVS: Topological Vector Space (p. 870).

WR, WLR, WVR, WHPR, SR, SLR, SVR, SHPR, UR, ULR, UVR, UHPR: Different forms of regularity. See Definition 6.2.3; see also "regular" in the index.

WPLS_{*}, SOS: See Definitions 6.1.1 and 6.1.3.

wpls_{*}, sos: Classes of discrete-time systems. See Definition 13.3.1.