Teknillinen korkeakoulu Londen / Mikkola

Mat-1.150 Reaalianalyysi

Välikoe 1, 29.3.1999 klo 12–15, vastauksin (Real analysis, first mid-term examination with answers)

Kirjoita selvästi jokaiseen koepaperiin eri riveille:
1) opintojakson nimi, välikokeen numero, päiväys;
2) opiskelijanumero+kirjain, TEKSTATEN sukunimi alleviivattuna, kaikki etunimet;
3) koulutusohjelma (Ke,Ko,M,Mk,P,R,S,Tf,Ti,Tu);
4) mahdolliset entiset nimet ja koulutusohjelmat;
5) nimikirjoitus.

Please write clearly on every sheet on separate lines:
1) course title, "first mid-term examination", date
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Tehtävät kannattaa tehdä alkaen helpoimmasta/nopeimmasta.

Tehtävä 1. (a) Määrittele käsite σ-algebra. (b) Olkoon $\mathfrak{M}$ joukon X σ-algebra. Olkoon Y topologinen avaruus. Olkoon f funktio $X \to Y$. Määritellään $\Omega = \{E \mid E \subset Y,\ f^{-1}(E) \in \mathfrak{M}\}$. Osoita, että Ω on Y:n σ-algebra.

(a) Define the concept σ-algebra. (b) Let $\mathfrak{M}$ be a σ-algebra on X. Let Y be a topological space. Let f be a function $X \to Y$. Define $\Omega = \{E \mid E \subset Y,\ f^{-1}(E) \in \mathfrak{M}\}$. Show that Ω is σ-algebra on Y.

Answer: `Points: 6666666655554.` [1]

[1] This problem has nothing to do with the fact that Y is a topological space. See [R2: p. 7] for f^{-1} of sets. We use here standard formulas $f^{-1}(E^c) = [f^{-1}(E)]^c$ and $f^{-1}(\cup_n A_n) = \cup_n f^{-1}(A_n)$ etc. (they are easy to verify, e.g., $f^{-1}(\cup_n A_n) = \{x \in X \mid f(x) \in \cup_n A_n\} = \{x \in X \mid f(x) \in A_n \text{ for some } n\} = \{x \in X \mid x \in f^{-1}(A_n) \text{ for some } n\} =: \cup_n f^{-1}(A_n)$. If the student seemed to know these, I didn't require the proofs.

(a) [R2: 1.3]: $\mathfrak{M} \subset 2^X$ ($:= \{\text{subsets of } X\}$), (i) $X \in \mathfrak{M}$, (ii) $A \in \mathfrak{M} \implies A^c \in \mathfrak{M}$, (iii) $A_n \in \mathfrak{M}\ \forall n \implies \cup_{n=1}^{\infty} A_n \in \mathfrak{M}$. `(2 p.)`

(b) Clearly $\Omega \subset 2^Y$. (i) $f^{-1}Y = X \in \mathfrak{M}$, (ii) $f^{-1}(E^c) = [f^{-1}(E)]^c \in \mathfrak{M}$, when $f^{-1}(E) \in \mathfrak{M}$, (iii) $f^{-1}(A_n) \in \mathfrak{M}\ \forall n \implies f^{-1}(\cup_{n=1}^{\infty} A_n) = \cup_{n=1}^{\infty} f^{-1}(A_n) \in \mathfrak{M}$, QED. `(4 p.)`

Tehtävä 2. Olkoon $X \neq \emptyset$ kompakti ja $f : X \to \mathbb{R}$ alhaalta puolijatkuva. Osoita, että f saavuttaa minimin jossakin X:n pisteessä.

Let $X \neq \emptyset$ be compact and let $f : X \to \mathbb{R}$ be lower semicontinuous. Show that f achieves a minimum at some point of X.

Answer: `Points: 6666653222210.` "f is lower semicontinuous" means that $V_r := \{f > r\} := \{x \in X \mid f(x) > r\}$ is open $\forall r \in \mathbb{R}$ `(1 p.)`. $\beta := \inf f \geq -\infty$. If we had $X \subset \cup_{M>\beta} V_M$, then a finite subcover of the open cover $\{V_M \mid M > \beta\}$ would cover X, i.e., $X \subset V_{M_1} \cup \ldots \cup V_{M_n}$ for some $M_1, M_2, \ldots, M_n > \beta$, hence $f(x) > \min_{j \leq n} M_n > \beta\ \forall x$, which is a contradiction ($\inf_x f(x) \geq \min_{j \leq n} M_n > \beta := \inf_x f(x)$).

Thus, $\exists a \in X \setminus \cup_{M>\beta} V_M$, so we have $f(a) = \beta$, QED.[2][3][4]

Tehtävä 3. Olkoon μ X:n positiivinen mitta ja $\mu(X) < \infty$. Olkoon $f \in L^\infty(\mu)$ ja $\|f\|_\infty > 0$. Määritellään $\alpha_n = \int_X |f|^n \, d\mu$ ($n = 1, 2, \ldots$). Osoita, että $\lim_{n\to\infty} \frac{\alpha_{n+1}}{\alpha_n} = \|f\|_\infty$.

Let μ be a positive measure on X and $\mu(X) < \infty$. Let $f \in L^\infty(\mu)$ and $\|f\|_\infty > 0$. Define $\alpha_n = \int_X |f|^n \, d\mu$ ($n = 1, 2, \ldots$). Show that $\lim_{n\to\infty} \frac{\alpha_{n+1}}{\alpha_n} = \|f\|_\infty$.

Answer: `Points: 2221111111110 (2° was hard).`
$0 = \alpha_n = \int_X |f|^n \, d\mu \implies |f|^n = 0$ a.e. $\implies f = 0$ a.e. $\implies \|f\|_\infty = 0$, so

[2] Because $f(a) \leq M\ \forall M > \beta$ implies that $f(a) \leq \beta$. Of course, $\beta \in \mathbb{R}$, because $f : X \to \mathbb{R}$.

[3] Alternative solution: the intersection K of compact [R2: 2.4], nonempty sets $\{f \leq r\} \subset_c X$ ($r > \beta$) is nonempty [R2: 2.6] (otherwise we had $\beta \geq \min_n r_n > \beta$), and $a \in K \implies f(a) = \beta$. Alternative 2: If we didn't have a minimum, we would have $X \subset \cup_{x\in X} W_x$, where $W_x := \{y \mid f(y) \leq f(x)\}$...

[4] Zorn's lemma wasn't of much help here. Nice try, XX.

we must have $\alpha_n > 0$ `(1 p.)` (hence $\mu(X) > 0$), for any n. On the other hand, $\alpha_n \le \int_X \|f\|_\infty^n \, d\mu = \mu(X)\|f\|_\infty^n < \infty$ `(1 p.)`.

We will now show that $\limsup_n \frac{\alpha_{n+1}}{\alpha_n} \le \|f\|_\infty$ [1°] and then that $\liminf_n \frac{\alpha_{n+1}}{\alpha_n} \ge \|f\|_\infty$ [2°], which shows that these must be equal, hence $\exists \lim_n \frac{\alpha_{n+1}}{\alpha_n} = \|f\|_\infty$.

1° $\alpha_{n+1} \le \int_X \|f\|_\infty |f|^n \, d\mu = \|f\|_\infty \alpha_n$,[5] so $\frac{\alpha_{n+1}}{\alpha_n} \le \|f\|_\infty$ $\forall n$ `(1 p.)`, hence $\limsup_n \frac{\alpha_{n+1}}{\alpha_n} \le \|f\|_\infty$.

2°[6] Let $0 < R < \|f\|_\infty$. Set $E := \{|f| \ge R\}$. Now $\mu(E) > 0$ (because $\|f\| :=$ sup of those R for which this holds [R2: 3.7]) and $\alpha_{n+1} \ge \int_E |f|^{n+1} \ge \int_E R^{n+1} = \mu(E)R^{n+1} > 0$.

Moreover, $\alpha_n/\alpha_{n+1} \le \int_E |f|^n/\alpha_{n+1} + \int_{E^c} f^n/\alpha_{n+1} \le \int_E |f|^n/(R\int_E |f|^n) + (\mu(E)R)^{-1} \int_{E^c}(|f|^n/R^n) \to R^{-1} + 0$, by dominated convergence [R2: 1.34], because $(|f(x)|/R)^n \to 0$ $\forall x \in E^c$ and $(|f|/R)^n < 1$ $\forall n$ and $1 \in L^1(E)$. Thus, $\limsup_n \frac{\alpha_n}{\alpha_{n+1}} \le R$, hence $\liminf_n \frac{\alpha_{n+1}}{\alpha_n} \ge R$. Because $R < \|f\|_\infty$ was arbitrary, $\liminf_n \frac{\alpha_{n+1}}{\alpha_n} \ge \|f\|_\infty$.[7] `(3 p.)`

Tehtävä 4. Olkoon H Hilbert-avaruus ja $S \subset H$. Osoita (a) $S^{\perp\perp}$ on H:n suljettu aliavaruus, (b) $S^{\perp\perp}$ on pienin H:n suljettu aliavaruus, joka sisältää joukon S.

Let H be a Hilbert space and $S \subset H$. Show that (a) $S^{\perp\perp}$ is a closed subspace of H, (b) $S^{\perp\perp}$ is the smallest closed subspace of H that contains the set S.

Answer: `Points: 5444433333221. I gave max 4 p. if you assumed` S `to be a subspace.`

(a) Let $x \in H$. If $\langle x, y\rangle = 0 = \langle x, y'\rangle$, then $\langle x, \alpha y + y'\rangle = 0$, hence $x^\perp := \{y \in H \,|\, \langle x, y\rangle = 0\}$ is a subspace of H (clearly $0 \in x^\perp$ too). If $x^\perp \ni y_n \to y$, then $\langle x, y\rangle = \lim_n \langle x, y_n\rangle = 0$ ($\langle x, \cdot\rangle$ is continuous, by [R2: 4.6]), i.e., $y \in x^\perp$, so $x^\perp$ is a closed subspace of H, thus so is $M^\perp := \cap_{x\in M} x^\perp$ for any $M \subset H$. In particular, so is $S^{\perp\perp} := (S^\perp)^\perp$ (an intersection of subspaces is clearly a subspace; an intersection of closed sets is closed). `(2 p.)`

(b) 1° $S \subset S^{\perp\perp}$. Proof: If $x \in S$, then $\langle x, y\rangle = 0$ $\forall y \in S^\perp$, i.e., $x \in S^{\perp\perp}$, QED.

2° $\bar{S} \subset S^{\perp\perp}$. Proof: Let $x \in \bar{S}$. Choose $\{x_n\} \subset S$ s.t. $x_n \to x$. If $y \in S^\perp$, then $\langle x, y\rangle = \lim_n \langle x_n, y\rangle = 0$, hence $x \perp y$. Thus, $x \in S^{\perp\perp}$, QED.

3° $S^{\perp\perp}$ is smallest: Let M be a closed subspace of H and $S \subset M$. Take ortogonal projections $P, Q \in \mathcal{L}(H)$ s.t. $P(H) \subset M$, $Q(H) \subset M^\perp$, and $P + Q = I$ [R2: 4.11].

Let $x \in S^{\perp\perp}$. Then $0 = \langle x, Qx\rangle = \langle Px + Qx, Qx\rangle = \langle Px, Qx\rangle + \langle Qx, Qx\rangle = 0 + \|Qx\|^2$, i.e., $x = Px + 0 \in M$. Thus, $S^{\perp\perp} \subset M$, QED. `(4 p.)`

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[5] The values of $|f|$ in the null set $\{|f| > \|f\|_\infty\}$ don't affect integrals.

[6] A second approach for 2°: show first that $\|f\|_n \to \|f\|_\infty$ (not too easy), and then use Hölder with $p = (n+1)/n$ to show that $\alpha_n/\alpha_{n+1} \le \mu(X)^{1/(n+1)}/\|f\|_{n+1} \to 1/\|f\|_\infty$.

[7] If $a > R$ for all $R < b$, then $a \ge b$ (otherwise $R := (a+b)/2$ would satisfy $a < R < b$, a contradiction). This is a standard approach, remember this!