Teknillinen korkeakoulu Londen / Mikkola

Mat-1.150 Real Analysis

Mid-term examination 2/2, December 4, 2002 at 14–17 (hall D) **with answers** (4 pages, unverified version 2002-12-02)

Grades: 37–48⇒5, 26–36⇒4, 20–25⇒3, 16–19⇒2, 13–15⇒1.

Notes on scoring: In "Show" exercises, definitions etc. not required (and you can get points for implicit definitions etc.). "By theorem x.yy" may be replaced with "by a theorem in the book" (if it is sufficiently described and/or its assumptions are sufficiently checked). Usually theorems prior to the one to be proved may be used in the proof but not those that were proved using it (nor an almost same theorem).

Exercise 1. Assume that μ and λ are measures on a σ-algebra $\mathfrak{M}$, μ is positive and λ is complex. Show that (a) and (b) are equivalent:

(a) $\lambda \ll \mu$;

(b) To every $\epsilon > 0$ corresponds a $\delta > 0$ s.t. $|\lambda(E)| < \epsilon$ for all $E \in \mathfrak{M}$ with $\mu(E) < \delta$.

Results: 0123456p=006(11)216 students

Answer: (See Theorem 6.11 of [R2].)

1° *"(b)⇒(a)":* $\lambda \ll \mu$ means that $\mu(E) = 0 \implies \lambda(E) = 0$ ($E \in \mathfrak{M}$) `(1 p.)`. Assume (b). Let $E \in \mathfrak{M}$. If $\mu(E) = 0$, then $\mu(E) < \delta\ \forall\delta > 0$, hence then $|\lambda(E)| < \epsilon\ \forall\epsilon > 0$, i.e., $|\lambda(E)| = 0$. Since $E \in \mathfrak{M}$ was arbitrary, (a) holds `(1 p.)`.

2° *"¬(b)⇒ ¬(a)":* Assume that (b) is not true, i.e., that there is $\epsilon > 0$ for which there is no $\delta > 0$.... Thus, there are sets $E_n \subset \mathfrak{M}$ s.t. $\mu(E_n) < 2^{-n}$ but $|\lambda(E_n)| \geq \epsilon$, hence $|\lambda|(E_n) \geq \epsilon$. Put

$$A_n := \cup_{i=n}^{\infty} E_i, \quad A := \cap_{n=1}^{\infty} A_n$$

Then $\mu(A_n) < 2^{-n+1}$, $A_n \supset A_n n + 1$, and so Theorem 1.19(e) shows that $\mu(A) = 0$ and that

$$|\lambda|(A) = \lim_{n\to\infty} |\lambda|(A_n) \geq \epsilon > 0,$$

since $|\lambda|(A_n) \geq |\lambda|(E_n)$. Consequently, $|\lambda| \ll \mu$ is not true, hence (a) is not true, by Proposition 6.8(e), QED.[1]

Scoring: Significant progress in 2°: `(1 p.)`.

Errors: 2° "Assume ¬(b), i.e., that $\exists\epsilon > 0, \exists E \in \mathfrak{M}$ s.t. $\forall\delta$ we have $\mu(E) < \delta$". Wrong, "$\neg\forall\epsilon > 0\ \exists\delta > 0\ \forall E \in \mathfrak{M}\ P(E)$" = "$\exists\epsilon > 0\ \forall\delta > 0\ \exists E \in \mathfrak{M}\ \neg P(E)$".

Exercise 2. Let $(X, \mathfrak{M}, \mu)$ be a finite positive measure space. Let $p \in [1, \infty)$, and let Φ be a bounded linear functional on $L^p(\mu)$. For measurable $E \subset X$, let χ_E be the characteristic function of E.

Define $\lambda(E) := \Phi(\chi_E)$ for all $E \in \mathfrak{M}$. Show that λ is a complex measure.

Results: 0123456p=020(18)213 students

Answer: (See the last nine rows on p. 127 of [R2].) Since $\|\chi_E\|_p = \mu(E)^{1/p} < \infty$ for all $E \in \mathfrak{M}$, λ is well defined. Obviously[2] $\lambda : \mathfrak{M} \to \mathbb{C}$, hence we only have to show that λ is countably additive `(1 p.)`. This becomes easier if we first establish finite additivity.

When A and B are disjoint, we have $\chi_{A\cup B} = \chi_A + \chi_B$, hence $\lambda(A \cup B) = \Phi(\chi_{A\cup B}) = \Phi(\chi_A + \chi_B) = \lambda(A) + \lambda(B)$. By induction, it follows that λ is additive `(2 p.)`.

Let $E = \cup_{i\in\mathbb{N}} E_i$, where $E_i \in \mathfrak{M}$ are disjoint. Set $A_k := E_1 \cup ... \cup E_k$, and note that

$$\|\chi_E - \chi_{A_k}\|_p = [\mu(E \setminus A_k)]^{1/p} \to 0 \quad (k \to \infty), \tag{1}$$

because $\mu(E \setminus A_k) = \mu(\cup_{j=k+1}^{\infty} E_k) = \sum_{j=k+1}^{\infty} \mu(E_k) \to 0$ (since $\sum_{j=1}^{\infty} = \mu(E) < \infty$) `(2 p.)`. Since Φ is continuous, it follows from (1) that $\lambda(A_k) \to \lambda(E)$, i.e., that $\lambda(E) = \lim_{k\to\infty} \lambda(A_k) = \lim_{k\to\infty} \sum_{j=1}^{k} \lambda(E_k) =: \sum_{j=1}^{\infty} \lambda(E_k)$, QED `(1 p.)`.[3]

Remarks: 1. The point in the (second part of the) proof was that $\chi_{A_k} \to \chi_E$ in $L^p(\mu)$ and hence $\Phi(A_k) \to \Phi(E)$. This part was worth 3 points. "Since Φ is continuous" does not give any more points (but if you say "we should show that $\chi_{A_k} \to \chi_E$ in L^p", you get one of them; the other two from the proof).

2. You need not prove that λ is bounded, the definition (6.1) only requires

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[1] "Hence $|\lambda(A)| > 0$". Not true, but there must be some (possibly other) $A' \in \mathfrak{M}$ s.t. $\lambda(A') \neq 0$ although $\mu(A') = 0$, by Proposition 6.8(e).

[2] Since $\Phi : L^p(\mu) \to \mathbb{C}$.

[3] *Quod erat demonstrandum* (lat.), what was to be shown ("mot." in Finnish), "////" in [R2], "□" usually.

that it is $\mathfrak{M} \to \mathbb{C}$ (this is obvious, since $\Phi \in L^p(\mu) \to \mathbb{C})$[4] and countably additive (although it is bounded, by Theorem 6.4).

You neither need to show that $\lambda(\emptyset) = 0$ (the axioms of a positive measure require that $\lambda(E) < \infty$ for some $E \in \mathfrak{M}$; an equivalent requirement is that $\lambda(\emptyset) = 0$ (or as well that $\lambda \not\equiv \infty$), but a complex measure may not have ∞ as its value).

Errors: "By Theorem 6.16 of [R2], there is $g \in L^q$ (with $p^{-1} + q^{-1} = 1$ s.t. $\Phi(f) = \int_X fg\,d\mu$ ($f \in L^p(\mu)$). Therefore, $\lambda(E) = \int_E g\,d\mu$, so ... (several details (using Theorem 1.38) removed) it follows that λ is countably additive." (This is circular reasoning (kehäpäätelmä), since the result of this exercise was used to prove Th6.16.) Nevertheless, I did not subtract points since I considered that one should not be required to know that this is part of the proof of Th6.16. Fortunately, it made the exercise only slightly easier.

2. "By linearity, $\Phi(\sum_{k\in\mathbb{N}} \chi_{E_k}) = \sum_{k\in\mathbb{N}} \Phi(\chi_{E_k})$" (but otherwise OK) `(3 p.)` (or 3-1=2p if 6.16 used). Linearity only applies to a sum of two (or finitely many, by induction) vectors, not of "infinite sums" (which mean the limits of finite sums).

3. "$\int_{\cup E_j} g\,d\mu = \sum_j \int_{E_j} g\,d\mu$" `(-3 p.)`. You must prove this (hint, use Theorem 1.38).[5]

4. "λ must be countably additive" `(1- p.)`.

5. "$\sum_k \mu(A_k)^{1/p} \le [\sum_k \mu(A_k)]^{1/p}$" wrong (if $p > 1$, then $(1/2)^{1/p} + (1/2)^{1/p} > (1/2+1/2)^{1/p}$).

Exercise 3. Show that if the function $f : [a,b] \to \mathbb{C}$ is absolutely continuous and $1 \le p < \infty$, then also the function $|f|^p$ is absolutely continuous.

[4]So here we do not even need the fact that Φ is bounded. Even if Φ was unbounded (i.e., discontinuous), its values would have to be in $\mathbb{C}$, by the definition of a function $L^p(\mu) \to \mathbb{C}$ (a linear functional on L^p means a linear function $L^p \to \mathbb{C}$; in particular, the value ∞ (and other elements $\notin \mathbb{C}$) is excluded from its range).

[5]Theorem 1.27 of [R2] ("$\int\sum = \sum\int$") only applies to nonnegative-valued functions, but a similar one holds for complex-valued ones with the additional condition that $f \in L^1$; use Th1.38 to prove this. (This is typical that claims hold (a) when $f \ge 0$, and (b) when f is complex and in L^1, e.g., this and Fubini.)

Answer: *Results: 0123456p=1247534 students* [6] [7] Let $\epsilon > 0$ be given. Since $|f|$ is continuous on the compact set $[a,b]$, $M := \max_{[a,b]} |f|$ exists. Set $\epsilon' := \epsilon/pM^{p-1} > 0$ (case $f \equiv 0$ is trivial and omitted). Since f is absolutely continuous, there is $\delta > 0$ s.t. whenever $n \in \{1,2,...\}$ and $[\alpha_i,\beta_i] \subset [a,b]$ are disjoint[8] intervals ($i = 1,2,...,n$) with $\sum_{i=1}^n |\beta_i - \alpha_i| < \delta$, then $\sum_{i=1}^n |f(\beta_i) - f(\alpha_i)| < \epsilon'$.

Set $g(t) := t^p$. Whenever $0 \le r < s < \infty$, there is $\xi \in (r,s)$ s.t.

$$g(s) - g(r) = g'(\xi)(s-r) = p\xi^{p-1}(s-r) \le ps^{p-1}(s-r) = ps^{p-1}|r-s|,$$

by the Mean Value Theorem. Therefore, we have

$$(\dagger) \quad |g(s) - g(r)| \le pM^{p-1}|s-r| \quad (s,r \in [0,M]).$$

Moreover, $|f(t)|^p = g(|f(t)|)$ for all $t \in [a,b]$, hence, given any n, $\{\alpha_i,\beta_i\}_{i=1}^n$ as above, we have

$$\begin{aligned}\sum_{i=1}^n \big||f(\beta_i)|^p - |f(\alpha_i)|^p\big| &= \sum_{i=1}^n |g(|f(\beta_i)|) - g(|f(\alpha_i)|)| \\ &\overset{\dagger}{\le} \sum_{i=1}^n pM^{p-1}\big||f(\beta_i)| - |f(\alpha_i)|\big| \le pM^{p-1}\epsilon' = \epsilon.\end{aligned}$$

[6]**Proof 2:** `(5- p.)` (sketch, partial) 1. $|f|$ is AC on $[a,b]$ (proof:...); 2. $t \mapsto t^p \in$ AC on $\mathbb{R}_+$ (proof: ...), hence on the range of $|f|$; 3. If g is AC on $[a,b]$ and h is AC on the range of g, then $h \circ g$ is AC on $[a,b]$. Proof: "we have $\sum |h(g(\beta_i)) - h(g(\alpha_i))| < \epsilon$ when $\sum |g(\beta_i) - g(\alpha_i)| < \delta' := \delta_{h,\epsilon}$, hence when $\sum |\beta_i - \alpha_i| < \delta_{g,\delta'}$". Wrong: this is OK if g is (real and) monotonic, so that $(g(\alpha_i), g(\beta_i))$ are disjoint, but not in general.

[7]**Proof 3:** (sketch, partial) 1. $t \mapsto |t|^p \in$ AC (proof: ...), 2. $h, g \in$ AC $\Rightarrow h \circ g \in$ AC proof: $(h \circ g)'(t) = h'(g(t))g'(t)$ for those t for which $g'(t)$ and $h'(g(t))$ exist, hence a.e. (this is not true: e.g., $g \equiv 0$ and $\nexists h'(0)$, however, one could show (omitted here) that the conclusion is nevertheless true). But $\int_a^t (h \circ g)'(t)\,dt = ... \overset{Th7.26}{=} \int_{g(a)}^{g(t)} h'(s)\,ds = (h \circ g)(t) - (h \circ g)(a)$ $\forall t \in [a,b]$, hence $h \circ g$ is AC on $[a,b]$ (if $g \in \mathrm{AC}([a,b])$ and h is AC on the range of g). Moreover, g should be 1-1 in Th7.26 (or 7.26(16)). `(5- p.)`

[8]`(`$\frac{1}{4}$ `p.)` Without the word "disjoint" in the definition, the (AC) function $f(x) := x^{1/2}$ would not be absolutely continuous on $[0,1]$ (it is AC, since $f' \in L^1([0,1])$ and $f = \int f'$). Indeed, take some $\epsilon > 0$. Given any $\delta > 0$, we can take any $N \in \mathbb{N}+1$, $\alpha_i = 0$, $\beta_i = \delta/2N$ ($i = 1,...,N$), to obtain that $\sum_{k=1}^N |f(\beta_i) - f(\alpha_i)| = N(\delta/2N)^{1/2} = (\delta/2)^{1/2}N^{1/2} > \epsilon$ for N big enough.

Since $\epsilon > 0$ was arbitrary, $|f|^p$ is absolutely continuous.

Comments: 1. "$||a|-|b|| \le |a-b|$": OK, you need not prove this.
2. "It is a known theorem that P holds": usually OK, if 1. the proof of the 'known theorem' does not depend on the theorem to be proved here, and 2. P is not too close what you were asked to show in the exercise (e.g., if you are asked to prove that the Dominated Convergence Theorem holds for the Lebesgue measure, you may not use the fact that it holds for any positive measure).

Scoring: (of incomplete answers) AC-def.: `(1 p.)`, case $p = 1$: `(1 p.)`, significant progress on case $1 < p < \infty$: `(1 p.)`.

1. Case $p = 1$ and defition (possibly implicitly) and the rest also OK except that (a) `(4 p.)`: Only case $p \in \mathbb{N}+1$ solved.[9]
(b) `(3 p.)`: used the non-fact that "$|a|^p - |b|^p \le |a-b|$ when $|a|,|b| \le 1$" (take $a = 1 < p$, $b = 1/2$ to get a counter-example) or "$b^p - a^p \le M^{p-1}(b-a)$ when $M = b \ge a > 0$" (take $b = 2 = p = M$, $a = 1$); (reason: no points given for non-facts).
(c) `(5- p.)`: Case $|f(\beta_i)|, |f(\alpha_i)| \le 1$ solved correctly and shown why only case $|f(\beta_i)|, |f(\alpha_i)| \ge 1$ remains.[10]
(d) `(3 p.)` "Set $g(t) := t^p$, $h := g \circ |f|$. Then $h(x) - h(a) = \int_a^x h\,dt = \int_a^x p|f|^{p-1}\frac{d}{dt}|f|\,dt$" This is OK if you know that h is AC, but you must prove it first.

2. `(2 p.)`: AC def. correctly, some slight sense in the rest.[11]
3. `(0 p.)`: [Th7.18] used (poorly), because $|f|^p$ need not be nondecreasing.
4. "$\phi(E) := \int_E |f|^p\,d\mu \implies \phi \ll \mu \implies |f|^p$ is AC" no, should be "$\int_a^t |f|^p\,dm$ is AC" (if complex-valued, i.e., if $< \infty$, i.e., if $|f|^p \in L^1([a,b])$), and this does not help us at all.

Exercise 4. Assume that $f : I \to X$, where X is a Banach space and that $I = [a,b]$ is a closed, bounded interval (equipped with the Lebesgue measure).

(a) Explain what is meant by saying that f is *Bochner integrable* and define the *Bochner integral* of f.

(b) Show that if f is measurable and $\|f\|$ is integrable, then f is Bochner-integrable.

Results: 0123456p=0154682 students

Answer: **(a)** `(4 p.)` (See [ABHN], p. 9.) A function $f : I \to X$ is called *Bochner integrable* if there exists simple functions[12] $\{g_n\}$ s.t. $g_n \to f$ pointwise a.e. and $\int_I \|g_n - f\|_X\,dm \to 0$, as $n \to \infty$.[13] `(2 p.)`

If $\{g_n\}$ are as above, the *Bochner integral* of f is $\int_I f\,dm := \lim_{n\to\infty} \int_I g_n\,dt$ `(1 p.)`, where the Bochner integral of a simple function $g := \sum_{k=1}^n x_k \chi_{E_k}$ is given by $\int_I g\,dm := \sum_{k=1}^n x_k m(E_k)$. `(1 p.)`[14]

(b) `(2 p.)` (See [ABHN], p. 10, rows 8–15, but with "2" in place of "$(1+1/n)$").[15]

Since f is measurable, there are simple functions $\{h_n\}$ s.t. $h_n \to f$ on $I \setminus N$, where $m(N) = 0$. Then also $g_n := h_n \chi_{\{t \in I \,\big|\, \|h_n(t)\|_X \le 2\|f(t)\|_X\}}$ are simple($\frac{2}{5}$ `p.`), $g_n(t) \to f(t)$ $\forall t \in I \setminus N$, $\|g_n\|_X \le 2\|f\|_X$ (hence $\|g_n - f\| \le 3\|f\|$) everywhere, and $3\|f\|_X \in L^1(I)$ Therefore, $\lim_n \int_I \|g_n - f\|_X\,dm = \int_I \lim_n \|g_n - f\|_X\,dm = \int_I 0\,dm = 0$, as $n \to \infty$, by the (scalar) Lebesgue Dominated Convergence Theorem. **Scoring: (a)** 1. "$\int_I \|g_n - f\|_X\,dm \to 0$" not required: `(-1 p.)`

2. "f is measurable and $\|g_n - f\|_1 \to 0$ for some sequence $\{g_n\}$ of simple functions" `(1+ (of 2) p.)`.[16]

(b) 1. "Since f is measurable, there are simple functions $\{h_n\}$ s.t. $h_n \to f$ a.e. We must then show that $\int_I \|h_n - f\|_X\,dm \to 0$." `(1- p.)`[17]

[9] Some used case $p = 1$ and induction (and showed that $f, g \in \mathrm{AC} \Rightarrow fg \in \mathrm{AC}$); some used $a^p - b^p = \sum_{k=0}^{p-1} a^k b^{p-1-k}$ ($p \in \mathbb{N}$).

[10] If you had applied this to f/M, you would have obtained full 6 points.

[11] E.g., δ depends on n although it should do for any n.

[12] These are measurable by definition [ABHN, p. 6].

[13] It follows that f is measurable. Moreover, one can show (using (b)) that the former requirement is superfluous. Finally, one can show that we could as well allow $\{g_n\}$ to be countably-valued and measurable.

[14] $\int_E f\,dm := \int_I \chi_E f\,dm$ when $E \subset I$ is measurable, but this was not explained in [ABHN].

[15] The proof in the book `(1 p.)` is incorrect. If (e.g., $f \ne 0$ is simple and) $h_n \equiv f(1+2/n)$, then $\{g_n\}$ does not converge to f pointwise. (Note: this measures whether you just remembered the "proof" or understood it.)

[16] We must have $g_n(t) \to f(t)$ with the same sequence (which is not necessarily the case here). Actually, this weaker definition is equivalent (and used in some books), since some subsequence converges a.e.

[17] Counter-example: If $I = [0,1]$, $h_n(t) = nt\chi_{[0,1/n]}$, then $h_n \to 0$ everywhere but $\int_I \|h_n - 0\|_X\,dm = 1/2 \not\to 0$.

2. "$\|g_n - f\| \leq 3\|f\|$ a.e. for n big enough" `(1- p.)` (see "1.").
3. "Assume that $\|f_n - f\|_1 \to 0$..." `(0 p.)`.
4. "We can choose $\{g_n\}$ s.t. $\|g_n\| \leq \|f\|(1+1/n)$." (but otherwise OK) `(1 p.)`