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**Mat-1.150 Real Analysis**

**Mid-term examination 1/2, November 11, 2002** at 9–12 am (hall R3) **with answers** (3 pages, semiverified version 2002-11-21)

"+" means something that I did not require (but may give the higher number if you are exactly in the middle of two results). My rough guess is that getting at least 9 (resp. 12, 14, 17, 20) points (or 1 less) here might indicate the student being most probably a grade 1 (resp. 2, 3, 4, 5) student. It was easy to make all kinds of errors, so the ones getting less than 8 points still have chances.

**Important:** Read the "comments" and "errors", particularly the "errors" at excercise 4. Otherwise you keep doing the same errors over and over again (also in the second mid-term examination).

**Valitustilaisuus:** (you may see your paper and correct scoring) pe 29.11. klo 14.15 U322:ssa (jos aika on sinulle mahdoton, voit kysellä muulloinkin, mieluiten aikaisintaan to 28.11. ja livenä).

**Exercise 1.** Prove Fatou's Lemma.

**Answer:** *Results: 4651376 (0-6p)* (See the proof of Theorem 1.28, p. 23 of [R2].)

**Fatou's Lemma:** $\int_X \liminf_{n\to\infty} f_n \, d\mu \le \liminf_{n\to\infty} \int_X f_n \, d\mu$ *(1) when* $f_n : X \to [0,\infty]$ *are measurable* $\forall n \in \mathbb{N}$. `(1 p.)`,[1]

Set $g_k(x) := \inf_{i\ge k} f_i(x)\ \forall k, x$. Then $g_k \le f_k$, hence $\int_X g_k \, d\mu \le \int_X f_k \, d\mu$ (3) `(1 p.)`. Also, $0 \le g_1 \le g_2 \le \ldots$ `(1 p.)` and each $g_k$ is measurable (being infimums of measurable functions) `(1 p.)`, and $g_k(x) \to \liminf_n f_n(x)$, as $k \to \infty$ `(1 p.)`. Thus, the Monotone Convergence Theorem shows that (the left-side of) (3) tends to (the left-side of) (1), as $k \to \infty$ `(1 p.)`.[2]

**Errors.** *1. "$f_n$ integrable measurable" (or "$f_n \in L^1(X)$"):* $f_n$ need not be integrable. In almost all theorems of [R2] the functions are required to be measurable, but usually they are assumed to be either $[0,\infty]$-valued or integrable, not both (in Fubini's Theorem, either assumption is enough!). It is easy to remember that in Fatou's Lemma (and in many other results) the functions must be $[0,\infty]$-valued (because infimums cannot be taken on complex-valued sequences).

[1] Naturally, you get this point without the statement if your proof is complete *or* if it makes it obvious that you know the statement.

[2] Since $\lim = \liminf = \limsup$ whenever lim exists.

**Exercise 2.** Let $\{u_n\}_{n=1}^\infty$ be an orthonormal set in a Hilbert space $H$. Show that each of the following three conditions on $\{u_n\}_{n=1}^\infty$ implies the other two:

(i) $\{u_n\}_{n=1}^\infty$ is a maximal orthonormal set in $H$.

(ii) The set $P$ of all finite linear combinations of members of $\{u_n\}_{n=1}^\infty$ is dense in $H$.

(iii) The equality $\sum_{n=1}^\infty |\hat{u}(n)|^2 = \|x\|_H^2$ holds for every $x \in H$.

**Answer:** *Results: 5326484 (0-6p)* See the proof of Theorem 4.18, pp. 85–86 of [R2]. (Note that here $\hat{u}(n) := \hat{x}(n) := (x, u_n)$.)

**Scoring:** "(i)⇒(ii)⇒(iii)⇒(i)": (max) 2 p. for each implication (sum=6). "(i) ⟺ (ii) ⟺ (iii)": (max) 1.5 p. for each implication (sum=6). If Theorem 4.17 was formulated, we allowed to use it without its proof (though this made the proofs quite simple). Also other results prior to 4.18 were allowed (e.g., 4.14).

**Errors:** Note: *maximal orthonormal set* means "not strictly contained in another orthonormal set".

*1. "If $\bar{P} \ne H$, then there is $x \in H$ s.t. $0 \ne x \perp \bar{P}$ due to the properties of Hilbert spaces"* `(-1 p.)` if no proof (e.g., a simple "$Px + Qx$" explanation is enough).

*2. "$\lim_{n\to\infty} \sum_{k=1}^\infty r_{n,k} = \sum_{k=1}^\infty \lim_{n\to\infty} r_{n,k}$":* (where, e.g., $r_{n,k} := |(x_n, u_k)|^2$). Wrong: $\lim_n \lim_N \ne \lim_N \lim_n$ is possible in general (though it happens to be true in this case) (in general, use Mon. Conv. or Dom. Conv. Th. or show that $|(x_n, u_k)|^2 \to |(x, u_k)|^2$ in $\ell^1(\mathbb{N})$; here none of these seems easy, see 4.17 instead). `(-1 p.)`

*3. "If (iii) holds, then $\exists \sum_{n=1}^\infty \hat{x}(n) u_n = x\ \forall x \in H$".* You must prove this first.[3]

*4. "$(\lim_{n\to\infty} a_n, \lim_{n\to\infty} b_n) = \lim_{n\to\infty}(a_n, b_n)$":* You must prove this.

**Exercise 3.** Consider $\mathcal{C}([0,1])$ with the sup-norm. Define a set $M$ by

[3] (I cannot even find this result in the book, although "(v) $\exists \sum_{n=1}^\infty \hat{x}(n) u_n = x\ \forall x \in H$" could be added to Th4.18 since it obviously implies (ii)). A Proof of (iii)⇒(v): Assume (iii). Set $x_n := \sum_{k\le n} \hat{x}(n) u_n$. Then, by (iii), $\|x - x_n\|^2 = \sum_k |(x - x_n, u_n)|^2 = \sum_{k>n} |(x, u_n)|^2 =: \sum_{k>n} |\hat{x}(n)|^2 \to 0$, as $n \to \infty$, hence $x = \lim_{n\to\infty} x_n$ in $H$, QED.

$$M := \{f \in \mathcal{C}([0,1]) \,\big|\, \int_0^{1/2} f(t)\,dt - \int_{1/2}^1 f(t)\,dt = 1\}.$$

Show that $M$ is **(a)** closed and **(b)** convex, and that **(c)** $M$ does not contain an element of smallest norm.

**Answer:** *Results: 106(10)645 (0-6p)* **(a)** `(2 p.)` (Here $\|f\|_{\mathcal{C}} := \sup_{t\in[0,1]}|f|$.)[4] [5] 1° *We have* $\Lambda \in \mathcal{C}([0,1])^*$, *where* $\Lambda f := \int_0^{1/2} f - \int_{1/2}^1 f$. Proof: Set $\mathcal{C} := \mathcal{C}([0,1])$. The map $\Lambda : \mathcal{C} \to \mathbb{C}$ is linear, because so is $\int$. Obviously,

$$|\Lambda f| \le \frac{1}{2}\|f\|_{\mathcal{C}} + \frac{1}{2}\|f\|_{\mathcal{C}} \quad (f \in \mathcal{C}), \tag{1}$$

hence $\|\Lambda\|_{\mathcal{C}([0,1])^*} \le 1 < \infty$, hence $\Lambda \in \mathcal{C}^* := \mathcal{L}(\mathcal{C},\mathbb{C})$.

2° *M is closed.* Proof: We have $M = \Lambda^{-1}(\{1\}) = \{f \in \mathcal{C} \,|\, \Lambda f = 1\}$ and $\{1\}$ is closed, hence also $\Lambda^{-1}(\{1\})$ is closed,[6] because $\Lambda$ is continuous (by 1°).

**(b)** `(1 p.)` *M is convex.* Proof: Let $f, g \in M$, $\lambda \in [0,1]$. Then $\Lambda(\lambda f + (1-\lambda)g) = \lambda\Lambda f + (1-\lambda)\Lambda g = \lambda\cdot 1 + (1-\lambda)\cdot 1 = 1$,[7] hence $\lambda f + (1-\lambda)g \in M$, QED.

**(c)** `(3 p.)`[8] [9] 1° *We have* $\inf_{f\in M}\|f\|_{\mathcal{C}} \le 1$. Proof: Let $\epsilon \in (0,1/2)$ be arbitrary. Define

$$g(t) := \begin{cases} 1, & \text{when } 0 \le t \le \frac{1}{2} - \epsilon; \\ 1 - \frac{t - (\frac{1}{2}-\epsilon)}{\epsilon}, & \text{when } \frac{1}{2} - \epsilon < t < \frac{1}{2} + \epsilon \\ -1, & \text{when } \frac{1}{2} + \epsilon \le t \le 1. \end{cases} \tag{2}$$

Then $g \in \mathcal{C}$, $\|g\|_{\mathcal{C}} = 1$, and

$$\Lambda g > \int_0^{1/2-\epsilon} 1 + \int_{1/2-\epsilon}^{1/2} 0 - \left(\int_{1/2}^{1/2+\epsilon} 0 + \int_{1/2+\epsilon}^1 (-1)\right) = 1 - 2\epsilon. \tag{3}$$

Consequently, $\Lambda\gamma g = 1$ for $\gamma := 1/\Lambda g < (1-2\epsilon)^{-1}$; in particular, $\gamma g \in M$. But $\|\gamma g\|_{\mathcal{C}} = \gamma < (1-2\epsilon)^{-1}$, where $\epsilon \in (0,1/2)$ was arbitrary, hence $\inf_{f\in M}\|f\|_{\mathcal{C}} \le 1$.[10]

2° $\|f\|_{\mathcal{C}} > 1$ *for all* $f \in M$, *so that there is no element of smallest norm.* Proof: Assume that $f \in M$ and $\|f\|_{\mathcal{C}} \le 1$. Then $0 = \Lambda f - 1 = \int_0^{1/2}(f-1)\,dm - \int_0^{1/2}(-f-1)\,dm$. But the real parts of both integrands are $\le 0$, hence they must be 0 a.e., hence everywhere (being continuous). But then $f(t) = \begin{cases} 1, & \text{for } t \in [0,1/2], \\ -1, & \text{for } t \in [1/2,1]; \end{cases}$ hence then $f \notin \mathcal{C}$, a contradiction.

Thus, $\|f\|_{\mathcal{C}} > 1$ for all $f \in M$, so that the infimum ($= 1$, by this and 1°) is achieved by no $f \in M$, QED.

**Comments:** (a) If $\Lambda$ is continuous, then $\Lambda^{-1}(E)$ is open (resp. closed) for open (resp. closed) sets $E$, by definition (resp. by $\Lambda^{-1}(E^c) = (\Lambda^{-1}(E))^c$). (b) If $\Lambda$ is linear, then $\Lambda^{-1}(E)$ is convex (resp. a subspace) whenever $E$ is (this also holds for $\Lambda(E)$).

(a) "$\lim_n \int f_n = \int \lim_n f_n$" with no proof nor explanation $\Longrightarrow$ only 1 point. Note also that $f$ need not be real-valued.

(c) "$\mathcal{C}$ is not complete". It is complete (Banach space) w.r.t. to the sup-norm, not w.r.t. pointwise seminorms (w.r.t. the topology of pointwise convergence,

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[4] **Proof 2:**(A set in a metric space is closed iff it is sequentially closed, by [R1]. +) Let $\{f_n\} \subset M$ and $f_n \to f$ in $\mathcal{C}$, as $n \to +\infty$. Set $R := \sup_I |f| + 1$, where $I := [0,1]$, and choose $N$ s.t. $\sup_I |f - f_n| < 1$ for $n \ge N$. Then $|f_n(x)| \le R$ for all $x \in I$, $n \ge N$. Since $R \in \Lambda^1([0,1/2])$ and $R \in \Lambda^1([1/2,1])$, we can apply the Dominated Convergence Theorem to obtain that $\Lambda f_n \to \Lambda f$ as $n \to \infty$. But $\Lambda f_n = 1\ \forall n$, hence $\Lambda f = 1$, hence $f \in M$, QED. (Note that $R$ had to be independent on $n$ (for D.C.Th.)!) **Comments:** You need not prove that $M$ (or $\mathcal{C}$) is complete, it suffices to assume that $M \ni f_n \to f \in \mathcal{C}$ in $\mathcal{C}$ and show that $f \in \mathcal{C}$! (However, $M$ is complete since it is closed in $\mathcal{C}$ and $\mathcal{C}$ is complete.)

[5] Note also that continuous functions are Borel-measurable, hence Lebesgue-measurable. +

[6] If $A$ is closed and $f$ is continuous, then $A^c$ is open and $f^{-1}(A) = [f^{-1}(A^c)]^c$ is closed, because $f^{-1}(A^c)$ is open.

[7] And $\lambda f + (1-\lambda)g \in \mathcal{C}$ +.

[8] If $\mathcal{C}$ were a Hilbert space, a unique smallest element would exist, by 4.10 of [R2].

[9] **Proof 2:** To obtain a contradiction, assume that $f \in M$, $r := \|f\| = \inf_{g\in M}\|g\|$. Assume that $f(1/2) \ne r$ (the other case is analogous and hence omitted). Then $f$ can be redefined so that still $\|f\| = r$ but $\Lambda f > 1$ (because (***)); divide $f$ by $\Lambda f$ to obtain a contradiction. (The part (***) seems to be longer than proof 1, hence not recommended.)

[10] Since $\le (1-2\epsilon)^{-1}$ for all $\epsilon \in (0,1/2)$.

as one notes by letting $\epsilon \to 0$ in 1°; this topology is not normable, not even metrizable, though yet would make $\mathcal{C}$ a TVS, see [R3]).

**Scoring of (c):** $\inf_{f\in M}\|f\| \geq 1$ `(1 p.)` $\inf_{f\in M}\|f\| \leq 1$ `(1 p.)`, $\|f\| > 1\ \forall f \in M$ (i.e., $\nexists$min)`(1 p.)` (if sufficiently proved; possibly less if only verbally/graphically sketched). Four students "proved" that if there is a minimal element, then one can construct an even smaller one but only received 1 p. each due to severe errors.

**Exercise 4.** Suppose that $\{\Lambda_n\}_{n=1}^\infty$ is a sequence of bounded linear transformations from a normed linear space $X$ to a Banach space $Y$. Suppose that $\|\Lambda_n\| \leq M < \infty$ for all $n$, and suppose that there is a dense set $E \subset X$ s.t. $\{\Lambda_n x\}_{n=1}^\infty$ converges for each $x \in E$.

Prove that $\{\Lambda_n x\}_{n=1}^\infty$ converges for each $x \in X$.

**Answer:** *Results: 000011111222222223333344455666666*

Let $x \in X$ be arbitrary. Since $Y$ is complete, it suffices to show that $\{\Lambda_n x\}$ is a Cauchy-sequence.

Let $\epsilon > 0$ be arbitrary. Set $\delta := \epsilon/4M$. Choose $e \in E$ such that $\|x-e\|_X < \delta$. Since $\{\Lambda_n e\}$ is a Cauchy sequence (being convergent), there is $N \in \mathbb{N}$ such that

$$\|\Lambda_n e - \Lambda_m e\| < \epsilon/3 \quad \forall n, m \geq N. \tag{4}$$

Then, for all $n, m \geq N$, we have

$$\begin{aligned} \|\Lambda_n x - \Lambda_m x\| &= \|\Lambda_n x - \Lambda_n e + \Lambda_n e - \Lambda_m e + \Lambda_m e - \Lambda_m x\| && (5)\\ &\leq \|\Lambda_n(x-e)\| + \|\Lambda_n e - \Lambda_m e\| + \|\Lambda_m(e-x)\| && (6)\\ &\leq \|\Lambda_n\|\delta + \epsilon/3 + \|\Lambda_m\|\delta \leq \epsilon/4 + \epsilon/3 + \epsilon/4 < \epsilon. && (7) \end{aligned}$$

Since $\epsilon > 0$ was arbitrary, we have shown that $\{\Lambda_n x\}$ is a Cauchy sequence, hence[11] it converges in $Y$, QED.

**Answer 2 (sketch):** Let $x \in X$ be arbitrary. Choose $\{e_k\} \subset E$ s.t. $e_k \to x$ as $k \to \infty$. Set $y_k := \lim_{n\to\infty} \Lambda_n e_k$ ($k \in \mathbb{N}$). ... $\|y_k - y_j\| < \epsilon$ ... $\{y_k\}$ Cauchy ... $y_k \to Y$ as $k \to \infty$ $\Lambda_n x \to y$, as $n \to \infty$, QED.

**Remarks:** We showed for an *arbitrary* $x \in X$ that $\{\Lambda_n x\}_{n=1}^\infty$ is a Cauchy sequence. We did this by showing that for *any* $\epsilon > 0$ there is $N \in \mathbb{N}$ such that for *all* $n, m \geq N$ we have $\|\Lambda_n x - \Lambda_m x\| < \epsilon$. Note that you must find $N$ for an arbitrary $\epsilon > 0$, and that the inequality must then hold for all $n, m \geq N$. This is the definition of a Cauchy sequence. (Note that $N$ may depend on $x$ and $\epsilon$ but not on $n, m$.)

**Do not do this:** Some people made a great amount of work to show that $\Lambda_n x_k \to \Lambda_n x$, as $k \to \infty$ (when $\{x_k\} \subset X$ was s.t. $x_k \to x$ as $k \to \infty$). This is trivial, since $\Lambda_n$ is continuous. Further unnecessary work was done as follows:

*"Let $\{x_n\} \subset E$ be s.t. $x_n \to x$, and pick a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ s.t. $\|x_{n_k} - x\| < 2^{-k}$ $\forall k \in \mathbb{N}$."* This is not wrong but stupid (unnecessarily much work but no reduction of grading). $\bar{E} = X$ means that any open set in $X$ contains an element of $E$; in particular, one can pick some $e_k \in B_{2^{-k}}(x) \cap E$ for all $k \in \mathbb{N}$, where $B_r(x) := \{z \in X \mid \|z - x\| < r\}$. But then $\{e_k\} \subset E$ and $e_k \to X$ as $k \to \infty$, so there really is a sequence in $E$ converging to $X$, and $\|e_k - x\| < 2^{-k}$ $\forall k \in \mathbb{N}$. Thus, since you are picking the sequence, you can originally require that $\|x_n - x\| < 2^{-n}$, you do not need to take any subsequences.

**Errors:** Otherwise OK but "$\forall \epsilon > 0\ \exists N$ s.t. $\forall n, m > N$ we have" missing at $\|\Lambda_n x - \Lambda_m x\| < \epsilon$ and no evidence of understanding that it should be there $\Rightarrow$ 4 points (out of 6).

"$\|\Lambda_n x - \Lambda_m x\| < 2M\tilde{\epsilon} + 2\epsilon' =: \epsilon$" No! No! No! $\epsilon > 0$ should be arbitrary and you should show that there is $N \in \mathbb{N}$ s.t. we get below this $\epsilon$ for all $n, m > N$. You cannot define $\epsilon > 0$, any $\epsilon > 0$ should do (with some $N = N_\epsilon$).

*"Let $n, m$ be so big that $\|\Lambda_n e - \Lambda_m e\| < \epsilon/3$ for all $e \in E$".* The convergence need not be uniform on $E$ (hence "big" may depend on $e$, i.e., it may be that no $N$ is big enough for all $e \in E$), equivalently, $\{\Lambda_n e\}$ need not be uniformly Cauchy on $E$.

*"Let $\Lambda := \lim_n \Lambda_n$" (max 2 p.).* It was stated that for each $e \in E$, there is $y_e \in Y$ s.t. $\Lambda_n e \to y_e$, as $n \to \infty$, not that there exists $\Lambda \in \mathcal{L}(X, Y)$ s.t. ...

*"Set $\Lambda(x) := \lim_{n\to\infty} \Lambda_n x$ ($x \in E$). Obviously, $\Lambda : E \to Y$ is linear ... $\|\Lambda\| \leq M$. By Hahn–Banach Th., extension $\tilde{\Lambda} \in \mathcal{L}(X, Y)$...* Wrong, you must have $Y = \mathbb{C}$ in order to use the H–B Th. [5.16]. For general $Y$, it does not hold! (Cf. Lemma A.3.11 of [Mikkola2002]; e.g., $I \in \mathcal{L}(H^1, H^1)$ has no extension in $\mathcal{L}(L^1, H^1)$.)

[11] Because $Y$ is a Banach space, hence complete.