Mat-1.150 Real Analysis: 6th exercise, Wed 2002-10-23, U322 (5 pages)

`http://www.math.hut.fi/teaching/realanal/`

Exercise 6.1 ([5.4] *(a1)–(b1): average, not long; (b2) difficult*):
Let $r \in (0,1)$. Define

$$C^\alpha := \{f : [0,1] \to \mathbb{C} \,\big|\, \sup_{t,s\in[0,1],\ t\neq s} \frac{|f(t)-f(s)|}{|t-s|^\alpha} < \infty\}.$$

Define a norm on C^α by

$$\|f\|_\alpha := \sup_{0\le t\le 1} |f(t)| + \sup_{t,s\in[0,1],\ t\neq s} \frac{|f(t)-f(s)|}{|t-s|^\alpha}. \tag{1}$$

(a1) Show that this is really a norm.

(a2) Show that C^α ("the space of Hölder continuous functions") is a Banach space (with this norm).

(b) Consider the subspace

$$h^\alpha := \{f \in C^\alpha \,\big|\, \lim_{\delta\to 0+} \sup_{|t-s|\le\delta,\ t\neq s,\ t,s\in[0,1]} \frac{|f(t)-f(s)|}{|t-s|^\alpha} = 0\}.$$

(b1) Show that h^α ("the space of little-Hölder continuous functions") is a Banach space with the norm inherited from C^α.

(b2) (Difficult) Let $\theta > \alpha$, $\alpha \in (0,1)$. Show that h^α is the closure of C^θ in C^α.

Exercise 6.2 ([5.4]([5.21]) *easy, quick, important*):
[R2:Exe5.8a] Let X be a normed space and let $X^* := \mathcal{L}(X,\mathbb{C})$ be its dual with its usual norm

$$\|f\|_{X^*} := \sup_{\|x\|_X\le 1,\ x\in X} |fx| \qquad f \in X^*.$$

Show that X^* is a Banach space.

Exercise 6.3 ([Chapter 5] *quite difficult; average length*):
Let X, Y be Banach spaces and let $D \subset X$ be a subspace. A linear map[1] $L : D \to Y$ is said to be *closed* iff its *graph* $\{(x, Lx) \,\big|\, x \in D\}$ is a closed subspace of $X \times Y$; equivalently, L is closed iff $x_n \to x$ and $Lx_n \to y$ imply that $x \in D$, $y = Lx$.

(a) Assume $L : D \to Y$ is closed. (a1) Show that D need not be a Banach space w.r.t. the $\|\cdot\|_X$ norm.[2] (a2) Show that D is a Banach space w.r.t. the *graph norm* $\|x\|_D := \|x\|_X + \|Lx\|_Y$.

(b) Is the operator $\frac{d}{dt}$ with domain of definition $D := \mathcal{C}^1([0,1]) \subset \mathcal{C}([0,1]) =: X =: Y$ closed?

Exercise 6.4 ([5.8] *rather easy and very short*):
Let X be a normed linear space and suppose that $A \subset X$ is *weakly bounded.*[3] Show that A is *bounded.*[4]

Remarks on **weak and weak* topology***:* Let X be a set, and let $\mathcal{F}$ be a family of mappings $f : X \to Y_f$, where Y_g is a topological space for each $f \in \mathcal{F}$. Then the *topology induced by the family* $\mathcal{F}$ is the weakest topology on X that makes each $f \in \mathcal{F}$ continuous; equivalently, the one whose base consists of the finite intersections of the set $\{f^{-1}V \,\big|\, f \in \mathcal{F},\ V \in \mathcal{T}_{Y_f}\}$.

In most applications, you need not understand the above definition, just to use some known consequences (see below). (The definition is actually not that difficult, but you might prefer, e.g., the form in [R3], or just get first used to these topologies and reread the definition a bit later.)

The *weak topology of* X is the one induced by its dual X^* when X is a normed space (or some other TVS), i.e., the weakest topology that makes mappings $x \mapsto \Lambda x$ continuous $X \to \mathbb{C}$, for all $\Lambda \in X^*$. The *weak* topology of* X^* is the one induced by dual X, i.e., the weakest topology that makes mappings $\Lambda \mapsto \Lambda x$

[1] Map, operator and function mean exactly the same (also in Finnish: kuvaus, operaattori, funktio), although linear ones are often called *maps* if bounded and *operators* if unbounded, just by convention.

[2] The dot "$\cdot$" refers to the variable; e.g., $e^{a\cdot}$ refers to function $x \mapsto e^{ax}$.

[3] This means that for each $x^* \in X^*$, we have $\sup_{x\in A} |x^*(x)| < \infty$.

[4] In a metric space, this means that A is contained in some ball $B_r(a) := \{x \in X \,\big|\, d(x,a) < r\}$. Since X is a normed space, an equivalent condition is that $\sup_{x\in A} \|x\|$.

continuous $X^* \to \mathbb{C}$, for all $x \in X$.[5]

In the future, for most applications you will probably only need the following on a weak topology:
1. Weak topology is weaker than the original topology.
2. $x_n \to x$ weakly iff $\Lambda x_n \to \Lambda x$ for all $\Lambda \in X^*$.
3. $f : Q \to X$ is weakly continuous iff $\Lambda \circ f$ is continuous for all $\Lambda \in X^*$ (for any set Q and function f).

Actually, claims 2. and 3. apply to any topology induced by a collection of mappings (as above). In particular:
1. Weak* topology of X^* is weaker than the original (norm) topology of X^*.
2. $\Lambda_n \to \Lambda$ weak* iff $\Lambda_n x \to \Lambda\ \forall x \in X$.
3. $f : Q \to X^*$ is weak* continuous iff $q \mapsto f(q)x$ is continuous $Q \to \mathbb{C}$ for all $x \in X$.

Claim 1. applies to any topology induced by *continuous* mappings (by [R3:3.11], the weak and weak* topologies of infinite-dimensional normed spaces are not normable, hence *strictly* weaker than the original ones). It is easy to prove all above claims.

The weak* topology of an infinite-dimensional normed space is not metrizable, by [R3:Exe3.15], thus, e.g., density need not imply sequential density in a weak* topology. The weak and weak* topologies of $\mathbb{C}^n$ (or $\mathbb{R}^n$) equal the original one. See, e.g., [R3:Chapter3] for more on these topologies.

Note: one can define a topology of pointwise convergence for $\mathcal{L}(X,Y)$ too, but then one meets further difficulties [TID], hence this is rarely done.

Solution to 6.1:

(a1) 1° *We have $f \in C^\alpha \Rightarrow \|f\|_\alpha < \infty$:* Let $f \in C^\alpha$. Set $M := \sup_{t,s\in[0,1],\ t\neq s} \frac{|f(t)-f(s)|}{|t-s|^\alpha}$. Then $|f(t) - f(0)| \le |t-0|^\alpha M \le M$, for all $t \in [0,1]$, hence then $\|f\|_\alpha \le (|f(0)| + M) + M < \infty$.

2° *C^α is a vector space:* This follows from (i) and (ii) below.[6]

3° *Definition [R2:5.1] is satisfied, i.e., $\|\cdot\|_\alpha : C^\alpha \to [0,\infty)$ is[7] a norm:*
(iii) $\|f\| = 0 \Rightarrow \sup|f| = 0 \Rightarrow f = 0$.
(ii) Obviously, $\|zf\| = |z|\|f\|\ \forall z \in \mathbb{C}$.
(i) $\|f+g\| \le \|f\| + \|g\|$, because[8] $\sup_{a\in A}(F(a) + G(a)) \le \sup_{a\in A} F(a) + \sup_{a\in A} G(a)$ for any functions $F, G : A \to [-\infty,\infty]$ (note that $|(f+g)(t) - (f+g)(s)| := |f(t) + g(t) - (f(s) + g(s))| \le |f(t) - f(s)| + |g(t) - g(s)|$).

(Recall that norms can be defined on vector spaces only, by [5.1].)

(a2) By (a1), C^α is a normed space (from (i) and (ii) one can verify that it indeed is a vector space; indeed, $f, g \in C^\alpha \Rightarrow zf, f+g \in C^\alpha$, so C^α is a vector subspace of the vector space $\mathbb{C}^{[0,1]} := \{f \mid f : [0,1] \to \mathbb{C}\}$).[9] Thus, we only have to show that C^α is complete.

(When proving that some metric space is complete, one typically finds first a "candidate" for the limit (in spaces of functions often a pointwise limit) and then shows that it actually is the limit in the topology considered.)

Let $\{f_n\} \subset C^\alpha$ be a Cauchy sequence. Obviously, then f_n is a Cauchy sequence in $\mathcal{C}([0,1];\mathbb{C})$, hence $f_n \to f$ uniformly for some $f \in \mathcal{C}([0,1];\mathbb{C})$.[10] Thus, it only remains to show that $\|f_n - f\|_\alpha \to 0$.

Given $\epsilon > 0$, there is $N \in \mathbb{N}$ s.t. $\|f_n - f_m\|_\alpha < \epsilon/3\ \forall n, m \ge N$. Given any $t, s \in [0,1]$, $t \neq s$, we have

$$\frac{|(f-f_n)(t) - (f-f_n)(s)|}{|t-s|^\alpha} = \lim_{m\to\infty} \frac{|(f_m - f_n)(t) - (f_m - f_n)(s)|}{|t-s|^\alpha} \le \epsilon/3,$$

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[5] Remark: use X^{**} in place of X to get the weak topology of the normed space X^*).

[6] An exact proof goes as follows: $\mathbb{C}^{[0,1]} := \{f \mid f : [0,1] \to \mathbb{C}\}$ is obviously a vector space (this is well known; if you do not believe, check the axioms on [R2:2.1]). Since $C^\alpha \subset \mathbb{C}^{[0,1]}$, it suffices to show that $f, g \in C^\alpha$, $z \in \mathbb{C} \Rightarrow zf, f+g \in C^\alpha$, which follows easily from (ii) and (i) above. Thus, C^α is a vector subspace of $\mathbb{C}^{[0,1]}$, hence a vector space (an alternative proof would consist of showing the axioms on [R2:2.1] directly, without using $\mathbb{C}^{[0,1]}$).

Recall that $(f+g)(x) := f(x)+g(x)$ and that $(zf)(x) := zf(x)$ and note that $0 : x \mapsto 0$ ($x \in [0,1]$) is the zero element of this vector space.

[7] Here we used 1°.

[8] Proof: $(F(b) + G(b)) \le \sup_{a\in A} F(a) + \sup_{a\in A} G(a)$ for each $b \in A$.

[9] Actually we cheated a bit in (a1): one should first verify that C^α is a vector space (which we did here), before one can state that $\|\cdot\|_\alpha$ is a norm. Recall from [5.1] that *norms* are defined on vector spaces only.

[10] Proof: The set $\mathcal{C}_b(A) := \mathcal{C}_b(A;\mathbb{C})$ of bounded continuous functions $A \to \mathbb{C}$ is a Banach space with norm $\|f\| := \sup_{t\in A}|f(t)|$, by [R1:Th7.15], for any topological space A. (Actually, even $\mathcal{C}_b(A;X)$ is a Banach space for any (real or complex) Banach space X.) When A is compact, any continuous function $A \to X$ is bounded, by [R1:4.15], hence then $\mathcal{C}_b = \mathcal{C}$.

for any $n \geq N$. Therefore,

$$\|f - f_n\|_\alpha := \sup_{0 \leq t \leq 1} |(f - f_n)(t)| + \sup_{t,s \in [0,1],\ t \neq s} \frac{|(f - f_n)(t) - (f - f_n)(s)|}{|t - s|^\alpha}$$
$$\leq \epsilon/3 + \epsilon/3 < \epsilon$$

for any $n \geq N$, hence $f_n \to f$ in C^α as $n \to \infty$. Since $\{f_n\}$ was arbitrary, C^α is complete.

(b1) As in (a1), we can verify that h^α is a vector subspace of C^α. Therefore, $\|\cdot\|_\alpha$ makes it a normed space, so we only have to show that it is complete (equivalently, a closed subspace of C^α). Let $\{f_n\} \subset h^\alpha$ be a Cauchy sequence, and let $f \in C^\alpha$ be its limit — we only need to show that $f \in h^\alpha$. But

$$|f(t) - f(s)| \leq |f(t) - f_n(t) + f_n(s) - f(s)| + |f_n(t) - f_n(s)| \tag{2}$$
$$\leq \|f - f_n\|_\alpha |t - s|^\alpha + |f_n(t) - f_n(s)|. \tag{3}$$

Let $\epsilon > 0$ be given. Choose n s.t. $\|f - f_n\|_\alpha < \epsilon/2$. Choose $\delta > 0$ s.t. $|f_n(t) - f_n(s)| \leq |t - s|^\alpha \epsilon/2$ when $|t - s| < \delta$ (this exists since $f_n \in h^\alpha$). We conclude from this and (2) that $|f(t) - f(s)| \leq \epsilon |t - s|^\alpha$ whenever $|t - s| < \delta$. Since $\epsilon > 0$ was arbitrary, we conclude that $f \in h^\alpha$.

(b2) (This means that $\overline{C^\theta} = h^\alpha$.) Obviously,[11] $C^\theta \subset h^\alpha$. By (b1), h^α is closed, hence $\overline{C^\theta} \subset h^\alpha$. Thus, we only need to show that $h^\alpha \subset \overline{C^\theta}$, i.e., that C^θ is dense in h^α.

1° *Case* $f \in h^\alpha(\mathbb{R}; \mathbb{C})$ (defined analogously to $h^\alpha([0,1]; \mathbb{C})$ above): For each $n \in \mathbb{N}$, choose $h_n \in \mathcal{C}^\infty(\mathbb{R})$ s.t.

$$\operatorname{supp} h_n \subset (-1/n, 1/n), \quad \|h_n\|_{L^1} = 1, \tag{4}$$

and[12] set $f_n(x) := \int_{\mathbb{R}} f(y) h_n(x - y)\, dy$, $x \in \mathbb{R}$, $n \in \mathbb{N}$. One can show that [13] $f_n \in \mathcal{C}_b^\infty(\mathbb{R})$ (has bounded derivatives of all orders) and that $\|f - f_n\|_\alpha \to 0$. Since f was arbitrary, we conclude that $\mathcal{C}_b^\infty$ is dense in C^α, hence so is C^θ (obviously, $\mathcal{C}_b^\infty \subset \mathcal{C}_b^1 \subset C^\theta$). We omit the details, although they would not be too difficult nor very long. The following lemma can be used:

Lemma If $f, f_t \in \mathcal{C}^1(J \times J'; \mathbb{C})$ and $F(t) := \int_I f(t,s)\, ds$, where $J, J' \subset \mathbb{R}$ are open and $I \subset J'$ is an interval, then $F \in \mathcal{C}^1(J; \mathbb{C})$ and $F' = \int_I f_t(t,s)\, ds$.

(Proof: Apply [R1:9.42] to the real and imaginary parts of f.)

2° *Case* $f \in h^\alpha([0,1]; \mathbb{C})$*:* (This is our case, but we shall reduce this to the above.) Extend f to $\mathbb{R}$ by setting

$$\tilde{f}(x) := \begin{cases} f(0), & \text{for } x \leq 0, \\ f(x), & \text{for } x \in [0,1], \\ f(1), & \text{for } x \geq 1. \end{cases}$$

Obviously, $\tilde{f} \in h^\alpha(\mathbb{R}; \mathbb{C})$ with same $\|\cdot\|_\alpha$ norm (and same $\|\cdot\|_{\Lambda^\infty}$ norm) as f has. Define $\{\tilde{f}_n\} \subset \mathcal{C}_c^\infty$ as in 1° and set $f_n := \tilde{f}_n|_{[0,1]}$. The rest of the proof goes as in 1°.

Remarks: Actually, we could have any interval I in place of $[0,1]$ and any Banach space X in place of $\mathbb{C}$ above (with essentially the same proof). These spaces are further treated in e.g., [Lunardi], pp. 3–. See also exercise 11.3 of 1999 for more on $C^\alpha =: \mathcal{C}^{0,\alpha} =: \mathrm{Lip}_\alpha$.

Solution to 6.2: 1° X^* is a normed space [5.1]:

(iii) $\|f\| = 0 \implies |fx| = 0\ \forall x \implies f = 0$ (the null function $0x = 0\ \forall x \in X$).

(ii) $\sup_{\|x\| \leq 1} |(\alpha f)(x)| = \sup_{\|x\| \leq 1} |\alpha||fx| = |\alpha| \sup_{\|x\| \leq 1} |fx| = |\alpha| \|f\|$.

[11] Proof: Set $M := \|f\|_\theta$. Given $\epsilon > 0$, choose $\delta := (\epsilon/M)^{(\theta-\alpha)^{-1}} > 0$. Then $|f(t) - f(s)|/|t - s|^\alpha \leq M|t - s|^{\theta - \alpha} \leq M\delta^{\theta - \alpha} < \epsilon$ whenever $|t - s| < \delta$. We conclude that $f \in h^\alpha$.

[12] Such functions are often called mollifiers. By them one often shows the density of smooth functions in some function space. $f_n := f * h_n$ below is a *convolution* that smooths f. See, e.g., Chapter 2 of [Adams] or Chapter 2 of [TID for details and for the existence of such functions.

[13] When $f \in \mathcal{C}$ and $g \in \mathcal{C}_c^1$, we have $\exists D(f * g) = f * Dg$. One can easily deduce this from the Lemma below.

(Here $\alpha f \in X^*$ is the mapping $x \mapsto \alpha(fx) \in \mathbb{C}$.)
(i) $|(f+g)(x)| := |fx+gx| \le |fx|+|gx| \le (\|f\|+\|g\|)\|x\|$, hence $\|f+g\| \le \|f\|+\|g\|$.

2° X^* is complete:[14] Let $\{f_n\}$ be a Cauchy sequence (i.e., $\forall \epsilon > 0\, \exists N_\epsilon$ s.t. $\|f_n - f_m\| < \epsilon\, \forall n, m \ge N_\epsilon$). If $x \in X$, then $\|f_n x - f_m x\| < \epsilon\, \forall n, m \ge N_{\epsilon/\|x\|}$, hence $\{f_n x\}$ is also a Cauchy sequence. Since $\mathbb{C}$ is complete, the number $f(x) := \lim_n f_n x$ exists.

2.1° $f \in X^*$ Obviously, $f : X \to \mathbb{C}$ is linear (since $f(\alpha x) := \lim_n f_n \alpha x = \alpha \lim_n f_n x = \alpha f(x)$ etc.). It remains to show that f is bounded (i.e., continuous).

Let $\epsilon > 0$. Choose some $n \ge N_\epsilon$. Now[15] $|fx| = |f_n x + fx - f_n x| \le |f_n x| + |fx - f_n x| = |f_n x| + |\lim_{m\to\infty} f_m x - f_n x| = |f_n x| + \lim_{m\to\infty} |(f_m - f_n)x| \le \|f_n\|\|x\| + \epsilon\|x\|\ \forall x$. Therefore, $\|f\| \le \|f_{N_\epsilon}\| + \epsilon < \infty$, hence f is bounded, i.e., $f \in X^*$.

2.2° *$f_n \to f$ in X^*:* We just showed above that $|(f-f_n)x| \le \epsilon\|x\|\ \forall x$, hence $\|f - f_n\| \le \epsilon$ (when $n \ge N_\epsilon$). Thus, $f_n \to f$ (w.r.t. the norm $\|\cdot\|_{X^*}$), QED.

Remarks: The same proof (with substitutions $|\cdot| \mapsto \|\cdot\|$) shows that $\mathcal{L}(X,Y)$ is a Banach space whenever X is a normed space and Y is a Banach space (if Y is merely a normed space, then so is $\mathcal{L}(X,Y)$ (unless $\dim X < \infty$)).

Solution to 6.3:

(a1) E.g., use the example in (b) and note that $\mathcal{C}^\infty$ is dense in $\mathcal{C}([0,1])$.

(a2) Obviously, $\|\cdot\|_D$ is a norm (it satisfies Definition [R2:5.1]), hence D becomes a normed space, so that we only need to show that it is complete.

Assume that $\{x_n\} \subset D$ is a D-Cauchy sequence. Then $\{x_n\}$ is a X-Cauchy sequence (use same N_ϵ for any $\epsilon > 0$), hence $x_n \to x$ for some $x \in X$, as $n \to \infty$. Similarly, $\{Lx_n\}$ is a Cauchy sequence, hence $Lx_n \to y$ for some $y \in Y$. Since L is closed, $x \in D$ and $y = Lx$, from which we observe that $\|x - x_n\|_D \to 0$, i.e., $\{x_n\}$ has the limit x in D, QED.

(b) Set $I := [0,1]$, $X := \mathcal{C}(I)$ (with sup-norm). Define $A : D \to X$ by $Af := f'$ $(f \in D)$. [16] [17]

Obviously, $\mathcal{D}(A) := D$ is a vector subspace of X and A is linear.[18]

Let $f_n \to f$ ja $Af_n \to g$ in $\mathcal{C}(I)$. We only have to show that necessarily $\exists Af = g$ (i.e., $f \in \mathcal{D}(A)$ & $Af = g$, i.e., $\exists f' = g$).

Let $x \in I$ and $\epsilon' > 0$ be arbitrary. Set $\epsilon := \epsilon'/4$. Choose $\delta > 0$ s.t. $|g(x) - g(y)| < \epsilon \Longleftarrow |x - y| < \delta$ & $x, y \in I$. Let $|h| \in]0, \delta[$ & $x + h \in I$; it suffices to show that $|\frac{f(x+h)-f(x)}{h} - g(x)| < \epsilon'$ (OK also at the endpoints $x = 0 \lor x = R$).

Choose N_ϵ s.t. $\|Af_n - g\| < \epsilon$ & $\|f_n - f\| < \epsilon|h|\ \forall n \ge N_\epsilon$. Set $n := N_\epsilon$. There is[19] $\xi = \xi_{f_n,x,h} \in]x, x+h[$ s.t. $|\frac{f_n(x+h)-f_n(x)}{h} - g(x)| \le |f_n'(\xi) - g(x)|$,

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[14] This is the usual way: 0. take an arbitrary Cauchy sequence $\{f_n\}$, 1. guess the limit f, 2. show that $f \in X^*$, 3. show that $f_n \to f$. Usually when you have to prove a metric space to be complete this is the simplest way.

[15] Recall from [R1] that $\lim_n$ may be taken out of a continuous function and from [R2:Chapter5] that the following functions are continuous on a Banach space: $\|\cdot\|$, $+$ and $-$.

[16] Remarks: we mean one-sided derivates at the endpoints 0 and 1. Recall also that functions in X are bounded and uniformly continuous, by [R1], since I is compact.

[17] *Example:* A non-open operator: If $X := L^2(I) =: Y$, $D := \mathcal{C}(I)$, $Af := f(0) \in Y$ (a constant function), then $F_n \to 1$ and $AF_n \equiv F_n(0) = 0 \to 0$ but $A1 = 1 \ne 0$, so that A is not closed (here F_n is given by 1 for $t > 1/n$, nt for $t \le 1/n$). *Note:* it is obvious that any continuous linear operator is closed.

[18] Note: A is not continuous. Proof: Let $g_n(t) = (1 - nt)\chi_{[0,1/n]} \in X$, $f_n(t) := \int_{1/n}^t g_n(r)\,dr \in \mathcal{C}^1(I)$. Then $\|f_n\| = f_n(0) = 1/2n \to 0$, although $\|Af_n\| = \|g_n\| = 1 \not\to 0$, hence $\|A\| = \infty$.

[19] This follows from the Mean Value Theorem [R1] if the functions are real. Even for complex functions (or for functions with values in any Banach space), the "Mean Value Inequality" given below is true (Lemma B.5.6, p. 944 of [Mikkola2002]; the proof is simple); alternatively, apply the MVT to real and imaginary parts and add "2" suitably to the formulas below to get 6ϵ in place of 4ϵ.

hence

$$\begin{aligned}
&|\frac{f(x+h)-f(x)}{h}-g(x)| \\
\leq\ &|\frac{f_n(x+h)-f_n(x)}{h}-g(x)|+|\frac{f(x+h)-f_n(x+h)}{h}|+|\frac{-f(x)+f_n(x)}{h}| \\
<\ &|f_n'(\xi_{n,h})-g(x)|+\epsilon+\epsilon \leq |f_n'(\xi_{n,h})-g(\xi_{n,h})|+|g(\xi_{n,h})-g(x)|+\epsilon+\epsilon \\
<\ &\epsilon+\epsilon+\epsilon+\epsilon=\epsilon'.
\end{aligned}$$

Since x and ϵ' were arbitrary, $\exists f'=g$, QED.

Remarks: (This goes out of the scope of this course but may be useful in the future.)

Any interval would do in place of $[0,1]$ (as long as it is compact or we require the functions and their derivatives to be bounded). Moreover, $\mathbb{C}$ could be replaced by any Banach space. The above proof can also be modified to show that $\mathcal{C}^1(I;B)$ (or $\mathcal{C}^k(I;B)$) is complete for any Banach space B.

Indeed, if $\{f_n\}$ is a Cauchy sequence in $\mathcal{C}^1(I)$ (with norm $\|f\|_{\mathcal{C}^1} := \|f\|_{\mathcal{C}} + \|f'\|_{\mathcal{C}} := \sup|f|+\sup|f'|$), then $\{f_n\}$ and $\{f_n'\}$ are Cauchy in $\mathcal{C}$, hence $f_n\to f$ and $f_n\to g$ (since $\mathcal{C}=\mathcal{C}_0$ is complete, by, e.g., [R2: 3.17]). By our solution above, $g=f'$, hence $\|f_n-f\|_{\mathcal{C}^1}=\|f_n-f\|_{\mathcal{C}}+\|f_n'-f'\|_{\mathcal{C}}\to 0$ i.e., $f_n\to f$ in $\mathcal{C}^1$. Thus, $\mathcal{C}^1(I)$ is indeed complete! By induction, so is $\mathcal{C}^k(I)$. In fact, also $X:=\mathcal{C}_b^k(I;B)$ is complete (the above proof applies mutatis mutandis), whenever B is a Banach space (and $I\subset\mathbb{R}^k$ open or something similar, so that the derivative means the gradient). Also the subspaces $\mathcal{C}_{bu}^k(I;B)$ and $\mathcal{C}_0^k(I;B)$ of $\mathcal{C}_b^k(I;B)$ are closed (i.e., complete, i.e., Banach spaces).

Solution to 6.4: One easily verifies that $\|j_x\|_{X^{**}} = \|x\|_X$ for all $x\in X$, where $j_x(\Lambda):=\Lambda x$ for all $\Lambda\in X^*$.

Assume that A is not bounded in X (hence in X^{**}). Then, by The Uniform Boundedness Theorem [R2:Th5.8] (applied to X^* in place of X), there is some $\Lambda\in X^*$ (in fact, a dense G_δ of them) s.t. $\sup_{x\in A}|j_x(\Lambda)| := \sup_{x\in A}|\Lambda x|=\infty$, hence then A is not weakly bounded, a contradiction. Thus, A must be bounded.