Mat-1.150 Real Analysis: 3rd exercise, Wed 2002-10-2, U322 (3 pages)

`http://www.math.hut.fi/teaching/realanal/`

1st mid-term examination (1. välikoe) 11.11. (11th Nov) at 9–12,
2nd mid-term examination (2. välikoe) 4.12. (4th Dec) at 14–17.
Details will be given later.

Exercise 3.1 ([3.7] *average level, rather short*):
Let μ be a positive measure on X. Let $\{f_n\}$ be a sequence of measurable functions $X \to \mathbb{C}$. In probability theory it is said that the sequence $\{f_n\}$ converges to f *in measure* (μ) if $\forall \epsilon > 0\ \exists N \in \mathbb{N}$ s.t.

$$n \geq N \implies \mu\left(\{x \,\big|\, |f_n(x) - f(x)| > \epsilon\}\right) < \epsilon.$$

Let $\mu(X) < \infty$. Prove that If $f_n \to f$ a.e., then $f_n \to f$ in measure.

Exercise 3.2 ([3.6] *easy, very short, useful*):
Let $0 < p < \infty$. Assume that $f : X \to \mathbb{C}$ is measurable and that $\int_X |f|^p \, d\mu < \infty$. Show that for each $\epsilon > 0$, we have

$$\mu(\{x \,\big|\, |f(x)| \geq \epsilon\}) \leq \epsilon^{-p} \int_X |f|^p \, d\mu.$$

(In probability theory, this inequality is known as the *Čebyšev inequality.*)[1]

Exercise 3.3 ([3.1] *average level, not long*):
Let $-\infty \leq a < b \leq \infty$. Show that for each convex $\phi :]a,b[\to \mathbb{R}$ has at each point $t \in]a,b[$ a right derivative ($\frac{d^+}{dt}\phi(t)$) and a left derivative ($\frac{d^-}{dt}\phi(t)$), and that $\frac{d^-}{dt}\phi(t) \leq \frac{d^+}{dt}\phi(t)$.[2][3]

[1]Pafnuti Tšebyšev, 1821–94. In Finnish (particularly non-scientific) literature one often uses the replacement č↦tš.

A measure μ is a probability measure iff $\mu(X) = 1$, in which case $\mu(E)$ equals "the probability that some $x \in E$ happens" (e.g., $X =$ die "d6" (cube with numbers 1–6), $E = \{2,5,6\}$). Thus, much of probability theory consists of corollaries of real analysis for the case $\mu(X) = 1$.

[2]*Hint:* $\frac{\phi(u)-\phi(t)}{u-t} \leq \frac{\phi(v)-\phi(u)}{v-u}$, when $t < u < v$ [R2:3.1(2)].

[3]*Example:* the inequality might be strict: $\phi : x \mapsto |x|$ is convex $\mathbb{R} \to \mathbb{R}$ but $\frac{d^-\phi}{dt}(0) := \lim_{s\to t-} \frac{|s|-0}{s-0} = \lim_{s\to t-} \frac{(-s)-0}{s-0} = -1 < 1 = \frac{d^+\phi}{dt}(0)$. (Obviously, if(f) the right and left derivate are equal, ϕ has a derivative at t.)

Exercise 3.4 ([1.39] *easy to follow, difficult to find*):
Assume that $\mu(X) = 1$. Show that if f is integrable on X, then[4]

$$\int_X f \, d\mu = 0 \quad \Longleftrightarrow \quad \int_X |1 + zf(x)| \, d\mu(x) \geq 1 \text{ for all } z \in \mathbb{C}.$$

Exercise 3.5 ([1.34] *quite easy and short*):
[R2:Exe2.25] (a) Find the smallest constant $c \in \mathbb{R}$ s.t.

$$\ln(1 + e^t) < c + t \quad (0 < t < \infty).$$

(b) Does the limit

$$\lim_{n\to+\infty} \frac{1}{n} \int_0^1 \ln\left(1 + e^{nf(x)}\right) \, dx$$

exist for every real $f \in L^1([0,1])$? If it exists, what is it?

Solution to 3.1: 1° By the assumption, we have $f_n(x) \to f(x)\ \forall x \in E^c =: Y$, where $\mu(E) = 0$, i.e., $\mu(Y) = \mu(X)$. Let $\epsilon > 0$ be arbitrary.

We should now find $N \in \mathbb{N}$ s.t. $\mu(A_n) < \epsilon\ \forall n \geq N$, where $A_n := \{\,\big|\, |f_n(x) - f(x)| > \epsilon\}$. We shall prove even more (in 2°), namely that $\mu(\cup_{k=N}^\infty A_k) = \mu(X) - \mu(B_N) < \epsilon$, where

$$B_n := X \setminus \cup_{k=n}^\infty A_k = \cap_{k=n}^\infty A_k^c.$$

2° *We have $Y \subset \cup_{n\in\mathbb{N}} B_n$.* Proof: When $x \in Y$, there is $n_x \in \mathbb{N}$ s.t. $|f_k(x) - f(x)| < \epsilon\ \forall k \geq n_x$, hence $x \in B_{n_x}$.

3° *We have $\mu(\cup_n B_n) = \mu(X)$,* because $\mu(Y) \leq \mu(\cup_n B_n) \leq \mu(X) = \mu(Y)$, because $Y \subset \cup_n B_n \subset X$.

4° Obviously, $B_1 \subset B_2 \subset B_3 \subset \cdots$, hence $\lim_{n\to\infty} \mu(B_n) = \mu(\cup_{n=1}^\infty B_n) = \mu(X)$ [R2:1.19d]. Therefore, $\mu(A_n) \leq \mu(\cup_{k=n}^\infty A_k) = \mu(X) - \mu(B_n) \overset{n\to\infty}{\to} 0$, which leads to the existence of the N we were looking for (i.e., $\exists N\ \forall n \geq N$ ($\mu(A_n) < \epsilon$)), QED.

[4]Here $\int_X g(x)\, d\mu(x) := \int_X g \, d\mu$, i.e., by "$d\mu(x)$" we mean "$d\mu$" but we also express that x is the variable in the integrand. This is the standard convention.

Solution to 3.2: Set $A_\epsilon := \{x \,\big|\, |f(x)| \geq \epsilon\}$.

$$\mu(A_\epsilon) = \int_{A_\epsilon} 1\, d\mu = \epsilon^{-p} \int_{A_\epsilon} \epsilon^p\, d\mu \leq \epsilon^{-p} \int_{A_\epsilon} |f|^p\, d\mu \leq \epsilon^{-p} \int_X |f|^p\, d\mu,$$

QED. (In fact, the inequality holds for all measurable f, since the right-hand-side is infinite for $f \notin L^p(\mu)$.)

Solution to 3.3: Any convex function ϕ satisfies, for all $\lambda \in [0,1]$, $x, y \in]a,b[$, the inequality

$$\begin{aligned} \phi(x + \lambda(y-x)) = &\ \phi((1-\lambda)x + \lambda y) && (1)\\ \leq &\ (1-\lambda)\phi(x) + \lambda\phi(y) = \phi(x) + \lambda(\phi(y) - \phi(x)). && (2) \end{aligned}$$

Moreover, $t < u \leq v < b \implies \frac{\phi(u)-\phi(t)}{u-t} \leq \frac{\phi(v)-\phi(t)}{v-t}$ (*), i.e., the slope of a secant is increasing, because (here we set $\lambda := \frac{u-t}{v-t}$)

$$\begin{aligned} \frac{\phi(u)-\phi(t)}{u-t} &= \frac{\phi(t+\lambda(v-t)) - \phi(t)}{\lambda(v-t)} && (3)\\ &\leq \frac{\phi(t) + \lambda[\phi(v)-\phi(t)] - \phi(t)}{\lambda(v-t)} = \frac{\lambda[\phi(v)-\phi(t)]}{\lambda(v-t)}. && (4) \end{aligned}$$

Let $a < t < b$. Since the above slope is increasing (i.e., decreasing when $u \to t+$), we have "lim = inf", i.e.,

$$\frac{d^+\phi}{dt}(t) := \lim_{u \to t+} \frac{\phi(u)-\phi(t)}{u-t} = \inf_{u>t} \frac{\phi(u)-\phi(t)}{u-t} < \infty, \qquad (5)$$

in particular, the limit exists (since any nonempty set has an infimum). Recall the formula [R2:3.1(2)]:

$$\frac{\phi(t)-\phi(t')}{t-t'} \leq \frac{\phi(u)-\phi(t)}{u-t} \quad (t' < t < u). \qquad (6)$$

Fix $t' \in]a,t[$ and let $u \to t+$ to obtain that $\frac{\phi(t)-\phi(t')}{t-t'} \leq \frac{d^+\phi}{dt}(t)$, in particular, that $\frac{d^+\phi}{dt}(t) > -\infty$. This shows that $\frac{d^+\phi}{dt}(t) \in \mathbb{R}$.

In the same way (or by applying the above to $t \mapsto \phi(-t)$), we observe that $\frac{d^-}{dt}\phi(t) = \sup_{s<t} \frac{\phi(t)-\phi(s)}{t-s} \in \mathbb{R}$. By (6), we have that $\frac{\phi(t)-\phi(s)}{t-s} \leq \frac{d^+}{dt}\phi(t)\ \forall s \in]a,t[$, hence $\frac{d^-}{dt}\phi(t) \leq \frac{d^+}{dt}\phi(t)$.

Note: $\exists \phi'(t) \iff \frac{d^-}{dt}\phi(t) = \frac{d^+}{dt}\phi(t)$.

Solution to 3.4: 1° *The latter condition is necessary:*

$$\int_X |1+zf|\, d\mu \geq \left| \int_X (1+zf)\, d\mu \right| = \left| 1 + z\int_X f\, d\mu \right| = 1.$$

2° *The latter condition is sufficient:* For any $z \neq 0$, write z as $r\mathrm{e}^{i\theta}$, where $r > 0$ and $\theta \in [0, 2\pi)$, and divide the condition by r to obtain that (recall that $\int_X 1\, d\mu = \mu(X) = 1$)

$$\int_X \frac{|1 + r\mathrm{e}^{i\theta} f(x)| - 1}{r}\, d\mu(x) \geq 0. \qquad (7)$$

Make a simple calculation[5] to obtain that

$$\frac{|1+rw|-1}{r} = \frac{2\,\mathrm{Re}\, w + r|w|^2}{|1+rw|+1}$$

($\to 2\,\mathrm{Re}\, w/2 = \mathrm{Re}\, w$ as $r \to 0+$) so that

$$\lim_{r \to 0+} \frac{|1 + r\mathrm{e}^{i\theta} f(x)| - 1}{r} = \mathrm{Re}[\mathrm{e}^{i\theta} f(x)].$$

Moreover,

$$\left| \frac{|1 + r\mathrm{e}^{i\theta} f(x)| - 1}{r} \right| \leq |f(x)|.$$

Therefore, by (the definition $\int (u+iv) := \int u + i\int v$ [1.31] and) Lebesgue's Dominated Convergence Theorem,

$$\begin{aligned} \mathrm{Re}\left(\mathrm{e}^{i\theta} \int_X f\, d\mu \right) &= \int_X \mathrm{Re}\left(\mathrm{e}^{i\theta} f(x)\right)\, d\mu(x) && (8)\\ &= \lim_{r \to 0+} \int_X \frac{|1 + r\mathrm{e}^{i\theta} f(x)| - 1}{r}\, d\mu(x) \geq 0, && (9) \end{aligned}$$

[5] $(|1+r\,\mathrm{Re}\, w| - 1)(|1+r\,\mathrm{Re}\, w| + 1) = |1+r\,\mathrm{Re}\, w|^2 - 1^2 = (1+r\,\mathrm{Re}\, w)^2 + r^2(\mathrm{Im}\, w)^2 - 1 = 2r\,\mathrm{Re}\, w + r^2((\mathrm{Re}\, w)^2 + (\mathrm{Im}\, w)^2) = r(2\,\mathrm{Re}\, w + r|w|^2)$.

by (7), and this holds for all $\theta \in [0, 2\pi)$. By choosing θ so that

$$e^{i\theta} \int_X f\, d\mu = -\left| \int_X f\, d\mu \right|$$

one obtains that $\mathrm{Re}(-\left|\int_X f\, d\mu\right|) \geq 0$, so that

$$\int_X f\, d\mu = 0.$$

Solution to 3.5: (a) $\ln(1 + e^t) < c + t \iff 1 + e^t < e^{c+t} = e^c e^t \iff e^{-t} + 1 < e^c$. Since $e^{-\cdot}$ is decreasing, this holds for all $t > 0$ iff $2 = e^0 + 1 \leq e^c$. Thus, $c = \ln 2$ is the smallest possible choice. [6]

(b) Let $f \in L^1([0,1])$ and $x \in [0,1]$. Set $g_n(x) := \frac{1}{n} \ln[1 + \exp(nf(x))]$ ($n \in \mathbb{N}_+$). We have $g_n(x) > \frac{1}{n} \ln[1 + 0] = 0$ for all n. If $f(x) \leq 0$, then $g_n(x) \leq \ln[1+1]/n \to 0$, as $n \to +\infty$. If $f(x) > 0$, then[7]

$$\lim_{n\to+\infty} g_n(x) = \lim_{n\to+\infty} \frac{\ln[1 + e^{nf(x)}]}{n} = \text{“}\frac{\infty}{\infty}\text{''} = \lim_{n\to+\infty} \frac{[e^{nf(x)} \cdot f(x)]/[1 + e^{nf(x)}]}{1} \tag{10}$$

$$= \lim_{n\to+\infty} \frac{f(x)}{1 + e^{-nf(x)}} = f(x)/(1+0) = f(x). \tag{11}$$

Thus, $g(x) := \lim_{n\to+\infty} g_n(x) = f^+(x) := \max(0, f(x))$ for all $x \in [0,1]$.

By (a), we have $g_n(x) < (c + nf(x))/n = c/n + f(x) =: G(x)$, hence $|g_n(x)| \leq G(x)$ for all $x \in [0,1]$ and $n \in \mathbb{N}_+$. But $G \in L^1$, hence we can apply Lebesgues Dominated Convergence Theorem [1.34] to obtain that the following limit exists:

$$\lim_{n\to+\infty} \int_0^1 g_n(x)\, dx = \int_0^1 \lim_{n\to+\infty} g_n(x)\, dx = \int_0^1 f^+(x)\, dx.$$

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[6] *Alternative solution:* differentiate $\log(1 + e^t) - t$ to observe that it is strictly decreasing and that hence it is smaller than $\log(1 + e^0) - 0 =: c$ (and that c is the smallest possible choice).

[7] Recall L'Hospital's Rule from [R1]: *if* the limit is of form $\pm\infty/\infty$ or $0/0$, then we obtain the same limit even if we first differentiate the numerator and the denominator (w.r.t. n). Note: here we actually compute the limit as $\mathbb{R}_+ \ni n \to +\infty$, but since it exists (as we show here), the limit $\mathbb{N}_+ \ni n \to +\infty$ also exists and has the same value.