Mat-1.150 Real Analysis: 2nd exercise, Wed 2002-9-25, U322 (3 pages)

`http://www.math.hut.fi/teaching/realanal/`

Note: "s.t." := "such that".

Exercise 2.1 ([1.12] *short but abstract*):
Let $f_n : X \to \mathbb{R}$ be measurable $\forall n \in \mathbb{N}$. Show that

$$S := \{x \in X \,|\, \lim_{n\to\infty} f_n(x) \text{ exists and is finite }\}$$

is a measurable set ($S \in \mathfrak{M}$).[1]

Solution: $S(n,k) := \cap_{i,j>n}\{x \in X \,|\, |f_i(x) - f_j(x)| < 1/k\} \in \mathfrak{M}$.[2]
For each $x \in X$ we have $x \in S \iff \{f_i(x)\}_{i=1}^{\infty}$ a Cauchy sequence in $\mathbb{R}$ (by [R1])[3] $\iff \forall k \in \mathbb{N}_+\ \exists n \in \mathbb{N}\ \forall i,j > n\ [|f_i(x) - f_j(x)| < 1/k] \iff \forall k \in \mathbb{N}_+\ \exists n \in \mathbb{N}\ [x \in S(n,k)] \iff \forall k \in \mathbb{N}_+\ [x \in \cup_{n\in\mathbb{N}} S(n,k)] \iff x \in \cap_{k\in\mathbb{N}_+}\cup_{n\in\mathbb{N}} S(n,k)$. Therefore, $S = \cap_{k\in\mathbb{N}_+}\cup_{n\in\mathbb{N}} S(n,k)$, hence $S \in \mathfrak{M}$ (recall that countable unions and intersections of measurable sets are measurable), QED.

Exercise 2.2 ([1.18] *(a)& (b) quite short and easy*):
[R2:Exe1.6] Let X be an uncountable set, let $\mathfrak{M}$ be the *cofinite σ-algebra in X*, i.e., the collection of all sets $E \subset X$ s.t. E or E^c is at most countable.
(a) Show that $\mathfrak{M}$ is a σ-algebra in X.
(b) Define $\mu(E) = 0$ if E is countable, $\mu(E) = 1$ if E^c is countable. Show that μ is a measura on $\mathfrak{M}$.
(c) Can you describe the measurable functions?

[1] Hint 1: $S(n,k) := \cap_{i,j>n}\{x \in X \,|\, |f_i(x) - f_j(x)| < 1/k\} \in \mathfrak{M}$. Hint 2: Read the first few lines of the solution.

[2] Proof of measurability: each of the sets is the preimage of the open set $B(0,1/k) =]-1/k, 1/k[$ by the measurable function $f_i - f_j$, and the index set $\{i,j \in \mathbb{N} \,|\, i,j > n\} \subset \mathbb{N}^2$ of the intersection is countable, since $\mathbb{N}^2$ is, by [R1].

[3] In $\mathbb{R}^n$ (or in any complete metric space), a sequence $\{x_n\}$ converges iff it is a Cauchy sequence, i.e., iff $\forall \epsilon > 0\ \exists n \in \mathbb{N}$ s.t. for all $i,j > n$ we have $d(x_n, x_m) < \epsilon$, i.e., iff, for any $\epsilon > 0$, the tail of the sequence is within ϵ from some x_n onwards. Obviously, "for all $\epsilon > 0$" can be replaced by "for $\epsilon = 1/k$ for all $k \in \mathbb{N}_+ := \{1,2,3,\ldots\}$".

Solution: Example: if $X = \mathbb{R}$, then $[0,1] \notin \mathfrak{M}$ but $\{3\}, \mathbb{Q} \in \mathfrak{M}$.

(a) The axioms [1.3] are satisfied: (i) $X^c = \emptyset$ is at most countable, hence $X \in \mathfrak{M}$.
(ii) Let $A \in \mathfrak{M}$ and set $B := A^c$. Then A or A^c is at most countable, hence so is B^c or B, hence $B \in \mathfrak{M}$.
(iii) Let $A := \cup_{n=0}^{\infty} A_n$, where $A_n \in \mathfrak{M}\ \forall n$. 1° If $A_n \preceq \mathbb{N}$ for all n, then $A \preceq \mathbb{N}$, by [R1]. 2° Otherwise $A_{n_0} \succ \mathbb{N}$ for some $n_0 \in \mathbb{N}$, hence then $A^c \subset A_{n_0}^c \preceq \mathbb{N}$ (since $A_{n_0} \subset A$), so that $A \subset \mathfrak{M}$.

(b) Definition [1.18] is satisfied: μ maps $\mathfrak{M}$ to $[0,\infty]$, and, if A is as in (iii) above and $A_n \cap A_m = \emptyset$ whenever $n \neq m$, then, in case "1°", we have $\mu(A) = 0 = \sum_n 0 = \sum_n \mu(A_n)$, and, in case "2°", we have $A_n \subset A_{n_0}^c$ for all $n \neq n_0$ (i.e., $A_n \cap A_{n_0} = \emptyset$), hence then $A_n \preceq A_{n_0}^c \preceq \mathbb{N}$, hence $\mu(A_n) = 0$, for all $n \neq n_0$, so that

$$\sum_{n=0}^{\infty} \mu(A_n) = \mu(A_{n_0}) + \sum_{n=0}^{\infty} 0 = 1 + 0 = \mu(A).$$

(c) *A function $f : X \to Y$ is measurable iff f is constant a.e. (i.e., iff f is constant outside some at most countable subset of X).* This holds at least if $Y \subset [-\infty, +\infty]$ or $Y = \mathbb{C}$ or $Y \subset \mathbb{R}^n$ or $Y \subset \mathbb{C}^n$.[4]

<u>Proof:</u> 1° *"If":* Let there be $c \in Y$ s.t. $E := f^{-1}(\{c\}^c) \preceq \mathbb{N}$. Let $V \subset Y$ be open. If $c \notin V$, then $f^{-1}(V) \subset f^{-1}(\{c\}^c) = E \preceq \mathbb{N}$, hence then $f^{-1}(V) \in \mathfrak{M}$; otherwise, $[f^{-1}(V)]^c = f^{-1}(V^c) \subset f^{-1}(\{c\}^c) = E \preceq \mathbb{N}$, hence then again $f^{-1}(V) \in \mathfrak{M}$. Since V was arbitrary, f is measurable.

2° *"Only if":* (This is the tricky part.) 2.1° *Case where $f : X \to [-\infty, \infty]$:* By [1.17], there is a sequence $\{s_n\}$ of simple measurable functions s.t. $s_n(x) \to f(x)\ \forall x \in X$. Write $s_n := \sum_k \alpha_{n,k}\chi_{E_{n,k}}$ so that $\cup_k E_{n,k} = X$ and $E_{n,k} \cap E_{n,k'} = \emptyset$ ($k \neq k'$) and $\alpha_{n,k} \neq \alpha_{n,k'}$ ($k \neq k'$), for all n. Obviously, exactly one $E_{n,k}$ is uncountable for each n;[5] denote this one by k_n.

[4] The "if" claim and its proof obviously apply to any topological space. The converse holds at least in spaces where measurable functions can be approximated by simple measurable functions; the latter property is called "strong measurability". (These properties are obviously equivalent for $Y = \mathbb{R}^n$ and $Y = \mathbb{C}^n$.)

[5] $\cup_k E_{n,k} = X$ implies that at least one is. If two of them are, say k and k', and we choose an open $V \subset [-\infty, \infty]$ s.t. $\alpha_{n,k} \in V \not\ni \alpha_{n,k'}$, then both $s_n^{-1}V \supset E_{n,k}$ and

Set $E := \cap_n E_{n,k_n}$, so that $E^c = \cup_k E^c_{n,k_n} \preceq \mathbb{N}$. For each $x \in E$, we have $f(x) = \lim_n s_n(x) = \lim_n \alpha_{n,k_n}$; in particular, the limit $c := \lim_n \alpha_{n,k_n}$ exists and $f(x) = c$ in E, hence a.e.

2.2° *Case where* $f : X \to \mathbb{C}$*:* By 2.1°, there are sets E and E' and constants c and c' s.t. $E^c, (E')^c \preceq \mathbb{N}$ and $\operatorname{Re} f(x) = c$ $(x \in E)$ and $\operatorname{Im} f = c'$ $(x \in E')$, hence $f(x) = c + ic'$ for all $x \in E \cap E'$, hence a.e.[6]

2.3° *Case where* $f : X \to \mathbb{C}^n$*:* Apply 2.2° to $f_1, \ldots, f_n$.

Remarks: Note how in 2° we first noted that our claim obviously holds for simple measurable functions, and then used [1.17] to extend it to all measurable functions. Recall to consider this trick when you need to prove a claim for measurable functions!

(We first did 2° more directly, using a countable dense subset of Y and the fact that in a compact metric space any sequence has a convergent subsequence. This proof was thrice as long.)

Exercise 2.3 ([1.19] *difficult, medium length*):
[R2:Exe3.16] Let $\{f_n\}_{n=1}^\infty$ be a sequence of measurable functions $X \to \mathbb{C}$ s.t. $\lim_{n\to\infty} f_n(x)$ exists for all $x \in X$. Let $\mu(X) < \infty$. Prove *Egoroff's theorem* ("Jegorovin lause"): $\forall \epsilon > 0\ \exists E \in \mathfrak{M}$ s.t. $\mu(X \setminus E) < \epsilon$ and $\{f_n\}$ converges uniformly on E. (Hint: see the hint on Exercise 3.16 in [R2].)

Solution: [7] $S(n,k) := \cap_{i,j>n}\{x \in X \mid |f_i(x) - f_j(x)| < 1/k\} \in \mathfrak{M}$.[8] Denote the limit of $\{f_n(x)\}$ by $f(x)$, for all x. Let $x \in X$. By the assumption, $\{f_n(x)\}$ is a Cauchy sequence, hence $\forall k \in \mathbb{N}_+\ \exists n \in \mathbb{N}_+$ s.t. $x \in S(n,k)$ (this is the definition of a Cauchy sequence, in an equivalent form; cf. Exercise 2.1/2002 above). Therefore, $\cup_{n\in\mathbb{N}_+} S(n,k) = X$ for all $k \in \mathbb{N}_+$.

Therefore, $\mu(S(n,k)) \overset{n\to\infty}{\to} \mu(X)$, by [R2:1.19d] (because $S(n,k) \subset S(n+1,k)\ \forall n,k$) for all $k \in \mathbb{N}_+$. Consequently, there are numbers n_k s.t. $\mu(S(n_k,k)) > \mu(X) - 2^{-k}\epsilon$. Let $E := \cap_{k\in\mathbb{N}_+} S(n_k,k)$. Now we can complete the proof:[9]

1^o *$\{f_n\}$ converges uniformly on E:* $\forall 1/k > 0\ \exists n := n_k$ s.t. $\forall x \in E\ [i,j > n_k \implies |f_i(x) - f_j(x)| < 1/k]$, because $E \subset S(n_k,k)$. Therefore, $|f_i(x) - f(x)| = \lim_{j\to\infty} |f_i(x) - f_j(x)| \le 1/k$. Consequently, $\forall 2/k > 0\ \exists n_k\ \forall x \in E\ [i > n_k \implies |f_i(x) - f(x)| < 2/k]$. Thus, given $\epsilon > 0$, we can take $k > 2/\epsilon$ and use n_k the show that the definition of uniform convergence is satisfied.[10]

2^o $\mu(E^c) = \mu(\cup_k S(n_k,k)^c) \le \sum_k \mu(S(n_k,k)^c) < \sum_k 2^{-k}\epsilon = \epsilon$, QED.

Exercise 2.4 ([1.35] *not very easy to find nor short*):
Let $\{f_n\}_{n=0}^\infty$ be a sequence of integrable functions on $X := \mathbb{R}$ s.t. $f_n \to f$ a.e. Using Egoroff's theorem, show that if f is integrable and if $\int_X f_n\,d\mu \to \int_X f\,d\mu$, then, for all $\epsilon > 0$, there exists a measurable set $A \subset X$ with $\mu(A) < \infty$, an integrable function $g \ge 0$, and an integer $N \in \mathbb{N}$ s.t. for all $n \ge N$ we have

$$|\int_{X\setminus A} f_n|\,d\mu < \epsilon, \quad |f_n| \le g \text{ on } A.$$

Solution: 1° *For any integrable function h and any $\epsilon > 0$, there are $M < \infty$ and $B \in \mathfrak{M}$ s.t. $\mu(B) < \infty$, $\int_{X\setminus B} |h|\,d\mu < \epsilon$, and $|h| \le M$ on B.*

Proof: Let $E_n := \{x \in X \mid 1/n \le |h(x)| \le n\}$, so that $X = \cup_{n\in\mathbb{N}} E_n$ and $E_1 \subset \cdots$. Then $|h| = \lim_{n\to\infty} h_n$, where $h_n := |h|\chi_{E_n}$, and $0 \le h_1 \le h_2 \le \cdots$, so that $\int_X |h|\,d\mu = \lim_{n\to\infty} \int_X h_n\,d\mu$, by the Monotone Convergence

$(s_n^{-1}V)^c \supset E_{n,k'}$ are uncountable, so that $s_n^{-1}V$ is not measurable, hence then s_n is not measurable, a contradiction.

[6] We have $(E \cap E')^c = E^c \cup (E')^c \preceq \mathbb{N}$.

[7] Condition "$\mu(X) < \infty$" is necessary: If $X = \mathbb{R}\,\&\,\mu = m\,\&\,f_n(x) := x/n$, then $f_n(x) \to 0\ \forall x$, but, given $\epsilon = 1$, for any n we have $|f_n(x) - 0| > \epsilon$ for all $x \in]n,\infty[$, hence $\mu(X \setminus E) = \infty$ if $|f_n - 0| < \epsilon$ on E.

Similarly, by choosing $X =]0,1[\ \&\ f_n(x) := 1/nx$ we observe that $E = X$ will not always do (because then we would have $x < 1/n \implies f_n(x) > 1$); however, then we could take, e.g., $E =]\epsilon/2, 1[\subset X$.

[8] The measurability of $S(n,k)$ follows from the facts that each of the sets is the preimage of the open ball $B(0,1/k)$ by the measurable function $f_i - f_j$, and that $\{i,j \in \mathbb{N} \mid i,j > n\} \subset \mathbb{N}^2$ is countable.

[9] Intuitively: the complement of each $S(n_k,k)$ is of measure less than $2^{-k}\epsilon$, hence $\mu(X \setminus E) < \epsilon$ [2^o]. On the other hand, for each ϵ of form $\epsilon = 1/k > 0$, the number n_k will do, hence the convergence is uniform in E [1^o].

[10] This way one could show that, in general, a convergent sequence converges uniformly iff it is "uniformly Cauchy".

Theorem [1.26]. Consequently,[11]

$$\int_{X\setminus E_n} |h|\,d\mu = \int_X |h|\,d\mu - \int_{E_n} |h|\,d\mu = \int_X |h|\,d\mu - \int_X h_n\,d\mu < \epsilon$$

for n big enough. Thus, $|h| \le n$ in E_n and $\int_{X\setminus E_n} |h|\,d\mu < \epsilon$. If we had $\mu(E_n) = \infty$, then $\int_X |h|\,d\mu \ge \int_{E_n} |h|\,d\mu \ge \int_{E_n} frac1n\,d\mu$, hence $\mu(E_n) < \infty$.[12]

2° Choose B and M as in 1° so that $\int_{X\setminus B} |f|\,d\mu < \epsilon/4$ and $|f| \le M$ on B and $\mu(B) < \infty$. By Egoroff's theorem,[13] there is $A \subset B$ s.t. $\mu(B \setminus A) < \epsilon/5M$ and $f_n \to f$ uniformly on A. Consequently, there is $N \in \mathbb{N}$ s.t. for all $n \ge N$ we have

$$|f_n - f| < \epsilon/6\mu(A) \text{ on } A, \quad \text{and} \quad |\int_X (f_n - f)\,d\mu| < \epsilon/7$$

(here we also used the original assumption $\int_X f_n \to \int_X f$). Thus, for any $n \ge N$, we have

$$\int_{X\setminus A} f_n = \textstyle\int_{X\setminus A}[f + (f_n - f)] = \int_{X\setminus A} f + \int_{X\setminus A}(f_n - f) \tag{1}$$

$$= \textstyle\int_{X\setminus B} f + \int_{B\setminus A} f + \int_X (f_n - f) - \int_A (f_n - f), \tag{2}$$

hence, by the Triangle Inequality ($|a+b| \le |a| + |b|$), we get

$$|\int_{X\setminus A} f_n| < \epsilon/4 + M\cdot\epsilon/5M + \epsilon/7 + \epsilon/6 < 4\epsilon/4 = \epsilon$$

for such n.[14] But for each $x \in A$, we have $|f_n(x)| \le |f(x)| + \epsilon/6\mu(A) \le M + \epsilon/6\mu(A) =: R$. Set $g := R\chi_A$ to complete the proof.[15]

[11] One easily shows that $\int_{A\cup B} f = \int_A f + \int_B f$ when $A, B \in \mathfrak{M}$, $A \cap B = \emptyset$.

[12] Since h is integrable, i.e., it is measurable and $\int_X |h|\,d\mu < \infty$.

[13] To get around the "a.e.", replace X by $X \setminus N$, where N is a null set s.t. $f_n \to f$ on $X \setminus N$.

[14] We used $4, 5, 6, 7$ instead of $4, 4, 4, 4$ so that the reader can easily see from where each ϵ term comes.

[15] Remark: in fact we were able to prove a stronger claim: for these A, $M < \infty$ and $N \in \mathbb{N}$, we have $\mu(A) < \infty$ and, for all $n \ge N$, $|\int_{X\setminus A} f_n|\,d\mu < \epsilon$ and $|f_n| \le M$ on A.

Exercise 2.5 ([2.8] *easy, short*):
(a) Show that the infimum of any collection of upper semicontinuous functions is upper semicontinuous. (b) Does this hold for the supremum?

Solution: (a) Let $f = \inf\{f_\alpha \mid \alpha \in \mathcal{A}\}$, where $f_\alpha : X \to [-\infty, \infty]$ are u.s.c., i.e., $\{f_\alpha < \beta\} := \{x \in X \mid f_\alpha(x) < \beta\}$ is open for all $\beta \in \mathbb{R}$.

Then $x \in \{f < \beta\} \iff f(x) < \beta \iff f_\alpha < \beta$ for some $\alpha \in \mathcal{A}$. Thus, $\{f < \beta\} = \cup_{\alpha\in\mathcal{A}}\{f_\alpha < \beta\}$ is open, by [1.2(iii)].

(b) No. E.g., let $X = \mathbb{R}$, $f_t = \chi_{\{t\}}$ for all $t \in (0, 1]$ and $f := \sup_{t\in(0,1]} f_t$. Then $f = \chi_{(0,1]}$ is not u.s.c. (nor l.s.c.) (because $\{f < 1\} = (0, 1]^c$ is not open), although f_t is u.s.c. for all t (because $\{f_t < \beta\} \in \{\emptyset, \{t\}^c, X\}$ is open).[16]

[16] One could even replace the functions f_t by suitable infinitely differentiable functions and still f would not be u.s.c.