Mat-1.150 Real Analysis: 9th exercise, Wed 2002-11-13, U322 (4 pages)

`http://www.math.hut.fi/teaching/realanal/`

1st mid-term examination: Results will be ready next week.
2nd mid-term examination 4.12. (4th Dec) at 14–17. in hall D.
final examination 19.12. (19th Dec) at 9–13 in halls L&N (you need not take this if you take the mid-term examinations; registration at webtopi.hut.fi recommended but not obligatory).

Exercise 9.1 ([7.12] *average level; short*):
Let $f \in L^1(\mathbb{R}^n)$ & $r \in \mathbb{R}_+$. Show that $x \mapsto [m(B(x,r))]^{-1} \int_{B(x,r)} f\,dm$ is continuous.[1]

Exercise 9.2 ([8.16] *quite easy and quick*):
Let $0 < p < \infty$. Assume that $f : \mathbb{R}^n \to [-\infty, \infty]$ belongs to *weak* $L^p(\mathbb{R}^n)$, i.e., that there is $c \geq 0$ s.t. $m(\{x \in \mathbb{R}^n \mid |f(x)| > \lambda\}) \leq c\lambda^{-p}\ \forall \lambda > 0$.
Show that $f \in L^q_{\text{loc}}(\mathbb{R}^n) := \{f \in L(\mathbb{R}^n) \mid f|_K \in L^q(K)\ \forall K \subset\subset \mathbb{R}^n\}$,[2] when $0 < q < p$.[3]

Exercise 9.3 ([7.5] *easy, short*):
Show that $f(x) := |x|^{-n}$ belongs to weak $L^1(\mathbb{R}^n)$ but not to $L^1(\mathbb{R}^n)$.

[1]**Hint:** $\int_{B(x,r)} f\,dm = \int \chi_{B(x,r)} f\,dm$, **Hint 2:** apply dominated convergence.

[2]Here $\forall K \subset\subset \mathbb{R}^n$ means "for all compact subsets K of $\mathbb{R}^n$". $f \in L(\mathbb{R}^n) := L(\mathbb{R}^n; \mathbb{C})$ means that f is measurable $\mathbb{R}^n \to \mathbb{C}$ (actually, $\mathcal{L}(\mathbb{R}^n)$ means such functions and $L(\mathbb{R}^n)$ the corresponding equivalence classes, so that L^p is a (vector) subspace of L for each $p \in (0, \infty]$).. $\mathrm{L}^q_{\text{loc}}$ is called the space of *locally integrable* functions.
Remark: $L^q_{\text{loc}} \subset L^r_{\text{loc}}$ whenever $0 < r \leq q \leq \infty$ (for $\mu = m$ and for any Borel-measure μ s.t. $\mu(K) < \infty$ for compact measurable K (if X is a Hausdorff space, then a compact set is closed, hence Borel-measurable)). We leave the simple proof to the reader (hint: use Hölder).

[3]**Hint 1:** You only need to study some arbitrary $K \subset\subset \mathbb{R}^n$. Set. $A := \{x \in K \mid f(x) > 1\}$, $\int_K = \int_A + \int_{A^c}$. **Hint 2:** apply [R2:8.16].

Important facts: Any mathematician should know the following two lemmas (and the polar coordinates of $\mathbb{R}^n$; see p. 176, Exe 8.6 of [R2]). Remember that you can find them from here!

Lemma 0.0.1 *Let $g : [0, \infty[\to [0, \infty]$ be a Borel function (i.e., Borel-measurable; note that any piecewise continuous function is a Borel function). Then*

$$\int_{\mathbb{R}^n} g(|x|)dx = \int_0^\infty g(r) a_n r^{n-1}\,dr, \tag{1}$$

where a_n is the "area" ($n-1$-dimensional measure) of the unit ball of $\mathbb{R}^n$.[4]

(Proof: [R2: p. 176, Exe 8.6].) By substituting $g(r) := \chi_{]R_1,R_2[} r^{-p}$, we obtain the following

Lemma 0.0.2

$$\int_{R_1<|z|<R_2} |z|^{-p}\,dz = a_n \int_{R_1}^{R_2} r^{-p} r^{n-1}\,dr = \frac{a_n}{n-p}\Big|_{R_1}^{R_2} r^{n-p} \tag{2}$$

(for $p \neq n$, or $a_n \log r$ for $p = n$).
Thus, $p < n \iff z^{-p} \in L^1(B_R)$, $\quad p > n \iff z^{-p} \in L^1(B_R^c)$.
In particular, $\frac{M}{1+|x|^a} \in L^p(\mathbb{R}^n)$ $\ 1 \leq p \leq \infty$, whenever $a > n$.

(Also here n is the dimension of the Euclidean space ($\mathbb{R}^n$).)

Exercise 9.4 ([7.5] *quite easy (with hints) and short*):
Prove the following statements concerning maximal functions.[5]
(a) $f, g \in L^1_{\text{loc}}(\mathbb{R}^k) \implies M(f+g)(x) \leq Mf(x) + Mg(x)$.
(b) $f \in L^1_{\text{loc}}(\mathbb{R}^k)$ & $\lambda \in \mathbb{R} \implies M(\lambda f)(x) = |\lambda| Mf(x)\ \forall x \in \mathbb{R}^k$.
(c) [Exe7.22b] $f \in L^1(\mathbb{R}) \not\Longrightarrow Mf \in L^1_{\text{loc}}(\mathbb{R})$.[6]

[4]The "area" of the ball B_r with radius r is $a_n r^{n-1}$; the "volume" (n-dimensional measure) of B_r equals $m(B_r) = v_n r^n$, where $0 < a_n, v_n < \infty$.

[5]**Hints**: (ii)&(iii): straight from the definitions. (iv): compute $Mf(x)\ \forall x \in]0, 1/4[$ and note that it is not a L^1 function.

[6]Recall that $L^1_{\text{loc}}(\mathbb{R}) := \{g \in L(\mathbb{R}) \mid \|g\|_{L^1(K)} := \int_K |g| < \infty\ \forall K \subset\subset \mathbb{R}\}$. **Hint 0:** $f(x) := \chi_{]0,1/2[}(x)/[x(\log x)^2]$. **Hint 1:** Compute $Mf(x)\ \forall x \in]0, 1/4[$ and note that it is not an L^1 function.

Exercise 9.5 ([7.20] *quite easy, short*):
Let $\alpha \geq 0$. A function $f : [a,b] \to \mathbb{C}$ is *Hölder-continuous with exponent* α (or *Lipschitz-continuous with exponent* α, i.e., $(f \in \mathcal{C}^\alpha([a,b]) := \mathcal{C}^{0,\alpha}([a,b]) =: \mathrm{Lip}_\alpha([a,b]))$, iff there is a constant $L_f \geq 0$ s.t.

$$|f(x) - f(y)| \leq L_f |x-y|^\alpha \ \forall x,y \in [a,b].$$

[R2: Exe7.11] Show the following: $f \in \mathrm{AC}([a,b])$ & $f' \in L^p([a,b])$ & $p \in (1,\infty) \implies f \in \mathcal{C}^{1-1/p}([a,b])$.[7]

Solution to 9.1: Since $[m(B(x,r))]^{-1}$ is a constant (for a fixed r), it suffices to show that g is continuous, where

$$g(x) := \int_{B(x,r)} f\,dm = \int_{\mathbb{R}^n} \chi_{B(x,r)} f\,dm; \tag{3}$$

i.e., we have to show that $g(x_n) \to g(x_0)$ for any sequence $\{x_n\}$ converging to some $x_0 \in \mathbb{R}^n$ [R1: 4.2].

Assume that $x_n \to x_0 \in \mathbb{R}^n$, as $n \to \infty$.

1° *We have* $(\chi_{B(x_n,r)}f)(y) \to (\chi_{B(x_0,r)}f)(y)$ $\forall y \in \mathbb{R}^n \setminus \partial B(x_0,r)$. Proof: Let $y \in \mathbb{R}^n \setminus \partial B(x_0,r)$. Let $\epsilon > 0$ be arbitrary. Then $|(\chi_{B(x_n,r)}f)(y) - (\chi_{B(x_0,r)}f)(y)| = 0 < \epsilon$ for $|x_n - x_0| < \delta := ||y-x_0| - r| > 0$. Indeed:

$$\begin{aligned} y \in B(x_0,r) &\implies \chi_{B(x_n,r)}(y) = 1 = \chi_{B(x_0,r)}(y) \\ &\quad \text{when } |x_n - x_0| < r - |y-x_0| =: \delta > 0; \\ y \in \bar{B}(x_0,r)^c &\implies \chi_{B(x_n,r)}(y) = 0 = \chi_{B(x_0,r)}(y), \\ &\quad \text{when } |x_n - x_0| < |y-x_0| - r =: \delta > 0. \end{aligned}$$

2° *g is continuous:* Proof: Set $A := \partial B(x_0,r)$, so that $m(A) = 0$ (e.g., by Lemma 0.0.1). We have $|\chi_{B(x,r)}f| \leq |f| \in L^1$ $\forall x \in \mathbb{R}^n$: this and 1° allow us to apply the Lebesgue's Dominated Convergence Theorem [1.34] to obtain that

$$\int_{\mathbb{R}^n} \chi_{B(x_n,r)} f\,dm = \int_{A^c} \chi_{B(x_n,r)} f\,dm \to \int_{A^c} \chi_{B(x_0,r)} f\,dm \tag{4}$$

$$= \int_{\mathbb{R}^n} \chi_{B(x_0,r)} f\,dm = \int_{B(x_0,r)} f\,dm, \tag{5}$$

as $n \to \infty$, QED.[8]

Solution to 9.2: Let $K \subset\subset \mathbb{R}^n$ be arbitrary. We only have to show that $\int_K |f|^q\,dm < \infty$. Set $g := |f|\big|_K$, $A := \{g > 1\} := \{x \in K \mid g(x) > 1\}$.

$$\int_K |f|^q\,dm = \int_K g^q\,dm = \int_A g^q\,dm + \int_{K\setminus A} g^q\,dm \leq \int_A g^q\,dm + m(K), \tag{6}$$

hence it suffices to prove that $\int_A g^q\,dm < \infty$.

$$\begin{aligned} \int_A g^q\,dm &= \int_K (\chi_A g)^q\,dm \overset{8.16}{=} \int_0^\infty m\{\chi_A g > t\} q t^{q-1}\,dt = \int_0^1 m(A) q t^{q-1}\,dt \\ &+ \int_1^\infty m\{g > t\} q t^{q-1}\,dt \leq m(A) + \int_1^\infty c t^{-p} q t^{q-1}\,dt < \infty, \end{aligned}$$

because $q - p - 1 < -1$.[9]

Solution to 9.3: 1° *f is in weak L^1.* Proof:

$$m(\{|f| > \lambda\}) = m(\{|x|^n < \lambda^{-1}\}) = m(\{|x| < \lambda^{-1/n}\}) \tag{7}$$

$$= m(B(0,\lambda^{-1/n})) = m(B_{\mathbb{R}^n}(0,1)) \cdot (\lambda^{-1/n})^n = v_n \lambda^{-1}, \tag{8}$$

[7] **Hints:** Recall that $f(b) - f(a) = \int_a^b f'(t)\,dt$ when f is AC.

[8] **Remark:** As noted in a previous exercise, Theorem 1.34 of [R2] as well as any (similar) results on limits of sequences actually apply to limits in $\mathbb{R}$ (or in any metric spaces), this just requires an application of [R1: 4.2]. Thus, we could have written "$x \to x_0$" above instead of using a sequence $\{x_n\}$, had we first extended [1.34] this way.

[9] Here we applied Theorem [R2:8.16] with $\phi(t) := t^q$. This is OK, because $\phi \in \mathrm{AC}(0,T)$ $\forall T > 0$ etc. (because $\phi(t) = \phi(0) + \int_0^t \phi'(s)\,ds$ & $\phi' \in L^1([0,T])$).

hence $\lambda m(\{|f| > \lambda\}) = v_n$ is bounded, i.e., f is in weak L^1.

2° $f \notin L^1$. Proof: By the lemma, we have[10]

$$\int_{\mathbb{R}^n} f\,dm \geq \int_{B(0,1)} f\,dm = a_n \int_0^1 r^{-n} r^{n-1}\,dr = a_n \Big|_0^1 \ln r = \infty. \tag{9}$$

Solution to 9.4: Notes: Let $f, g : X \to \mathbb{R}$ & $\lambda \geq 0$. Then $\sup_{x\in X}[f(x) + g(x)] \leq \sup_{x\in X} f(x) + \sup_{x\in X} g(x)$ ja $\sup_{x\in X} \lambda f(x) = \lambda \sup f(x)$ (easy exercises left to the reader). The definition of the *maximal function* $Mf : \mathbb{R}^k \to [0,\infty]$ of $f \in L^1_{\text{loc}}(\mathbb{R}^k;\mathbb{C})$ is[11]

$$Mf(a) := \sup_{0<r<\infty} m(B_r)^{-1} \int_{B(a,r)} |f|\,dm \ (a \in \mathbb{R}^k,\ f \in L^1_{\text{loc}}). \tag{10}$$

(a) $M(f+g)(a) := \sup_{0<r<\infty} m(B_r)^{-1} \int_{B(a,r)} |f+g|\,dm$

$$\leq \sup_{0<r<\infty} [m(B_r)^{-1} \int_{B(a,r)} |f|\,dm + m(B_r)^{-1} \int_{B(a,r)} |g|\,dm]$$

$$\leq \sup_{0<r<\infty} m(B_r)^{-1} \int_{B(a,r)} |f|\,dm + \sup_{0<r<\infty} m(B_r)^{-1} \int_{B(a,r)} |g|\,dm$$

$= Mf(a) + Mg(a)$, QED.

(b) $M(\lambda f)(a) := \sup_{0<r<\infty} m(B_r)^{-1} \int_{B(a,r)} |\lambda f|\,dm$

$$= \sup_{0<r<\infty} |\lambda| m(B_r)^{-1} \int_{B(a,r)} |f|\,dm$$

$$= |\lambda| \sup_{0<r<\infty} m(B_r)^{-1} \int_{B(a,r)} |f|\,dm = |\lambda| Mf(a), \text{ QED.}$$

(c) Summary of the proof below:

1^o $f \in L^1(\mathbb{R})$, where $f(x) := \chi_{]0,1/2[}(x)/[x(\log x)^2]$.

2^o $(Mf)(x) \geq (2x)^{-1}/\log x^{-1} =: g(x)\ \forall x \in]0,1/4[$.

3^o $\int_0^{1/4} g(x)\,dx = \infty$, hence $\int_0^{1/4} Mf(x)\,dx = \infty$. Thus, $Mf \notin L^1([0,1/4])$, hence $Mf \notin L^1_{loc}$, QED.

Details of the proof: 1^o *We have $f \in L^1$.* Proof:

$$\int_{\mathbb{R}} |f|\,dm = \int_0^{1/2} \frac{1}{t}(\log t)^{-2}\,dt = \Big|_0^{1/2} - (\log t)^{-1} \tag{11}$$

$$= -(\log 1/2)^{-1} - (-\infty)^{-1} = (-\log 1/2)^{-1} - 0 = (\log 2)^{-1} < \infty. \tag{12}$$

2^o Naturally, $\sup_{r>0} h(r,x) \geq h(x,x)$ and $B_x(x) = (0,2x)$ (for $x > 0$), hence from (10) we obtain that $(Mf)(x) \geq$

$$(B_x(x))^{-1} \int_0^{2x} |f| > (2x)^{-1} \int_0^x (\log t)^{-2} t^{-1}\,dt = (2x)^{-1} \Big|_0^x - (\log t)^{-1}$$

$$= (2x)^{-1}[-(\log x)^{-1} + (-\infty)^{-1}] = (2x)^{-1}/\log x^{-1} \quad \forall x \in (0,1/4),$$

3^o From 2° we obtain that (here we shall use the substitution $s = 1/x$, $x = 1/s, dx/ds = -s^{-2}$) $\int_0^{1/4} (Mf)(x)\,dx \geq \int_0^{1/4} 1/2x \log x^{-1}\,dx = \frac{1}{2}\int_4^\infty 1/s \log s\,ds = \frac{1}{2}\Big|_4^\infty \log(\log s) = \infty$.

Solution to 9.5: *Notes:* $f \in \text{AC} \implies \exists f'(x)$ a.e. [7.20], and $\exists f'(x)\ \forall x \implies f \in \text{AC}$ [7.21] (but $\exists f'(x)$ a.e. $\not\Longleftarrow f \in \text{AC}$ [7.16b]). AC-functions are exactly those that are the integrals of their derivatives (the simple proof is left to the reader).

Whenever $a \leq t \leq s \leq b$, we have

$$|f(s) - f(t)| = |\int_t^s f'| \leq \int_t^s |f' \cdot 1| \overset{\text{Hölder}}{\leq} \|f'\|_{L^p([t,s])} \|1\|_{L^q([t,s])} \leq \|f'\|_p |s-t|^{1/q}, \tag{13}$$

where $1/q := 1 - 1/p$, QED.

Remarks: In a previous exercise, we showed that $\mathcal{C}^\alpha := \text{Lip}_\alpha([a,b])$ is a Banach space with norm $\|f\| := M_f + \sup_x |f(x)|$ [R2:Exe5.11],

[10] Actually, no $|x|^p$, $p \in \mathbb{R}$ belongs to any L^q for any $q \in (0,\infty)$ and any $\mathbb{R}^n$, by Lemma 0.0.2.

[11] Actually, we can define Mf for any measurable f. However, $Mf(a) = \infty\ \forall a \in \mathbb{R}^k$ if $f \notin L^1_{\text{loc}}$. Proof: If $\int_K |f| = \infty$, choose r s.t. $K \subset B(a,r)$.

where M_f is the infimum of suitable L_f. It is not hard to show that $\mathcal{C}_b^1([a,b]) \subsetneq \mathrm{Lip}_1([a,b]) \subsetneq \mathrm{AC}([a,b]) \subsetneq \mathrm{BV}([a,b]) \cap \mathcal{C}([a,b])$.