Mat-1.150 Real Analysis: 8th exercise, Wed 2002-11-6, U322 (3 pages)

`http://www.math.hut.fi/teaching/realanal/`

1st mid-term examination (1. välikoe) 11.11. (11th Nov) at 9–12 (am), in hall R3 (rakennusosasto - Dept. of C.&E. Engineering, northeast of the main building, 2nd floor).

Requirements:

Chapter 1: Everything except: Theorem 1.36 and its proof.
Chapter 2: Everything except: Sec. 2.23, Th. 2.25, Proof of Urusohn's Lemma, Proof of Th. 2.14.
Chapter 3: Everything. Chapter 4: Everything.
Chapter 5: Everything except Sections 5.22–5.25.
Exercises 1.1–7.3 are related to the above.

2nd mid-term examination (2. välikoe) 4.12. (4th Dec) at 14–17. in hall D (main building).

Exercise 8.1 ([6.7] *quite easy, short*):
Let μ, ν and λ be complex measures on a σ-algebra $\mathfrak{M}$ and s.t. $\mu \perp \lambda$ and $\nu \perp \lambda$.
Show that[1] (a) $(\alpha\mu) \perp \lambda$ $\forall \alpha \in \mathbb{C}$ and (b) $\mu + \nu \perp \lambda$.

Exercise 8.2 ([6.10] *quite easy and short*):
Let λ, μ ja ν be finite positive measures[2] on $(X, \mathfrak{M})$.
(a) Assuming that $\nu \ll \mu \ll \lambda$, prove the "chain rule" $\frac{d\nu}{d\lambda} = \frac{d\nu}{d\mu}\frac{d\mu}{d\lambda}$.
(b) Assuming that $\nu \ll \mu$, prove that $\nu(\{x \in X \mid \frac{d\nu}{d\mu}(x) = 0\} = 0$.

Exercise 8.3 ([6.19] *quite easy and short*):
By Theorem 6.19 of [R2], we may identify $C[-\pi, \pi]$ with the space $M := M[-\pi, \pi]$ of complex regular Borel measures on $[-\pi, \pi]$.

Show that $\mu_n \to \mu$ weak* if $\{\mu_n\} \subset M[-\pi, \pi]$, $\sup_n \|\mu_n\| < \infty$, and $\hat{\mu}_n(j) \to \hat{\mu}(j)$ for all j (here $\hat{\mu}(j) := \int_{-\pi}^{\pi} \mathrm{e}^{-ijt}\, d\mu(t)$, as on p. 133).[3]

Exercise 8.4 ([Chapter 3 (or 6)] *easy, short; the result is important*):
Let $(X, \mathfrak{M}, \mu)$ be a σ-finite positive measure space, $1 \le p \le \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, and $f \in L^p(\mu)$. Show that

$$\|f\|_{L^p(\mu)} = \sup_{\|g\|_{L^q(\mu)} \le 1} \int_X |fg|\, d\mu. \tag{1}$$

Solution to 8.1: By the assumptions [6.7], there are sets $A, A', B, B' \in \mathfrak{M}$ s.t. μ is concentrated on A (i.e., $\mu(E) = \mu(E \cap A)$ $\forall E \in \mathfrak{M}$), λ is concentrated on B, and $A \cap B = \emptyset$, and ν is concentrated on A', λ on B' and $A' \cap B' = \emptyset$.[4] Case $f = 0$ is trivial, so we assume that $f \neq 0$.

(a) $(\alpha\mu)(E) := \alpha \cdot \mu(E) = \alpha \cdot \mu(E \cap A) =: (\alpha\mu)(E \cap A)$ $\forall E \in \mathfrak{M}$, hence also $\alpha\mu$ is concentrated on A. Since $A \cap B = \emptyset$, we have $\alpha\mu \perp \lambda$, QED.

(b) Since $\lambda(E) = \lambda(E \cap B')$ $\forall E \in \mathfrak{M}$ and $\lambda(F) = \lambda(F \cap B)$ $\forall F \in \mathfrak{M}$, we have $\lambda(E) = \lambda(E \cap B') = \lambda((E \cap B') \cap B) = \lambda(E \cap (B' \cap B))$ $\forall E \in \mathfrak{M}$, hence λ is concentrated on the set $B \cap B' =: B''$. On the other hand, $(\mu + \nu)$ is concentrated on $A'' := A \cup A'$, by the following:

$$(\mu + \nu)(E) := \mu(E) + \nu(E) = \mu(E \cap A) + \nu(E \cap A')$$

$$\overset{\ddagger}{=} \mu(E \cap (A \cup A')) + \nu(E \cap (A \cup A')) = (\mu + \nu)(E \cap (A \cup A')),$$

[1] **Hints:** (a) from the definition (same A and B will do). (b) from the definition (take sets A and B for μ and λ; and sets A' and B' for ν ja λ; set $A'' := A \cup A'$ & $B'' := B \cap B'$).

[2] Obviously, a positive measure is finite iff it is a complex measure, by [R2:1.18&6.1].

[3] **Hint 1:** use Theorem 4.25. **Hint 2:** start from the case $f = \mathrm{e}^{-ij\cdot}$.

[4] E.g., $X = [0, 1]$, $\mu = \delta_0$, $\nu = \delta_1$, $\lambda = m$, where $\delta_r(E) = 1$ if $r \in E$, $= 0$ if $r \notin E$. Then we can take $A = \{0\}$, $B = (0, 1]$, $A' = \{1\}$, $B' = [0, 1)$ (or $A = \{0, 1/5, 4/7, 99/100\}$, $B = (0, 1) \setminus \mathbb{Q}$ or anything s.t. $m(X \setminus B) = 0$ etc.). Note, in particular, that the sets A and B are not unique.

[Explanation for "‡": we have $\mu(E \cap (A' \setminus A)) = \mu((E \cap (A' \setminus A)) \cap A) = \mu(\emptyset) = 0$, hence $\mu(E \cap (A \cup A')) = \mu(E \cap A) + \mu(E \cap (A' \setminus A)) = \mu(E \cap A)$; a similar claim holds on ν.]

Thus, $(\mu + \nu) \perp \lambda$, because $A'' \cap B'' = (A \cap B'') \cup (A' \cap B'') \subset (A \cap B) \cup (A' \cap B') = \emptyset$, QED.

Solution to 8.2: Recall: if $\mu \ll \lambda$, then[5] $\exists! f \in L^1(\lambda)$ s.t. $\mu(E) \overset{6.10}{=} \int_E f \, d\lambda \ \forall E \in \mathfrak{M}$ [R2:6.10b], and for this f one denotes $\frac{d\mu}{d\lambda} := f$.

(a) Set $f := \frac{d\mu}{d\lambda}$, $g := \frac{d\nu}{d\mu}$, $h := \frac{d\nu}{d\lambda}$.[6]

$$\int_E h \, d\lambda \overset{6.10}{=} \nu(E) = \int_E g \, d\mu \overset{1.29}{=} \int_E gf \, d\lambda \quad \forall E \in \mathfrak{M},$$

joten $\int_E (h - gf) \, d\lambda = 0 \ \forall E \in \mathfrak{M}$, joten $h - gf = 0$ m.k.$[\lambda]$ [1.39b], QED.

(b) $N := \{x \in X \mid \frac{d\nu}{d\mu}(x) = 0\}$. $\nu(N) = \int_N \frac{d\nu}{d\mu} \, d\mu = \int_N 0 \, d\mu = 0$, QED.

Solution to 8.3: *Notes:* The exercise is written in a bit sloppy language; here we mean $\mathcal{C}(\mathbf{T})$, where $\mathbf{T}$ is the unit circle $\{z \in \mathbb{C} \mid |z| = 1\}$. As in 4.23 of [R2], we identify the functions through $t \mapsto e^{it}$ and thus use $f := F(e^{i\cdot}) \in \mathcal{C}([-\pi, \pi])$ in place of F (note that $f(-\pi) = f(\pi)$, so that $\mathcal{C}(\mathbf{T})$ corresponds (through an isometric isomorphism) to a subspace of $\mathcal{C}([-\pi, \pi])$; similarly, by M we actually mean $M(\mathbf{T})$ through a similar identification.[7]

Solution: Set $X := [-\pi, \pi)$. We are supposed to show that $\mu_n \to \mu$ weak*, i.e., that $\Lambda_{\mu_n}(f) \to \Lambda_\mu(f)$ for all $f \in \mathcal{C}(\mathbf{T})$, where $\Lambda_\mu(f) := \int_X f \, d\mu$ (thus, here $\Lambda : M(X) \to \mathcal{C}(X)^*$ is the identification (isometric isomorphism) mentioned in the exercise). Note[8] that $\|f\|_{\mathcal{C}(X)} := \sup |f| = \|f\|_\infty$.

[5] "$\exists!$" := "there exists a unique".

[6] $\frac{d\nu}{d\lambda}$ exists, because $\nu \ll \lambda$, which follows from the fact that $\ll$ is transitive (proof: $\lambda(E) = 0 \implies \mu(E) = 0 \implies \nu(E) = 0$).

[7] Without the identification "$\mu(e^{iE})$":= $\mu(E)$ from $\mu \in M(\mathbf{T})$ to "μ"∈"$M([-\pi, \pi])$" (where $e^{iE} := \{e^{it} \mid t \in E\}$), we would have $\hat{\mu}(j) := \int_{\mathbf{T}} z^j \, d\mu(z)$ etc.

1° *Case* $f = e^{-ik\cdot}$, $k \in \mathbb{Z}$: Now $\mu(f) := \int_X e^{-ikt} \, d\mu(t) =: \hat{\mu}(k)$. Thus, $\mu_n(f) = \hat{\mu}_n(k) \to \hat{\mu}(k) = \mu(f)$, by the assumption.

2° *Case* $f = \sum_{k=-n}^{n} \alpha_n e^{-ik\cdot}$, $n \in \mathbb{N}$, $\alpha_n \in \mathbb{C} \ \forall n$: This follows from 1°, by the linearity of μ and μ_n.

3° *General case* $f \in \mathcal{C}(\mathbf{T})$: Let $\epsilon > 0$ be given. Set $R := \max\{\|\mu\|, \sup_n \|\mu_n\|\} < \infty$, $\delta := \epsilon/3R$. By Theorem 4.25 of [R2], there is a polynomial[9] $P \in \mathcal{C}([-\pi, \pi])$ s.t. $\|f - P\|_\infty < \delta$. Choose N s.t. $|\Lambda_\mu P - \Lambda_{\mu_n} P| < \epsilon/4 \ \forall n \geq N$. Then (note that $|\Lambda_\mu f| \leq \|\Lambda_\mu\|_{\mathcal{C}(X)^*} \|f\|_\infty = \|\mu\| \|f\|_\infty$)

$$|\Lambda_\mu(f) - \Lambda_{\mu_n}(f)| \leq |\Lambda_\mu(f - P)| + |\Lambda_\mu P - \Lambda_{\mu_n} P| + |\Lambda_{\mu_n}(P - f)| \tag{2}$$
$$\leq R\delta + \epsilon/4 + R\delta = \epsilon/3 + \epsilon/4 + \epsilon/3 < \epsilon \tag{3}$$

for all $n \geq N$. Thus, $\Lambda_{\mu_n}(f) \to \Lambda_\mu(f)$, QED.

Remarks: From the Uniform Boundedness Theorem [5.8] it follows that also "only if" is true.[10]

As in the above footnote, one can easily show that every weak* (or weakly or strongly[11]) converging sequence is (norm) bounded in any normed space. On the

[8] We have sup = ess sup for continuous functions when, e.g., $\mu(V) > 0$ for all open sets V (in particular, when $\mu = m$ and $X \subset \mathbb{R}^d$ is open with the standard topology inherited from $\mathbb{R}^d$). Proof: Let $f \in \mathcal{C}(X)$ be real-valued (or apply this to $|f|$). 1° Obviously, $\sup f \geq \operatorname{ess\,sup} f$. 2° If $r < \sup f$, then $\operatorname{ess\,sup} f > r$, because $\mu(V_r) > 0$, where $V_r := \{x \in X \mid f(x) > r\}$ is open (being the preimage of the open set (r, ∞)). Thus, $\operatorname{ess\,sup} f \geq \sup f$ QED.

[9] Note that $(z \mapsto z^k) \in \mathcal{C}(\mathbf{T})$ corresponds to $(t \mapsto (e^{it})^k = e^{itk}) \in \mathcal{C}([-\pi, \pi])$, hence linear combinations of such functions (i.e., functions like the one in 2°) are called polynomials.

[10] Proof: Assume that $\mu_n \xrightarrow{*} \mu$. If $\sup_n |\Lambda_{\mu_n} f| = \infty$ for some $f \in \mathcal{C}(X)$, then we cannot have $\Lambda_{\mu_n} f \to \Lambda_\mu f$, hence there is $R < \infty$ s.t. $R \geq \|\Lambda_{\mu_n}\| = \|\mu_n\|_M$ for all n, by [5.8]. Take $f := e^{-ij\cdot}$ to observe that $\hat{\mu}_n(j) \to \hat{\mu}(j)$.

[11] "Strong" convergence means usually convergence in norm (i.e., the "usual" convergence). However, in case of $\mathcal{L}(X, Y)$-valued operators $\{L_n\}$ *uniform convergence* means convergence w.r.t. the norm ($\|L_n - L\|_{\mathcal{L}(X,Y)} \to 0$), *strong convergence* means "strong pointwise convergence" ($\|L_n x - Lx\|_Y \to 0 \ \forall x \in X$, i.e., $L_n x \to x \ \forall x \in X$) and *weak convergence* means "weak pointwise convergence" ($L_n x \to x$ weakly $\forall x \in X$, i.e.,

other hand, $\liminf_n \|f_n\| \ge \|f\|$ when $f_n \to f$ strongly, weakly or weak*.[12]

Solution to 8.4: From the Hölder inequality we obtain "$\ge$", hence we only need to establish "$\le$".

1° *Case* $p = 1$*:* Take $g \equiv 1 \in L^\infty$.

2° *Case* $f \ge 0$ *and* $1 < p < \infty$*:* An easy guess is that a good choice would be $g = cf^{p-1}$, so that $\int fg = c\int f^p = c\|f\|_p^p$. $p^{-1} + q^{-1} = 1 \implies q = p/(p-1)$.

We want to have $\|g\|_q = 1$. Now $\|f^{p-1}\|_q^q = \int_X (f^{p-1})^{p/(p-1)} = \|f\|_p^p$, hence $c := \|f^{p-1}\|_q^{-1} = (\|f^{p-1}\|_q^q)^{-1/q} = (\|f\|_p^p)^{-1/q} = \|f\|_p^{-(p-1)}$. Thus, $\int |fg| = c\int f^p = c\|f\|_p^p = \|f\|_p$, QED.

3° *Case* $1 < p < \infty$*:* Choose g for $|f| \ge 0$ as in 2°, so that $\int |fg| = \int |f|g = \|(|f|)\|_p = \|f\|_p$.

4° *Case* $p = \infty$*:* Only now we need σ-finiteness: we know that $X = \cup_{n=1}^\infty X_n$, where $\mu(X_n) < \infty\ \forall n$.

Let M be s.t. $\|f\|_\infty > M > 0$. Then $\mu(E) > 0$, where $E := \{x \in X \bigm| |f| > M\}$. Now $\mu(E) = \mu(\cup_{n=1}^\infty (E \cap X_n)) = \mu(\cup_{N=1}^\infty \cup_{n=1}^N (E \cap X_n)) = \lim_{N\to\infty} \mu(\cup_{n=1}^N (E \cap X_n))$ [R2:1.19d], hence $0 < \mu(E_N)$ for some $E_N := \cup_{n=1}^N (E \cap X_n)$. Therefore, $g := c\chi_{E_N}$ will do for $c := \mu(E_N)^{-1}$ (because $\mu(E_N) < \infty$, so that $\|g\|_1 = 1$), since $\int_X |fg|\,d\mu = c\int_{E_N} |f|\,d\mu \ge cM\mu(E_N) = M$. Because $M < \|f\|_\infty$ was arbitrary, the supremum in (1) equals (at least) $\|f\|_\infty$, QED.

Remarks: From the above proof we observe that if $p < \infty$, then μ need not be σ-finite and sup = max (i.e., max exists).

The result (1) (and some of its variants) is commonly used in mathematical literature explicitly and implicitly. For a generalization of this result, see Theorem B.4.12(b1) of [Mikkola2002]. By the theorem, (1) holds even if $\|f\|_p = \infty$, and we can require that, e.g., g is simple. If X is an open subset of $\mathbb{R}^k$ (or similar) and $\mu = m$ then (1) holds for any $f \in L^1_{\text{loc}}$ even if we require that $g \in \mathcal{C}_c^\infty(X)$. Moreover, corresponding results hold even if f has its values in a Banach (or Fréchet) space.

"Counter-example": Let $X = \{a, b\}$, $\mu(\{a\}) = 1$, $\mu(\{b\}) = \mu(X) = \infty$. Set $g = \chi_{\{b\}}$. Then, obviously, $L^1 = \{\alpha\chi_{\{a\}} \bigm| \alpha \in \mathbb{C}\}$, hence $\int_X fg\,d\mu = 0 < \|g\|_\infty = 1$ for all $f \in L^1$. Why?[13]

$\Lambda L_n x \to \Lambda Lx\ \forall x \in X\ \forall \Lambda \in Y^*$). Be careful!

[12] Let $r < \|f\|$. Choose $g \in X^*$ (or $g \in X$ if $f_n \to f \in X^*$ weak*) s.t. $\|g\| = 1$ and $|gf| > r$ [5.20] (or $\|\cdot\|_{X^*}$-def.). Then $\|f_n\| = \|g\|\|f_n\| \ge |gf_n| > r$ for n big enough.

[13] **Answer:** μ is not σ-finite.