Mat-1.150 Real Analysis: 7th exercise, Wed 2002-10-30, U322 (3 pages)

`http://www.math.hut.fi/teaching/realanal/` **1st mid-term examination** (1. välikoe) 11.11. (11th Nov) at 9–12 (am), in hall R3 (rakennusosasto - Dept. of C.&E. Engineering, northeast of the main building, 2nd floor); Chapters 1–5 of the book [R2] (details later).
2nd mid-term examination (2. välikoe) 4.12. (4th Dec) at 14–17. in hall D (main building).

Exercise 7.1 ([5.10] *average*):
Let X, Y be a Banach spaces. Let $X' := \mathcal{L}(X,\mathbb{C})$ and Y' be their dual spaces. Recall that X' is a Banach space with norm

$$\|f\|_{X'} := \sup\{|f(x)| \bigm| \|x\|_X = 1\}$$

(similarly for Y'). For any $L \in \mathcal{L}(X,Y)$, the *adjoint* operator $L^* \in \mathcal{L}(Y',X')$ is defined by

$$(L^*y^*)(x) := y^*(Lx) \quad (x \in X,\ y^* \in Y').$$

Show the following:
(a) $L^* \in \mathcal{L}(Y',X')$, as claimed above, and $\|L^*\| \le \|L\|$.
(b) If L is 1-1 onto a closed subspace of Y, then L^* is onto.
(c) If L is onto a dense subset of Y, then L^* is 1-1.

Exercise 7.2 ([5.20] *average*):
Let $X \ne \{0\}$ be a Banach space.[1] Define the *James map* $j : X \to X''$ by $j(x)(x^*) := x^*(x)$ $(x^* \in X',\ x \in X)$. Show that this map has the following properties: (a) j is 1-1 and linear; (b) $\|jx\| = \|x\|\ \forall x$ and $\|j\| = 1$; (c) $j(X)$ is a closed subspace of X''. Hint: use Theorem 5.20 of [R2].

---

[1] Note that $\|j\| = 0$ if $X = \{0\}$.

Exercise 7.3 ([5.16] *average*):
[R2:Exe5.6] Let $f \in M^* := \mathcal{L}(M,\mathbb{C})$, where M is a subspace of a Hilbert space H. **(a)** Show that there is a unique norm-preserving extension $\Lambda \in H^*$ of f. **(b)** Show that this extension vanishes on $M^\perp$.

Exercise 7.4 ([6.19] *average*):
[R2:Exe6.3] Let X be a locally compact Hausdorff space. Set $M := M(X) := \{\mu \bigm| \mu \text{ is a complex, regular Borel measure on } X\}$. Set $\|\mu\| := |\mu|(X)\ \forall \mu \in M$. Show that $M(X)$ is a Banach space.

Solution to 7.1: **(a)** Obviously, $L^*(\alpha y^*) = \alpha L^*(y^*)$ and $L^*(y^* + z^*) = L^*(y^*) + L^*(z^*)$ for all $y^*, z^* \in Y'$, $\alpha \in \mathbb{C}$, hence L^* is a linear map. Moreover,

$$\begin{aligned} \|L^*y^*\|_{X'} &:= \sup\{|L^*y^*(x)| \bigm| \|x\|_X \le 1\} = \sup\{|y^*(Lx)| \bigm| \|x\|_X \le 1\} && (1)\\ &\le \|y^*\|_{Y'}\|L\|_{\mathcal{L}(X,Y)} \quad (y^* \in Y), && (2) \end{aligned}$$

hence $\|L^*\|_{\mathcal{L}(Y',X')} \le \|L\|_{\mathcal{L}(X,Y)}$.[2]

(b) By Theorem 5.10 of [R2] (with $L(X)$ in place of Y)[3] , there is $\delta > 0$ s.t. $\|\Lambda x\| \ge \delta\|x\|\ \forall x \in X$.

Let $x^* \in X'$ be given. Define $f \in \mathcal{L}(L(X),\mathbb{C})$ by $f(Lx) = x^*(x)$. Since L is 1-1, it follows that f is well-defined and linear. Moreover,

$$|f(Lx)| = |x^*(x)| \le \|x^*\|\|x\| \le \|x^*\|\delta^{-1}\|Lx\|.$$

By the Hahn–Banach theorem, we can extend f to an element $y^* \in Y'$ s.t. $\|y^*\| \le \delta^{-1}\|x^*\|$. Then

$$L^*y^*(x) = y^*(Lx) = f(Lx) = x^*(x) \quad (x \in X),$$

[2] Actually, $\|L^*\| = \|L\|$. Proof W.l.o.g., assume that $\|L\| = 1$. Given $r < 1$, choose $x \in X$ s.t. $\|x\| = 1$ and $\|Lx\| > r$. By Theorem 5.20 of [R2], there is $y^* \in Y'$ s.t. $\|y^*\| = 1$ and $|y^*Lx| = \|Lx\| > r$. From $|L^*y^*(x)| = |y^*Lx| > r$ we conclude that $\|L^*y^*\| > r$, hence $\|L^*\| > r$. Since $r < 1$ was arbitrary, $\|L^*\| \ge 1$, QED.

[3] Note that, being closed, $L(X) \subset Y$ is complete, hence a Banach space.

hence $L^*y^* = x^*$ and we have shown that L^* is onto.

(c) If $L^*y^* = 0$, then $y^*(Lx) = 0\ \forall x$. Since Lx is dense in Y, this can only happen if $y^* = 0$. Thus, L^* is 1-1.

Remarks: See [R3:Chapter 4] for more on adjoints.

Solution to 7.2: **(a)** Obviously, j is linear. Assume that $x \neq 0$ and $j(x) = 0$ to obtain a contradiction. By Theorem 5.20, there is $x^* \in X'$ s.t. $x^*(x) = \|x\|$ and $\|x^*\| = 1$. But

$$0 = (jx)(x^*) := x^*(x) = \|x\|,$$

hence $x = 0$, a contradiction. Thus, j is 1-1.

(b) Let $x \in X \setminus \{0\}$. By Theorem 5.20, there is $x^* \in X'$ s.t. $x^*(x) = \|x\|$ and $\|x^*\| = 1$. But

$$\|x\| \geq \sup\{|y^*x| \mid \|y^*\|_{X'} \leq 1\} = \sup\{|(jx)(y^*)| \mid \|y^*\|_{X'} \leq 1\} \quad (3)$$
$$= \|jx\|_{X''} \geq |(jx)(x^*)| := |x^*(x)| = \|x\|. \quad (4)$$

Thus, $\|jx\|_{X''} = \|x\|_X\ \forall x \in X$.

(c) If $jx_n \to x'' \in X''$, then $\{x_n\}$ is a Cauchy sequence, by (b), hence then $x_n \to x$ for some $x \in X$. Consequently, $jx = \lim_{n\to\infty} jx_n = x''$. We conclude that $\overline{j(X)} \subset j(X)$, hence $j(X)$ is closed.

Remarks: Parts (a) and (b) hold for any normed space X. Part (c) holds iff X is complete. X is said to be *reflexive* if "$X = X''$" (meaning that $j(X) = X''$; vectors x and $j(x)$ are often identified).

Note: all Hilbert spaces are reflexive (and so are $L^p(\mathbb{R})$ for $1 < p < \infty$ (but not for $p \in \{1, \infty\}$), since $L^p(\mathbb{R})^{**} = L^q(\mathbb{R})^* = L^p(\mathbb{R})$, by [6.16]), and reflexive spaces are Banach spaces (since X'' is complete).

Solution to 7.3: **(a)** **Proof 1:** 1° *There is* $y_f \in \bar{M}$ *s.t.* $f(x) = \langle x, y_f\rangle\ \forall x \in M$. Proof 1a: As in the proof of theorem 4.12 of [R2]. Proof 1b: Let $\bar{f} \in \bar{M}^*$, be the unique continuous extension[4] of f to $\bar{M}$. Since $\bar{M}$ is a Hilbert space, y_f exists, by [4.12].

2° The functional $\Lambda x := \langle x, y_f\rangle\ \forall x \in H$ has the norm[5] $\|\Lambda\|_{H^*} = \|y_f\|_H = \|f\|_{M^*}$. Thus, it only remains to establish the uniqueness.

3° **Uniqueness:** Let P and Q be the orthogonal projections $P : H \to \bar{M}$ and $Q : H \to \bar{M}$ (so that $x = Px + Qx$ etc., as in [4.11]). Let $g \in H^*$ and $g_{|M} = f$. We now have to show that $g = \Lambda$ if $\|g\| = \|f\|$.

Set $h := g - \Lambda \in H^*$, so that $h_{|M} = 0$. Then $h = \langle \cdot, z\rangle$ for some $z \in H$, by [4.12], but $\langle x, z\rangle = \langle x, Pz\rangle + \langle x, Qz\rangle$ equals $0\ \forall x \in M$, hence $\forall x \in \bar{M}$. Consequently, $Pz = 0$ (take $x = Pz$ above), i.e., $z = Qz \in M^\perp$. Thus, $g(x) = \Lambda(x) + h(x) = \langle x, y_f\rangle + \langle x, z\rangle = \langle x, y_f + z\rangle\ \forall x \in H$. As in 2°, it follows that $\|g\|^2 = \|y_f+z\|^2 = \|P(y_f+z)\|^2 + \|Q(y_f+z)\|^2 = \|y_f\|^2 + \|z\|^2$, so that g can be norm-preserving only if $z = 0$, i.e., if $g = \Lambda$.

Proof 2: $\exists! \bar{f} \in M^*$ s.e. $\bar{f}x = fx\ \forall x \in M$ (proof: as 1° above). Let P and Q be as om 3°. Set $\Lambda x := \bar{f}Px$, so that $\|\Lambda x\| \leq \|\bar{f}\|\|Px\| \leq \|\bar{f}\|\|x\|\ \forall x \in H$, hence $\|\Lambda\| \leq \|\bar{f}\|$, hence $\|\Lambda\| = \|\bar{f}\|$. Obviously, $\Lambda z = \bar{f}Pz = 0$, when $z \in M^\perp$, i.e., when $z = Qx$.

Uniqueness: as 3° above.

(b) Since $y_f \in \bar{M}$ and $\bar{M}^\perp = M^\perp$ (by exercise 5.4/2002), we have $\Lambda z = \langle z, y_f\rangle = 0\ \forall z \in M^\perp$. (Alternative proof: $\langle z, y_f\rangle = \lim_n \langle z, y_n\rangle = 0$, where $M \ni y_n \to y_f$.)

Solution to 7.4: Let $\mathfrak{M}$ be the Borel σ algebra on X, i.e., the smallest σ algebra containing $\mathcal{T}_X$ [1.11]. *Regularity* [p. 130 & 2.15] of μ means that of $|\mu|$, i.e., that for all $E \in \mathfrak{M}$ we have[6]

---

[4] By the Hahn–Banach Theorem [5.16], $\bar{f}$ exists. By density, it is unique (i.e., if $\bar{f}, g \in \bar{M}^*$ and $\bar{f} = g (= f)$ on M, then $\bar{f} = g$ on $\bar{M}$ (proof: $\bar{f}x = \lim_n \bar{f}x_n = \lim_n gx_n = gx$).

(Actually, any bounded linear map $Y \to Z$ can be uniquely extended to the closure of Y as long as, e.g., Y is a normed space and Z is a Banach space, by Lemma A.3.10 of [Mikkola2002].)

[5] Proof: 1° $\|fx\| \leq \|y_f\|\|x\|\ \forall x \in M$, hence $\|f\| \leq \|y_f\|$. Similarly, $\|\Lambda\| \leq \|y_f\|$. 2° $\|y_f\| \leq \|f\|$, because $\|f\|\|y_f\| = \lim_n \|f\|\|y_n\| \geq \lim_n |fy_n| = \lim_n \langle y_n, y_f\rangle = \langle y_f, y_f\rangle = \|y_f\|\|y_f\|$, when $M \ni y_n \to y_f$. 3° Trivially, $\|f\| \leq \|\Lambda\|$.

[6] In principle, the condition (2) only needs be satisfied for those E for which $|\mu|(E) < \infty$, but in case of a complex measure μ this is the case for all $E \in \mathfrak{M}$.

(1) $|\mu|(E) = \inf\{|\mu|(V) \mid V \supset E \text{ is open}\}$, and
(2) $|\mu|(E) = \sup\{|\mu|(K) \mid K \subset E \text{ is compact}\}$.

1° *The set V of complex measures on $\mathfrak{M}$ is a normed space.* Proof:

1.1° *V is a vector space.* Proof: Obviously, $c\mu, \mu+\nu \in V$, if $\mu, \nu \in V$ and $c \in \mathbb{C}$ (see [6.5]). Obviously, also the other axioms [2.1] of a vector space are satisfied (with $0 := 0_V$ being the measure $0_V : E \mapsto 0$ ($E \in \mathfrak{M}$) etc.).

1.2° *V is a normed space.* Proof: By 1.1°, it only remains to verify the three axioms [5.1] of a norm:

(c) If $\|\mu\| = 0$, then $\mu(E) = 0\ \forall E \in \mathfrak{M}$, i.e., $\mu = 0$, because otherwise we would have $|\mu|(X) \geq |\mu|(E) \geq |\mu(E)| > 0$.

(b) $\|c\mu\| := |c\mu|(X) = \sup_{\{E_i\}} \sum_{i=1}^{\infty} |c\mu(E_i)| = \sup_{\{E_i\}} \sum_{i=1}^{\infty} |c||\mu(E_i)| = |c| \sup_{\{E_i\}} \sum_{i=1}^{\infty} |\mu(E_i)| = |c|\|\mu\|$ for any $c \in \mathbb{C}$.

(a) $\|\mu + \nu\| := |(\mu + \nu)|(X) := \sup_{\{E_i\}} \sum_{i=1}^{\infty} |(\mu + \nu)(E_i)| \leq \sup_{\{E_i\}} \sum_{i=1}^{\infty}[|\mu(E_i)| + |\nu(E_i)|] = \sup_{\{E_i\}} |[\sum_{i=1}^{\infty} |\mu(E_i)| + \sum_{i=1}^{\infty} |\nu(E_i)|] \leq \sup_{\{E_i\}} \sum_{i=1}^{\infty} |\mu(E_i)| + \sup_{\{E_i\}} \sum_{i=1}^{\infty} |\nu(E_i)| = |\mu|(X) + |\nu|(X) = \|\mu\| + \|\nu\|$.

2° *The set M is a normed space.* Proof: Since $M \subset V$, it suffices to show that M is a (vector) subspace of V, by 1°, i.e., that $c\mu$ and $\mu+\nu$ are regular whenever $\mu, \nu \in M$ and $c \in \mathbb{C}$.

2.1° Let $c \in \mathbb{C}$, $\mu \in M$. The (positive) measure $|c\mu|(E) = |c||\mu|(E)$ obviously satisfies the (regularity) conditions (1) and (2) (a nonnegative constant can be taken out of inf and sup; case $c = 0$ is trivial).

2.2° Let $\mu, \nu \in M$. Condition (1): Let $\epsilon > 0$ and $E \in \mathfrak{M}$. Choose open $V, V' \supset E$ s.t.

$$|\mu|(V) < |\mu|(E) + \epsilon/2, \ |\nu|(V') < |\nu|(E) + \epsilon/2. \tag{5}$$

Set $W := V \cap V'$. Then[7] $|\mu + \nu|(W) - |\mu + \nu|(E) = |\mu + \nu|(W \setminus E) \leq |\mu|(W \setminus E) + |\nu|(W \setminus E) \leq |\mu|(V \setminus E) + |\nu|(V' \setminus E) < \epsilon$. Condition (2) can be established similarly.

3° *M is complete, hence a Banach space.* By Theorem 6.19 of [R2], M is isometrically isomorphic to $\mathcal{C}_0(X)^*$, hence M is complete, because so is $\mathcal{C}_0(X)^*$.[8]

Remarks: In principle, the isomorphism used in 3° would provide the whole proof similarly. However, since the proof of Theorem 6.19 used the fact that M is a vector space, at least 1.1° and 2° should be included to complete that proof.

Note also that the space ("V") of complex measures on any σ-algebra is a normed space (the proof of 1° did not use the fact that $\mathfrak{M}$ was the Borel σ-algebra).

Is V complete? I do not know. Given a Cauchy sequence $\{\mu_n\}$, the sequence $\{\mu_n(E)\} \subset \mathbb{C}$ is obviously a Caucy sequence for each E, hence $\mu(E) := \lim_n \mu_n(E)$ exists. Obviously, if $\{\mu_n\}$ has a limit in V, then this limit must equal μ, but it does not seem easy to verify whether μ is a measure (and whether $\mu_n \to \mu$ in V).

[7] We have $\lambda(E \cup E') = \lambda(E) + \lambda(E')$ whenever $E \cap E' = 0$, for any positive measure λ, since measures are required to be (countably) additive [1.18a].

[8] By Exercise 5.8 of [R2] (i.e., our Exercise 6.2 of 2002), any dual space is complete. If $\{\mu_n\}$ is a Cauchy sequence in M and J is the isometric isomorphism $M \to \mathcal{C}_0(X)^*$, then $\{J\mu_n\}$ is a Cauchy-sequence in $\mathcal{C}_0(X)^*$, hence $\{J\mu_n\} \to \Lambda$ for some $\Lambda \in \mathcal{C}_0(X)^*$. But then $J^{-1}\Lambda$ is the limit of $\{\mu_n\}$ in M (since $\|\mu_n - J^{-1}\Lambda\|_M = \|J(\mu_n - J^{-1}\Lambda)\|_{\mathcal{C}_0(X)^*} = \|J\mu_n - \Lambda)\|_{\mathcal{C}_0(X)^*} \to 0$, as $n \to \infty$).