Mat-1.150 Real Analysis: 10th exercise, Wed 2002-11-20, U322 (4 pages)

`http://www.math.hut.fi/teaching/realanal/`

Last exercise 11: At 14.15 on 27.11. (27th Nov), i.e., 2 hours earlier.
1st mid-term examination: Results will probably be ready on Friday.
2nd mid-term examination 4.12. (4th Dec) at 14–17. in hall D. From [R2]: 6.1-6.13, 6.15-6.18, 7.1-7.11, 7.16-7.20. Moreover, 6.19 without its proof. Understand Fubini's Theorem (not proof). From [ABHN]: Sections 1.1 & 1.2 (except Corollary 1.1.3 and the proof of Th. 1.1.1).
Final examination 19.12. (19th Dec) at 9–13 in hall L (you need not take this if you take the two mid-term examinations; registration at webtopi.hut.fi recommended but not obligatory).
[ABHN] The second course book (cf. Exercises 10.4 and 11.1–11.4). Ask kalle.mikkola@iki.fi if you do not have access to [ABHN].
Sometimes some comments here refer to Appendix B of [Mikkola2002]:
`http://www.math.hut.fi/~kmikkola/research/thesis/contents/contents.html`
These are only meant for those wishing to get further insight, it is not a substitute to [ABHN].

Exercise 10.1 ([7.10] *trivial, quick*):
[R2:Exe7.1] Let $f \in L^1(\mathbb{R}^k)$. Show that $|f(x)| \leq (Mf)(x)$ at every Lebesgue point x of f.

Exercise 10.2 ([7.20] *quite easy and quick*):
[R2:Exe7.14]
(a) Show that fg is AC over $[a, b]$ whenever f, g are.
(b) Use (a) to state and prove a theorem on partial integration.

Exercise 10.3 ([7.20] *difficult, long; results (the ones below) are important*):
[R2:Exe7.12] Suppose that $\phi : [a, b] \to \mathbb{R}$ is nondecreasing.
(a) Show that there is a left-continuous nondecreasing f on $[a, b]$ s.t. $\{f \neq \phi\}$ is at most countable.
(b) Imitate the proof of Theorem 7.18 to show that there is a positive Borel measure μ on $[a, b]$ for which

$$f(x) - f(a) = \mu([a, x)) \quad (a \leq x \leq b). \tag{1}$$

(c) Deduce from (b) that $f'(x)$ exists a.e. $[m]$, $f' \in L^1(m)$ and

$$f(x) - f(a) = \int_a^x f'(t)\, dt + s(x) \quad (a \leq x \leq b), \tag{2}$$

where s is nondecreasing and $s'(x) = 0$ a.e. $[m]$.
(d) Show that **(d1)** $\mu \perp m$ iff $f' = 0$ a.e. $[m]$, **(d2)** $\mu \ll m$ iff f is AC on $[a, b]$.
(e) Prove that $\phi' = f'$ a.e. $[m]$.

Exercise 10.4 ([Bochner-integral] *difficult, quite long*):
Let X be a Banach space. Show that if $\int_a^b g\phi\, dm = 0$ for all $\phi \in \mathcal{C}_c^\infty(a, b)$, where $g \in L^1([a, b]; X)$, then $g(t) = 0$ a.e.

Solution to 10.1: By Theorem 7.10, for any Lebesgue point x of f we have[1]

$$|f(x)| \overset{7.10}{=} |\lim_{r\to 0} m(B_r)^{-1} \int_{B_r(x)} |f|\, dm| \leq \lim_{r\to 0} m(B_r)^{-1} \int_{B_r(x)} |f|\, dm \tag{3}$$

$$\leq \sup_{0<r<\infty} m(B_r)^{-1} \int_{B_r(x)} |f|\, dm =: (Mf)(x). \tag{4}$$

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[1] Alternative proof: one easily observes that x is a Lebesgue point of $|f|$ too, hence the first "$\leq$" and the term preceding it can be removed.

Solution to 10.2: **(a)** Since f,g are AC on $I := [a,b]$, hence continuous, and I is compact, $M := \max\{\max_I |f|, \max_I |g|, 1\} < \infty$. Given $\epsilon > 0$, there are $\delta_{f,\epsilon/2M}$, $\delta_{g,\epsilon/3M}$ s.t. ... But

$$\sum_{i=1}^n |(fg)(\beta_i) - (fg)(\alpha_i)| \le \sum_{i=1}^n (|f(\beta_i)||g(\beta_i) - g(\alpha_i)| + |g(\alpha_i)||f(\beta_i) - f(\alpha_i)|)$$
$$\le M\epsilon/3M + M\epsilon/2M < \epsilon,$$

when $\sum_i |\beta_i - \alpha_i| < \delta_{fg,\epsilon} := \min\{\delta_{f,\epsilon/2M}, \delta_{g,\epsilon/3M}\}$ and (α_i, β_i) are disjoint.

(b) We have

$$\frac{(fg)(t) - (fg)(t_0)}{t - t_0} = \frac{f(t)[g(t) - g(t_0)]}{t - t_0} + \frac{g(t_0)[f(t) - f(t_0)]}{t - t_0} \tag{5}$$
$$\to f(t_0)g'(t_0) + g(t_0)f'(t_0), \tag{6}$$

as $t \to t_0$, for those $t_0 \in I$ for which f', g' exist (since $f,g \in \mathrm{AC} \subset \mathcal{C}$), hence for a.e. $t_0 \in I$, by Theorem 7.20.[2] From this and Theorem 7.20 we deduce that $\Big|_s^t fg \overset{7.20}{=} \int_s^t (fg)' = \int_s^t f'g + \int_s^t fg'$ for all $s,t \in I$. Thus:

Theorem *If $f,g \in \mathrm{AC}([a,b])$, then $\int_s^t f'g = \Big|_s^t fg - \int_s^t fg'$ for all $s,t \in [a,b]$.*

Solution to 10.3: **(a)** Set $f(t) := \phi(t-) := \lim_{s\to t-} \phi(s) = \sup_{s<t} \phi(s)$ $(t \in [a,b])$. Obviously, $F(t) := \phi(t+) - \phi(t-) \ge 0$ $\forall t$ (set $\phi(b+) := \phi(b)$, $\phi(a-) := \phi(a)$).

$1°$ *We have $\phi(b) - \phi(a) \ge \sum_{t\in[a,b]} F(t)$.* Proof: Let $A' \subset [a,b]$ be finite but otherwise arbitrary; order the elements: $A' =: \{t_1, ..., t_N\}$, $t_1 < t_2 < ....$ Choose $\{s_k\}$ s.t. $a \le s_0 < t_1 < s_1 < t_2 < ... < t_N < s_N \le b$ (allow for $t_1 = s_0$ if $t_1 = a$ and for $t_N = s_N$ if $s_N = b$). Then $\phi(b) - \phi(a) \ge \phi(s_N) - \phi(s_0) = \sum_{k=1}^N [\phi(s_k) - \phi(s_{k-1})] \ge \sum_{k=1}^N [\phi(t_k+) - \phi(t_k-)] = \sum_{k=1}^N F(t_k) = \sum_{t\in A'} F(t)$, QED.

$2°$ *$E := \{F \ne 0\}$ is at most countable, and $\{f \ne \phi\} \subset E$.* Proof: By $1°$, E is at most countable.[3] If $F(t) = 0$, then $\phi(t-) = \phi(t) = \phi(t+)$ (since necessarily $\phi(t-) \le \phi(t) \le \phi(t+)$), hence then $f(t) := \phi(t-) = \phi(t)$.

(b) (Sketch) Set $H := \sum_{t\in E} F(t)\chi_{(t,\infty)} \in L^\infty([a,b])$ (since $\sum_{t\in E} F(t) < \infty$, the sum converges uniformly, hence in L^∞); obviously, H is nondecreasing. Set $G := f - H$, so that G is continuous and nondecreasing.[4] Set $g(x) := x + G(x)$ is strictly increasing. As on p. 147 of [R2], we observe that ν is a measure, where $\nu(E) := m(g(E))$. By the Radon-Nikodym theorem, $\nu = \mu_m + \mu_2$, where $h := d\mu_m/dm \in L^1(m)$, $\mu_m \ll m$, $\mu_2 \perp m$. Obviously, $G(x) - G(a) = \mu'([a,x))$ $(x \in [a,b])$, where $\mu' := \mu_2 + \mu_1$, $\mu_1(E) := \int_E (h-1)\,dm$. Set $\mu_d := \sum_{t\in E} F(t)\delta_t$, where $\delta_t(E) := 1$ if $t \in E$, $:= 0$ if $t \notin E$. One can easily verify that $f(x) - f(a) = \mu([a,x))$ $(x \in [a,b])$, where

$$\mu := \mu_1 + \mu_2 + \mu_d. \tag{7}$$

Comment: *these parts are called "absolutely continuous", "singular continuous" and "discrete" (or atomic).* [5]

(c) Set $\tilde f := h - 1 \in L^1(m)$, so that $\mu_1([a,t)) = \int_a^t \tilde f\,dm$ for all $t \in [a,b]$, $\tilde F(t) := f(a) + \int_a^t \tilde f\,dm$ $(t \in [a,b])$. Then for any

$$(f - \tilde F)(t + r) - (f - \tilde F)(t) = (\mu_2 + \mu_d)([t, t + r)) \qquad (t, t + r \in [a,b]) \tag{8}$$

hence $(f - \tilde F)'(t) = 0$ a.e. $[m]$, by Theorem 7.13 of [R2].[6] But $\tilde F' = \tilde f$ everywhere, by Theorem 7.11, hence $f' = \tilde f$ a.e. Set $s := f - \tilde F$ to complete (c).[7]

(d1) We have $\mu \perp m$ iff $\mu_1 = 0$, i.e., if $\int_E f'\,dm = 0$ $\forall E \in \mathfrak{M}$, i.e., iff $f' = 0$ a.e., by Theorem 1.39b.

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[2] Note that being a limit of measurable functions, a derivative of a measurable function is always measurable.

[3] If $A \subset [0,\infty)$, then $S := \sum_{a\in A} A$ means the supremum of sums of finite elements of A. If $S < \infty$, then $A_+ := \{a \in A || a > 0\}$ is at most countable. Proof: Each $A_n := \{a \in A \,|\, a > 1/n\}$ is countable (otherwise we can take kn elements of A_n to have a finite sum $> k$), hence so is $A_+ := \cup_{n\in\mathbb{N}_+} A_n$.

[4] It is not hard to verify these.

[5] By construction, μ_*, μ and ν are defined on all Lebesgue sets, hence on all Borel sets (hence they are Borel measures).

[6] Let $r \to 0+$, and note that $[t, t + r)$ is nicely shrinking. Similarly for $r \to 0-$.

[7] Since μ_2 and μ_d are positive measures, we observe from (8) that s is nondecreasing; note that $s' = f' - \tilde F' = \tilde f - \tilde f = 0$ a.e.

(d2) We have $\mu \ll m$ iff $\mu_2 = 0 = \mu_d$, i.e., iff $\mu = \mu_1$, i.e., iff $f = \tilde{F}$. By Theorem 7.18, $f = \tilde{F}$ iff f is AC.

(e) Let $t \in E^c$ be s.t. $f'(t)$ exists (this is true for a.e. t). Given $\epsilon > 0$, choose $\delta > 0$ for f' and $\epsilon/2$. Let $h > 0$ arbitrary but s.t. $h < \delta$ and $t + h \in (a, b)$. For any $r > 1$ s.t. $t + rh \in [a, b]$, we have

$$\frac{\phi(t+h) - \phi(t)}{h} - f'(t) = \frac{\phi(t+h) - f(t)}{h} - f'(t) \le \frac{f(t+rh) - f(t)}{h} - f'(t)$$
$$= [\frac{f(t+rh) - f(t)}{rh} - f'(t)]r + (r-1)f'(t) < r\epsilon/2 + \epsilon/3 < \epsilon, \tag{9}$$

(once we take $r > 1$ small enough and (e.g.) s.t. $\phi(t + rh) = f(t + rh)$). Similarly, with $r < 1$ we obtain that $\frac{\phi(t+h)-\phi(t)}{h} - f'(t) > -\epsilon$. Thus, the right derivative of ϕ at t equals $f'(t)$, use a similar computation for the left derivative.

Remarks: If f is BV and real, then $f = f_1 - f_2$, where f_1, f_2 are as above, so that similar results apply to any BV functions (treate $\mathrm{Re}\, f$ and $\mathrm{Im}\, f$ separately in the general case); see Exercise 7.13 of [R2] for details. (In the case of BV functions with values in a (Radon-Nikodym) Banach space, one needs a different kind of proof, see, e.g., [Kettler].)

Moreover, similar results also apply with $\mathbb{R}$ in place of $[a, b]$ (reduce the case of $\mathbb{R}$ to the case of $[-n, n]$, $n \in \mathbb{N}$).

Solution to 10.4: Instruction: Try to prove 1° yourself. Then read 2° and try to prove 3°–5° yourself. This way this exercise is still a bit difficult but not very difficult.

1° *Lemma: If $(Q, \mathfrak{M}, \mu)$ is a positive measure space and $f \in L^1(Q; X)$, then, for any $\epsilon > 0$, there is $\delta > 0$ s.t. $E \in \mathfrak{M}$ & $\mu(E) < \delta \Rightarrow \int_E \|f\|_X \, d\mu < \epsilon$.*

Proof: Assume that $f : Q \to [0, \infty)$ (replace f by $\|f\|$ if necessary). By the definition of (Lebesgue) $\int_Q$, there is a simple function $s \ge 0$ s.t. $\int_Q f \, d\mu - \int_Q s \, d\mu < \epsilon/2$. Set $M := \max_Q s < \infty$ and $\delta := \epsilon/3M$. If $\mu(E) < \delta$, then$\int_E f = \int_E (f - s) + \int_E s < \epsilon/2 + \int_E s \le \epsilon/2 + \mu(E)M < \epsilon$, QED.

2° *Lemma: If $K \subset V \subset \mathbb{R}^n$, where K is compact and V open, then there is $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^n; \mathbb{C})$ s.t. $\chi_K \le \phi \le \chi_V$.*

Proof: Choose[8] a compact K' and an open V' s.t. $K \subset (K')^o \subset V' \subset \overline{V'} \subset V$. By Urusohn's Lemma, there is $\widetilde{\phi} \in \mathcal{C}_c(\mathbb{R}^n)$ s.t. $\chi_{K'} \le \widetilde{\phi} \le \chi_{V'}$.

Choose[9] $J_1 \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ s.t. $J_1 \ge 0$, $J_1 = 0$ on B_1^c, and $\int_{B_1} J_1 \, dm = 1$. Set $J_r(s) := r^{-n} J_1(s/r)$ $(r > 0,\ s \in \mathbb{R}^n)$, so that[10] $\int_{\mathbb{R}^n} J_r(s) \, ds = 1$ $(r > 0)$. Then $(J_r * \widetilde{\phi})(t)$ depends on the values of $\widetilde{\phi}$ on $t + B_r$ only (since $J_r = 0$ on B_r^c); one easily verifies that $\chi_K \le \phi \le \chi_V$ when $\phi := J_r * \widetilde{\phi}$ for some small r ($r < \epsilon/3$ in the example in the footnote).

Finally, since $J_r \in \mathcal{C}_c^\infty$ and $\widetilde{\phi} \in L^1$, we have $J_r * \widetilde{\phi} \in \mathcal{C}_c^\infty$, QED.[11]

3° *If $Y \ne \{0\}$ is a separable Banach space, then there is $\{\Lambda_n\}_{n \in \mathbb{N}} \subset Y^*$ s.t. $\|\Lambda_n\| = 1$ $\forall n$ and $\|x\| = \sup_n |\Lambda_n x|$ $\forall x \in Y$.* Proof: Let $\{x_n\}$ be dense in Y. By [R2:5.20], for each $n \in \mathbb{N}$, there is $\Lambda_n \in Y^*$ s.t. $\|\Lambda_n\| = 1$ and $|\Lambda_n x_n| = \|x_n\|$. If $x \in Y \setminus \{0\}$, and $x_{n_k} \to x$ as $k \to \infty$, then $|\Lambda_{n_k} x| \le |\Lambda_{n_k} x_{n_k}| + |\Lambda_{n_k}(x_{n_k} - x)| \le \|x_{n_k}\| + \|x_{n_k} - x\| \to \|x\| + 0$, as $k \to \infty$. Therefore, $\sup_n |\Lambda_n x| \ge \|x\|$. But $|\Lambda_n x| \le \|x\|$ $\forall n$, hence $\sup_n |\Lambda_n x| = \|x\|$, QED.

4° *If $\Lambda g = 0$ a.e. for all $\Lambda \in X^*$, then $g = 0$ a.e.* Proof: W.l.o.g, we assume that $A := g([a, b])$ is separable,[12] hence so is the smallest closed

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[8] Sketch of a proof: set $\epsilon := d(K, V^c) > 0$, $K' := \{q \in V \,|\, d(q, K) \le \epsilon/3\}$, $V := \{q \in V \,|\, d(q, K) < 2\epsilon/3\}$.

[9] E.g., $J_1(t) := \chi_{B_1} \gamma \exp(-(1-t^2)^{-1})$ ("bell function"), where $t^2 := t \cdot t$ (inner product in $\mathbb{R}^n$) and $\gamma > 0$ is s.t. $\int_{B_1} J_1 \, dm = 1$ (here $B_1 := \{t \in \mathbb{R}^n \,|\, \|t\| < 1\}$.

[10] Proof: set $t := s/r$ so that $|ds/dt| = r^n$ (Jacobian $= \det rI = r^n$, as in p. 150 and Theorem 7.26 of [R2]) $\int_{\mathbb{R}^n} J_r(s) \, ds = \int_{B_r} J_r(s) \, ds = \int_{B_1} J_r(rt) r^n \, dt = \int_{B_1} J_1(t) r^{-n} r^n \, dt = 1$.

[11] Proof (sketch): $(d(J_r * \widetilde{\phi})/dt_k)(t) := \lim_{\mathbb{R} \ni h \to 0} \int_{\mathbb{R}^n} h^{-1}[J_r(t + he_k - s) - J_r(t - s)]\widetilde{\phi}(s) \, ds = \lim_{\mathbb{R} \ni h \to 0} \int_{\mathbb{R}^n} (dJ_r/dt_k)(t-s)\widetilde{\phi}(s) \, ds = ((dJ_r/dt_k) * \widetilde{\phi})(t)$, by the Dominated Convergence Theorem (since $|h^{-1}[J_r(t+he_k-s) - J_r(t-s)]\widetilde{\phi}(s)| \le M|\widetilde{\phi}(s)| \in L^1$, where $M := \sup |dJ_r/dt_k|$, by the Mean Value Theorem). by induction, $\partial^\alpha(J_r * \widetilde{\phi}) = (\partial^\alpha J_r) * \widetilde{\phi}$, where ∂^α is any combination of partial derivatives (and $J_r \in \mathcal{C}_c^\infty$ and $\widetilde{\phi} \in L^1$). Moreover, $f * g \in \mathcal{C}_{bu}$ whenever $f \in L^p$ and $g \in L^q$, as one easily proves using the Hölder Inequality.

[12] For a general measurable function f, $A' := f(N^c)$ is separable for some null set N, by Theorem 1.1.1 of [ABHN]. Set $g := f\chi_{N^c}$ and apply 4° to observe that $g = 0$ a.e., so that $f = 0$ a.e. too (if $g = 0$ on N'^c, also $N' \cup N$ is a null set and $f = g = 0$ on

subspace $Y \subset X$ containing A.[13] Obviously, $g \in L^1([a,b];Y)$.[14]

Let $\{\Lambda_n\} \subset Y^*$ be as in 3°, so that $\Lambda_n x = 0\ \forall n \in \mathbb{N} \implies x = 0$ (for all $x \in Y$). If $\Lambda_n g = 0$ a.e. $\forall n$, then $m(N) = 0$, where $N := \{g \neq 0\} = \cup_n \{\Lambda_n g \neq 0\}$ (because $m(\{\Lambda_n g \neq 0\}) = 0\ \forall n$). But any $\Lambda_n \in Y^*$ can be extended to an operator in X^*, by the Hahn–Banach Theorem, hence 4° holds.

5° *(The actual exercise)* Let $g \in L^1([a,b];X)$ be s.t. not $g = 0$ a.e. It suffices to show that then there is $\phi \in \mathcal{C}_c^\infty([a,b];\mathbb{C})$ s.t. $\int_a^b g\phi\, dm \neq 0$.

By 4°, there is $\Lambda \in X^*$ s.t. not $\Lambda g = 0$ a.e. Obviously, $\Lambda g \in L^1([a,b];\mathbb{C})$.[15] (We have now reduced the problem to the scalar case.) By Theorem 1.39b of [R2], $z := \int_E \Lambda g\, dm \neq 0$ for some $E \in \mathfrak{M}$. Set $r := |z| > 0$. Now we have solved the (scalar) problem for "$\phi := \chi_E$", it remains to take $\phi \in \mathcal{C}_c^\infty$ close to χ_E.

By 1°, there is $\delta > 0$ s.t. $\int_A |\Lambda g|\, dm < r/3$ whenever $m(A) < \delta$. Since m is regular [R2:2.20], there are a compact K and open V s.t. $K \subset E \subset V \subset \mathbb{R}^n$ and $m(V \setminus K) < \delta$. Let ϕ be as in 2°. Then (set $f := \Lambda g \in L^1$; for the last "=" recall that $z = \int_E f$)

$$\int_{\mathbb{R}^n} \phi f - z = \int_V \phi f - z = \int_K 1 f + \int_{V\setminus K} \phi f - z \tag{10}$$

$$= \int_E f - \int_{E\setminus K} f + \int_{V\setminus K} \phi f - z = -\int_{E\setminus K} f + \int_{V\setminus K} \phi f \tag{11}$$

has norm less than $|\int_{E\setminus K} f| + |\int_{V\setminus K} f| < 2r/3$, hence $|\int_{\mathbb{R}^n} \phi f| > |z| - 2r/3 = r/3 > 0$. But $\int \phi f = \int \phi \Lambda g = \int \Lambda \phi g = \Lambda \int \phi g$ (since $\int \Lambda = \Lambda \int$, by Proposition 1.1.6 of [ABHN]), hence $\int_{\mathbb{R}^n} \phi g\, dm \neq 0$, QED.

$(N' \cup N)^c$).

[13] Proof: Let Y_0 be the set of finite linear combinations of vectors in A and set $Y := \bar{Y}_0$. If Q is dense in A and at most countable, then the set of finite linear combinations $\sum_k \alpha_k q_k$ with $q_k \in Q$, $\alpha_k \in \mathbb{Q} + i\mathbb{Q}$ is at most countable and dense in Y.

[14] **Comment:** This way problems on strongly measurable functions can be reduced to separable Banach spaces. (To be exact, if $g : [a,b] \to X$, then $\tilde{g} : [a,b] \to Y$ defined by $\tilde{g}(t) := g(t)\ \forall t \in [a,b]$ is a different function, but one usually uses the same letter for each of them since the difference is insignificant in most operations.)

[15] If $g_n \to g$ in L^1 and g_n is simple and L^1 for all n, then the same holds for Tg_n and $Tg_n \to Tg$ in L^1 for any $T \in \mathcal{L}(X,Z)$, Z Banach.