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Mat-1.159 Harmonic Analysis

Mid-term examination 1/1, October 27, 2003 at 12–15 (hall N)

Notes: I gave 6 p. for Duoandikoetxea's answers even though they contained very little details (in particular, I did not require anything on measurability), except for parts that clearly were not "obvious" but "left for the reader" (e.g., Exercise 3). Some of you provided more details (here usually in parenthesis), usually incorrectly, but that did not lower your score ("no penalty for trying to be more exact than the others"), because probably some of the others did not understand the details either (however, if you did it very well and the details were important and nontrivial, it might have compensated some minor errors).

Exercise 1. Show that the Fourier transform is a continuous map from $\mathcal{S}$ to $\mathcal{S}$.

Exercise 2. Show that the Hilbert transform on $\mathcal{S}$ satisfies

$$m(\{x \in \mathbb{R} \mid |Hf(x)| > \lambda\}) \leq \frac{C}{\lambda}\|f\|_{\mathrm{L}^1}$$

for all $f \in \mathcal{S}$ and all $\lambda > 0$.

Exercise 3. Let $\sigma(x,\xi) \in S^0$, and let σ have a compact support in x. Show that then $T_\sigma : \mathrm{L}^2 \to \mathrm{L}^2$ continuously.

Exercise 4. Let Ω be an integrable function on S^{n-1} with zero average. Show that the Fourier transform of P.V.$\frac{\Omega(x')}{|x|^n}$ is given by

$$m(\xi) = \int_{S^{n-1}} \Omega(u) \left[\log\left(\frac{1}{|u\cdot\xi'|}\right) - i\frac{\pi}{2}\operatorname{sgn}(u\cdot\xi')\right] d\sigma(u),$$

where $\xi' := \xi/|\xi|$.

Solution to 1: [pp. 13–14] We have $\xi^\alpha D^\beta \widehat{f}(\xi) = C(D^\alpha x^\beta f)\widehat{\ }(\xi)$, hence $\|\xi^\alpha D^\beta \widehat{f}\|_\infty \leq |C|\|D^\alpha x^\beta f\|_1$, which in turn can be dominated by finite linear combination of seminorms of f.[1] ((+ The Fourier transform is obviously linear on $\mathcal{S}$; this and) the above boundedness imply that it is continuous $\mathcal{S} \to \mathcal{S}$.[2]

Alternative proof: If $\{f_n\} \subset \mathcal{S}$ is such that $f_n \to 0$ (i.e., all seminorms converge to zero), then $\widehat{f_n} \to 0$, by the above. This and linearity imply that the Fourier transform is continuous.[3]

Solution to 2: See pp. 52–53 of [Duo]. *Scoring:* "$g \leq 2\lambda$" not mentioned (and its use not explained): -1p. c_j-trick missing: -1p.

Remarks: These are the reasons for the following (in)equalities:
p. 52, line -3: $|\{|F| > \alpha\} \leq \alpha^{-p}\int |F|^p$ because $|\{|F| > \alpha\} = \alpha^{-p}\int_{|F|>\alpha}\alpha^p \leq \alpha^{-p}\int_{|F|>\alpha}|F|^p \leq \alpha^{-p}\int |F|^p$.
p. 52, line -1: $g \leq 2\lambda$. p. 53, line 1: $\int b = 0$

[1] Set $g := D^\alpha x^\beta f$. Now $(1+|x|^2)^n = (1+(\sum_{k=1}^n x_k^2)^n =: P(x)$ is a polynomial, hence $\|Pg\|_\infty \leq \sum_{j=1}^l p_j(g) = \sum_{j=1}^l q_j(f)$, where $q_j(f) = x^{\beta_j}D^{\alpha_j} f\ \forall j$. But $M := \|1/P\|_1 < \infty$, hence $\|g\|_1 \leq \|Pg\|_\infty \|P^{-1}\|_1 \leq M\sum_{j=1}^l q_j(f)$, QED. +

[2] (1) Recall that *if a map is linear*, then it is continuous iff it is bounded. In addition to maps between normed spaces, this also holds for maps between metrizable locally convex TVSs, such as $\mathcal{S}$. (2) However, it is easier to remember that for any metrizable spaces X, Y and any $F : X \to Y$, the map F is continuous iff $x_n \to x \Rightarrow F(x_n) \to F(x)$ [Rudin:PMA] (if F is linear, we may obviously take $x = 0$). (3) A third characterization (not as simple as (2)): If the topology in vector spaces X and Y is defined through a collection of seminorms, then a linear function $T : X \to Y$ is continuous if for each such seminorm p on Y, there is a collection of such seminorms $q_1, ..., q_l$ on X and $M < \infty$ s.t. $p(Tx) \leq M\sum_{j=1}^l q_j(x)\ \forall x \in X$ [Rudin:FA].

[3] This is what you should remember (also the converse holds)! You do not need to know details on the topology nor on the metric (which is not the supremum of the seminorms (which might be ∞)), see [Rudin:FA], Th 1.37–Rem 1.38 (and Chapter 7) for details (which you do not need to now).

Solution to 3: [pp. 113–114]. We assume that the support is compact in x *independent of* ξ, i.e., that for some compact $E \subset \mathbb{R}^n$ we have $\sigma(x,\xi) = 0$ for all $x \notin E$, $\xi \in \mathbb{R}^n$ (it seems that this is what the exercise stands for, although the wording is ambiguous).[4] Since T_a is obviously linear, we only hat to show that $\|T_a f\|_2 \leq M\|f\|_2\ \forall f \in \mathcal{S}(\mathbb{R}^n)$ for some $M < \infty$ (i.e., that T_a can be extended to an element of $\mathcal{B}(\mathrm{L}^2(\mathbb{R}^n)))$.[5]

(Def. of T_a and S^0: `(1 p.)`.) Set $\hat{\sigma}(q,\xi) := \int_{\mathbb{R}^n} \sigma(x,\xi)\mathrm{e}^{-2\pi i x\cdot q}\, dx$.[6] We have $(2\pi i q)^\alpha \hat{\sigma}(q,\xi) = \mathcal{F}(D_x^\alpha \sigma)$ `(1 p.)` (here $\mathcal{F}$ denotes the Fourier transform), hence $\|D_x^\alpha \sigma\|_1 \leq C_{0,\alpha} m(E)$ (we have $\|D_x^\alpha \sigma\|_\infty \leq C_{0,\alpha}\ \forall \alpha$, because $\sigma \in S^0$), hence $|q^\alpha \hat{\sigma}(q,\xi)| \leq C_\alpha\ \forall q,\xi \in \mathbb{R}^n$ for some C_α, $\forall \alpha \in \mathbb{N}^n$. `(1 p.)` One easily deduces from this that[7]

$$\sup_{q,\xi\in\mathbb{R}^n} |(1+|q|)^{+n+1}\hat{\sigma}(q,\xi)| =: A < \infty. \tag{1}$$

Now $(T_\sigma f) := \int \sigma(x,\xi)\widehat{f}(\xi)\mathrm{e}^{2\pi i x\cdot\xi}\, d\xi = \int\int \hat{\sigma}(q,\xi)e^{2\pi i q\cdot x}\widehat{f}(\xi)\mathrm{e}^{2\pi i x\cdot\xi}\, d\xi\, dq = \int (T^q f)(x)\, dq$, where $(T^q f)(x) = e^{2\pi i x\cdot q}(T_{\hat{\sigma}(q,\xi)} f)(x)$. But (1) implies that $\|T^q\|_{\mathrm{L}^2\to\mathrm{L}^2} \leq A(1+|q|)^{-n-1}$,[8] hence $\|T_\sigma f\|_2 = \|\int_{\mathbb{R}^n} T^q f\, dq\|_2 \leq \int_{\mathbb{R}^n} \|T^q f\|_2\, dq \leq \|f\|_2 \int_{\mathbb{R}^n} A(1+|q|)^{-n-1}\, dq = M\|f\|_2$.[9] `(3 p.)`

Scoring: Everyone failed in footnote 7, so I did not charge on that! Definitions: 1p.

Solution to 4: [p. 72–73] (better on pp. 267–269 of [Grafakos]) Notes: 1. "-1" OK since $\int \Omega = 0$. 2. "By the dominated convergence" is not obvious, but it is a (difficult) basic math course level exercise to show that the inner integrals define bounded ($|\int| \leq M\ \forall u$) functions (this would not be true if we would take the absolute value within the inner integrals); this together with the fact that $\Omega \in \mathrm{L}^1$ shows that $|\Omega \int| \leq M|\Omega| \in \mathrm{L}^1$, and we have our majorant (and Dom. Conv. Th. is hence applicable). 3. On line -3, p. 72 the last term appeared because we replaced $2\pi|u\cdot\xi|$ by 1 in two of the limits of the first two integrals.

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[4] Naturally, one should also make some measurability assumption on σ, so actually we only prove that the claim is true when σ is measurable enough (e.g., continuous).

[5] We do not have to know whether the direct definition is applicable to all $f \in \mathrm{L}^2$; by T_a one denotes the unique continuous extension of $T_a|_{\mathcal{S}}$, not the direct definition.

[6] As shown below, $\|\sigma\|_1 \leq C_{0,0}m(E) < \infty$, hence $\hat{\sigma}$ exists (and is $\mathcal{C}_0$ w.r.t. q).

[7] $(1+|q|)^{n+1} \leq (2+|q|^2)^{n+1} = (2+\sum_{k=1}^n q_k^2)^{n+1} =: P(q)$ which is a finite linear combination of terms q^α. Thus, $|P(q)\hat{\sigma}(q,\xi)| \leq A\ \forall q,\xi$.

[8] Recall that $\widehat{Sf} := F\widehat{f} \Rightarrow \|S\|_{\mathcal{B}(\mathrm{L}^2)} = \|F\|_\infty$ (by multiplier theory).

[9] Use the Minkowsky integral inequality ($\|\int F\|_p \leq \int \|F\|_p\ \forall p \in [1,\infty]$, for $p=2$) for the first inequality. Note that $\|\cdot\|_2$ refers to x, whereas $\int_{\mathbb{R}^n}$ is w.r.t. q.