Mat-1.159 Harmoninen analyysi: 9th exercise, Wed 2003-11-12, U345 (2 pages)

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By (7.6), we have $w \in A_p$ iff $w_Q (w^{-1/(p-1)})_Q^{p-1}$ is bounded, i.e., iff $w_Q^{1/(p-1)}(w^{-1/(p-1)})_Q$ is bounded ($1 < p < \infty$). By (7.16), we have $w \in A_\infty$ iff $\log w \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $w_Q \exp\left(-(\log f)_Q\right)$ is bounded.

Exercise 9.1 ([p. 139] *easy, quick*):
Show that if $w \in A_p$, then $w \in A_q$ for some $q < p$.[1]

Exercise 9.2 ([p. 125] *"⇒" easy, "⇐" average*):
Prove Corollary 6.13 of [Duo].[2]

Exercise 9.3 ([p. 135] *average*):
Let $f \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^n)$ be real-valued.[3] Let $p \in (1, \infty)$. Show that $\mathrm{e}^f \in A_p$ iff

$$C_1 := \sup_Q \frac{1}{|Q|} \int_Q \mathrm{e}^{f - f_Q}\, dm < \infty, \tag{1}$$

$$C_2 := \sup_Q \frac{1}{|Q|} \int_Q \mathrm{e}^{-(f - f_Q)/(p-1)}\, dm < \infty. \tag{2}$$

Hint: observe that if $\mathrm{e}^f \in A_p$, then $\frac{1}{|Q|}\int_Q \mathrm{e}^{f-f_Q}\, dm \le (\mathrm{e}^{-f/(p-1)})_Q^{p-1} (\mathrm{e}^f)_Q$.[4]

Exercise 9.4 ([p. 150] *easy, quick*):
Use Exercise 9.3 to show that $w \in A_p \Leftrightarrow w, w^{-1/(p-1)} \in A_\infty$ when $1 < p < \infty$.[5]

Exercise 9.5 ([p. 134] *quite easy*):
Show that $u(x) := \begin{cases} f(x) := \log(\frac{1}{|x|}), & \text{when } |x| < 1/e; \\ f(x) := 1, & \text{when } |x| \ge 1/e, \end{cases}$ satisfies $u \in A_1$.[6]

Solution to 9.1: This is shown in the proof of Corollary 7.6(1): as explained there, we get $(|Q|^{-1}\int_Q w^{1-q'})^{1/(1+\epsilon)} \le \tilde{C}|Q|^{-1}\int_Q w^{1-p'}$. Since $(p-1)/(1+\epsilon) = (p'-1)^{-1}/(1+\epsilon) = (q'-1)^{-1} = q-1$ (because $(p-1)(p'-1) = 1 = (q-1)(q'-1)$), by raising both sides to the power of $p-1$, we obtain that $(|Q|^{-1}\int_Q w^{1-q'})^{q-1)} \le [\tilde{C}|Q|^{-1}\int_Q w^{1-p'}]^{p-1} \le \tilde{C}^{p-1}C/w_Q$ (by (7.6)), hence $w \in A_q$.

Solution to 9.2: By the proof of Corollary 6.12, we have $|Q|^{-1}\int_Q |f(x) - f_Q|^p\, dx \le M_p \|f\|_*^p$, where $M_p := pC_1 C_2^{-p}\Gamma(p) = p! C_1 C_2^{-p}$ (for $p \in \mathbb{N}+1$). Therefore,[7] $|Q|^{-1}\int_Q \mathrm{e}^{\lambda|f(x)-f_Q|}\, dx = |Q|^{-1}\int_Q \sum_{n\in\mathbb{N}} \frac{(\lambda|f(x)-f_Q|)^n}{n!}\, dx \le$

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[1] Hint: see the proof of Cor7.6(1).

[2] Hint: $\mathrm{e}^t = \sum_{t\in\mathbb{N}} t^n/n!$; see p. 125.

[3] By p. 134 that any A_p weight is > 0 a.e., so this does not reduce generality.

[4] Hint: use Jensen's inequality and (7.6).

[5] Recall from p. 150 that $v \in A_\infty$ iff $\sup_Q w_Q \mathrm{e}^{(\log w^{-1})_Q} < \infty$, equivalently, iff $\exists \delta > 0$, $C < \infty$ s.t. for all cubes Q and all measurable $S \subset Q$ we have $v(S)/w(Q) \le C(|S|/|Q|)^\delta$.

[6] Hint: study balls $B(a, R)$ instead of Q's. If $|a| < 3R$, then $B(a, R) \subset B(0, 5R)$ etc.

[7] Recall from Theorem 1.27 of [Rudin:RCA] that $\int \sum f_n = \sum \int f_n$ for nonnegative measurable f_n.

$\sum_{n\in\mathbb{N}} \frac{\lambda^n}{n!} M_n \|f\|_*^n \le C_1 \sum_{n\in\mathbb{N}} \left(\frac{\lambda\|f\|_*}{C_2}\right)^n < \infty$, if $\lambda < C_2/\|f\|_*$ (or $f = 0$ (constant a.e.)), QED.

Solution to 9.3: 1.1° Set $\phi(t) := \mathrm{e}^{-t/(p-1)}$, so that $\phi'' > 0$, hence ϕ is convex. By Jensen's Inequality,[8] it follows that $|Q|^{-1}\int_Q \phi(f)\,dm \ge \phi(|Q|^{-1}\int_Q f\,dm) = \phi(f_Q)$, hence $C_{f,Q} := [|Q|^{-1}\int_Q \mathrm{e}^{-f(x)/(p-1)}\,dx]^{p-1} \ge \phi(f_Q)^{p-1} = \mathrm{e}^{-f_Q}$. Thus, $|Q|^{-1}\int_Q \mathrm{e}^{f-f_Q}\,dm = \mathrm{e}^{-f_Q}|Q|^{-1}\int_Q \mathrm{e}^f\,dm \le C_{f,Q}(\mathrm{e}^f)_Q$. But, by (7.6), $C_{f,Q}(\mathrm{e}^f)_Q$ is bounded iff $\mathrm{e}^f \in A_p$ (note that $1-p' = -1/(p-1)$).

1.2° Set $F := \mathrm{e}^f$. As above (set $\phi(t) := \mathrm{e}^t$), we have $F_Q \ge \mathrm{e}^{f_Q}$ hence $|Q|^{-1}\int_Q \mathrm{e}^{-(f-f_Q)/(p-1)}\,dm = \mathrm{e}^{f_Q/(p-1)}|Q|^{-1}\int_Q \mathrm{e}^{-f/(p-1)}\,dm = (\mathrm{e}^{f_Q})^{1/(p-1)}(F^{-1/(p-1)})_Q \le F_Q^{1/(p-1)}(F^{-1/(p-1)})_Q$, which is bounded iff $F \in A_p$.

2° "If": Assume that $C_1, C_2 < \infty$ (replace C_1 by 1 below if $C_1 = 0$). Since $C_1 \ge \mathrm{e}^{-f_Q}F_Q$, we have $\mathrm{e}^{f_Q} \ge F_Q/C_1$, hence $\infty > C_2^{p-1} \ge (\mathrm{e}^{f_Q})(F^{-1/(p-1)})_Q^{p-1} \ge C_1^{-1}F_Q(F^{-1/(p-1)})_Q^{p-1}$, hence F is A_p.

Solution to 9.4: We assume that $w > 0$ a.e. (since otherwise w cannot be a weight); it follows that $f := \log w : \mathbb{R}^n \to \mathbb{R}$ (a.e.).

1° "$\Rightarrow$": Since $A_r \subset A_\infty$ $\forall r \ge 1$ (Cor7.6), and $w \in A_p \Leftrightarrow w^{-1/(p-1)} \in A_{p'}$ (Prop7.2(2)), $w \in A_p$ implies that $w, w^{-1/(p-1)} \in A_\infty$.

2° "$\Leftarrow$": Assume that $w, w^{-1/(p-1)} \in A_\infty$. By (7.16) [Duo, p. 150], this implies that $f := \log w \in \mathrm{L}^1_{\mathrm{loc}}$ and that there are $C_1, C_2 < \infty$ s.t. $w_Q \le C_1\mathrm{e}^{f_Q}$ and $|Q|^{-1}\int_Q \mathrm{e}^{-f/(p-1)} \le C_2\exp\{|Q|^{-1}\int_Q -f/(p-1)\} = C_2\exp\{-f_Q/(p-1)\}$ for all cubes $Q \subset \mathbb{R}^n$. By Exercise 9.3, this implies that $w \in A_p$.

Solution to 9.5: By p. 134 of [Duo], it suffices to show that there is $C < \infty$ s.t. for all $a \in \mathbb{R}^n$, $R > 0$ and $x \in B(a,R)$, we have $u_B \le Cu(x)$.

1° *Case* $|a| > 3R$ *and* $|a|+R < 1/e$: We have $u(x) = -\log|x|$ and $|x| < 4a/3 \le 2(|a|-R)$ for $x \in B(a,R)$, hence $u(y) \le -\log(|a|-R) \le -\log(|x|/2) = \log 2 + u(x) \le 2u(x)$ $\forall x,y \in B(a,R)$, hence $u_B \le 2u(x)$ $\forall x \in B(a,R)$.

2° *Case* $|a| > 3R$ *and* $|a|+R \ge 1/e$: As in 1°.

3° *Case* $|a| \le 3R$ *and* $5R < 1/e$: Now $1/|B(a,R)| = cR^{-n}$ and $B(a,R) \subset B(0,5R)$, hence $u_B \le cR^{-n}\int_{B(0,5R)} u = c'R^{-1}\int_0^{5R}\log\frac{1}{r}\,dr = c'R^{-1}\|_0^{5R} r(1-\log r) = 5c'(1-\log 5R) \le 5c'(1+u(x)) \le 10c'u(x)$ $\forall x \in B(a,R)$.

4° *Case* $|a| \le 3R$ *and* $5R \ge 1/e$: As in 3°, $u_B \le c'R^{-1}(\int_0^{1/e} + \int_{1/e}^{5R}) = c'R^{-1}(2/e + 5R - 1/e) = c'(5 + 1/eR) \le c'(5+5) \le 10c'u(x)$ $\forall x$.

[8] Note that the measure $\mu := m/|Q|$ is a probability measure on Q, i.e., $\mu \ge 0$ and $\mu(Q)(Q) = m(Q)/|Q| = 1$.