Mat-1.159 Harmoninen analyysi: 8th exercise, Wed 2003-11-5, U345 (2 pages)

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Exercise 8.1 ([p. 120] *easy, quite quick*):
Compute the details of Example 6.7 (p. 120), i.e., that $(H\,\mathrm{sgn})(x) = \frac{2}{\pi}\log|x|$.[1]

Exercise 8.2 ([p. 120] *easy, quick*):
Show that the integral term in (6.4) is in BMO (for any $f \in \mathrm{L}^\infty(\mathbb{R}^n)$).[2]

Exercise 8.3 ([p. 127] *easy*):
Show that $\delta_0 - \delta_a \in \mathrm{H}^p(\mathbb{R})$ when $1/2 < p < 1$. Use the fact that $f \in \mathrm{H}^p \Leftrightarrow M_P f \in \mathrm{L}^p$, where $(M_P f)(x) := \sup_{t>0} |(f * P_t)(x)|$. (This is Definition 6.4.2 on p. 449 of [Grafakos]; it is equivalent to p. 127 of [Duo] by (6.4.11) of [Grafakos]).[3]

Exercise 8.4 ([p. 136] *quite easy and short*):
Show that $p < q \;\Rightarrow\; A_p \subset A_q$.[4]

Exercise 8.5 ([p. 125] *very easy, quick*):
Prove Corollary 6.14 of [Duo], i.e., assuming that $f : \mathbb{R}^n \to \mathbb{C}$ is measurable and

$$m\{x \in Q \bigm| |f(x) - f_Q| > \lambda\} \le C_1 \mathrm{e}^{-C_2\lambda/K}|Q| \;\forall Q, \lambda, \qquad (1)$$

show that $f \in \mathrm{BMO}$.[5]

Solution to 8.1: Now $\pi K(x,y) = 1/(x-y)$. Let $Q = [-a/2, a/2]$, $Q^* = [-a,a]$, $f_1 := \chi_{Q^*}\,\mathrm{sgn}$, $f_2 := \mathrm{sgn}(\cdot) - f_1$. Then, by (6.4), for a.e. $x \in (0, a/2)$, we have $(\pi H\,\mathrm{sgn})(x) = \pi H f_1(x) + \pi H f_2(x)$.

1° $\pi H f_1(x) :=$p.v.$\int_{-a}^{a} \frac{\mathrm{sgn}\, y}{x-y}\,dy = \lim_{\epsilon\to 0}(-\int_{-a}^{0} + \int_0^{x-\epsilon} + \int_{x+\epsilon}^{a})\frac{1}{x-y}\,dy = \lim_{\epsilon\to 0} |_{0,0,x+\epsilon}^{-a,x-\epsilon,a} - \log|x-y| = -\log\frac{a^2-x^2}{x^2}$.

2° $\pi H f_2(x) := \int_{\mathbb{R}}[(x-y)^{-1}+y^{-1}]f_2(y)\,dy = \lim_{N\to+\infty}(\int_a^N - \int_{-N}^{-a})[(x-y)^{-1} + y^{-1}]\,dy = \lim_{N\to+\infty}(|_{a,-a}^{N,-N})[-\log|x-y| + \log|y|] = \lim_{N\to+\infty}(|_{a,-a}^{N,-N})\log\frac{1}{|1-x/y|} = \lim_{N\to+\infty}\log|\frac{(1-x/a)(1+x/a)}{(1-x/N)(1+x/N)}| = \log|1-(x/a)^2|$.

3° Thus, $\pi H(f_1 + f_2) = \log(x^2/a^2) = 2\log x - 2\log a$. Since $K(-x,-y) = -K(x,y)$, we obviously have $(\pi H\,\mathrm{sgn})(x) = 2\log|x|$+constants for a.e. $x \in (-a/2, 0)$ too (by a change of variable in $\int$); since $a > 0$ was arbitrary, this is true for a.e. $x \in \mathbb{R}$.

Solution to 8.2: This goes exactly as in the proof of Theorem 6.6. (Note: the second "$\le$" uses Jensen's Inequality, after the third one the C should be C', and the last "$\le$" uses (5.11).)

Solution to 8.3: Set $g(x,t) := ((\delta_0 - \delta_a) * P_t)(x) = P_t(x) - P_t(x-a)$, so that $\pi g(x,t) = at(a-2x)/(t^2+x^2)(t^2+(x-a)^2)$. Set $F := \pi M_p(\delta_0 - \delta_a) = \sup_{t>0}|\pi g(\cdot,t)|$. We have to show that

[1] Hint: assume first that $x \in (0, a/2)$.

[2] Hint: see the proof of Theorem 6.6.

[3] Hint: show that $F(x) := (M_p(\delta_0 - \delta_a))(x) \le C/|x|$ for $x \in (-a/2, a/2)$ (so that $F \in \mathrm{L}^p(-a/2, a/2)$) and treate similarly the other parts of $\mathbb{R}$.

[4] Hint: compute $\|w^{-1}\|_{\mathrm{L}^{p'-1}(Q, \frac{dm}{|Q|})}$.

[5] Hint: adapt the proof of Cor 6.12.

$F \in \mathrm{L}^p(\mathbb{R})$. W.l.o.g., assume that $a > 0$.

Since $t/(t^2 + x^2) \leq 1/2|x|$, we have $F(x) \leq C/|x|$ when $x \in (-a/2, a/2)$ (similarly, $F(x) \leq \tilde{C}/|x-a|$ when $x \in (a/2, 3a/2)$) and $F(x) \leq (C'/|x|)(a-2x)/|x-a|^2$ when $x > 3a/2$ or $x < -a/2$ ($\leq C''$ when, in addition, $|x| \leq 2a$), and $F(x) \leq C'''/|x|^2$ when $|x| > 2a$. We conclude that $F \in \mathrm{L}^p$.

Solution to 8.4: 1° For $p = 1$, the proof is given on p. 136 (set $a_Q := \sup_{x \in Q} 1/w(x)$ and note that $(|Q|^{-1} \int_Q w^{1-q'})^{q-1} \leq (a_Q^{q'-1})^{q-1} = a_Q$, because $1/(p'-1) = p-1$. For $p > 1$, the claim obviously follows from 2° and 3°, because $p' - 1 \geq q' - 1$.

2° *We have $w \in A_p$ iff there is $C < \infty$ s.t. $\|w^{-1}\|_{\mathrm{L}^{p'-1}(Q, \frac{dm}{|Q|})} \leq C/w_Q$ $\forall Q$:* Recall that $w_Q := \int_Q w \, d\frac{m}{|Q|}$. Indeed, $\|w^{-1}\| = (|Q|^{-1} \int_Q w^{1-p'})^{1/(p'-1)}$, and $1/(p'-1) = p-1$, hence the above condition equals (7.6) [p. 135].

3° *If $\mu(X) = 1$, then $0 < p \leq q \leq \infty \Rightarrow \|g\|_p \leq \|g\|_q$ $\forall g$:* Assume that $q < \infty$ (otherwise this is obvious). Set $r := q/p$. Then $\|g\|_p^p = \int g^p \leq \|g^p\|_r \|1\|_{r'} = (\int (g^p)^r)^{1/r} = \|g\|_q^p$.

An alternative proof: By Theorem 7.1, M is weak (p,p) w.r.t. $w \, dm$. But obviously $M \in \mathcal{B}(\mathrm{L}^\infty(\mathbb{R}^n, w \, dm))$ (see p. 137), hence M is strong (hence weak) (q,q) w.r.t. $w \, dm$ for all $q \in (p, \infty)$, by the Marcinkiewicz Interpolation Theorem [p. 29], hence $w \in A_q$, by Theorem 7.1.

Solution to 8.5: By p. 28, $\int_Q |f - f_Q| \, dm = \int_0^\infty m\{x \in Q \bigm| |f(x) - f_Q| > \lambda\} \, d\lambda \leq \int_0^\infty C_1 \mathrm{e}^{-C_2 \lambda / K} |Q| \, d\lambda = |Q| C_1 K / C_2$ for any cube $Q \subset \mathbb{R}^n$, hence $\|f\|_* \leq C_1 K / C_2 < \infty$, QED.