Mat-1.159 Harmoninen analyysi: 6th exercise, Wed 2003-10-22, U345 (3 pages)

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1st mid-term examination (1. välikoe) ma 27.10.2003 12-15 (Oct 27): You only need to read the following results and proofs (and understand them and possess basic skills in real and functional analysis): Th 1.13, 2.2, 2.4, 3.2, Prop. 3.6, 4.3, Th 4.4, Prop 4.6, Th 5.1, the following theorem: Let $\sigma(x,\xi) \in S^0$, and let σ have a compact support in x. Then $T_\sigma : \mathrm{L}^2 \to \mathrm{L}^2$ continuously (cf. p. 113 of [Duoandikuotxea]). **2nd mid-term examination:** None. Instead, after the course, ask Stigu for a home exercise and return it to him.

Exercise 6.1 ([p. 101] *a bit difficult*):
Show that if two Calderón–Zygmund operators are associated with the same kernel, then their difference is a pointwise multiplication operator.[1]

Exercise 6.2 ([p. 98] *average*):
Show that the distribution "$|x|^{-z}(\phi)$" (the fourth formula row on p. 98) is homogeneous of degree $-z$ when $\operatorname{Re} z < n+1$, $z \neq n$.[2]

Exercise 6.3 ([p. 99] *very easy, straight-forward*):
Let A be a Lipschitz function on $\mathbb{R}$. Let

$$K(x,y) = \frac{1}{x-y+i[A(x)-A(y)]} \quad (x,y \in \mathbb{R}^2 \setminus \Delta) \tag{1}$$

Show that $K(x,y)$ is a standard kernel.

Exercise 6.4 ([p. 101] *quite easy*):
Prove Proposition 5.12.[3]

Exercise 6.5 ([p. 98]):
On p. 98 Duo says that the Fourier transform of $|x|^{-z}$, $0 < \operatorname{Re} z < n$, was already computed on p. 71. The truth is that on p. 71 only the case z real was considered. Try to extend the reasoning on p. 71 to a complex. Can you do it without using the fact that two analytic functions that coincide on a segment coincide everywhere?

Solution to 6.1: 1° Obviously, standard kernels form a vector space and so do CZO:s (pp. 99-100), and if $T_1, T_2 \in CZO$, then $T := T_1 - T_2 \in CZO$ with $K = K_1 - K_2$. Thus, now $K = 0$ and we have to show that $Tf = af\ \forall f \in \mathrm{L}^2$ for some function a (it follows that a is measurable and $\|a\|_\infty = \|T\|$).[4]

Sketch of the proof: 3° Find (define) a candidate for a, 5° show that $T\chi_A = a\chi_A$ (a.e.) when $m(A) < \infty$, 6° $Tf = af$ (a.e.) when f is simple, by linearity, hence for all $f \in \mathrm{L}^2$, by density and continuity, QED.[5]

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[1] Hint: see the Sketch of the proof below.

[2] Hint: read 0°; the rest is straight-forward.

[3] Hint: mimic the proof of Proposition 5.7; this is even simpler.

[4] *Measurability:* Let $V \subset \mathbb{C}$ be open; we have to show that $A := a^{-1}[V]$ is measurable. Now $a\chi_{B_R} = T\chi_{B_R} \in \mathrm{L}^2\ \forall R > 0$, hence $B_R \cap A = B_R \cap (a\chi_{B_R})^{-1}[V] \in \mathfrak{M}\ \forall R > 0$, hence $A = \cup_{k\in\mathbb{N}}(A \cap B_k) \in \mathbb{R}$. Since V was arbitrary, a is measurable.

$\|a\|_\infty = \|T\|_{\mathcal{B}(\mathrm{L}^2(\mathbb{R}^n))}$: Proof: Obviously, $\|T\| \le \|a\|_\infty$. If $E := \{x \mid |a(x)| > \|T\|\}$ has a positive measure, then $0 < m(E \cap B_R) < \infty$ for some $R > 0$, and $\|af\|_2 > \|T\|\|f\|_2$, where $f := \chi_{E\cap B_R} \in \mathrm{L}^2$, a contradiction.

[5] Note that we should add "a.e." to certain places below. To avoid problems with measurability and measures, we use countable unions etc. An alternative method for avoiding (some of these) problems is to use the Lebesgue values of a function, as in Exercise 7.23 of [R2] — this is a standard (though not linear)

2° By (2) of Definition 5.11, p. 100, we have $Tf(x) = \int 0 = 0$ for $x \notin \operatorname{supp} f$, $f \in \mathrm{L}^2_{\mathrm{c}}$ ("c" for compact support).

3° *How to find a candidate for a?* We should have $T\chi_A = a\chi_A$ for any measurable A s.t. $m(A) < \infty$ (i.e., s.t. $\chi_A \in \mathrm{L}^2$), hence $T\chi_A = a$ on A. Thus, we define our candidate as follows: set $E_k := B_{k+1} \setminus \overline{B_k}$ and define a on E_k by $T\chi_{E_k}$ (i.e. $a := \sum_k T\chi_{E_k}$; recall from 2° that $T\chi_{E_k} = 0$ outside $\bar{E}_k$).

4° *We have $T\chi_E = 0$ a.e. on E^c if $m(E) < \infty$:* By the regularity of m, there are compact $K_j \subset E$ s.t. $m(E \setminus K_j) < 1/j$, i.e., $m(E \setminus E_0) = 0$, where $E_0 := \cup_j K_j$. By 2°, $T\chi_{K_j} = 0$ on E^c. But $\chi_{K_j} \to \chi_E$ in L^2, as $j \to \infty$, hence $\chi_{E^c} T\chi_E = \chi_{E^c} \lim_{j\to\infty} T\chi_{K_j} = \lim_{j\to\infty} \chi_{E^c} T\chi_{K_j} = \lim_{j\to\infty} 0 = 0$ (in L^2, hence a.e., since $\chi_{E^c}\cdot$ is continuous on L^2), i.e., $T\chi_E = 0$ a.e. on E^c.

5° *We have $T\chi_A = a\chi_A$ for any measurable A s.t. $m(A) < \infty$:* Fix $k \in \mathbb{N}$. Obviously, $\chi_A = \chi_{A\cap E_k} + \chi_{A\setminus E_k}$. But $T\chi_{A\setminus E_k} = 0$ (a.e.) on $A \cap E_k$, hence $T\chi_A = T\chi_{A\cap E_k}$ on $A \cap E_k$. Similarly, $T\chi_{A\cap E_k} = T\chi_{E_k}$ on $A \cap E_k$, hence $T\chi_A = T\chi_{A\cap E_k} = T\chi_{E_k} =: a = a\chi_A$ (a.e.) on $A \cap E_k$. Since this holds for any $k \in \mathbb{N}$, we have $T\chi_A = a\chi_A$ a.e. on A $(= \cup_k (A \cap E_k))$, hence a.e. on $\mathbb{R}^n$, by 4°.

6° By the above footnote, $a \in \mathrm{L}^\infty(\mathbb{R}^n)$ (there 5° is enough, we need not have $Tf = af$ $\forall f \in \mathrm{L}^2$). Therefore, $a \in \mathcal{B}(\mathrm{L}^2(\mathbb{R}^n))$. Since T and a are linear and continuous on L^2, we have $(T - a)f = 0$ for all simple $f \in \mathrm{L}^2$, hence for all $f \in \mathrm{L}^2$, by density.

Solution to 6.2: 0° *Remarks:* The integral[6] $\int_\epsilon^1 |x|^{-z}\,dx = \int_\epsilon^1 r^{n-1} \int_{S^{n-1}} r^{-z}\,d\sigma\,dr = |S^{n-1}| \cdot |_\epsilon^1 r^{n-z}/(n-z) = \frac{|S^{n-1}|}{n-z}(1 - \epsilon^{n-z})$ does not converge, as $\epsilon \to 0+$, for $\operatorname{Re} z \geq n$. However, the fourth formula on p. 98 does make sense whenever $\operatorname{Re} z < n+1$, $z \neq n$ (since $|\phi(x) - \phi(0)|/x \leq \|\nabla\phi\|_\infty$ and $|x|^{-z+1} \in \mathrm{L}^1(B_1)$); we'll denote it by

$$T\phi := \int_{|x|<1} \frac{\phi(x) - \phi(0)}{|x|^z}\,dx + \int_{1<|x|} \frac{\phi(x)}{|x|^z}\,dx + \frac{|S^{n-1}|}{n-z}\phi(0) =: A + B + C$$

and show that $T\phi_\lambda = \lambda^{-z} T\phi$ $\forall \phi \in \mathcal{C}_{\mathrm{c}}^\infty(\mathbb{R}^n)$.

1° $\int_{a<|x|<b} |x|^{-z}\phi_\lambda(x) = \lambda^{-z} \int_{a/\lambda<|x|<b/\lambda} |x|^{-z}\phi(x)$. Proof: set $s := x/\lambda$, so that $\det dx/ds = \det \lambda I = \lambda^n$, and then replace s by x. Recall that $\phi_\lambda(x) := \lambda^{-n}\phi(x/\lambda)$.

2° Set $T\phi_\lambda := A_\lambda + B_\lambda + C_\lambda$. By 1° (with $\phi(\cdot) - \phi(0)$ in place of ϕ), we have $A_\lambda = \lambda^{-z} \int_{|x|<1/\lambda} |x|^{-z}[\phi(x) - \phi(0)]\,dx$, $B_\lambda = \lambda^{-z} \int_{|x|>1/\lambda} |x|^{-z}\phi(x)\,dx$, $C_\lambda = \lambda^{-n} C$. Thus, $A_\lambda + B_\lambda + C_\lambda$ does not look exactly like $\lambda^{-z}(A + B + C)$ $\forall \lambda > 0$, but yet it is:

Case $1/\lambda > 1$: when comparing A to A_λ we observe the additional term $D_\lambda := \lambda^{-z} \int_{1<|x|<1/\lambda} |x|^{-z}\phi(0)\,dx = \lambda^{-z}\phi(0)\frac{|S^{n-1}|}{n-z}(\lambda^{z-n} - 1) = (\lambda^{-n} - \lambda^{-z})C$ (whether $1/\lambda > 1$ or not). Thus, $A_\lambda + B_\lambda + C_\lambda = \lambda^{-z}(A + B) - D_\lambda + C_\lambda = \lambda^{-z}(A + B + C) = \lambda^{-z} T\phi$, hence $T\phi_\lambda = \lambda^{-z} T\phi$ $\forall \lambda > 0$. The case $1/\lambda \leq 1$ is similar.

Solution to 6.3: Now $|K(x,y)| \leq |x - y|^{-1}$, hence (5.12) holds ($\Delta := \{(x,x) \mid x \in \mathbb{R}\}$). Set $f(x,y) := 1/K(x,y)$. *Lipschitz* means Lip_1, i.e., $|A(x) - A(y)| \leq M|x - y|$ $\forall x, y \in \mathbb{R}$ for some $M < \infty$. Therefore, $|f(x,y)|^2 \leq |x - y|^2(1 + M^2)$, hence $a := |K(x,y) - K(x,z)| = |f(x,z) - f(x,y)|/|f(x,y)f(x,z)| = |f(y,z)|/|f(x,y)f(x,z)| \leq \sqrt{1+M^2}|y - z|/|x - y||x - z|$. When $|x - y| > 2|y - z|$, we have $|x - z| \leq |x - y| + |y - z| < 2|x - y|$, hence then $a \leq 2\sqrt{1+M^2}|y - z|/|x - y|^2$, hence (5.13) holds. Since $K(x,y) = -K(y,x)$, (5.14) is equivalent to (5.13).

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method to pick a uniquely determined representative of each equivalence class.

[6]See, e.g., Exercise 8.6 of [Rudin:RCA] for integrals in polar coordinates, like the one here.

Solution to 6.4: (In Proposition 5.12, one should assume that K is a standard kernel. Actually, it suffices that (5.12) holds and K is measurable (when (5.13) and (5.14) hold, the function K is obviously continuous, hence measurable (on $\mathbb{R}^n \times \mathbb{R}^n \setminus \Delta$)).)

1° *Assume that* $(Tf)(x) := \lim_{\epsilon\to 0+}(T_\epsilon f)(x)$ *exists for a.e.* $x \in \mathbb{R}^n$, *for any* $f \in \mathcal{C}_c^\infty(\mathbb{R}^n)$, where $T_\epsilon f := \int_{\epsilon<|x-y|} K(x,y)f(y)\,dy$ [p. 101].[7] Given $R > 0$, choose f s.t. $f \equiv 1$ on B_{R+1}. Then $\int_{\epsilon<|x-y|<1} K(x,y)\,dy = (T_\epsilon f)(x) - \int_{1<|x-y|} K(x,y)f(y)\,dy$ for each $x \in B_R$ (since $|x-y| < 1 \Rightarrow y \in B_{R+1}$), hence $\lim_{\epsilon\to 0+}\int_{\epsilon<|x-y|<1} K(x,y)\,dy = (Tf)(0) + \int_{1<|x-y|} K(x,y)f(y)\,dy$ exists for a.e. $x \in B_R$, hence for a.e. $x \in \mathbb{R}^n$ (since $R > 0$ was arbitrary).

2° *Assume that* $x \in \mathbb{R}^n$ *is s.t.* $L(x) := \lim_{\epsilon\to 0+}\int_{\epsilon<|x-y|<1} K(x,y)\,dy$ *exists.* Then $(Tf)(x) = \lim_{\epsilon\to 0+}\int_{\epsilon<|x-y|<1} K(x,y)[f(y)-f(x)]\,dy + f(x)L + \int_{1<|x-y|} K(x,y)f(y)\,dy =: A_1 + A_2 + A_3$ exists, because (A_2 and A_3 exist and) the limit A_1 exists, because $|K(x,y)[f(y) - f(x)]| \le C|x-y|^{-n}\|\nabla f\|_\infty |y-x|$, which is integrable over $\{y \in \mathbb{R}^n \mid |x-y| < 1\}$.[8]

Solution to 6.5: The proof on p. 71 still applies without changes.

We explain here some details of that proof: 1. If f is rotation-invariant, then so is $\widehat{f}$, as one easily deduces from (1.17). 2. $\int_0^\infty \mathrm{e}^{-\pi r^2} r^b\,dr = \int_0^\infty \mathrm{e}^{-x}(x/\pi)^{b/2}(x\pi)^{-1/2}/2\,dx = (\pi^{-b/2-1/2}/2)\int_0^\infty \mathrm{e}^{-x}x^{(b-1)/2}\,dx = \frac{1}{2}\pi^{-\frac{1+b}{2}}\Gamma(\frac{1+b}{2})$ at least when $\operatorname{Re} b > 0$. (Note: $r^{s+it} = r^s \mathrm{e}^{it\ln r}$.)

Remarks: What about using holomorphicity directly to the formula $(|x|^{-z})\widehat{\ } = \ldots$, which was known for real $z \in (0, n)$? The right-hand-side is a holomorphic function of z on a certain open connected subset of $\mathbb{C}$ containing $V := \{z \mid 0 < \operatorname{Re} z < n\}$, so if we can show that the same holds for the left-hand-side, the equality extends to V. However, although $z \mapsto (\chi_{B_1}|\cdot|^{-z})\widehat{\ }$ can quite easily be shown to be holomorphic on V (using, e.g., the Morera Theorem), for $z \mapsto (\chi_{B_1^c}|\cdot|^{-z})\widehat{\ }$ this is not obvious.

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[7] This exists, because $|K| > C/\epsilon$ for $|x-y| > \epsilon$, by (5.12).

[8] As shown in Exercise 5.4, $f \in \mathcal{C}^\infty(V) \Rightarrow g(\cdot) := [f(\cdot) - f(x)]/(\cdot - x) \in \mathcal{C}^\infty(V)$, for any $x \in V$ (when $V \subset \mathbb{R}^n$ is open), and the same holds with $\mathcal{S}$ or $\mathcal{C}_c^\infty$ in place of $\mathcal{C}^\infty$. Similarly, $f \in \mathcal{C}^{k+1}(V) \Rightarrow g \in \mathcal{C}^k(V)$.