Mat-1.159 Harmoninen analyysi: 5th exercise, Wed 2003-10-15, U345 (2 pages)

`http://www.math.hut.fi/teaching/harmanal/`

1st mid-term examination (1. välikoe) ma 27.10. 12-15 (Oct 27). **2nd mid-term examination** (2. välikoe) to 18.12. 13-16 (Dec 18).

Exercise 5.1 ([p. 79] *trivial*):
Show that in Lemma 4.11 of [Duo], to get $\|\Delta_\epsilon\|_1 \le C'_q\|\Omega\|_q$, it suffices to show that $\|\Delta_1\|_1 \le C'_q\|\Omega\|_q$.

Exercise 5.2 ([p. 71] *"homogeneous $-n$": easy, "zero average" ?*):
Show that if $f \in \mathcal{C}^\infty(\mathbb{R}^n \setminus \{0\})$ is homogeneous of degree 0 with zero average on S^{n-1}, then $\frac{\partial^n f}{\partial x_i^n}$ is homogeneous of degree $-n$ and has zero average on S^{n-1}.

Exercise 5.3 ([p. 78] *very easy*):
Show that the functions κ_j^* obtained in the proof of Lemma 4.10 is odd.

Exercise 5.4 ([p. 80] *a bit difficult*):
Find all $u \in \mathcal{S}'(\mathbb{R}^n)$ that satisfy $\triangle u = 0$.[1]

Exercise 5.5 ([p. 72] *a bit difficult*):
Let $\Omega \in \mathrm{L}^1(S^{n-1})$ be nonzero, with mean value zero. Let $T_\Omega(f)$:=P.V.$\frac{\Omega(x')}{|x|^n} * f$, $0 \ne f \ge 0$, $f \in \mathrm{L}^1(\mathbb{R}^n)$. Show that $T_\Omega(f) \notin \mathrm{L}^1(\mathbb{R}^n)$.[2]

[1] Hint: show that $E \subset \mathrm{Ker}(\hat{u})$ (see the proof).

[2] Hint: Show that $\widehat{T_\Omega(f)}$ is not continuous at zero. See (4.4). Assume (first?) that $f \in \mathcal{S}$.

Solution to 5.1: We have $\Delta_\epsilon(x) = \epsilon^{-n}\Delta(\epsilon^{-1}x)$ [p. 79], hence $\int_{\mathbb{R}^n} |\Delta_\epsilon(x)|\,dx = \epsilon^{-n}\int_{\mathbb{R}^n} |\Delta_1(\epsilon x)|\,dx = \int_{\mathbb{R}^n} |\Delta_1(y)|\,dy = \|\Delta_1\|_1$, because $y := \epsilon I x$ implies that $\frac{dx}{dy} = \epsilon^{-1}I$, hence $|\det\frac{dx}{dy}| = \epsilon^{-n}$.

Solution to 5.2: 1° Assume that $f(rx) = r^a f(x)$ for all $x \in \mathbb{R}^n$ (i.e., that f is *homogeneous* of degree a). Then $f_i(rx) = \lim_{h\to 0} h^{-1}[f(rx+he_i) - f(rx)] = \lim_{h\to 0} r^{-1}(h/r)^{-1}[r^a f(x+(h/r)e_i) - r^a f(x)] = r^{-1}r^a f_i(x)$, hence $f_i := df/dx_i$ is homogeneous of degree $a-1$. By induction, $D_i^k f$ is homogeneous of degree $a-k$, for any $k \in \mathbb{N}$.

2° (Not done.)

Solution to 5.3: This is a consequence of the following: *If $f_n(x) = -f_n(-x)$ $\forall x \in \mathbb{R}^n$, and $f_n \to f$ in L^p, $p \in [1,\infty]$, then f is odd.*

To be exact, the equivalence class f contains an odd function; naturally, it also contains non-odd functions, since one can redefine f on a null set and still have $f_n \to f$ in L^p.

Proof: Then $f_n \to f$ a.e. (replace $\{f_n\}$ by a subsequence if necessary), say on N^c, where $m(N) = 0$, hence $f(x) = \lim_n -f_n(-x) = -f(-x)$ when $x, -x \notin N$, i.e., $x \notin N \cup (-N) =: N'$. Thus, the function $\chi_{\mathbb{R}^n\setminus N'} f$ is odd and in the equivalence class of f.

Solution to 5.4: (For a different, less exact proof, see p. 123 of [Grafakos].) Obviously, polynomials of order 0 or 1 are such. We shall show that there are no others. For any $\phi \in \mathcal{S}(\mathbb{R}^n)$, $u \in \mathcal{S}'(\mathbb{R}^n)$, we have[3] $\triangle u(\phi) = \sum_k D_k^2 u(\phi) :=$

[3] We have $u(\phi) = \hat{u}(\mathcal{R}\mathcal{F}\phi)$ $\forall u \in \mathcal{S}'$, $\phi \in \mathcal{S}$ (because $\mathcal{R}\mathcal{F}\mathcal{F} = I$, $\mathcal{R}\mathcal{F} = \mathcal{F}\mathcal{R}$,

$\sum_k u((-D_k)^2\phi) = u(\triangle\phi) = \hat{u}(\mathcal{R}\mathcal{F}\triangle\phi) = \hat{u}(\mathcal{R}a|\xi|^2\widehat{\phi})$, where $a := -4\pi^2$, because $\widehat{D_k\phi}(\xi) = 2\pi i\xi_k\widehat{\phi}(\xi)$ and $\triangle = \sum_k D_k^2$. (Here $\mathcal{F}f := \widehat{f}$, $\mathcal{R}f(x) := f(-x)$ ("Reflection").)

Now $\triangle u = 0$, i.e., $0 = u(\triangle\phi)\ \forall\phi$, i.e., $0 = \hat{u}(|\xi|^{-2}\widehat{\phi}(\xi))\ \forall\phi$ i.e., $0 = \hat{u}(|x|^{-2}\phi(x))\ \forall\phi\in\mathcal{S}(\mathbb{R}^n)$ (here x is the variable). But if $\phi\in E$, where

$$E := \{\phi\in\mathcal{S}\,\big|\,\phi(0)=0,\ (D_k\phi)(0)=0\ (k=1,...,n)\},$$

then $|x|^{-2}\phi \in \mathcal{C}^\infty$, by Lemma A below, [4]hence then $\psi_\phi := |x|^{-2}\phi \in \mathcal{S}$,[5] hence then $\hat{u}(\phi) = \hat{u}(|x|^2\psi_\phi) = 0$. Thus, $E \subset \operatorname{Ker}(\hat{u})$. But $E = \cap_{k=0}^n \operatorname{Ker}(\Lambda_k)$, where[6] $\Lambda_0 := \delta_0 = \hat{1}$, $\Lambda_k := \hat{x}_k = -(2\pi i)^{-1}D_k\delta_0$ (i.e., $\Lambda_k(\phi) = -(D_k\phi)(0)$). Therefore, $\hat{u} = \sum_{k=0}^n \alpha_k\Lambda_k$,[7]hence $u(x) = \alpha_0 1 + \sum_{k=1}^n \alpha_k x_k$, a polynomial of order 0 or 1.

---

and $u(\widehat{\phi}) = \hat{u}(\phi)\ \forall u,\phi$, and $\mathcal{F}$ is a linear bounded bijection $\mathcal{S}\to\mathcal{S}$ and $\mathcal{S}'\to\mathcal{S}'$ [p. 14–15]).

[4]Lemma A: *If $f\in\mathcal{C}^k(V)$, $V\subset\mathbb{R}^n$ is open, $0\in V$, $0\le j\le k$, and $(D^\alpha f)(0)=0$ whenever $|\alpha|\le j$, then $g_\alpha\in\mathcal{C}^{k-j}(V)$, where $g_\alpha(x) := f(x)/x^\alpha\ \forall x\ne 0$ (and $g_\alpha(0) := \lim_{x\to 0} g_\alpha(x) = D^\alpha f(0)/\alpha!$).*

Proof: (Note that g is trivially $\mathcal{C}^k(V\setminus\{0\})$.) 1° *Case $n=1=j$:* Since $f(0)=0$, we have (set $t := s/x$) that $f(x) = \int_0^x f(s)\,ds = \int_0^1 f(xt)x\,dt$, hence $g(x) := f(x)/x = \int_0^1 f(xt)\,dt$, hence $g\in\mathcal{C}^{k-1}(V)$.

2° *General case:* This follows from 1° (keep the other coordinates constant) and induction.

Remark: By the same proof, assumptions $f\in\mathcal{C}^{\alpha+\beta}(V)$, $(D^\gamma f)(0)=0\ \forall\gamma\le\alpha$ imply $g_\alpha\in\mathcal{C}^\beta(V)$. We may even allow for Banach-valued functions. Note also that there is a multi-index Taylor Theorem, from which we would have obtained the proof directly.

[5]All derivatives are linear combinations of $|x|^{-j}D^\alpha\phi$ times polynomials, hence bounded, because $\phi\in\mathcal{S}$.

[6]$\hat{\delta}(\phi) := \delta(\widehat{\phi}) := \widehat{\phi}(0) := \int\phi = 1(\phi)$, hence $\hat{\delta}=1$, hence $\hat{1} = \mathcal{R}\delta = \delta$ (because $\mathcal{R}\delta(\phi) := \delta(\mathcal{R}\phi) := (\mathcal{R}\phi)(0) = \phi(0) = \delta(\phi)\ \forall\phi\in\mathcal{S}$).

[7]By Lemma 3.9 of [Rudin:FA], $E := \cap_{k=1}^n\operatorname{Ker}(\Lambda_k)\subset\operatorname{Ker}(L) \implies L = \sum_k\alpha_k\Lambda_k$. An alternative proof follows from the fact that $(\Lambda_1,...\Lambda_n) : E^\perp\to\mathbb{C}^n$ is linear and injective, hence any linear $L' : E^\perp\to\mathbb{C}$ equals $\alpha\circ(\Lambda_1,...\Lambda_n)$ for some linear $\alpha:\mathbb{C}^n\to\mathbb{C}$, i.e., for the inner product with some $\alpha\in\mathbb{C}^n$.

Solution to 5.5: If $f\in\mathcal{S}(\mathbb{R}^n)$, then $\widehat{T_\Omega f} = mf$, where $m := m_\Omega$ is from (4.4) of [Duo] (actually, [Duo] does not prove this completely but p. 267–269 of [Grafakos] do).

1° *m is not a constant:* We have $\hat{\delta}=1$. Since $\mathcal{F}$ is injective on $\mathcal{S}'(\mathbb{R}^n)$, this implies that m is a constant, say $c\in\mathbb{C}$, iff $T_\Omega = c\delta$ (i.e., $T_\Omega(\phi) = c\phi(0)\ \forall\phi\in\mathcal{S}(\mathbb{R}^n)$). Next we shall construct $\widetilde{\phi}\in\mathcal{S}(\mathbb{R}^n)$ s.t. $\delta\widetilde{\phi} = 0 \ne T_\Omega\widetilde{\phi}$ to complete the proof of 1°.

Set $g(x) := \Omega(x')$ $(x\ne 0)$, where $x' := x/|x|$. Using polar coordinates we observe that $0\ne\Omega\in\mathrm{L}^1(S^{n-1})$[8] implies that $0\ne g\in\mathrm{L}^1(A_R)$ for any $R>1$, where $A_R := B_R\setminus\overline{B_{1/R}}$ (annulus), $B_R := \{x\,\big|\,|x|<R\}$. Therefore, the same is true for $F(x) := g(x)/|x|^n$, hence there is $\widetilde{\phi}\in\mathcal{C}_c^\infty(A_2)$ s.t. $0\ne\int_{A_2}F(x)\phi(x)\,dx = \int_{\mathbb{R}^n}F\phi\,dm$.[9]We conclude that $(F*\widetilde{\phi}) =$P.V.$(F*\widetilde{\phi}) = T_\Omega\widetilde{\phi}$ (near the origin) satisfies $(F*\widetilde{\phi})(0)\ne 0$ (and is continuous near the origin, being there $\in\mathrm{L}^1*\mathrm{L}^\infty$), hence $T_\Omega\widetilde{\phi}\ne 0$, although $\delta(\widetilde{\phi}) := \widetilde{\phi}(0) = 0$, hence $T_\Omega\ne c\delta$ for any $c\in\mathbb{C}$, QED.

2° *m is discontinuous at 0:* By (4.4), $m(\xi) = m(\xi')$, hence m does not converge as $|\xi|\to 0$ unless m is a constant,[10] which is not possible, by 1°.

3° Since f is continuous and $0\ne f\ge 0$, we have $\widehat{f}(0) = \int_{\mathbb{R}^n} f > 0$, hence $m\widehat{f}$ is not continuous hence $m\widehat{f}\ne\hat{g}$ for all $g\in\mathrm{L}^1$. But $m\widehat{f} = \widehat{T_\Omega f}$,[11]hence $T_\Omega f\notin\mathrm{L}^1$, QED.

---

[8]Here $0\ne\Omega$ means that not $\Omega=0$ a.e. on S^{n-1}; similarly for g.

[9]If $V\subset\mathbb{R}^n$ is open and $F\in\mathrm{L}^p(V)$ (or $F\in\mathrm{L}^1_{\mathrm{loc}}(V)$), then $\|F\|_{\mathrm{L}^1(V)} = \sup_{\phi\in\mathcal{C}_c^\infty(V),\ \|\phi\|_q\le 1}|\int_V F\phi\,dm|$, by, e.g., Theorem B.4.12(b2) of [M02] (which would allow one to replace m by any locally finite regular measure and to treate (Banach space) vector-valued functions; moreover, $\mathcal{C}_c^\infty$ can be replaced by suitable simple functions, by (b1))). Now $F\ne 0\implies\|F\|_1\ne 0\implies\int F\phi\ne 0$ for some $\phi\in\mathcal{C}_c^\infty\subset\mathcal{S}$.

[10]We should have $\|m(\xi)-m(0)\| = \|m(r\xi)-m(0)\|\to 0$, as $r\to 0+$, since m is radial, hence $m(\xi) = m(0)$, for any $\xi\in\mathbb{R}^n$.

[11]At least if $f\in\mathcal{S}$. It is not clear whether $T_\Omega f$ is defined at all if $f\notin\mathcal{S}$.