Mat-1.159 Harmoninen analyysi: 4th exercise, Wed 2003-10-8, U345 (2 pages)

`http://www.math.hut.fi/teaching/harmanal/`

Note: if X is a σ-algebra and $a, b : X \to \mathbb{R}$ are measurable, then the set

$$\{(x, r) \in X \times \mathbb{R} \,\big|\, a(x) < r < b(x)\} \subset X \times \mathbb{R} \tag{1}$$

is (product) measurable. (The same holds with $\leq$ in place of one or both $<$'s.)[1] One often needs this fact when using the Fubini Theorem (since the integrand has to be product measurable!), as in Exercise 3.3. (where we did not say why χ_A is measurable).

Exercise 4.1 ([p. 59] *easy, quick*):

Show that $\|S_{-\infty,b}f\|_p \leq C_p\|f\|_p$, when $\widehat{S_{-\infty,b}f} := \chi_{(-\infty,b)}\widehat{f}$ and $1 < p < \infty$.[2]

Exercise 4.2 ([p. 51] *"$\Rightarrow$" easy, "$\Leftarrow$" average*):

Use the relation $\widehat{Hf} = (-i\,\mathrm{sgn})\widehat{f}$ (i.e., $Hf = \mathcal{F}^{-1}(-i\,\mathrm{sgn}\,\widehat{f})$), valid for $f \in \mathrm{L}^2(\mathbb{R})$, to extend the definition of the Hilbert transform to any function $f : \mathbb{R} \to \mathbb{C}$ with at most polynomial growth at infinity. Then show that $H(\cos x) = \sin x$.

Exercise 4.3 ([p. 70] *trivial*):

Verify that the integral (actually, P.V.$\int$)

$$-\int_{\mathbb{R}^2} \frac{f(y_1, y_2)(x_1 - y_1)}{[(x_1 - y_1)^2 + (x_2 - y_2)^2]^{3/2}}\, dy_1\, dy_2 \tag{2}$$

is of form

$$\int_{\mathbb{R}^n} \frac{\Omega(y')}{|y|^n} f(x - y)\, dy \tag{3}$$

for some Ω defined on the unit sphere of $\mathbb{R}^n$, where $y' := y/|y|$.

Exercise 4.4 ([p. 51] *average*):

Let $\phi \in \mathcal{S}(\mathbb{R})$. Show that $\lim_{N\to+\infty}$ P.V.$\int_{\mathbb{R}} \frac{e^{2\pi iNx}}{x}\phi(x)\, dx = \phi(0)\pi i$.[3]

Exercise 4.5 ([p. 65] *too difficult*):

Show that if $f, Hf \in \mathrm{L}^1$, then $\phi * Hf = H(\phi * f)\ \forall \phi \in \mathcal{S}$.

Solution to 4.1: (Thus, we are asked write out the "obvious changes" mentioned on p. 59 of [Duo].)

Set $M_b f := e^{2\pi ib\cdot} f$, so that $\widehat{M_b f}(\xi) = \int_{\mathbb{R}} e^{-2\pi ix\xi} e^{2\pi ixb} f(x)\, dx = \widehat{f}(\xi - b)$. Set $T_b f := iM_b H M_{-b} f$, $g := M_{-b} f$, so that

$$\begin{aligned}\widehat{T_b f}(\xi) &= i\widehat{Hg}(\xi - b) = \mathrm{sgn}(\xi - b)\widehat{g}(\xi - b) = \mathrm{sgn}(\xi - b)\widehat{f}(\xi) \\ &= \begin{cases} \widehat{f}(\xi), & \text{when } \xi > b; \\ -\widehat{f}(\xi), & \text{when } \xi < b. \end{cases}\end{aligned}$$

Thus, $(f - T_b f)/2 =: S_{-\infty,b} f$ satisfies $\widehat{S_{-\infty,b}f} = \chi_{(-\infty,b)}\widehat{f}$. But M_b and H are continuous on L^p, hence so are T_b and $S_{-\infty,b}$

---

[1] The functions $f(x, r) := b(x) - r$ and $g(x, r) := r - a(x)$ are product measurable (since a measurable function of one variable is obviously product measurable), hence so is the set $f^{-1}((0, \infty)) \cap g^{-1}((0, \infty)) =(1)$.

[2] Hint: modify slightly the proof on pp. 58–59 of [Duo].

[3] P.V.$\int_{\mathbb{R}} \frac{e^{2\pi iNx}}{x}\phi(x)\, dx = Hf_N(0) = \int_{\mathbb{R}} \widehat{Hf_N}\, dm$ if we set $f_N := \ldots$.

Solution to 4.2: One easily verifies that $T_f : \phi \mapsto \int_{\mathbb{R}} f\phi\, dm$ is in $\mathcal{S}'(\mathbb{R})$, hence f can be identified with the tempered distribution T_f.

But what does this exercise (= Exe4.1.3(a) of [Grafakos]) mean by Hf? That $\widehat{Hf}(\phi) := (-i\operatorname{sgn}\widehat{f})(\widehat{\phi})$? (Recall that $\widehat{f}(\tilde{\widehat{\phi}}) = f(\phi)\ \forall f \in \mathcal{S}',\ \phi \in \mathcal{S}$; cf. p. 13 of [Grafakos].)

But $\widehat{f} \in \mathcal{S}'$ (it need not exist as a function), and the multiplication of tempered distributions by functions has been defined for continuous functions only ($gT(\phi) := T(g\phi)$ when $g \in \mathcal{C}^\infty$ and its derivatives have at most polynomial growth, so that $g\phi \in \mathcal{S}$). Thus, I cannot understand the exercise (neither did Stigu nor Juha), perhaps there is an error in it. E.g., it seems that one could use the P.V. formula instead of the Fourier formula.

Solution to 4.3: By a change of variable (set $z := y - x$ and then replace z by y), we see that (2) equals

$$-\int_{\mathbb{R}^2} \frac{y_1}{(y_1^2 + y_2^2)^{3/2}} f(x-y)\, dy = \int_{\mathbb{R}^2} \frac{-y_1/|y|}{|y|^2} f(x-y)\, dy, \tag{4}$$

which equals (3) for $\Omega(y') := -y_1/|y| = -(y')_1$.

Solution to 4.4: Set $f_N(x) := \mathrm{e}^{-2\pi i N x}\phi(-x) \in \mathcal{S}$, so that $\widehat{f_N}(\xi) = \phi(\widetilde{\xi + N}) = \widehat{\phi}(-N-\xi)$.

From pp. 50-51 of [Duo] we observe that $H_\epsilon\phi \to H\phi$ everywhere, when $\phi \in \mathcal{S}$. But

$$(\pi H f_N)(0) = \text{ P.V. } \int_{\mathbb{R}} \frac{f_N(0-x)}{x}\, dx = \text{ P.V. } \int_{\mathbb{R}} \frac{\mathrm{e}^{2\pi i N x}}{x}\phi(x)\, dx =: I_N.$$

For $N > \max|\operatorname{supp}\phi|$ we have $(Hf_N)(0) = \int_{\mathbb{R}} \mathrm{e}^0 \widehat{Hf_N} = \int_{\mathbb{R}} -i\operatorname{sgn}(\xi)\widehat{\phi}(-N-\xi)\, d\xi = i\int_{\mathbb{R}} \widehat{\phi}(-N-\xi)\, d\xi$ (since $\phi(-N-\xi) = 0$ for $\xi \ge 0$) $= i\int_{\mathbb{R}} \widehat{\phi}(\xi)\, d\xi = i\phi(0)$, hence $I_N = i\pi\phi(0)$ (for these N). In particular, $\lim_{N\to+\infty} I_N = i\pi\phi(0)$.

Solution to 4.5: I made a "proof" using the following facts but it appeared faulty. Probably the task is very difficult. There is a reference to a proof by Calderón and Capri (1984) on p. 65. However, I list these facts anyway for future use:

1.1° *If $f \in \mathrm{L}^1(\mu)$, then $\forall\epsilon > 0\ \exists\delta > 0$ s.t. $\mu(E) < \delta \implies \int_E |f|\, d\mu < \epsilon$.* (Exercise 1.??? of [Rudin:RCA].)

1.2° *There is $C \in (0,\infty)$ s.t. $m\{|Hf| > r\} < C\|f\|/r\ \forall r > 0\ \forall f \in \mathrm{L}^1(\mathbb{R})$.* (Since H is weak $(1,1)$, by [p. 56].)

1.3° *Let $f \in \mathrm{L}^p(\mathbb{R})$, $1 \le p < \infty$. Then $H_\epsilon f \to Hf$ a.e., as $\epsilon \to 0+$. Moreover, $H_\epsilon f \to Hf$ in L^p if $1 < p < \infty$ and in measure if $p = 1$.* [pp. 55–56]. Here $H_\epsilon f(x) := \frac{1}{\pi}\int_{|y|>\epsilon} \frac{f(x-y)}{y}\, dy$.

1.4° *Egorov's Theorem: If $\mu(X) < \infty$ and $f_n(x) \to f(x)$ a.e. and $f_n : X \to \mathbb{C}$ are measurable, then $\forall\epsilon > 0\ \exists E \in \mathfrak{M}$ s.t. $\mu(E) < \epsilon$ and $\{f_n\}$ converges uniformly on $X\setminus E$.* (Exercise 3.16 of [Rudin:RCA].)

1.5° $\mu\{|f| > \epsilon\} \le \|f\|_1/\epsilon\ \forall\epsilon > 0$: Set $E := \{|f| > \epsilon\}$. Then $\mu(E) = \int_E 1 \le \int_E |f|/\epsilon \le \|f\|_1/\epsilon$.

1.6° $\|f * g\|_p \le \|f\|_1\|g\|_p\ \forall f \in \mathrm{L}^1\ g \in \mathrm{L}^p$: [p. 17].