Mat-1.159 Harmoninen analyysi: 2th exercise, Wed 2003-9-24, U345 (4 pages)

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There will be two mid-term examinations (dates to be decided later). These exercises will not give you any bonus, but it is very useful to at least try them before reading the answers.

By "[p. 11]" we refer to page 11 of Duoandikoetxea. After the exercise number (see below) it refers to a page that you should reach before trying the exercise. "s.t." := "such that".

$\widehat{f}(k) := \int_T f(x)e^{-2\pi ikx}\,dx$ for $f \in L^1(T)$ [p. 2]; $T := [0,1]$[1] (*Fourier coefficients*); $\widehat{f}(\xi) := \int_{\mathbb{R}^n} f(x)e^{-2\pi ix\cdot\xi}\,dx$ for $f \in L^1(\mathbb{R}^n)$ [p. 11] (Fourier transform). In different books, you see 2π in different formulas/places (e.g., in the inverse formula only), but if $\mathcal{F}_1$ and $\mathcal{F}_2$ are such transforms, we always have $(\mathcal{F}_1\widehat{f})(\xi) = r(\mathcal{F}_2\widehat{f})(t\xi)$, where $r, t > 0$ are some universal constants, so there is no significant difference.

Exercise 2.1 ([p. 18] *quite easy, average*):
Prove[2] that in $L^p(\mathbb{R})$, we have $\lim_{R\to\infty} \sigma_R f = f$ [Duo: p. 18], where $1 \le p < \infty$, $(\sigma_R f)(x) := \frac{1}{R}\int_0^R S_t f(x)\,dt$, $\widehat{S_R f} := \chi_{(-R,R)}\widehat{f}$.

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[1] Note: by functions on T we mean functions $\mathbb{R} \to \mathbb{C}$ with period 1!

[2] Hint: Minkowsky's Integral Inequality.

Exercise 2.2 ([p. 27] *quite short*):
Show that the set[3]

$$E := \{f \in L^p(X,\mu) \mid \lim_{t\to t_0} T_t f(x) \text{ exists a.e. }\}$$

is closed in $L^p(X,\mu)$. Here $\{T_t\}$ is a family of linear operators on $L^p(X,\mu)$ such that $T^* := \sup_t |T_t f|$ is weak (p,q), and $q \in [1,\infty)$, $p \in [1,\infty]$.

Exercise 2.3 ([p. 31] *easy, short*):
"Formulate and prove a correct version of Lemma 2.6." (There is a simple counter-example.)

Exercise 2.4 ([p. 13] *very easy and short*):
Let $\{\phi_n\}_{n=1}^\infty \subset \mathcal{S}'$, $\phi_n \to \phi$ in $\mathcal{S}'$, $g \in \mathcal{S}$. Show that $\phi_n * g \to \phi * g$ in $\mathcal{S}'$.[4]

Exercise 2.5 ([p. 42] *A bit difficult*):
Show that the expression (2.16) defines a norm on $L\log L$, that is, the norm

$$\|f\| := \inf\{\lambda > 0 \mid \int_\Omega \frac{|f(x)|}{\lambda}\log^+\frac{|f(x)|}{\lambda}\,dx \le 1\}.$$

(Recall that $\log^+(f) := \max\{0, \ln f\}$.)[5]

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[3] Hint: the proof is sketched on pp. 27–28.

[4] Hint: use the definition $(\phi * g)(u) := \phi(\widetilde{g} * u)$, where $\widetilde{g}(x) := g(-x)$.

[5] Hint: Read the definitions of I and L from the solution. If you need even more help, read a bit more of the solution.

Solution to 2.1: In fact, the function $\hat{f}$ (hence the functions $S_R f$ and $\sigma_R f$ too) can only be defined if $f \in \mathrm{L}^r$ for some $r \in [1,2]$, so we must add this extra assumption (if $p \in (2,\infty)$).[6] By pp. 17–18 of [Duo], in that case we have $\sigma_R f(x) = (F_R * f)(x)$, where[7]

$$F_R(x) = R^{-1}\int_0^R D_t(x)\,dt = \frac{\sin^2(\pi R x)}{R(\pi x)^2} \geq 0, \tag{1}$$

$D_R(x) := \int_{-R}^R \mathrm{e}^{2\pi i x\xi}\,d\xi = \sin(2\pi R x)/\pi x$ (hence $|D_R(x)| \leq D_R(0) = 2R$).

We have $\|F_R\|_1 = \int_{\mathbb{R}} F_R(x)\,dx = 1$, because $\int_{\mathbb{R}} \sin^2(t)/t^2\,dt = 1$. Moreover, $\lim_{R\to+\infty}\int_{|x|>\delta} F_R(x)\,dx \leq \lim_{R\to+\infty} R^{-1}\int_{|x|>\delta}(\pi x)^{-2}\,dx = 0\ \forall\delta > 0$. Since $\int F_R = 1$, by Minkowski's Integral Inequality we have

$$\|\sigma_R f - f\|_p = \|F_R * f - f\|_p \tag{2}$$
$$= \|\int_{\mathbb{R}} F_R(t)[f(\cdot - t) - f(\cdot)]\,dt\|_p \tag{3}$$
$$\leq \int_{\mathbb{R}} F_R(t)\|f(\cdot - t) - f(\cdot)\|_p\,dt \tag{4}$$
$$\leq \int_{|t|<\delta} F_R(t)\|f(\cdot - t) - f(\cdot)\|_p\,dt + \int_{|t|>\delta} F_R(t) 2\|f\|_p\,dt < \epsilon, \tag{5}$$

when we choose first $\delta > 0$ small enough (since $\|f(\cdot - t) - f(\cdot)\|_p \to 0$ as $|t| \to 0$) and then R big enough, QED.

Details: Given $\epsilon > 0$, choose $\delta = \delta_\epsilon > 0$ s.t. $|t| < \delta \implies \|f(\cdot - t) - f(\cdot)\|_p < \epsilon/2$ (this is possible by a real analysis exercise (hint: first let $f \in \mathcal{C}_c$, then use density in L^p), since $p < \infty$). Then choose R_ϵ s.t. $\int_{|x|>\delta} F_R(x)\,dx < \epsilon/2\|f\|_p$ for all $R > R_\epsilon$. By the above, it follows that $\|\sigma_R f - f\|_p < \epsilon/2 + \epsilon/2 = \epsilon\ \forall R > R_\epsilon$, QED.

[6] Actually, we show below that $F_R * f \to f$ even without this extra assumption.

[7] To prove this, assume that $f \in \mathcal{C}_c(\mathbb{R})$ (or that $f \in \mathrm{L}^1$) and use the Fubini Theorem. Then let $\mathcal{C}_c \ni f_n \to f$ in L^r and note that $\widehat{f}_n \to \widehat{f}$ in $\mathrm{L}^{r'}$, hence $\|\chi_{(-t,t)}(\widehat{f}_n - \widehat{f})\|_1 \leq \|\chi_{(-R,R)}\|_r\|\widehat{f}_n - \widehat{f}\|_{r'} \to 0$ (by Hölder), hence $\|S_t(f_n - f)\|_\infty \to 0$ uniformly for $t \in (0,R)$. Similarly, $F_R * f_n \to F_R * f$ uniformly (by Hölder), because $F_R \in \mathrm{L}^{r'}$.

Solution to 2.2: We rewrite the proof of Theorem 2.2 (p. 27) as indicated below it: Set $S_t f(x) := \limsup_{t\to t_0} T_t f(x) - \liminf_{t\to t_0} T_t f(x)$, so that $E = \{f \in \mathrm{L}^p \mid S_t f = 0 \text{ a.e. }\}$ and $S_t(f+g) \leq S_t f + S_t g$.

Let $E \ni f_n \to f \in \mathrm{L}^p(X,\mu)$, as $n \to +\infty$ (so that $S_t f_n = 0$ a.e. for all n). Then, for all $\lambda > 0$, we have (here $\{F > 0\} := \{x \in X \mid F(x) > 0\}$ etc.)

$$\mu(\{S_t f > \lambda\}) \leq \mu(\{S_t(f - f_n)(x) + S_t f_n(x) > \lambda\}) \tag{6}$$
$$\leq \mu(\{S_t(f - f_n)(x) + S_t f_n(x) > \lambda\}) \leq \mu(\{S_t(f - f_n)(x) > \lambda\}) + 0 \tag{7}$$
$$\leq \mu(\{2T^*(f - f_n)(x) > \lambda\}) \leq [C\|f - f_n\|_p/(\lambda/2)]^q \to 0 \tag{8}$$

(the last "$\leq$" by the definition of "weak (p,q)"), as $n \to \infty$, hence $\mu(\{S_t f > \lambda\}) = 0$ But $\{S_t f > 0\} = \cup_k\{S_t f > 1/k\}$, hence $\mu(S_t f > 0) = 0$, hence $f \in E$, QED.

Solution to 2.3: Let $\mathcal{I}$ be a collection of open intervals in $\mathbb{R}$ and let K be a compact set contained in their union. Then there exists a finite subcollection $\{I_0, ..., I_n\} \subset \mathcal{I}$ s.t. $K \subset \cup_j I_j$ and $\sum_j \chi_{I_j}(x) \leq 2\ \forall x \in \mathbb{R}$.

(The word "open" is new. Otherwise we could have $I_n = [2^{-n-1}, 2^{-n}]$ ($n \in \mathbb{N}$), $I_{-1} = [-1, 0]$, $[-1,1] \subset \cup_n I_n$ but there is no finite subcover of $[-1,1]$.)

Proof: 1° By compactness, some finite subcollection $\mathcal{I}' \subset \mathcal{I}$ covers K. Remove from $\mathcal{I}'$ an interval that is contained in the union of any other two elements of $\mathcal{I}'$ and repeat this as long as such intervals exist. Denote the resulting subcollection by $\mathcal{I}''$. Obviously, the subcollection still covers K.

2° *Let $x \in K$. Then $\sum_{I\in\mathcal{I}''} \chi_I(x) \le 2$.* Proof: To obtain a contradiction, assume that $I_k :=]a_k, b_k[\subset \mathcal{I}''$ and $x \in]a_k, b_k[$ for $k = 1, 2, 3$. Choose l, m s.t. $a_l = \min_j a_j$ and $b_m = \max_j b_m$. Then the third interval $\subset I_l \cup I_m$, a contradiction, QED.

Solution to 2.4: Note: Here $\phi * g \in \mathcal{S}'$ means (by definition see [Grafakos, p. 114]) the map $u \mapsto \phi(\widetilde{g} * u)$, where $\widetilde{g}(x) := g(-x)$.

Now $(\phi_n * g)(u) := \phi_n(\widetilde{g} * u) \to \phi(\widetilde{g} * u) = (\phi * g)(u)$ for all $u \in \mathcal{S}$, i.e., $\phi_n * g \to \phi * g$ in $\mathcal{S}'$, QED.

Remarks: 1. The definition of $*$ makes sense, because *for any $u, v \in \mathcal{S}(\mathbb{R}^n)$, we have $u * v \in \mathcal{S}(\mathbb{R}^n)$*, by p. 98 of [Grafakos].

2. By, e.g., taking Fourier transforms, one could show that $\phi * g$ coincides with the $\mathcal{C}^\infty$ function $x \mapsto \phi(\tau_x \widetilde{g})$, (which in turn coincides with the usual $\phi * g$ if ϕ corresponds to a function).

Solution to 2.5: We shall show that $\|\cdot\| := \|\cdot\|_{L\log L(\Omega)}$ is a norm on X, the space of (equivalence classes of) measurable functions $\Omega \to \mathbb{C}$ for which this norm is finite. (The zero class consists of functions that are zero a.e. Of course, we assume that $\Omega \subset \mathbb{R}^n$ is measurable.)

Set $I(f,\lambda) := \int_\Omega \phi(|f(x)|/\lambda)\,dx$, $\phi(t) := t\log^+ t$, $L(f) := \{\lambda > 0 \,|\, I(f,\lambda) \le 1\}$, so that $\|f\| = \inf L(f)$. Recall that $\log^+(t) := \max\{0, \ln t\}$.

1° *Some properties:* Obviously, $I(f,\lambda)$ is decreasing, hence $L(f) \subset (0,\infty)$ is an interval (empty iff $\|f\| < \infty$) for each measurable f.

2° *$\|\alpha f\| = |\alpha|\|f\|$:* Obviously, $I(\alpha f, |\alpha|\lambda) = I(f,\lambda)$, hence $L(\alpha f) = |\alpha| L(f) := \{|\alpha|\lambda \,|\, \lambda \in L(f)\}$, hence $\|\alpha f\| = \inf L(\alpha f) = |\alpha|\|f\|$.

3° *$\|f\| = 0 \implies f = 0$ a.e.:* Assume that $\|f\| = 0$, i.e., that $L(f) = (0,\infty)$. It suffices to show that $m(\{|f| > \epsilon\}) = 0$ for any $\epsilon > 0$ (since then $m(\{f \ne 0\}) = 0$, because $\{|f| > 0\} = \cup_{n=1}^\infty \{|f| > 1/n\}$).

Let $\epsilon > 0$. Set $E := \{x \in \Omega \,|\, |f(x)| > \epsilon\}$. Then $1 \ge I(f, \epsilon/r) \ge \int_E \phi(r)\,dm = m(E)\phi(r)$ $\forall r > 0$, hence $m(E) \le 1/\phi(r) \to 0$, as $r \to +\infty$, hence $m(E) = 0$, QED.

4° *$\|f+g\| \le \|f\| + \|g\|$:* Let $f, g \in X$. Assume that $\alpha \in L(f)$, $\beta \in L(g)$. We shall show that $\alpha+\beta \in L(f+g)$, so that $\|f+g\| \le \alpha+\beta$. It follows that $\|f+g\| \le \inf L(f) + \inf L(g) =: \|f\| + \|g\|$.

Set $F := f/\alpha$, $G := g/\beta$, $h := (f+g)/(\alpha+\beta)$, $\theta := \alpha/(\alpha+\beta)$, so that $h = \theta F + (1-\theta)G$. Then $I(f+g, \alpha+\beta) = \int_\Omega \phi(|h|)\,dm$. But $\phi : t \mapsto t\log^+ t$ is convex[8] hence[9] so is $\phi(|\cdot|) : \mathbb{R}^n \to [0,\infty)$, hence $\phi(|h(t)|) \ge \theta\phi(|F(t)|) + (1-\theta)\phi(|G(t)|)$ $\forall t \ge 0$, hence $\int_\Omega \phi(|h|)\,dm \le \theta\int_\Omega \phi(|F(t)|)\,dt + (1-\theta)\int_\Omega \phi(|G(t)|)\,dt = \theta I(f,\alpha) + (1-\theta)I(g,\beta) \le 1$. But $I(f+g,\alpha+\beta) = \int_\Omega \phi(|h|)\,dm \le 1$ implies that $\alpha+\beta \in L(f+g)$, QED.

5° We have shown that $\|\cdot\|$ is a norm on $\Omega \subset \mathbb{R}^n$. As an encore, we shall show that $\|\cdot\|$ *is complete* on X, i.e., that X is a Banach space.

5.1° *We have $\|f\| < \mathrm{e}^{-n-k} \implies m(\{|f| \ge \mathrm{e}^{-n}) \le 1/k\mathrm{e}^k$.* Proof: if $\|f\| < \mathrm{e}^{-n-k}$, then $I(f, \mathrm{e}^{-n-k}) \le 1$, hence

[8] Because ϕ is AC (i.e., $\exists\phi'$ a.e. and $\phi(s) - \phi(r) = \int_r^s \phi'$) and $\phi' \ge 0$ a.e.

[9] Because $\phi(|\theta x + (1-\theta)y|) \le \phi(\theta|x| + (1-\theta)|y|) \le \theta\phi(|x|) + (1-\theta)\phi(|y|)$.

$1 \geq \int_\Omega \phi(|f|/\mathrm{e}^{-n-k})\,dm \geq m(\{|f| \geq \mathrm{e}^{-n}) \phi(\mathrm{e}^{-n}/\mathrm{e}^{-n-k}) = m(\{|f| \geq \mathrm{e}^{-n}) k\mathrm{e}^k$, QED.

5.2° Let $\{f_n\} \subset X$ be a Cauchy-sequence. Take a subsequence $\{g_n\}$ if necessary so that $\|g_n - g_m\| < e^{-2N}$ for $n, m \geq N$ (take a subsequence if necessary). Then $m(\{|g_n - g_m| > e^{-N}\}) \leq Ne^{-N}$ for $n, m \geq N$, by 5.1°, hence "$\{g_n\}$ is a Cauchy sequence in measure" (cf. [Rudin:RCA], Exercise 3.18, p. 74), hence some subsequence converges a.e. to a function, say $f : \Omega \to \mathbb{C}$, by 5.3°. Let $\{f_{n_j}\}_{j\in\mathbb{N}}$ be the subsequence.

Given $\epsilon > 0$ choose N s.t. $\|f_n - f_m\| < \epsilon$ (hence $I(f_n - f_m, \epsilon) \leq 1$) for $n, m \geq N$. Since ϕ is continuous, we have $I(f - f_m, \epsilon) = \int_\Omega \phi(|f - f_m|/\epsilon) = \int_\Omega \liminf_j \phi(|f_{n_j} - f_m|/\epsilon) = \liminf_j \int_\Omega \phi(|f_{n_j} - f_m|/\epsilon) \leq 1$, hence $\epsilon \in L(f - f_m)$, hence $\|f - f_m\| \leq \epsilon$. We conclude $f - f_m \in X$, hence $f \in X$ (by 4°). Since $\epsilon > 0$ was arbitrary, we have $f_m \to f$ as $m \to \infty$, QED.

5.3° *If $\{f_n\}$ is a Cauchy-sequence in measure,*[10] *then some subsequence of $\{f_n\}$ converges a.e.*[11] Proof: Take a subsequence if necessary so that $\forall n, m \geq N$ $m(\{x \mid |f_n(x) - f_m(x)| > 2^{-N}) < 2^{-N}$. The set $E := \{x \mid f_n(x) \text{ does not converge}\}$ is measurable.[12] For each $k \geq 1$, set

$$E_k := \cup_{j=0}^\infty \{|f_{k+j} - f_{k+j+1}| > 2^{-k-j}\}. \tag{9}$$

Then $m(E_k) \leq \sum_{j=0}^\infty 2^{-k-j} \leq 2^{-k+1}$. But $x \notin E_k \implies |f_{k+m}(x) - f_{k+n}(x)| \leq \sum_{j=m}^{n-1} |f_{k+j} - f_{k+j+1}| le 2^{-k-m+1}$ when $n > m \geq k$, hence then $f_j(x)$ converges (given ϵ choose m s.t. $2^{-k-m-1} < \epsilon$ to have $|f_j - f_n| < \epsilon$ for $j, n \geq k + m$).

Therefore, $f_n(x)$ converges for all $x \notin \cap_k E_k$. But $m(\cap_k E_k) \leq 2^{-k+1} \forall k$, hence $m(\cap_k E_k) = 0$, QED.

Remarks: 1. Everything above applies mutatis mutandis whenever ϕ is a convex, increasing function s.t. $\phi(0) = 0$ and $\phi(t) \to +\infty$ as $t \to +\infty$ (cf. p. 42, where resulting Banach spaces are referred as *Orlicz spaces*).

2. *We have $X \subset \mathrm{L}^1_{\mathrm{loc}}$.* Proof: Given $f \in X$, choose $\lambda \in L(f)$, $r := \lambda/\mathrm{e}$ to observe that $\infty > I(f, \lambda) \geq \int_E \phi(|f|/\lambda)\,dm \geq \int_E |f|/\lambda\,dm$, hence $\int_E |f|\,dm < \infty$, where $E := \{x \in \Omega \mid |f| > \lambda\mathrm{e}\}$. If $K \subset \Omega$ is compact, $\int_{K\setminus E} |f| \leq m(K)\lambda\mathrm{e} < \infty$, hence $\int_K \leq \int_{K\setminus E} + \int_E < \infty$, QED.

[10] This means that $\forall\epsilon > 0\ \exists N\ \forall n, m \geq N\ m(\{x \mid |f_n(x) - f_m(x)| > \epsilon) < \epsilon$.

[11] This holds (with the same proof) also for any measure space in place of Ω and for functions having their values in any Banach space.

[12] Since $g = \limsup_n f_n(x) - \liminf_n f_n(x)$ is measurable and $E = g^{-1}(\{0\}^c)$.