Mat-1.159 Harmoninen analyysi: 10th exercise, Wed 2003-11-19, U345 (2 pages)

`http://www.math.hut.fi/teaching/harmanal/`

Note: the next (11th) exercises are the last ones (Nov 26).

Exercise 10.1 ([p. 160] *average, short, quite good*):
Let $\psi \in \mathcal{S}(\mathbb{R}^n)$ satisfy $\psi(0) = 0$. Assume that $\sum_{j\in\mathbb{Z}} |\psi(2^{-j}\xi)|^2 = C_1\ \forall \xi \neq 0$. Set $\widehat{S_j f} := \psi(2^{-j}\xi)\widehat{f}(\xi)\ \forall \xi$. Use, e.g., the polarization identity to show that $\int_{\mathbb{R}} \sum_{j\in\mathbb{Z}} (S_j f)\overline{S_j g}\, dx = C_2 \int f\bar{g}\, dm$.

Exercise 10.2 ([p. 165] *easy, short, good*):
Prove Hörmander Theorem 8.10, i.e., that if $\sup_{j\in\mathbb{Z}} \|m(2^j\cdot\psi)\|_{\mathrm{L}^2_a} < \infty$, then T_m is bounded on $\mathrm{L}^p(\mathbb{R}^n)$, $1 < p < \infty$, where $\widehat{T_m f} = m\widehat{f}$. Assume that the theorem holds for $p \geq 2$.[1]

Exercise 10.3 ([p. 165] *average, quite short*):
Let $m \in \mathcal{C}^k(\mathbb{R}^n \setminus \{0\})$ be real-valued, $k \geq \lfloor \frac{n}{2} \rfloor + 1$. Assume that $|D^\alpha m(\xi)| \leq C_\alpha |\xi|^{-|\alpha|}\ \forall \xi \in \mathbb{R}^n \setminus \{0\},\ \forall |\alpha| \leq k$. Is $\mathrm{e}^{im(\cdot)}$ a multiplier on $\mathrm{L}^p(\mathbb{R}^n)$?[2]

Exercise 10.4 ([p. 168] *straight-forward*):
By Cor5.2.5 (p. 361) of [Grafakos], m is a multiplier on $\mathrm{L}^p(\mathbb{R}^n)$, $p \in (1,\infty)$, if $m \in \mathcal{C}^n(\mathbb{R}^n \setminus \{0\})$ and there is $A < \infty$ s.t. for all $k \in \{1,...,n\}$ and $j_1,...,j_k \in \{1,...,n\}$ we have $|(D_{j_1}...D_{j_k}m)(\xi)| \leq A|\xi_{j_1}|^{-1}...|\xi_{j_k}|^{-1}\ \forall \xi \in \mathbb{R}^n \setminus \{0\}$. Deduce from this that $m := [|\xi_1|^r + |\xi_2|^s]^{it}$ is a multiplier on $\mathbb{R}^2$.[3]

Exercise 10.5 ([p. 166] *average*):
Show that the operators T_j (down on page 166) are bounded on $\mathrm{L}^2(w)$.[4]

Solution to 10.1: Lemma 10.1A: If H, G are (complex) Hilbert spaces, $T \in \mathcal{B}(H,G)$ and $\|Tf\| = C\|f\|\ \forall f \in H$, then $\langle Tf, Tg\rangle = C^2\langle f, g\rangle\ \forall f \in H$.
Proof: $A, B \in \mathcal{B}(H)$ & $\langle f, Af\rangle = \langle f, Bf\rangle\ \forall f \Rightarrow \langle f, Ag\rangle = \langle f, Bg\rangle\ \forall f, g$, by Theorem 12.7. of [Rudin:FA]. Set $A := T^*T,\ B := C^2$.[5]

1° Apply Lemma 10.1A to $H = \mathrm{L}^2(\mathbb{R}^n)$, $G := \mathrm{L}^2(\mathbb{R}^n;\ell^2)$, $T := S$, $C := \sqrt{C_1}$, to observe that $\langle Sf, Sg\rangle_G = C_1\langle f, g\rangle_H\ \forall f, g \in H$, by 2°.[6]

2° Set $(Sf)_j := S_j f$, so that $\|Sf\|_2^2 = \|\widehat{Sf}\|_2^2 = \int_{\mathbb{R}^n} \|\widehat{Sf}(\xi)\|_{\ell^2}^2\, d\xi = \int_{\mathbb{R}^n} \sum_j |\psi(2^{-j}\xi)\widehat{f}(\xi)|^2\, d\xi = C_1 \int_{\mathbb{R}^n} |\widehat{f}(\xi)|^2\, d\xi = C_1\|\widehat{f}\|_2^2 = C_1\|f\|_2^2$.

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[1] Hint: $\|T_m f\|_p = \sup\{\langle T_m f, g\rangle \mid \|g\|_q \leq 1\} = ... = \sup\{\langle f, T_{\bar{m}} g\rangle \mid \|g\|_q \leq 1\} \leq ...$

[2] Hint: use induction to get a "rough" upper estimate of $|D^\alpha \mathrm{e}^{im}|$.

[3] Just use straight-forward computation to get A for $D_1 m$, $D_{11}m$, $D_{12}m$ (cases $D_2 m$ and $D_{22}m$ are similar), using the facts that if $z \in \mathbb{C} \setminus \{0\}$, then $D(f^z) = zf^{z-1}f'$ and hence $D|x|^z = z|x|^z/x\ \forall x \in \mathbb{R} \setminus \{0\}$.

[4] Use (3.9) (and Theorem 3.2, p. 51–52) to get a bound C s.t. $\|S_{a,b}\| \leq C\ \forall a, b$, then Minkowski Integral Inequality to T_j (line -8 on p. 166).

[5] Proof of Th 12.7.: Let $B = 0$ w.l.o.g. Since $\langle (f+g), A(f+g)\rangle = 0$, we have $2\,\mathrm{Re}\,\langle f, Ag\rangle = 0$. Replace f by if to observe that $\mathrm{Im}\,\langle f, Ag\rangle = 0$ too. Since f, g were arbitrary, $A = 0$, QED.

[6] Note that $\langle f, g\rangle_{\mathrm{L}^2(\mathbb{R}^n;F)} := \int_{\mathbb{R}^n} \langle f(x), g(x)\rangle_F\, dx$ and that the (vector-valued) Plancherel Theorem holds for $\mathrm{L}^2(\mathbb{R}^n;F)$ for any Hilbert space F, with the same proofs as in the scalar case. See, e.g., http://www.math.hut.fi/~kmikkola/research/thesis/ [M02] for details and references.

Solution to 10.2: Let $p^{-1} + q^{-1} = 1,\ q \geq 2$. Obviously, also $\bar{m}$ satisfies the condition, hence $\|T_{\bar{m}} f\|_q \leq C_q \|f\|_q\ \forall f \in \mathcal{S}(\mathbb{R}^n)$. But $\langle f, T_m^* g\rangle = \langle Tf, g\rangle := \int_{\mathbb{R}^n} T_m f \bar{g}\, dm = \langle m\widehat{f}, \widehat{g}\rangle = \langle \widehat{f}, \bar{m}\widehat{g}\rangle$, i.e., $T_m^* = T_{\bar{m}}$. Now $\|T_m f\|_p = \sup\{\langle T_m f, g\rangle \,\big|\, \|g\|_q \leq 1\} = \sup\{\langle f, T_{\bar{m}} g\rangle \,\big|\, \|g\|_q \leq 1\} \leq \|f\|_p C_q \cdot 1 = C_q \|f\|_p$, QED.[7]

Solution to 10.3: Let $\alpha \in \mathbb{N}^n$ be s.t. $|\alpha| \leq \lfloor \frac{n}{2} \rfloor + 1$. Set $C := \max\{1, \max_{\beta \leq \alpha} C_\beta\}$. Set $F(\xi) := \mathrm{e}^{m(\xi)}$. Then $D_j F = (D_j m) F$, hence, by induction, $D^\alpha F = F \sum_{j=1}^{r_\alpha} i^{k_j} \prod_{l=1}^{n_j} D^{\beta_{j,l}} m$, where $r_\alpha \leq |\alpha|!$, $n_j \leq |\alpha|$, $\sum_l \beta_{j,l} = \alpha\ \forall j$.

Therefore, $\|D^\alpha F(\xi)\| \leq \sum_{j=1}^{r_\alpha} \prod_{l=1}^{n_j} C |\xi|^{-|\beta_{j,l}|} \leq \sum_{j=1}^{r_\alpha} C^{|\alpha|} |\xi|^{-|\alpha|} \leq |\alpha|! C^{|\alpha|} |\xi|^{-|\alpha|}\ \forall \xi \in \mathbb{R}^n \setminus \{0\}$, Since $|\alpha| \leq \lfloor \frac{n}{2} \rfloor + 1$ was arbitrary, F is an L^p multiplier for any $p \in (1, \infty)$, by Corollary 8.11 (p. 165).

Solution to 10.4: 1° *If $z \in \mathbb{C} \setminus \{0\}$, then $D(f^z) = z f^{z-1} f'$ and $D|x|^z = z|x|^z/x\ \forall x \in \mathbb{R} \setminus \{0\}$.* Proof: $D\mathrm{e}^{z \log f} = \frac{zf'}{f} \mathrm{e}^{z \log f} = z f' f^{z-1}$. But $|x|' = \operatorname{sgn} x = |x|/x$.

2° Now $D_1 m(\xi) =: m_1(\xi) = it[]^{it-1} r |\xi_1|^r / \xi_1$, hence $|m_1(\xi)| \leq tr/|\xi_1|$ (because $|[]^{it}| = 1$ and $[] \geq |\xi_1|^r$). Moreover, $m_{12}(\xi) = it(it-1)[]^{it-2} r |\xi_1|^r s |\xi_2|^s / \xi_1 \xi_2$, hence $|m_{12}| \leq C/|\xi_1 \xi_2|$. Finally, $m_{11}(\xi) = it(it-1)[]^{it-2} r^2 |\xi_1|^{2r} / \xi_1^2 - \xi_1^{-2} r |\xi_1|^r []^{it-1} + r(r-1)|\xi_1|^r \xi_1^{-2} []^{it-1}$, hence $|m_{11}| \leq C' |\xi_1|^{-2}$. Similarly (exchange 1 and 2), $|m_2| \leq C''/|\xi_2|$ and $|m_{22}| \leq C'''/|\xi_2|^2$, hence the corollary applies, QED.

(Cases $t = 0$, $r = 0$ and $s = 0$ are almost trivial, so we assumed that $t, r, s \neq 0$.)

Remarks: Note that Theorem 8.14 of [Duo] implies Cor5.2.5 of [Grafakos] for $n = 2$ (with bounds $2A \log 2$, $2A \log 2$, $2A (\log 2)^2$, because $2 \int_{2^j}^{2^{j+1}} A t_1^{-1}\, dt_1 = 2A \log \frac{2^{j+1}}{2^j}$).

[7] By, e.g., Theorem B.4.12(b1), p. 935 of [M02], this "duality" method is applicable (even in the vector-valued case, which we do not need here).

Solution to 10.5: We shall prove more than required: that T_j is bounded on $\mathrm{L}^p(w)$ $(1 < p < \infty)$. By p. 166, we have

$$(T_j f)(x) = m(2^j)(S_j f)(x) + (S_{t,2^{j+1}} f)(x) \int_{2^j}^{2^{j+1}} (S_{t,2^{j+1}} f)(x)\, dm(t).$$

Obviously $M_a f := \mathrm{e}^{i 2\pi a \cdot} f$ satisfies $\|M_a f\|_p = \|f\|_p$, hence $\|S_{(a,b)} f\|_{\mathrm{L}^p(w)} \leq \|H\|_{\mathcal{B}(\mathrm{L}^p(w))} \|f\|_{\mathrm{L}^p(w)}\ \forall a, b \in \mathbb{R}$, by (3.9) [p. 58]. Set $C_{w,p} := \|H\|_{\mathcal{B}(\mathrm{L}^p(w))} < \infty$ (Theorem 3.2, p. 51–52), $C_m := \max\{\|m\|_\infty, \sup\{\|m\|_{\mathrm{BV(I)}} \,\big|\, I \text{ is dyadic}\}\} < \infty$. Then, by the Minkowski Integral Inequality,[8]

$$\begin{aligned} \Big\| \int_{2^j}^{2^{j+1}} (S_{t,2^{j+1}} f)(x)\, dm(t) \Big\|_p &\leq \int_{2^j}^{2^{j+1}} \|(S_{t,2^{j+1}} f)\|_p\, d|m|(t) && (1) \\ &\leq \int_{2^j}^{2^{j+1}} C_{w,p} \|f\|_{\mathrm{L}^p(w)}\, d|m|(t) && (2) \\ &\leq C_m C_{w,p} \|f\|_{\mathrm{L}^p(w)}, && (3) \end{aligned}$$

hence $\|T_j f\|_{\mathrm{L}^p(w)} \leq 2 C_m C_{w,p} \|f\|_{\mathrm{L}^p(w)}$.

[8] Recall (from, e.g., Exercises 7.12 and 7.13 of [Rudin:RCA]) that if m is of bounded variation (normalized to be right-continuous, which requires a redefinition on a countable set only), then there is a unique complex Borel measure μ s.t. $\mu(I) := \int_I dm(t)$ (the Stieltjes integral) for any interval I (conversely, given complex Borel measure μ, the limit $f(x) := \lim_{t \to x+} \mu((-\infty, t))$ exists $\forall x$ and f is normalized and BV). Moreover, the total variation T_m (where $T_m(t) - T_m(a) := \|m\|_{\mathrm{BV}((a,t])}$) satisfies $|\mu|(I) := \int_I d|m|(t)$, where $d|m|(t)$ means $dT_m(t)$. Thus, we can read the Stieltjes integral $\int_I g\, dm$ as the (equivalent) Lebesgue integral $\int_I g\, d\mu$ etc., hence Minkowski is applicable. For any complex measure μ and any $g \in \mathrm{L}^1(\mu)$ (which means that $g \in \mathrm{L}^1(|\mu|)$) we have $|\int_E g\, d\mu| \leq \int_E |g|\, d|\mu|$ (proof: write g as $G|g|$, $\int_E d\mu = \int_E h\, d|\mu|$ with $|G| = |h| = 1$ a.e. (Radon–Nikodym Theorem), etc.).