Mat-1.159 Harmoninen analyysi: 1th exercise, Wed 2003-9-17, U345 (3 pages)

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There will be two mid-term examinations (dates to be decided later). These exercises will not give you any bonus, but it is very useful to at least try them before reading the answers.

By "[p. 11]" we refer to page 11 of Duoandikoetxea. After the exercise number (see below) it refers to a page that you should reach before trying the exercise.

$\widehat{f}(k) := \int_T f(x)\mathrm{e}^{-2\pi ikx}\,dx$ for $f \in \mathrm{L}^1(T)$ [p. 2]; $T := [0,1]$[1] (*Fourier coefficients*); $\widehat{f}(\xi) := \int_{\mathbb{R}^n} f(x)\mathrm{e}^{-2\pi ix\cdot\xi}\,dx$ for $f \in \mathrm{L}^1(\mathbb{R}^n)$ [p. 11] (Fourier transform). In different books, you see 2π in different formulas/places (e.g., in the inverse formula only), but if $\mathcal{F}_1$ and $\mathcal{F}_2$ are such transforms, we always have $(\mathcal{F}_1\widehat{f})(\xi) = r(\mathcal{F}_2\widehat{f})(t\xi)$, where $r, t > 0$ are some universal constants, so there is no significant difference.

Exercise 1.1 ([p. 11] *easy, quick, useful*):
Show that (a) if $g(x) = c^{-n}f(c^{-1}x)$, then $\widehat{g}(\xi) = \widehat{f}(c\xi)$; (b) if A is an orthogonal transformation, then $\widehat{f(A\cdot)}(\xi) = \widehat{f}(A\xi)$.

Exercise 1.2 ([p. 2] *easy, quick, good*):
Let P be a trigonometric polynomial on T. (a) Show that $P(x) = \sum_k \widehat{P}(k)\mathrm{e}^{2\pi ikx}$. (b) Let $f \in \mathrm{L}^1(T)$ Prove that $(f * P)(x) = \sum \widehat{P}(k)\widehat{f}(k)\mathrm{e}^{2\pi ikx}$.

Exercise 1.3 ([p. 16] *easy, important*):
Prove: When $f \in \mathrm{L}^p(T)$, $1 \le p \le 2$, we have $\widehat{f} \in \ell^{p'}$ and $\|\widehat{f}\|_{p'} \le \|f\|_p$.[2]

Exercise 1.4 ([p. 9] *average*):
Suppose that f is a differentiable function on T whose derivative $f' \in \mathrm{L}^2(T)$. Prove that $f \in A(T)$ and that $\|f\|_A \le \|f\|_1 + \frac{1}{2\pi}(\sum_{j\ne 0} j^{-2})^{-1/2}\|f'\|_2$.[3]

Exercise 1.5 ([p. 10] *A bit difficult*):
(a) Show that $f \in \mathrm{L}^1(T)$, $f_n(t) := f(nt)$, $n \in \{1,2,3,\dots\}$ imply that $\widehat{f}_n(m) = \begin{cases} f(m/n), & \text{when } n|m; \\ 0, & \text{when } n\not| m. \end{cases}$

(b) Show that if $f \in \mathrm{L}^1(T)$ and $g \in \mathrm{L}^\infty(T)$, then $\lim_{n\to\infty}\int_T f(t)g(nt)\,dt = \widehat{f}(0)\widehat{g}(0)$.[4]

Solution to 1.1: **(a)** Now $\int_{x\in\mathbb{R}^n} \mathrm{e}^{-2\pi ix\cdot\xi}c^{-n}f(c^{-1}x)\,dx = \int_{y\in\mathbb{R}^n}\mathrm{e}^{-2\pi icy\cdot\xi}f(y)\,dy = \widehat{f}(c\xi)$, QED.

Remarks: (Here $c > 0$; for general $c \in \mathbb{R}\setminus\{0\}$ we should

[1] Note: by functions on T we mean functions $\mathbb{R}\to\mathbb{C}$ with period 1!

[2] Hint: The Riesz–Thorin Interpolation Theorem 1.19.

[3] Hint: partial integration.

[4] Hints: (a) $\sum_{l=0}^{n-1} a^l = (1-a^n)/(1-a)$; (b) start with f being a trigonometric polynomial.

have $g(x) = |c|^{-n}f(c^{-1}x)$.) The Jacobian matrix of $x = cy$ is $\frac{\mathrm{d}x}{\mathrm{d}y} = cI$, whose determinant (the *Jacobian*) is $|\frac{\mathrm{d}x}{\mathrm{d}y}| = c^n$. Recall: we have $\int_{g[\Omega]} f(x)\,dx = \int_\Omega f(g(y))||\frac{\mathrm{d}x}{\mathrm{d}y}||\,dy$, where $\frac{\mathrm{d}x}{\mathrm{d}y} = (\frac{\mathrm{d}g_i}{\mathrm{d}y_j})_{ij}$ is the *Jacobian matrix* of g, provided that, e.g., $g : \Omega \to \mathbb{R}^n$ is differentiable and 1-1, $\Omega \subset \mathbb{R}^n$ is open and f is measurable $\mathbb{R}^n \to [0,\infty]$ or integrable $\mathbb{R}^n \to \mathbb{C}$, by Theorem 7.26 of [Rudin:RCA].[5]

(b) $\widehat{f\circ A}(\xi) = \int_{\mathbb{R}^n} f(Ax)\mathrm{e}^{-2\pi i x\cdot\xi}\,dx = \int_{\mathbb{R}^n} f(y)\mathrm{e}^{-2\pi i A^{-1}y\cdot\xi}||A^{-1}||\,dy = \int_{\mathbb{R}^n} f(y)\mathrm{e}^{-2\pi i A^T y\cdot\xi}\,dy = \int_{\mathbb{R}^n} f(y)\mathrm{e}^{-2\pi i y\cdot A\xi}\,dy = \widehat{f}(A\xi)$, because $|A^{-1}| = 1/|A| = \pm 1$.[6]

Remarks: For historical reasons, *orthogonal transformation* means $A \in \mathbb{R}^{n\times n}$ s.t. $A^T = A^{-1}$ (equivalently, a unitary $A \in \mathbb{R}^{n\times n}$).[7] Note that $x\cdot y = y^T x$, hence $A^T x\cdot y = y^T A^T x = (Ay)^T x = x\dot{A}y$. In general (the complex case), $x\cdot y := y^* x$ ("orthogonal" means here "real unitary").

Solution to 1.2: **(a)** Set $e_k(x) := \mathrm{e}^{2\pi i k x}$, so that $\widehat{e_k}(j) = \delta_{jk}$ [p. 2]. If $P = \sum_{k=-N}^N \alpha_k e_k$, then $\hat{P}(j) = \sum_{k=-N}^N \alpha_k \widehat{e_k}(j) = \alpha_j$, hence $P = \sum_{k=-N}^N \alpha_k e_k = \sum_{k=-N}^N \hat{P}(k)e_k$.

(b) 1° $(f*\mathrm{e}^{2\pi i k\cdot})(x) := \int_T f(t)\mathrm{e}^{2\pi i k(x-t)}\,dt = \mathrm{e}^{2\pi i k x}\int_T f(t)\mathrm{e}^{-2\pi i k t}\,dt = \mathrm{e}^{2\pi i k x}\widehat{f}(k)$.

2° By (a), $(f*P)(x) = (f*\sum_k \hat{P}(k)e_k)(x) = \sum_k \hat{P}(k)(f*e_k)(x) = \sum_k \hat{P}(k)\mathrm{e}^{2\pi i k x}\widehat{f}(k)$, by 1°, QED.

Remarks: Recall that $(f*g)(x) := \int_T f(t)g(x-t)\,dt = \int_T f(x-t)g(t)\,dt$ (the *convolution* on T). It is linear in f (and in g).

Solution to 1.3: We have $\|\widehat{f}\|_\infty \le \int_T |f(x)||\mathrm{e}^{-2\pi i k x}|\,dx = \|f\|_1$ and $\|\widehat{f}\|_2 = \|f\|_2$. By the famous Riesz–Thorin Interpolation Theorem 1.19, $\hat{}$ is linear $\mathrm{L}^1 + \mathrm{L}^2 = \mathrm{L}^1 \to \ell^\infty + \ell^2 = \ell^\infty$, and $\|\widehat{f}\|_q \le M\|f\|_p$, when we set ($p_0 = 1$, $p_1 = 2$, $q_0 = \infty$, $q_1 = 2$, $M_0 = 1$, $M_1 = 1$ and) $\theta := (1^{-1} - p^{-1})/(1^{-1} - 2^{-1}) = 2 - 2/p \in [0,1]$, $q^{-1} = (1-\theta)/\infty + \theta/2 = \theta/2 = 1 - 1/p =: 1/p'$ (hence $q = p'$), and $M := M_0^{1-\theta}M_1^\theta = 1$, QED.

Remarks: Recall that $1/p + 1/p' = 1$ and that $\|\widehat{f}\|_{\ell^{p'}} := [\sum_k |\widehat{f}(k)|^{p'}]^{1/p'}$ ($p' < \infty$); $\|\widehat{f}\|_\infty := \sup_k |\widehat{f}(k)|$ (i.e. $\ell^p := \ell^p(\mathbb{Z})$ equals $\mathrm{L}^p(\mathbb{Z},\sigma)$, where σ is the counting measure so that $\int_E f\,d\sigma = \sum_{k\in E} f(k)$.) Trivially, the R-T Th is valid for $\theta \in \{0,1\}$ too.

Solution to 1.4: Obviously, $\|\widehat{f}(0)\| = |\int_T f| \le \|f\|_1$. For $k \ne 0$, we have $\widehat{f}(k) = \int_T f(t)e^{-2\pi i k t}\,dx = |_0^1 f(t)\frac{e^{-2\pi i k t}}{-2\pi i k} - \int_0^1 f'(t)\frac{e^{-2\pi i k t}}{-2\pi i k} = 0 + \frac{1}{2\pi i k}\int_0^1 f'(t)e^{-2\pi i k t}\,dx = \frac{1}{2\pi i k}\widehat{f'}(k) = g(k)\widehat{f'}(k)$ (because we require that $f(0) = f(1)$ for all functions $T \to \mathbb{C}$), where $g(k) := \frac{1}{2\pi i k}$ for $k \ne 0$, $g(0) = 0$. Therefore, $\|f\|_{A(T)} := \|\widehat{f}\|_1 := \sum_k |\widehat{f}(k)| \le \|f\|_1 + \|g\|_2\|\widehat{f'}\|_2$ (by the Hölder Inequality) $= \|f\|_1 + \frac{1}{2\pi}[\sum_{k\ne 0} k^{-2}]^{1/2}\|f'\|_2$ [Th1.9, p. 9].

Remarks: We assumed too much: it suffices that $f : T \to \mathbb{C}$ is absolutely continuous (this is the case if, e.g., f is differentiable everywhere). (In a Real Analysis exercise (from [Rudin:RCA]) we showed that the partial integration formula is valid for all AC

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[5] Note: $\int_{-1}^1 f(x)\,dx := \int_{(-1,1)} f(x)\,dx = \int_{(0,2)} f(1-y)|-1|\,dy =: \int_0^2 f(1-y)|-1|\,dy$. On $\mathbb{R}^1$ one often uses, instead of the above, a directed interval and does not take the absolute value of the Jacobian: $\int_{-1}^1 f(x)\,dx = \int_2^0 f(1-y)(-1)\,dy = \int_0^2 f(1-y)\,dy$. Such formulas would be too complex for more general sets than intervals, thus it is simpler to use the set $g[\Omega]$ (without any "positive direction") and the absolute value of the Jacobian.

[6] Because $1 = I = |A||A^{-1}| = |A||A^T| = |A|^2$.

[7] By induction, one can easily show that these are exactly the rotations plus the multiplication of some coordinates by -1 (I did not check this).

functions.)

Solution to 1.5: (a) (Case $n = 0$ is trivial.) 1° Let $m = kn$. Then (set $y := nx$) $\widehat{f_n}(m) = \int_0^1 f(nx)\mathrm{e}^{-2\pi i m x}\,dx = \int_0^n f(y)\mathrm{e}^{-2\pi i k y} n^{-1}\,dy = n^{-1}\sum_{l=0}^{n-1}\int_l^{l+1} = n^{-1}\sum_{l=0}^{n-1}\int_0^1 = n^{-1}\sum_{l=0}^{n-1}\widehat{f}(k) = \widehat{f}(m/n)$.

2° Let $m = kn + j$, $j \in \{1, \ldots, n-1\}$. $\widehat{f_n}(m) = \int_0^1 f(nx)\mathrm{e}^{-2\pi i(kn+j)x}\,dx = \int_0^n f(y)\mathrm{e}^{-2\pi i(k+j/n)y} n^{-1}\,dy = n^{-1}\sum_{l=0}^{n-1}\int_l^{l+1} = n^{-1}\sum_{l=0}^{n-1}\int_0^1 f(y)\mathrm{e}^{-2\pi i(k+j/n)(y+l)}\,dy = n^{-1}\sum_{l=0}^{n-1}\mathrm{e}^{-2\pi i j l/n}\int_0^1 f(y)\mathrm{e}^{-2\pi i(k+j/n)y}\,dy = 0$, because $\sum_{l=0}^{n-1}\mathrm{e}^{-2\pi i j l/n} = \sum_{l=0}^{n-1} a^l = (1-a^n)/(1-a) = 0$, where $a := \mathrm{e}^{-2\pi i j/n}$ (hence $a^n = \mathrm{e}^{-2\pi i j} = 1$).

(b) 1° Assume that f is a trigonometric polynomial ($\sum_{k=-N}^{N}$). Set $h_n(t) := g_n(-t)$ ($t \in \mathbb{R}$, $n \in \mathbb{N}$). Then $I_n := \int_T f(t)g(nt)\,dt = \int_T h_n(-t)f(t)\,dt = (h_n * f)(0) = \sum_k \widehat{f}(k)\widehat{h}_n(k)$, by exercise 2(b). But $\widehat{h}_n(k) = \widehat{g}_n(-k) = 0$ for $n > |k| \neq 0$ by (a), hence $I_n = \widehat{f}(0)\widehat{h}_n(0) = \widehat{f}(0)\widehat{g}(0)$ for $n > N$.

2° Set $M = \max\{1, \|f\|_1, \|g\|_\infty\}$. Given $\epsilon > 0$, choose a trigonometric polynomial P s.t. $\|f - P\|_1 < \epsilon/3M$ (possible, by Cor1.11, p. 10). Choose then N s.t. $\int_0^1 P(t)g(nt)\,dt = \hat{P}(0)\widehat{g}(0)$ for $n > N$. Then (note that $\|g_n\|_1 := \int_0^1 |g_n|\,dm \leq \|g_n\|_\infty$)[8]

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\begin{aligned}
\Big|\int_T f(t)g(nt)\,dt - \widehat{f}(0)\widehat{g}(0)\Big| &\leq \Big|\int_T [f(t)-P(t)]g(nt)\,dt\Big| + |\hat{P}(0) - \widehat{f}(0)||\widehat{g}(0)| && (1)\\
&\leq \|f-P\|_1\|g_n\|_\infty + \|P-f\|_1\|g_n\|_\infty && (2)\\
&\leq 2M\epsilon/3M < \epsilon. && (3)
\end{aligned}
$$

[8]We had to find some N for an arbitrary $\epsilon > 0$. If you first pick n and then some P for this n, you'll probably fail; we picked P first and only then N.