Helsinki University of Technology Londen/Mikkola

Mat-1.151 Complex Analysis (= kompleksianalyysi = funktioteoria)
Mid-term examination 2/2, 23.11.1998 at 16–19; reprint with answers

Note: you may return either this or the final examination, not both. Answer in English tai suomeksi eller på svenska.

Please write clearly on every sheet on separate lines:
1) course title, date
2) student number+letter, IN BLOCK LETTERS, surname underlined, all given names
3) study programme (Ke,Ko,M,Mk,P,R,S,Tf,Ti,Tu);
4) any former names and study programmes
5) signature.

1. An important result concerning entire functions is the Weierstrass Factorization Theorem

$$f(z) = z^m e^{g(z)} \prod_{n=1}^{\infty} E_{n-1}\left(\frac{z}{a_n}\right). \quad (1)$$

(a) Explain what the symbols and fuctions on the right side of (1) are.
(b) Indicate how (1) is proved (at least outline the proof, not all details are needed).

Solution: (scores: 444556666) *Notice: the whole examination is pretty tough (but students made them well anyway), hence the limits for the grades will be low.*

(a) (2 points) P. 266, Th8.2.4.: Explain E_n, a_n, g and m.

Let f an entire function. Suppose that f vanishes to order m at 0, $m \geq 0$. Let $\{a_n\}$ be the other zeros of f, listed with multiplicities. Then there is an entire function g s.t.

$$f(z) = z^m e^{g(z)} \prod_{n=1}^{\infty} E_{n-1}\left(\frac{z}{a_n}\right). \quad (1)$$

Here $E_p(z) := (1-z)\exp(z + z^2/2 + ... + z^p/p)$.

(b) (4 points) **Lemma** *Let $\{a_n\}$ have no (finite) accumulation points. Then $\prod_{n=1}^{\infty} E_{n-1}(z/a_n)$ converges uniformly on compact subsets of $\mathbb{C}$ to an entire function F. The zeros of F are the points $\{a_j\}$ (1/2 points).*

Proof: (1 point) Take N s.t. $|a_n| > 2r\ \forall n \geq N$ (OK, because only a finite number of them are in $\bar{D}(0,2r)$). $|1 - E_p(z)| \leq |z|^{p+1}$, when $|z| \leq 1$, so then $\sum_{n=N}^{\infty} |1 - E_{n-1}(z/a_n)| \leq \sum_{n=N}^{\infty} |z/a_n|^n \leq \sum_{n=N}^{\infty} 1/2^n < \infty$ on $\bar{D}(0,r)$, so the product converges normally to an entire function F and its zeros are those of $E_{n-1}(z/a_n)$ [Th8.1.9], which are clearly $\{a_n\}$ (clearly $E_n(w) = 0 \iff w = 1$), QED.

Lemma applies, because zeros of $f \not\equiv 0$ don't have accumulation points. Now $z^m F$ has the same zeros as f, so $f/z^m F$ is an entire and nonvanishing function, hence $e^g = f/z^m F$ for some entire g, QED.

2. Use Runge's Theorem to show that there is a sequence of holomorphic polynomials p_k such that $p_k(0) = 1$ for all k and $p_k \to 0$ pointwise on $\mathbb{C} \setminus \{x + i0 \,|\, x \geq 0\}$ as $k \to \infty$.

Solution: (345666666) This is Exa12.1.4, pp. 360–361.

3. Let U be a bounded open connected set and suppose that f is holomorphic on U. Show that then

$$\oint_\gamma f(z)\,dz = 0$$

for any closed curve $\gamma : [0,1] \to U$ such that $\mathrm{Ind}_\gamma(P) = 0$ for all points $P \in \mathbb{C} \setminus U$.

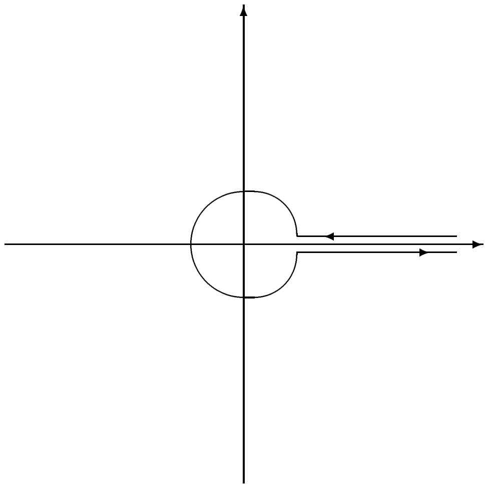
You may assume that Cauchy's Integral formula holds for a square Q:

$$\frac{1}{2\pi i} \int_{\partial Q} \frac{f(w)}{w-z}\,dw = \begin{cases} f(z), & \text{if } z \in \mathrm{Int}\,Q; \\ 0, & \text{if } z \notin \bar{Q}. \end{cases}$$

Solution: (345666666) This is the proof of Th11.2.4, pp. 338–340.[1] (There very many stupid obvious mistakes, despite of them I gave 6 points for many — but some not-completely obvious ones were penaltied.)

[1] Note that there's a finite number sides (of Q_js) in C_δ. After a side there is always

4. Let $\epsilon \in (0, 2\pi)$, let C_ϵ be as in the picture below. For fixed $z \in \mathbb{C}$, define $H_\epsilon(z) = \int_{C_\epsilon} \frac{(-w)^{z-1}e^{-w}}{1-e^{-w}}\,dw$. Using the fact that $\Gamma(z)\zeta(z) = \int_0^\infty \frac{t^{z-1}e^{-t}}{1-e^{-t}}\,dt$, for $\operatorname{Re} z > 1$, show that the zeta-function ζ can be extended analytically to $\mathbb{C} \setminus \{1\}$ by $\zeta(z) = -\frac{H_\epsilon(z)}{2i\sin(\pi z)\Gamma(z)}$.



The picture for problem 4.: The contour C_ϵ goes (from infinity) to the left with $\operatorname{Re} z = \delta > 0$, then (almost) around the circle $\partial D(0, \epsilon)$ and, finally, to the right with $\operatorname{Re} z = -\delta$ (to infinity). Here $\epsilon, \delta > 0$ are small.

Solution: (222334455) Prop15.2.3, p. 460–462 (pretty hard).

Writing the integral out (with t^z as $e^{z \ln t}$) etc.: 1 point. Tedious calculations: 2 points. Why $\epsilon \to 0+$ (and $\delta \to 0+$) OK? ($H_{\epsilon,\delta}$ is constant) (1 point) Why $0, -1, -2, \ldots$ OK? (poles and zeros simple, cancel) (1 point) Rest of the analytic continuation: 1 point. Morera not required (+).

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another (with a turn to right/none/left; we define to go to the right whenever possible, so the next side is always unique), and the finite number of sides guarantees that soon we will be at the same side again, hence C_δ is closed (although it may consist of several (possibly disjoint) cycles, but there's at most a finite number of them). Because all sides are oriented so that U_δ is on the left, we can nowhere go to the wrong direction. Thus integration over C_δ is well defined.