Helsinki University of Technology Londen/Mikkola

Mat-1.151 Complex Analysis (= kompleksianalyysi = funktioteoria)
Mid-term examination 1 / 26.10.1998 at 16–19

Please write clearly on every sheet on separate lines:
1) course title, date
2) student number+letter, IN BLOCK LETTERS, surname underlined, all given names
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This version contains also the answers. The part of the book corresponding to this examination is pp. 1–241 (approximately Chapters 1–7), excluding the proof of the Riemann Mapping Theorem.

1. Let $U \subseteq \mathbb{C}$ be open. Let $f : U \to \mathbb{R}$ be continuous. Suppose that for each $\bar{D}(P, r) \subseteq U$

$$f(P) \le \frac{1}{2\pi} \int_0^{2\pi} f(P + re^{i\theta})\, d\theta \qquad (*).$$

Show that f is subharmonic.

Solution: See the first part of the proof of Prop7.7.4, p.231–232.

Because of some misunderstandings, I'll repeat the proof here[1]

Let's assume that $(*)$ holds but f is not subharmonic, i.e., for some $\bar{D}(P, r) \subset U$ and for some real-valued,[2] harmonic h defined[3] on a neighborhood of $\bar{D}$ ($D := D(P, r)$) we have $f \le h$ on ∂D but $f(Q) > h(Q)$ for some $Q \in D$.[4]

$g := f - h \in \mathcal{C}(\bar{D})$, $g \le 0$ on ∂D but $g(Q) > 0$. Thus, $M := \max_{\bar{D}} g > 0$.[5]
$K := \{z \in \bar{D} \mid g(z) = M\}$ is closed[6], hence it's compact.[7]

On the other hand $K \cap \partial D = 0$. The rest can be done in several ways, two of which are given below:

Way 1) Take $z \in \partial K$ and a $R \in (0, d(z, \partial D)$ so small that $g(z') < M$ for some $z' \in \partial D(z, R)$.[8] Then $(*)$ and the continuity of g imply that $g(z) < M$ (details in the book), a contradiction.

Way 2) If $z \in K$ and we take $D(z, R) \subset D$, Then $(*)$ and the continuity of g imply that $g(z') = M$ for all z' (details in the book), hence K is open. Because K is both open and closed and nonempty, $K = \partial D$ (by the definition of connectedness), a contradiction.

Points: The definition of subharmonicity or its contrahypothesis gives one point. A (useful) bit more gave another. I assumed elementary topology to be known (so "K is compact" is (almost) OK, without explanations etc., although requiring a short explanation would have been a reasonable possibility). Two students had a wrong definition but almost correct proof, both got 3 points.

2. Let $U \subseteq \mathbb{C}$ be a connected open set. Let $f : U \to \mathbb{C}$ be holomorphic. Let $Z = \{z \in U \mid f(z) = 0\}$. Show that if Z has an accumulation point in U, then $f(z) \equiv 0$.

Solution: This seems to have been hard to remember/invent, as one could guess (try to remember to prove first that $z_0 \in E$, then $E = U$)). See the proof of Th3.6.1, pp. 93–94. Roughly it goes as follows:

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[1] Topological details are assumed to be known and can be found in [Rudin:PMA].

[2] The condition $f \le h$ says "f and h are real and $f \le h$", so I have allowed omitting this word. Note that subharmonic functions are always real, unlike harmonic ones (whose real and imaginary parts are real and harmonic).

[3] Note: "for some h", not "for all h". The counter assumption of $\forall h\ [P(h)]$ ("$P(h)$ is true for all h") is $\neg\forall h\ [P(h)]$ ("$P(h)$ does not hold for all h"), i.e., $\exists h\ [\neg P(h)]$ ("$P(h)$ does not hold for some h"). By induction, we get: $\neg\forall h\ \forall g\ \exists f\ \forall F\ [P(f, g, h, F)] \iff \exists h\ \neg\forall g\ \exists f\ \forall F\ [P(f, g, h, F)] \iff \exists h\ \exists g\ \neg\exists f\ \forall F\ [P(f, g, h, F)] \iff \exists h\ \exists g\ \forall f\ \neg\forall F\ [P(f, g, h, F)] \iff \exists h\ \exists g\ \forall f\ \exists F\ [\neg P(f, g, h, F)]$ (by common sense you should notice this even without the above formal calculation).

[4] Some students have taken $Q = P$, but that assumption is not justified.

[5] A continuous function has a maximum on a compact set.

[6] Because $K = g^{-1}[\{M\}]$ and preimage of a closed set is closed (because $g \in \mathcal{C}$)

[7] A set $K \subset \mathbb{R}^n$ (or $K \subset \mathbb{C}^n$) is compact iff it is closed and bounded.

[8] $z \in \partial K$ means that you can find a $z' \in K^c$ arbitrarily near z.

1. Let $Z \ni z_k \to z_0 \in U$. We will prove $z_0 \in E := \{z \in U \mid \left(\frac{\partial}{\partial z}\right)^n f(z) = 0 \; all n\}$.

Suppose that $\left(\frac{\partial}{\partial z}\right)^j f(z_0) \neq 0$ for some j and let n_0 be least such j. Define $g(z) := \frac{f(z)}{(z-z_0)}^{n_0} = \sum_{j=n_0}^{\infty} \left(\frac{\partial}{\partial z}\right)^j f(z_0) \frac{(z-z_0)^{j-n_0}}{j!}$ (where n_0 is the multiplicity of the zero of f in z_0), for which $0 \neq g(z_0) = \lim_k g(z_k) = 0$, a contradiction, hence $z_0 \in E$.

2. Then show that series expansion makes E open and E's definition makes E closed, so by the connectedness of U, $E = U$ (because E is nonempty).[9]

Points: Three points for 1. and 2. each.

3. Let f be holomorphic on $D(P,r) \setminus \{P\}$ and have a pole at P.
(a) Show that $1/f$ has a removable singularity at P.
(b) Let k be the order of the zero of $1/f$ at P. Prove that $(z-P)^k f$ has a removable singularity at P.

Solution: This is easy. The pole means that $\exists \lim_{z\to P} |f(P)| = \infty$ [p.107]. Because the zeros of a holomorphic[10] function f form a discrete set Z, the function $g := 1/f$ is defined and holomorphic in the open set[11] $U \setminus \{P\}$, where $U := D(P,r) \setminus Z$.

(a) $\lim_{z\to P} |g(z)| = \lim_{z\to P} 1/|f(z)| = 0$, hence $\lim_{z\to P} g(z) = 0$. Thus, g is bounded near P, hence g has a removable singularity at P [p.107].[12][13]

(b) So $1/f(z) =: g(z) = (z-P)^k h(z)$ on U,[14] where $h \in H(U)$ and $h(P) \neq 0$. $F(z) := (z-P)^k f(z) = 1/h(z)$ on $U \setminus \{P\}$ (remember that f and hence also F is not defined at P), so $\lim_{z\to P} f(z) = 1/h(P)$, therefore F has a removable singularity at P.

Points:

4. Construct a linear fractional transformation which maps $D(0,1)$ onto the half-plane $\operatorname{Re} z < 0$.

Solution: Cayley's transformation $f(z) := \frac{z-i}{z+i}$ maps $\{\operatorname{Im} z > 0\}$ conformally onto $D(0,1)$, hence its inverse g maps $D(0,1)$ conformally onto $\{\operatorname{Im} z > 0\}$. Thus, $F := ig$ maps $D(0,1)$ conformally onto $\{\operatorname{Re} z < 0\}$ (because $h : z \mapsto iz$ maps $\{\operatorname{Im} z > 0\}$ conformally onto $\{\operatorname{Re} z < 0\}$).[15]

But what is g and is ig a LFT? $w = f(z) \iff (z+i)w = (z-i) \iff z(w-1) = -i - iw \iff z = -i\frac{w+1}{w-1} =: g(w)$, hence $F(w) := ig(w) = \frac{w+1}{w-1}$, which is an LFT (of the form $(aw+b)/(cw+d)$ with $ad - bc \neq 0$, i.e., well-defined and nonconstant [p.186]), QED.

A second way is to write $F(w) := (aw+b)/(cw+d)$ with $ad - bc \neq 0$ and solve for a, b, c, d (and refer to the theory in the book). *Note that our solution is not unique.*

Points: We didn't require the definition of LFT, but that with nothing more gave 1 point. 3 points for the above solution without specifying f (and hence g, F).

[9] You cannot claim without proof, that $\forall z \in E \; \forall D(z,r) \subset U [D(z,r) \subset E]$ implies $E = U$, although it seems intuitively plausible (-1 point).

[10] "Holomorfinen" kirjoitetaan yhdellä f-kirjaimella. Se myös lausutaan kuten kirjoitetaan, samoin "vampyyri" ja "kampanja" (muttei "tulepa tänne"). Harjoittele!

[11] U is open, because Z is closed.

[12] A second way: take $D(P,R) \subset U$ s.t. $|f| > 1$ on $D(P,R)$. Then $|g| < 1$ on $D(P,R)$, QED.

[13] One often extends $\mathbb{R}$ to $\bar{\mathbb{R}} := \mathbb{R} \cup \{+\infty, -\infty\}$ with a certain metrizable topology making limit theory to the one you've been taught at high school (and making $\bar{\mathbb{R}}$ compact). The same cannot be done for $\mathbb{C}$ in a reasonable way, but one often defines $\bar{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ (metrizable and compact too), as done in the book.

[14] In fact, we should write $\hat{g}(z) = (z-P)^k h(z)$, where $\hat{g} \in H(U)$ is the holomorphic extension of g, but usually mathematicians trust that the reader understands that here g means the extension of the original g.

[15] Because $y > 0 \iff \operatorname{Re} i(x+iy) = -y < 0$, where $z := x + iy$; also h is holomorphic and has a holomorphic inverse $z \mapsto -iz$.