Mat-1.151 Complex Analysis (kompleksianalyysi), 9th (= last) exercise Wed 18.11.1998 at 12.15–14, U344 (2 page(s))

This is the *last exercise*!!!

The book: Sections 13.1, 15.1, 15.2, 15.3, 15.4.

What applications do schlicht functions have? At least one can use them for solving the Bieberbach conjecture, according to Ville Havu...

Exercise IX.1 ([p.386] *quite easy, quite short*):
[Exe13.9] Let S be the family of all schlicht functions. Fix an integer $k > 1$. Prove that there is a number $C_k > 0$ s.t. if $S \ni f = z + \sum_{k=2}^{\infty} a_k z^k$, then $|a_k| \le C_k$.[1]

Solution: We know that $h := \frac{1}{f} = \frac{1}{z} + \sum_{k=0}^{\infty} b_k z^k$ and that $\sum_{k=1}^{\infty} k|b_k|^2 \le 1$ [Th13.1.3], hence $|b_k| \le 1\ \forall k \ge 1$. Take some $z_0 \in \mathbb{D}$ with $f(z_0) \ne 0$. In some neighborhood of z_0 the Laurent series for h and f converge absolutely and uniformly and whatever we might wish, so at least there[2]

$$1 \equiv h(z)f(z) = (\frac{1}{z} + \sum_{k=0}^{\infty} b_k z^k)(z + \sum_{k=2}^{\infty} a_k z^k)$$

$$= 1 + \sum_{k=1}^{\infty} a_{k+1} z^k + \sum_{k=1}^{\infty} b_{k-1} z^k + \sum_{k=0}^{\infty} [\sum_{j=0}^{k} a_j b_{k-j}] z^k.$$

The uniqueness of coefficients in that (or any) neighborhood (on the left and on the right) gives us (note that $a_0 = 0$, $a_1 = 1$) z^0: $1 = 1$; z^1: $0 = a_2 + b_0 + [0 + a_1 b_0]$, or $b_0 = -a_2/2$, hence $|b_0| = |a_2|/2 \le 1$ [Th13.1.5]. So now we know that $|b_k| \le 1\ \forall k$. Moreover, we have $|a_k| \le C_k$ with $C_0 = 0,\ C_1 = 1,\ C_2 = 2$. z^k: $0 = a_{k+1} + b_{k-1} + \sum_{j=0}^{k} a_j b_{k-j}$, so $|a_{k+1}| \le |b_{k-1}| + \sum_{j=0}^{k} C_j |b_{k-j}| \le 1 + \sum_{j=0}^{k} C_j =: C_{k+1}$. Thus we get a recursive formula for all k, QED.[3]

[1] **Hint:** Use Th13.1.3 and Th13.1.5 to get $|b_k| \le 1\ \forall k \ge 0$, then study $f \cdot \frac{1}{f} \equiv 1$.

[2] See, e.g., [Rudin:PMA:Th3.50]. Note that we need this only in one point (let's say z_0), so it would be enough to have absolute convergence there. Oops, no, we need really a neighborhood, so that Taylor's Theorem gives us the uniqueness.

[3] You could easily calculate this out, as well you might get a better limit.

Exercise IX.2 ([p.454] *not hard*):
[Exe 15.5] Show that $\frac{1}{\Gamma(z)\Gamma(-z)} = \frac{-z\sin(\pi z)}{\pi}$ (for $z \in \mathbb{Z}$) (use, e.g., Prop15.1.9). Conclude that $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$. (*Hint: use the lemma below.*)

Solution: Formula $\frac{1}{\Gamma(z)} = z e^{\gamma z} \prod_{j=1}^{\infty} \left[(1 + z/j)e^{-z/j}\right]$ gives us $\frac{1}{\Gamma(z)\Gamma(-z)} = -z^2 e^{\gamma(z-z)} \prod_{j=1}^{\infty} \left[(1+z/j)e^{-z/j}\right] \prod_{j=1}^{\infty} \left[(1-z/j)e^{+z/j}\right] = -z^2 \prod_{j=1}^{\infty}(1 - z^2/j^2)$. Now we should show $\sin \pi z$ equal to $\frac{-\pi}{z} \frac{1}{\Gamma(z)\Gamma(-z)} = \pi z \prod_{j=1}^{\infty}(1 - z^2/j^2)$, and this follows from the Lemma below.

Now $\Gamma(z)\Gamma(1-z) = \Gamma(z)(-z)\Gamma(-z) = \frac{\pi}{\sin \pi z}$ (note: thus $\Gamma(1/2)^2 = \pi$, i.e., $\Gamma(1/2) = \sqrt{\pi}$ (because clearly $\Gamma \ge 0$ on $\mathbb{R} \setminus (-\mathbb{N}_0)$), QED.

Lemma $\sin \pi z = \pi z \prod_{j=1}^{\infty}(1 - z^2/j^2)\ \forall z \in \mathbb{C}$.

Proof: See [Rudin:RCA:Exe15.4,p.316].[4]

Exercise IX.3 ([p.447] *very easy, short*):
[Exe 15.3] Fix $k \in \mathbb{N}_+$. What does the Prime Number Theorem say about how many primes there are between N^k and N^{k+1} as $N \to \infty$?

Solution: $\pi(N^{k+1}) - \pi(N^k) \approx \frac{N^{k+1}}{\ln N^{k+1}} - \frac{N^k}{\ln N^k} = \frac{N^{k+1}}{(k+1)\ln N} - \frac{N^k}{k \ln N} = \frac{N^{k+1}}{(k+1)\ln N}(1 - \frac{k+1}{kN}) \approx \frac{N^{k+1}}{(k+1)\ln N} =: a_N \approx \pi(N^{k+1})$. So almost all primes $\le N^{k+1}$ are $\ge N^k$.

$\approx$ is not an exact guarantee for our claim, so let's give one:
$\frac{N^k}{\ln N^k} / \frac{N^{k+1}}{\ln N^{k+1}} = (k+1)/Nk \to 0$, so

$$\frac{\pi(N^k)}{\pi(N^{k+1})} = [\frac{N^k}{\ln N^k} / \frac{N^{k+1}}{\ln N^{k+1}}][\pi(N^k)/\frac{N^k}{\ln N^k}]/[\pi(N^{k+1})/\frac{N^{k+1}}{\ln N^{k+1}}] \to 0 \cdot 1 \cdot 1 = 0,\ QED.$$

Exercise IX.4 ([p.457] *quite easy, short*):

[4] Both functions are entire and have the same simple zeros $z \in \mathbb{Z}$ (the product converges uniformly by Th8.1.9, and by the same Th8.1.9 the zeros are the zeros of the terms). Thus their quotient is an entire, nonvanishing function. If we could show that both of them are periodic and their quotient is bounded on a strip, then the quotient would be bounded, hence constant, and we should then show that this constant is 1. Unfortunately, this is harder than the way [Rudin:RCA:Exe15.4,p.316] suggests.

[Exe 15.19] Prove that if $\operatorname{Re} z > 1$, then

$$\zeta(z)^2 = \sum_{m=1}^{\infty} \frac{d(m)}{m^z},$$

where $d(m)$ is the number of divisors of m and ζ is the zeta-function.

Solution: $(\sum_{n=1}^{\infty} a_n)(\sum_{n=1}^{\infty} b_n) = \sum_{n=1}^{\infty} \sum_{k=1}^{n} a_k b_{n-k+1}$ [Rudin:PMA:Th3.50,p.74]. Thus, $\zeta(z)^2 := [\sum_{n=1}^{\infty} \frac{1}{n^z}]^2 = \sum_{n=1}^{\infty} \sum_{k=1}^{n} \frac{1}{k^z} \frac{1}{(n-k+1)^z}$

$$= \sum_{n=1}^{\infty} \sum_{k=1}^{n} \frac{1}{[k(n-k+1)]^z}. \ (*)$$

For some $m \in \mathbb{N}_+$, how many times does $\frac{1}{m^z}$ appear in the formula?

For each divisor k of m we have exactly one $\frac{1}{m^z}$ in the sum, namely the one with $n = k + m/k - 1$ (because then and only then $k(n-k+1) = k(m/k) = m$).[5] Thus, $\sum_{m=1}^{\infty} \frac{d(m)}{m^z}$ $(**)$ is series $(*)$ rearranged, QED.[6]

Exercise IX.5 (*very easy, very short*):
[Rudin:FA:Exe 9.8] (a) If $a > 0$, $ax = 1$, prove that $1 - a < a[x] \leq 1$, where $[x] := \lfloor x \rfloor := \max\{n \in \mathbb{Z} \,|\, n \leq x\}$. (b) Deduce from this that $e^x K(x)$ is bounded (as asserted in the proof of [Rudin:FA:Th9.12]),[7] where $K(x) := 2k(x) - k(x-1) - k(x-\lambda)$, $\lambda \in \mathbb{R}$, $k(x) := [e^x]e^{-x}$.

Solution: (a) $1 = ax \geq a[x] > a(x-1) = 1 - a$.
(b) $1 \geq k(x) > 1 - e^{-x}$, by (a), so $e^x K(x) \geq e^x\{2 - 2e^{-x} - 1 - 1\} = -2$ and $e^x K(x) \leq e^x\{2 - (1 - e^{-(x-1)}) - (1 - e^{-(x-\lambda)})\} \leq e^1 + e^\lambda < \infty$, QED.

[5] And there are no other terms in $(*)$, because $m = k(n-k+1)$ is divisible by k.

[6] To be exact: Fix z with $\operatorname{Re} z > 0$. The series converges absolutely, hence so does the one for $\zeta(z)^2$, hence does any rearrangement of it (to the value $\zeta(z)^2$), QED. (Note that there clearly is a bijection between the terms of $(*)$ and $(**)$, so they are rearrangements of each other.)

[7] This exercise is all too easy, but it is part of the alternative proof of the Prime Number Theorem given in [Rudin:FA].