Mat-1.151 Complex Analysis (kompleksianalyysi), 8th exercise Wed 11.11.1998 at 12.15–14, U344 (3 page(s))

(Exercise IX (Wed 18.11.98) will be the last one.)

The book: Sections 11.1, 11.2, 11.3, the statement of Th11.4.1, 12.1. & the statement of Th12.2.1

Note: If U has 2 holes, then γ is homologous to 0 without being homotopic to a constant curve (cf. Lemma 11.2.3), when γ goes around hole 1, then around hole 2, then backwards around hole 1, and, finally, backwards around hole 2. Sec11.5 (skipped) examines the commutativity of this kind of routes.

Note: Runge's Theorem 11.4.3 (or 12.1.1): $f \in H(U)$ is the normal limit of polynomials, if U is bounded, connected and "without holes".

Note: The proofs here are not always rigorous. It is just that you should write down much more details to get them exact (or find a more clever way to solve them).

Simple connectivity: If U is open, we may define $\pi_1(U)$ as in pp. 334–335 (cf. VIII.1) and say that U is *topologically simply connected* iff $\#\pi_1(U) = 1$, i.e., iff $\pi_1(U)$ contains only the neutral element $[e]$, the equivalence class of the constant curve, i.e., iff all closed curves in U are homotopic. Thus, for any $E \subset \mathbb{C}$, we define that E is topologically simply connected iff all closed curves in E are homotopic. Clearly simply connected $\Longrightarrow$ path connected $\Longrightarrow$ connected (but the converse does not hold).

In Th11.4.1 it is proved, that a bounded, connected open $U \subset \mathbb{C}$ is simply connected iff $\mathbb{C} \setminus U$ is connected, i.e., iff U does not have holes (which would be the bounded components of $\mathbb{C} \setminus U$. However, this is not true for general $E \subset U$, as the following example shows:

Let $K' \subset \mathbb{C}$ be the union of $S := \{t + i\sin(1/t) \mid t \in (0,1]\}$ ("the topologist's sine curve" (sp?)) and $[-1,1]i$.[1] The set K' is compact and connected (because $K' = \bar{S}$ and S is connected, because it is the continuous image of a connected set $(0,1]$), but not path connected.[2]

Let $K := K' \cup K''$, where K'' consists of line elements $i \to 1+2i$ and $1+2i \to 1 + i\sin 1$. Then K is compact (union of two compact sets) and path connected (because it can be parametrized by $\Gamma : (0,2] \to \mathbb{C}$ (or $\Gamma : (0,1] \to \mathbb{C}$ as well, if we rescale the parametrization of S)) and even simply connected.[3]

Still K has a hole: $\mathbb{C} \setminus K = U_1 \cap U_2$, where U_1 is the plane outside K and U_2 is the hole inside K.[4]

Examinations: Second half-term examination and full examination are on Monday 23th, at 16–19, as was informed earlier. There will not be this kind of too hard plane geometry problems, we may use some intuition and we will concentrate on things studied in the book, not on extra knowledge contained in these exercises.

Exercise VIII.1 ([p.310] *not hard*):
[Exe11.22] Let U be a domain (a connected open set).[5] Suppose that $\Gamma : [0,1] \to U$ is a (fixed) continuous curve with $\Gamma(0) = P \in U$ and $\Gamma(1) = Q \in U$. For each continuous, closed curve $\gamma : [0,1] \to U$ with $\gamma(0) = P = \gamma(1)$, define $T(\gamma) := \Gamma \cdot \gamma \cdot \Gamma^{-1} : [0,1] \to U$ by

$$(\Gamma \cdot \gamma \cdot \Gamma^{-1})(t) := \begin{cases} \Gamma(1-4t), & \text{if } 0 \le t \le 1/4; \\ \gamma(2(t-1/4)), & \text{if } 1/4 < t \le 3/4; \\ \Gamma(4(t-3/4)), & \text{if } 3/4 < t \le 1. \end{cases}$$

Then clearly $(T\gamma)(0) = Q = (T\gamma)(1)$. Show that

(a) (with Γ fixed) the equivalence class $[T\gamma]$ in $\pi_{1,Q}(U)$ (which means $\pi_1(U)$ based at Q) depends only on $[\gamma] \in \pi_{1,P}(U)$.

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[1] Any point $0 + iy$ with $-1 \le y \le 1$ would be enough to make the set connected and compact and non-path connected.

[2] Because any $\gamma : [0,1] \to K'$ with $\gamma(0) = 0 \wedge \gamma(1) = 1 + i\sin 1$ would be uniformly continuous (because it is continuous and $[0,1]$ is compact, see [Rudin:PMA]). Take $b := \sup\{t \mid \operatorname{Re}\gamma(t) = 0\} < 1$. For any $\delta > 0$ we have points $t_1, t_2 \in [b, \min\{1, b+\delta\}]$ s.t. $\operatorname{Im}\gamma(t_1) = 1 \wedge \operatorname{Im}\gamma(t_2) = -1$, so γ is not uniformly continuous, a contradiction, QED.

[3] If γ_1 is a closed curve in K, then it can't go through the "non-path-continuity gap" from $[-1,1]i$ to the right, so there can be no loops. Thus we may shrink γ_1 to a point (a constant curve), do the same for any γ_2 and then move these together along Γ. Thus, any two closed curves in K are homotopic, i.e., K is topologically simply connected.

[4] Non-rigorous proof: $\bar{U}_2 \subset U_2 \cup K$ and $\bar{U}_1 \subset U_1 \cup K$ have no common points, so they are maximal connected subsets (i.e., connected components) of U. (E.g., if $x + yi \in \bar{U}_2$, then 1∘ if $x = 0$, then clearly $y \in [-1,1]$, so $z \in K' \subset K$. 2∘ if $x > 0$, then it is enough to study the half-plane $\operatorname{Re} z > x/2$, which contains only points of S with $t \in [x/2, 1]$, and in that compact set the curve is uniformly continuous, hence clearly $y \ge \sin x$, and so $x + iy \in K \cup U_2$. Thus, $\bar{U}_2 \subset K \cup U_2$. Similarly, $\bar{U}_1 \subset K \cup U_1$.)

[5] Any path connected $U \subset \mathbb{C}$ would do (because Γ must exists but more is not needed). An open connected set is always path connected.

(b) the mapping $S : [\gamma] \to [T\gamma]$ is a group homomorphism $\pi_{1,P}(U) \to \pi_{1,Q}(U)$.
(c) $\pi_{1,P}(U)$ is isomorphic to $\pi_{1,Q}(U)$, for all $P, Q \in U$.

Solution: Note that, by (c), $\pi_{1,P}(U)$ is independent of P and may thus be called $\pi_1(U)$, as is done in the book.

(a) If $\gamma_1 \sim \gamma_2$ (i.e., $[\gamma_1] = [\gamma_2]$ in $\pi_{1,P}(U)$) via H (cf. p. 334), then $T\gamma_1 \sim T\gamma_2$ (i.e., $[T\gamma_1] = [T\gamma_2]$ in $\pi_{1,Q}(U)$) by h given below, hence S is well-defined ($S[\gamma]$ means one element of $\pi_{1,Q}(U)$ only).[6]

$$h(s,t) := \begin{cases} \Gamma(1-4t), & \text{if } 0 \le t \le 1/4; \\ H(s,(2t-1/4)), & \text{if } 1/4 < t \le 3/4; \\ \Gamma(4(t-3/4)), & \text{if } 3/4 < t \le 1. \end{cases}$$

(b) Clearly $Te \sim e$,[7] so $S[e] = [e]$. $T\gamma_1 \cdot T\gamma_2 = \Gamma \cdot \gamma_1 \cdot \Gamma^{-1} \cdot \Gamma \cdot \gamma_2 \cdot \Gamma^{-1} \sim \Gamma \cdot \gamma_1 \cdot \gamma_2 \cdot \Gamma^{-1} \sim T(\gamma_1 \cdot \gamma_2)$ (cf. pp. 334–335), hence $S[\gamma_1 \cdot \gamma_2] := [T(\gamma_1 \cdot \gamma_2)] = [T\gamma_1 \cdot T\gamma_2] =: [T\gamma_1] \cdot [T\gamma_2] =: S[\gamma_1] \cdot S[\gamma_2]$, so S (but not T) is a group homomorphism.[8]

(c) Let $V\gamma := \Gamma^{-1} \cdot \gamma \cdot \Gamma$. Then, as proved above for T (but with the roles of P and Q exchanged), $W : [\gamma] \mapsto V[\gamma]$ is a group homomorphism $\pi_{1,Q} \to \pi_{1,P}$. Moreover, clearly $T(V\gamma) \sim \gamma\ \forall \gamma \in \pi_{1,Q}$, i.e., $SW = I$, similarly $WS = I$, so S is a group isomorphism (a bijective homomorphism, whose inverse mapping is also a homomorphism) $\pi_{1,P} \to \pi_{1,Q}$, QED.

[6]The continuity of h should be intuitively clear (and is easy to prove, because H is uniformly continuous, because $[0,1]^2$ is compact): it deforms γ_1 (i.e., the middle part of $T\gamma_1$) to γ_2 with the start and end fixed (to what is common for $T\gamma_1$ and $T\gamma_2$).

[7]To be exact, $Te_P \sim e_Q$, where $e_P \equiv P$ and $e_Q \equiv Q$. Just let Te_P deform to e_Q via Γ, it is even very easy to write down the formula.

[8]You should know this, if you have studied algebra (group theory): If G and H are groups and $S : G \to H$ satisfies $Sa \cdot Sb = S(a \cdot b)$ and $Se_G = e_H$, then S is a group homomorphism, i.e., it preserves the group structure, because $Sa \cdot Sa^{-1} = S(a \cdot a^{-1}) = Se_G = e_H$, so $Sa^{-1} = (Sa)^{-1}$ (and associativity is clearly preserved and rest of the group structure ($\cdot$ and neutral element) is assumed above).

Note that $Se_G = e_H$ is necessary: Although $Se_G \cdot Sa = S(e_G \cdot a) = Sa\ \forall a \in G$ says that Se_G is the neutral element of $S[G]$, and hence S is a group homomorphism to $S[G]$, e_G still need not be equal to e_H: take, e.g., $G = \mathbb{C}^{2\times 2} = H$ and $Sg := [1,0;0,0]\ \forall g$, so that $Sa \cdot Sb = [1,0;0,0] = S(a \cdot b)\ \forall a,b$, but $Se_G = [1,0;0,0] \neq [1,0;0,1] = I = e_H = e_g$.

Exercise VIII.2 ([p.342]):
[Exe 11.10] Let $K \subset \mathbb{C}$ be compact and topologically simply connected. Is K^o necessarily simply connected? (b) If not, what reasonable extra condition makes it simply connected?

Solution: Remember, that a simply connected set is a connected set with no holes.[9]

No, e.g., let $K := \bar{D}(2,1) \cup \bar{D}(-2,1) \cup [-2,2]$, the "eye-glasses set", then $K^o = D(2,1) \cup D(-2,1)$ is not even connected.

(b) *"K^o is connected"* might be one, but it is not easy to prove, because the simple connectivity of K does not imply that $\mathbb{C} \setminus K$ were conneced (unlike for open sets, by Th11.4.1), see the beginning of this paper. Thus we don't give an answer here.

Exercise VIII.3 ([p.365]):
[Exe 12.4] Let $\gamma : [0,1) \to \mathbb{C}$ be a nonself-intersecting, piecewise linear curve s.t. $\lim_{t\to 1-} \gamma(t) = \infty$. Prove that there is a sequence of entire functions $\{h_j\}$ s.t. $\lim_{j\to\infty} h_j(\gamma(t)) = 1/(1-t)\ \forall t \in [0,1)$ but $h_j \to 0$ uniformly on compact subsets of $V := \mathbb{C} \setminus \Gamma$, where $\Gamma := \gamma[[0,1)]$.

Solution: Hint: use Th12.2.1 (note: you need not study Sec12.2).

1° Let $K_k := \{z \mid d(z,\Gamma) \in [\frac{1}{k},\infty)\} \cap \bar{D}(0,k)$ for $k \in \mathbb{N}_+$. K_k is compact, because it is the intersection of a closed set[10] and a compact set.

2° $\mathbb{C} \setminus K_k$ **is connected**: Let G be the connected component of $\mathbb{C} \setminus K_k$ containing Γ. $G_1 := \bar{D}(0,k)^c$ is connected and contains points of Γ[11], hence

[9]In fact it may contain holes, if you cannot go around them, see the example in the beginning of the paper.

If the set has at least 2 connected components, then clearly closed curves in each don't deform to each other, because they would have to meet the boundary of the components. Thus a simply connected set is connected.

If there are holes, then closed curves going around such holes (if possible, cf. the example) don't deform to a constant curve.

[10]Preimage of a closed set by a continuous function is closed and $\{z \mid d(z,\Gamma) \in [\frac{1}{k},\infty)\} = d(\cdot,\Gamma)^{-1}[\frac{1}{k},\infty)$.

[11]All $\gamma(t)$ for $t > 1-\delta$ for some $\delta > 0$, because $\gamma(1-) = \infty$.

$G_1 \subset G$. $G_2 := \{z \mid d(z, \Gamma) \in [0, \frac{1}{k})\} \supset \Gamma$ is clearly a connected set too, so $G_2 \subset G$ too and hence $\mathbb{C} \setminus K_k = G_1 \cup G_2 \cup G$ is connected.

3° $K'_k := \{\gamma(t) \mid t \in [0, 1 - 1/k]\}$ is compact, hence so is $K''_k := K_k \cup K'_k$. Clearly $G' := \mathbb{C} \setminus K''_k = (\mathbb{C} \setminus K_k) \setminus K'_k$ is connected.[12]

4° Define f_k on K''_k s.t. $f_k|_{K_k} = 0$ and $f_k(\gamma(t)) = 1/(1-t)$ on K'_k. Then f_k is continuous on K''_k and holomorphic (zero) on $(K''_k)^o = K^o_k$, thus, by Mergelyan's Th12.2.1, there is a polymial p_k s.t. $|p_k - f_k| < 1/k$.

5° $\{p_k\}$ has desired properties: (i) $p_k(\gamma(t)) = 1/(1-t)\ \forall k > 1/(1-t)$ (i.e., $\forall 1 - 1/k > t$, i.e., $K'_k \ni \gamma(t)$). (ii) Let $K \subset \mathbb{C} \setminus \Gamma$ be compact and $\epsilon > 0$. $R := \max\{|z| \mid z \in K\}$, $r := \min\{d(z, \Gamma) \mid z \in K\} > 0$. If $k > R$ and $1/k < r$ and $1/k < \epsilon$ (i.e., $k > \max(R, 1/r, 1/\epsilon)$), then $K \subset K_k$, hence then $|p_k(z)| = |p_k(z) - f_k(z)| < 1/k < \epsilon\ \forall z \in K$. Thus, $p_k(z) \to 0$ uniformly on K, QED.

[12] Because $G = \mathbb{C} \setminus K_k$ is connected and now we just make a new hole into it. Because K'_k is piecewise linear, this is pretty clear (I guess this is true anyway (this is the converse of the Jordan Curve Theorem), but I guess it were hard to prove in the general case).