Mat-1.151 Complex Analysis (kompleksianalyysi), 7th exercise Wed 4.11.1998 at 12.15–14, U344 (2 page(s))

The book: Sections 10.1–10.3, 10.6, 13.3, 13.4.

Note: Little Picard Theorem 10.5.5: the range a nonconstant entire function covers the whole $\mathbb{C}$ except possibly one value. The same happens near an essential singularity of a function, by Th10.5.6.

1st half-term examination: Ch 1–7 except 7.9 and the proof of the Riemann Mapping Theorem.

2nd half-term examination: 7.9 (only the statement of Th7.9.1), 8.1, 8.2, 8.3 (up to p. 271), 9.1, 9.2 (not the proof of Th9.2.1), 10.1, 10.2, 10.3, 10.4, 10.5 (only what Picards 2 theorems say), 10.6, 11.1, 11.2, 11.3, 11.4 (only what Th11.4.1 states), 12.1, 12.2 (only what Th12.2.1 states), 13.1, 13.3, 13.4 (H^p Spaces are important), 15.1, 15.2, 15.3, 15.4.

Exercise VII.1 ([p.310]):
What does the monodromy theorem tell us about the function $\sqrt{z}$?

Solution: Note first that *any holomorphic extension of a branch of* $\sqrt{\ }$ *is a branch of* $\sqrt{\ }$: if $U, V \subset \mathbb{C}$ are open, $f(z)^2 = z$ on U, $V \supset U$ and $g \in H(V)$ is s.t. $g = f$ on U, then $g(z)^2 = z$ holds in a non-discrete subset U of V (and both sides define a function $\in H(V)$), hence it holds on the whole V.[1]

Take now $W := \mathbb{C} \setminus \{0\}$. (f, U) admits unrestricted continuation in W for any $U = D(P, r) \subset W$ and any $f \in H(U)$ with $f(z)^2 = z$, but to make things easy we take $U = D(2, 1)$ and $f(re^{i\theta}) := r^{1/2}e^{i\theta/2}$ with $\theta \in (-\pi, \pi]$, so to prove W to be OK it is enough to note that if γ is a path in W and $\alpha(t) := \arg\gamma(t)$ is chosen is such a way that it is continuous and $\alpha(0) = 0$ ($= \operatorname{Arg}\gamma(0) = \operatorname{Arg} P = \operatorname{Arg} 2$), then $|\gamma(t)|^{1/2}e^{i\alpha/2}$ is the extension we want[2], so W is really OK.[3]

The Monodromy Theorem 10.3.3 [p.309] says that if γ_0, γ_1 are homotopic curves (i.e., if "$\gamma_1 - \gamma_0$" goes zero times around zero (its winding number around zero is zero))[4] from P to a point Q in W, then the analytic continuations of (f, U) along γ_0 and γ_1 are equal, in particular, their extensions around Q agree (i.e., they define the same branch of $\sqrt{\ }$).

[1] So the magic was in the fact that $w^2 \in H(V)$, even $\in H(\mathbb{C})$.

[2] To be exact, for $U_t = D(P_t, r_t)$ (where $P_t = \gamma(t)$) take $f_t(re^{i\theta}) := r^{1/2}e^{i\theta/2}$ with $\theta \in (\alpha(t) - \pi, \alpha(t) + \pi]$.

[3] But $0 \in W \implies W$ is not OK, because no extension of f cannot have even a real derivative at 0. Thus, $\mathbb{C} \setminus \{0\}$ is the maximal W. (I admit, there's a bit of intuition here used, but it would just take some time to make it an exact (and longer) argument.

Exercise VII.2 ([p.302] *not hard, not very long*):
[Exe 10.11] Consider the function

$$g(z) = \int_0^\infty t^z \frac{1}{1 + t^4}\, dt.$$

(a) For which values of z does this formula define g as an analytic function?
(b) Can one use integration by parts to continue the function analytically to a larger open set?

Solution: (a) $f(z, t) := t^z \frac{1}{1+t^4} \in H(\mathbb{C})$[5]. $g(z) := \int_0^\infty f(z, t)\, dt$. As in Exa10.1.2, p. 301, $\int_0^1 |f(z)|\, dt \le \int_0^1 t^{\operatorname{Re} z}\, dt$ (and $\ge \int_0^1 t^{\operatorname{Re} z}/2\, dt$) converges iff $\operatorname{Re} z > -1$.[6]

$\int_1^\infty |f(z)|\, dt \le \int_1^\infty t^{\operatorname{Re} z}\frac{1}{t^4}$ (and $\ge \int_1^\infty t^{\operatorname{Re} z}\frac{1}{2t^4}$) converges iff $\operatorname{Re} z - 4 < -1$, i.e., $\operatorname{Re} z < 3$. So the integral converges absolutely in $U_0 := \{\operatorname{Re} z \in (-1, 3)\}$.

By Morera's theorem, $g(z) \in H(U_0)$, because $\oint_\gamma g(z)\, dz = \oint_\gamma \int_0^\infty f(z)\, dt\, dz = \int_0^\infty \oint_\gamma f(z)\, dz\, dt = \int_0^\infty 0\, dt = 0$[7] for all closed γ

[4] One could show the converse: if this winding number is nonzero, then we get different branches. Similarly, there is a $g \in H(W)$ s.t. $g = f$ on U iff $W \subset \mathbb{C} \setminus \{0\}$ is s.t. no closed curve on W goes around 0. Cf. also [Rudin:RCA:Th13.11].

[5] Remember that $t^z := e^{z \ln t} \in H(\mathbb{C})$ has no problems, because $t \in \mathbb{R}_+$. But z^t would be non-unique (e.g., $z^{1/2}$).

[6] We won't examine the improper integral $\lim_{\epsilon\to 0+} \int_\epsilon^1$, because that is pretty hard and examining analyticity of such a non-absolute converging integral is even harder. The same decision was made in Exa10.1.2.

[7] If you don't believe that the order of integration may be reversed, use the definition $\oint_\gamma F(z)\, dz := \int_0^1 F(\gamma(t))\gamma'(t)\, dt$ and basic calculus theorems about reversing the order of integration.

in ${U_0}$[8].[9]

(b) The answer is "yes", by 2° below.

1° $f(z) = \frac{1}{4}t^{z-3}[\ln(1+t^4)]'$, so to have the term $\Big|_0^\infty \frac{1}{4}t^{z-3}[\ln 1 + t^4]$ finite, we should have $\operatorname{Re} z - 3 < 0$ (for ∞) and $\operatorname{Re} z - 3 \geq 0$ (for 0), so this way we cannot get anywhere.

2° $f(z) = [\frac{t^{z+1}}{z+1}]'(1+t^4)^{-1}$, hence $g(z) = \Big|_0^\infty \frac{t^{z+1}}{z+1}(1+t^4)^{-1} + \int_0^\infty 4t^3\frac{t^{z+1}}{z+1}(1+t^4)^{-2}\,dt = 0 + \frac{4}{z+1}\int_0^\infty t^{z+4}(1+t^4)^{-2}\,dt$ for $z \in U_0$, and the latter integral converges for $\operatorname{Re} z + 4 > -1$ and $\operatorname{Re} z + 4 - 2\cdot 4 < -1$, i.e., for $\operatorname{Re} z \in (-5, 3)$ and can be seen to be analytic there (as in (a)), so there we have an analytic extension of g to a larger open set.[10]

Exercise VII.3 ([p.330] *easy, short*):
[Exe 10.19] Suppose $f : \mathbb{C} \to \mathbb{C} \cup \{\infty\}$ is a doubly periodic meromorphic function, with periods 1 and i, that is holomorfic on $U := \mathbb{C} \setminus \{m+ni \mid m, n \in \mathbb{Z}\}$. Prove that the residue of f at each $m + ni$ is $= 0$. (Hint: Integrate f aroud the unit square centered at $m + ni$.)

Solution: So $f(z + i) = f(z) = f(z + 1)$ $\forall z \in \mathbb{C}$.

By Th4.2.4 [p.113], $a_{-1} = \frac{1}{2\pi i}\oint_{\partial D(m+ni,1/2)} f(z)\,dz$. By Cauchy's Integral Theorem [p.62],[11] $a := \oint_{\partial D(m+ni,1/2)} f(z)\,dz = \oint_\gamma f(z)\,dz$, where γ is the unit square centered at $m + ni$. So we have to prove that $a = 0$.

By the periodicity of f, we have $\int_{[m+ni-\frac{1}{2}i-\frac{1}{2},m+ni-\frac{1}{2}i+\frac{1}{2}} f = \int_{[m+ni+\frac{1}{2}i-\frac{1}{2},m+ni+\frac{1}{2}i+\frac{1}{2}} f = -\int_{[m+ni+\frac{1}{2}i+\frac{1}{2},m+ni+\frac{1}{2}i-\frac{1}{2}} f$, so the bottom integral and the top integral cancel each other. Similarly, the left and the right integral cancel each other, so $a = 0$, QED.

Notes: In fact we might have as well integrated around $m + ni = 0$ only, because of the periodicity. Note that either $f \equiv \mathbb{C}$ or f has a (non-simple, because $a_{-1} = 0$) pole at each $m + ni$ (clearly each of the same multiplicity and completely identical), or f is entire and hence constant [p.324].

Exercise VII.4 ([p.398] *quite easy, short*):
[Exe 13.1] Let $N := H^0$ be the class of functions $f \in H(D(0,1))$ with

$$a_f := \sup_{0<r<1} \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f_r|\,d\theta < \infty,$$

where $\log^+ t := \max\{\log t, 0\}$ and $f_r(\theta) := f(re^{i\theta})$. Prove that $f \in H^p$, $p > 0$ implies $a_f < \infty$.

Solution: We call N the Nevanlinna class [Rudin:RCA:17.7] and define $\|f\|_0 := e^{a_f}$ (not a norm). Note that then $H^s \subset H^p$ for $\infty \geq s \geq p \geq 0$, by the result to be shown.

$F(t) := \log^+ t^{1/p} = \frac{1}{p}\log^+ t$ is convex and nondecreasing on $[1, \infty)$, hence on $\mathbb{R}$. Now, by Jensen's inequality [Rudin:RCA:Th3.3]

$$\frac{1}{2\pi}\int_0^{2\pi} \log^+ |f_r|\,d\theta = \frac{1}{2\pi}\int_0^{2\pi} F(|f_r|^p)\,d\theta \leq F[\frac{1}{2\pi}\int_0^{2\pi} |f_r|^p\,d\theta]$$

$$= \log^+[\frac{1}{2\pi}\int_0^{2\pi} |f_r|^p\,d\theta]^{1/p} = \log^+ \|f_r\|_p \leq \log^+ \|f\|_p < \infty$$

(hence $\|f\|_0 \leq \exp(\log^+ \|f\|_p) \leq \max(\|f\|_p, 1)$), QED.

Exercise VII.5 ([p.397] *very easy, very short*):
[Exe 13.5] Let $f \in H(\mathbb{D})$, where $\mathbb{D} := D(0,1)$. Let $0 < p < \infty$. Suppose that there is a harmonic function u on the unit disc s.t. $|f|^p \leq u$. Prove that $f \in H^p$.

Solution: $\|f_r\|_p^p := \frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p \leq \frac{1}{2\pi}\int_0^{2\pi} u(re^{i\theta}) = u(0)$, by the mean value property. Thus, $\|f\|_p^p := \sup_r \|f_r\|_p^p \leq u(0) < \infty$, so $f \in H^p$.

Notes: The converse holds too [Rudin:RCA:Exe17.3].

[8] Another way to prove this would be by noting that $\int_a^{1/a} f(z,t) \in H(U_0)$ by differentiation under the integral sign (or by Morera's Theorem) and then noting that the convergence to $g(z)$ is normal.

[9] One also uses Morera's theorem to prove Laplace transform $\hat{h}(z) := \int_0^\infty e^{-zt}h(t)\,dt$ to be holomorphic in that right half-plane, where the integral converges absolutely (containing $\operatorname{Re} z > r_h := \inf\{r \in \mathbb{R} \mid |e^{-rt}h(t)| \text{ is bounded}\}$).

[10] It seems that by induction we could extend g for the whole $\{\operatorname{Re} z < 3\}$ (not checked, might be wrong).

[11] Because the curves can be continuously deformed to each other in U.