Mat-1.151 Complex Analysis (kompleksianalyysi=funktioteoria), 5th exercise Wed 21.10.1998 at 12.15–14, U344 (2 page(s))

The book: Chapter 7 (so now we have read Chapters 1–7, skipping just some hardest proofs, e.g., you don't need to know the (very hard) proof of the Riemann Mapping Theorem).

Exercise V.1 ([p.209] *easy,short*):
[Exe7.12] Prove: If u and u^2 are real-valued and harmonic on a connected open set U, then u is a constant.

Solution: $\partial_{xx}u^2 = \partial_x 2uu_x = 2u_x^2 + 2uu_{xx}$, so $0 = \triangle u^2 = 2u_x^2 + 2uu_{xx} + 2u_y^2 + 2uu_{yy} = 2u0 + 2(u_x^2 + u_y^2)$, i.e., $u_x \equiv 0 \equiv u_y$ (because they are real). Thus, $u \equiv C$.

Exercise V.2 ([p.209] *not long, hard to guess*):
[Exe 7.30] Let u be a positive harmonic function on D_1 and suppose that $u(0) = a$. How large can $u(3/4)$ be? How small can $u(3/4)$ be? What is the best possible bound? What function realizes that bound?

Solution: By Harnack Inequality [p.226],

$$\frac{R-3/4}{R+3/4}a \le u(3/4) \le \frac{R+3/4}{R-3/4}a$$

for any $R<1$ (because then u is positive and harmonic on a neighborhood of $\bar{D}(0,R)$). Thus, the equation also holds for the limits (when $R \to 1-$), i.e., $a/7 \le u(3/4) \le 7a$.

Now we will show by examples, that these bounds are the best ones: We would like to have "$u(z) = \frac{1+x}{1-x}$" on $(-1,1)$, because that would realize our bound ($u(0) = 1 \wedge u(3/4) = 7$) (and another function "$u(z) = \frac{1-x}{1+x}$" for the other bound)

But $u>0$ implies that u is real, so $u = \operatorname{Re} F$ for some $F \in H(D_1)$ [p.209] (on the other hand each such u is harmonic [p.209]). Let's guess that

$$u(z) = \operatorname{Re}\frac{1+z}{1-z} = \operatorname{Re}\frac{(1+z)(1-\bar{z})}{|1-z|^2} = \operatorname{Re}\frac{1+z-\bar{z}-|z|^2}{|1-z|^2} = \frac{1-x^2-y^2}{(1-x)^2+y^2} > 0$$

on D_1, yes, that it is, because then $u(x+0i) = \frac{1-x^2}{(1-x)^2} = \frac{(1+x)(1-x)}{(1-x)^2} = \frac{1+x}{1-x}$, so really $u(0)=1$ and $u(3/4)=7$. Similarly, taking $u(z) = \operatorname{Re}\frac{1-z}{1+z}$ we get a positive, harmonic function on D_1 with $u(0) = 1 \wedge u(3/4) = 1/7$.

Exercise V.3 ([p.233] *very easy, pretty short*):
[Exe 7.46] Let u be a real-valued harmonic function on $U \subset \mathbb{C}$ and let $\phi : \mathbb{R} \to \mathbb{R}$ be convex (not necessarily nondecreasing). Prove that $\phi \circ u$ is subharmonic.

Solution: This is almost exactly Exa7.7.6: A convex function is continuous [Rudin:Th3.2][1], so $\phi \circ u$ is continuous. A function is subharmonic iff it is real-valued and continuous and satisfies the mean value inequality [Prop7.7.4], so we only have to check the inequality:

If $\bar{D}(P,r) \subset U$ then[2] $(\phi \circ u)(P) := \phi(f(P)) = \phi(\frac{1}{2\pi}\int_0^{2\pi} u(P+re^{i\theta})\,d\theta) \le \frac{1}{2\pi}\int_0^{2\pi} \phi(u(P+re^{i\theta}))\,d\theta =: \frac{1}{2\pi}\int_0^{2\pi}(\phi\circ u)(P+re^{i\theta})\,d\theta$, where the inequality follows from Jensen's Inequality [Rudin:Th3.3][3], QED.[4]

Exercise V.4 ([p.231] *very easy, very short*):
[Exe 7.64] True of false: if f is subharmonic on a domain U, then so is $|f|$?

Solution: $|f|(P) := |f(P)| \le |\frac{1}{2\pi}\int_0^{2\pi} f(P+re^{i\theta})\,d\theta| \le \frac{1}{2\pi}\int_0^{2\pi}|f(P+re^{i\theta})|\,d\theta$ for each $\bar{D}(P,r) \subset U$, hence $|f|$ is subharmonic [Prop7.7.4]? Wrong: the first inequality need not hold (for negative f. Let's find a f for which that doesn't hold an show that this gives us a counterexample.

Clearly f need to be strictly subharmonic (otherwise that "inequality" would be an equality) and at least somewhere negative, so let $f(z) = x^2 - 1$.

---

[1] But if ϕ were convex on some $[a,b]$ only, then it could be discontinuous at a and b (add any constants!).

[2] Unlike the book, by $\subset$ we denote being a subset and by $\subsetneq$ being a strict subset. This has become a standard notation, because you usually don't need $\subsetneq$.

[3] Rudin uses the Lebesgue measure divided by 2π, i.e., $\mu := m/2\pi$. Similarly, Jensen's Inequality implies that $\phi(\frac{1}{b-a}\int_a^b u) \le \frac{1}{b-a}\int_a^b \phi\circ u$, whenever u is real and ϕ is convex (at least on $f[(a,b)]$).

[4] Note that in Exa7.7.6 one of the equalities is an inequality (because there u may be non-harmonic), and that requires that ϕ is nondecreasing (i.e., in that case u may be any subharmonic function; note that subharmonic functions are always real unlike harmonic ones). A "counter-example" is given on p. 233.

Because $f \in \mathcal{C}^2$, f is subharmonic iff $\triangle f \geq 0$ [Exe7.41&7.69b],[5] and now $\triangle f = 2$, so f is subharmonic (on $\mathbb{C}$).

Take $P = 0$ and $r = 1$. $\frac{1}{2\pi}\int_0^{2\pi} |f(0 + re^{i\theta})|\, d\theta = \frac{1}{2\pi}\int_0^{2\pi} |[\cos\theta]^2 - 1|\, d\theta = \frac{1}{2\pi}\int_0^{2\pi}\{1 - [\cos\theta]^2\}\, d\theta = 1 - \frac{1}{2\pi}\int_0^{2\pi}[\cos 2\theta + 1]/2\, d\theta = 1 - 1/2 = 1/2 < 1 = |f|(0)$, so $|f|$ is not subharmonic, QED.

Exercise V.5 ([p.241] *try at least the beginning of the problem*):
[Exe 7.53] The Perron method for solving the Dirichlet problem is somewhat nonconstructive. In $\mathbb{R}^2_+$, you can instead use a conformal mapping to transform the problem to one on the disc. Solve $\triangle u = 0$, $u(x, 0) = 0$, $u(0, y) = y$ on $U := \{(x, y) \,|\, x > 0, y > 0\}$.

Warning: Stop when you notice that you can't go further, this is an ill-posed problem.

Solution: U can be mapped conformally to the upper half-plane by[6] $z \mapsto z^2$ and that in turn to D_1 by the Cayley transform $H(z) := \frac{z-i}{z+i}$, so $f(z) := \frac{z^2-i}{z^2+i}$ maps $U \to D_1$.

$f(x + 0i) = \frac{x^2-i}{x^2+i}$ belongs to the lower unit semicircle and $f(0 + yi) = \frac{y^2+i}{y^2-i}$ to the upper one. Let $v(f(z)) := u(z)\ \forall z \in \partial U$ (i.e., $v(w) := u(f^{-1}(w))\ \forall w \in \partial D_1$). Then we have a problem: along both arcs we have $v(-1) = v(f(0)) = 0$, but $v(1) = v(f(\infty)) = 0$ along the lower arc and $= \infty$ along the upper arc.

If you calculate out this $v(w) := u(f^{-1}(w))$, you get (unless I made a mistake) $v(w) = \sqrt{2\,\mathrm{Im}\, w}/|1 - w|$, so Th7.3.4 gives $v(w) = (1 - |w|^2)\frac{1}{2\pi}\int_0^{2\pi} \frac{\sqrt{2\sin\theta}}{|1-e^{i\theta}|}|w - e^{i\theta}|^{-2}\, d\theta\ \forall w \in \partial D_1$.

Because f is holomorphic on U and v is harmonic on $f[U]$, $u := v \circ f$ is holomorphic on U [Lemma7.3.2]. Because f is continuous on $\bar{U}$ and v is continuous on $f[\bar{U}] = \bar{D}_1 \setminus \{1\}$, u is continuous on $\bar{U}$, so it is a solution of the Dirichlet problem.

But did we cheat? Is v really such? Th7.3.4 says nothing about this (because v is not continuous on ∂D_1 as Th7.3.4 assumes), but if you go the proof of Th7.3.4 through, you see that also in this case things are as we said (the other possibility is to check that this $u := v \circ f$ that we defined above is really harmonic on U and continuous on $\bar{U}$, but it takes all too much time).

If you don't want to go through that proof, note that [Rudin:11.6&11.7,p.233–234] says that v is really harmonic, because $v \in L^1(\partial D_1)$ (it is bounded except near 1 and $v(e^{i\theta} \sim [\cot\theta]^{1/2} \in L^1$ near 1, details omitted). You should still check the continuity as in the proof of Th7.3.4 or [Rudin:Th11.23] which together with the preservation of angles under a conformal map to see that u is nontangentially continuous on ∂U and deduce then that it is continuous). Rudin is just a bit too hard for a first-timer for a course book ("RCA" is his worst or at least most cryptic book), though it is a good reference manual.

[5] Remember, that $\triangle f = 0$ iff f is harmonic, so $f > 0$ implies that the mean value inequality is strict somewhere.

[6] The holomorphic inverse is the square root with positive imaginary part. Remember that any holomorphic 1-1 function defined on a domain set is conformal to its range, by p.181 and the Open Mapping Theorem (which says that the range is a domain too). The same result is given in [Rudin:Th10.33] too.