Mat-1.151 Complex Analysis (kompleksianalyysi), 4th exercise Wed 14.10.1998 at 12.15–14, U344 (2 page(s))

The book: pp. 159–191 (from Ch5&6) correspond to these exercises.

Exercise IV.1 ([p.170] *very easy, very short*):
[Exe5.10(a)] How many zeros does $f(z) := z^8 + 5z^7 - 20$ have on $D_6 := D(0,6)$.

Solution: $g(z) := z^8$. $|f-g| = |5z^7-20| \leq 20+5\cdot 6^7 < 6^8 = |g| \leq |f|+|g|$ on ∂D_6, so $Z_f = Z_g = 8$, by Rouché's Theorem [5.3.1] ($g = (z-0)^8$ has a zero of multiplicity 8 at 0).

Exercise IV.2 ([Ch2] *not hard,quite short*):
[Exe 5.6] Let $f : D_1 \to \mathbb{C}$ be analytic and nonvanishing ($D_1 := D(0,1)$). Show that f has a well-defined analytic logarithm on D_1 by first showing that the equation $g' = f'/f$ has a solution g.

Solution: By a logarithm we mean that $e^g = f$ on D_1 (remember that "analytic" and "holomorphic" are synonymes). "Nonvahishing" means that $f(z) \neq 0$ $\forall z$ $(\in D_1)$.

Define $h := f'/f \in H(D_1)$. By Th2.3.3, $h = H'$ for some $H \in H(D_1)$ (for a generalization for more general sets than D_1, see [Rudin:Th13.11]). Let's fix the constant correctly: take $c \in \mathbb{C}$ s.t. $e^c = f(0)/e^{H(0)}$ (by the next exercise) and define $g(z) := H(z) + c$ $\forall z$, so that $e^{g(0)} = e^{H(0)}e^c = f(0)$ and $g' = H' = h = f'/f$.

Now we have a nice candidate g, let's prove that $e^g = f$ on the whole D_1: $F := fe^{-g}$, so that $F' = f'e^{-g} - g'fe^{-g} = f'e^{-g} - f'e^{-g} = 0$, hence $fe^{-g} \equiv C$, i.e., $f = Ce^g$, and $C = f(0)e^{-g(0)} = 1$, QED.

Generalization: This is true for any simply connected open set, not just D_1, see [Rudin:Th13.11(h)].

Exercise IV.3 ([p.5] *very easy, very short*):
Show that if $w \neq 0$, then $\exists z \in \mathbb{C}$ s.t. $e^z = w$.

Solution: Let $r := \ln|w| \in \mathbb{R}$ and $\theta := \operatorname{Arg} w$, then $e^{r+i\theta} = |w|e^{i\theta} = w$, QED.

Hmm, were we to hasty? Let $q := w/|w|$. By $\operatorname{Arg} w$ we mean the unique number $\theta \in [0, 2\pi)$ satisfying $e^{i\theta} = q$ [p.5] (hence $\operatorname{Arg} w = \operatorname{Arg} q$). This does exist, because $e^{i\theta} := \cos\theta + i\sin\theta$ maps $[0, 2\pi)$ 1-1 onto ∂D_1 (an easy calculus exercise). So the proof was OK (see [Rudin:"R&C",p.2] for a more accurate version).

Exercise IV.4 ([p.169] *not too hard, neither long*):
(This is approximately Exe5.5.) Let $f_j \in H(D_1)$ have k roots (counting multiplicities) in $D_1 := D(0,1)$ $\forall j$ and let $f_j \to f$ normally (i.e., uniformly on compact sets $K \subset D_1$). (a) Show that f may have no roots. (b) What reasonable added hypothesis on $\{f_j\}$ guarantees that f has at least k roots? (c) -"- exactly k roots?

Solution: (a) $f_j := z - (1 - 1/j) \to z - 1 =: f$ uniformly ($|f - f_j| = 1/j \to 0$ uniformly on D_1), but f has no zeros on D_1 although f_j has (namely $1 - 1/j$).

(b) "$\exists R \in (0,1)$ s.t. the zeros of f_j are in D_R for each j". Proof:

1° Assume that $r \in (R, 1)$ is s.t. f has no zeros on ∂D_r. $\epsilon := \min_{\partial D_r} |f| > 0$ (because a continuous function has a minimum (and a maximum) on a compact set [Rudin:Principles of Modern Analysis]). Take j s.t. $|f_j - f| < \epsilon$ on $\bar{D}_r$. Then $|f(z) - f_j(z)| < \epsilon \leq |f(z)| \leq |f(z)| + |f_j(z)|$ on ∂D_r, hence also f has k zeros in D_r [Rouché], QED.

2° If f has an infinite number of zeros in D_1, then it has at least k roots, and otherwise we can let $R' := |z_{f,\max}|$ be the greatest one of the absolute values of the zeros of f and note that by taking any $r > \max(R, R')$ with $r < 1$ we see that also f has k zeros in D_r, by 1°, hence also then f has at least k zeros.

(c) "$\exists R \in (0,1)$ s.t. the zeros of f_j are in D_R for each j and $f \not\equiv 0$". Proof:[1][2]

[1] Note that if we did add the assumption "f has a finite number of zeros", the proof would be contained in the proof of (b).

[2] Another way to proof this is to work as in the proof of Hurwitz's Theorem [Th5.3.3]: We assume that $f \not\equiv 0$ but f has zeros P_1, ..., P_m, whose multiplicity is $\sum n_l > k$ and we will derive a contradiction (note that if f has an infinite number of zeros, we only need to examine, e.g., just $k+1$ of them).

For each $l \leq m$, take $r'_l > 0$ s.t. there are no other zeros of f in $D(P_l, r'_l)$ and take $r_l := r'_l/2$, so that the discs $D_l := D(P_l, r_l)$ are disjoint[3] Now (the number of zeros of f_j in

Let $Z_f := \{z \in D_1 \mid f(z) = 0\}$. Let $R' \in (R, 1)$ be arbitrary. Choose $R'' \in (R', 1)$. Because $f \not\equiv 0$, Z_f is discrete in D_1, hence $Z_f \cap \overline{D_{R''}}$ is finite, so we may take $r \in (R', R'')$ s.t. $\partial D_r \cap Z_f = \emptyset$. By 1°, f has exactly k zeros in D_r, so f has at most k zeros in $D_{R'} \subset D_r$. Because $R' \in (R, 1)$ was arbitrary, F has at most k zeros in D_1 (and k zeros in D_r), so f has exactly k zeros in D_1, QED.[4]

Note: So the assumption in (b) implies that f either has k zeros in $\bar{D}_R$ (see the footnote) and no other zeros in D_1, or $f \equiv 0$ (which is the case, e.g., when $f_j = f_1/j$).

This is intuitively clear: the zeros of a nonzero function in H move "continuously", by Rouché's theorem: If $H(U) \ni f \not\equiv 0$ and $f(P) = 0$ with multiplicity k, then $\forall \epsilon > 0\ \exists \delta > 0$ s.t. $|g - f| < \delta \implies g$ has k zeros in $D(P, \epsilon)$ (proof: Take $r > 0$ s.t. f has no zeros in $\bar{D}_r \setminus \{P\}$. $\delta := \min_{\partial D_r} |f|$. $|g-f| < \delta \implies |g-f| < |f| \leq |f|+|g|$ on ∂D_r, so also g has k zeros in D_r [Rouché]).

Exercise IV.5 ([p.191] *quite easy, short*):
Let $U := \{z \mid \operatorname{Im} z > 0\}$. Characterize all conformal maps $U \to U$.

Solution: Note that if f and g are conformal, then so is $f \circ g$. $\operatorname{Aut}(U_1, U_2) := \{$conformal maps $U_1 \to U_2\}$, $H :=$Cayley transform$\in \operatorname{Aut}(U, D)$ [Th6.3.6]. $M :=\operatorname{Aut}(D) =$Möbius transformations $\omega\phi_a$ (with $|\omega| = 1 > |a|$) [Th6.2.3].

Let $f \in \operatorname{Aut}(U)$. Now $F := H \circ f \circ H^{-1} \in \operatorname{Aut}(D) = M$ and $f = H^{-1} \circ F \circ H$; on the other hand, each such f is in $\operatorname{Aut}(U)$. Thus, $\operatorname{Aut}(U) = \{H^{-1} \circ F \circ H \mid F \in M\}$.[5]

D_l) $Z(f_j, D_l) = \frac{1}{2\pi i} \oint_{\partial D_l} \frac{f_j'}{f_j} \to \frac{1}{2\pi i} \oint_{\partial D_l} \frac{f'}{f} = Z(f, D_l)$ (as in the proof of [Th5.3.3,p.171]) and $Z(f_j, D_l)$ is an integer, hence $Z(f_j, D_l) = Z(f, D_l)$ for $j \geq N_l$, for some $N_l \in \mathbb{N}$.

Take $N := \max_{l=1,\ldots,m} N_l$. Then f_N has $Z(f, D_l) = n_l$ zeros in D_l, hence f_N has at least $\sum_l n_l > k$ zeros, a contradiction, QED.

[4] Of course, all k zeros of f are in $\bar{D}_R$ (because if $z \in Z_f$ and $|z| > R$, we could take $r \in (R, |z|)$ so that ∂D_r would have no zeros of f and then all k zeros of f would be in D_r, a contradiction).

[5] I considered writing this out writing $F = \omega\phi_a$ out with arbitrary $|\omega| = 1 > |a|$, and adding the formulas of H and $H^{-1}(z) = i\frac{1+w}{1-w}$, but the formula (with ω and a as parameters, as for $\operatorname{Aut}(U)$), didn't seem to simplify easily and neither Mathematica could simplify it, so I didn't bother, although I think it can be somewhat simplified (with only basic calculus skill, but it would take some time).