Mat-1.151 Complex Analysis (kompleksianalyysi), 3rd exercise Wed 7.10.1998 at 12.15–14, U344 (2 page(s))

The page numbers mentioned after the exercise numbers tell you how far (at least) you should read the book before making the exercise. The footnotes usually explain things that you don't need to know.

Errata: p. 143 5th last line: "and that f has a pole at ∞" should be "and that f has a pole or a removable singularity at ∞". Note also that $(***)$ on page 138 should have many more details (although it is correct).

Exercise III.1 ([p.109] *quite easy, short*):
[Exe 4.3] Let $f \in H(U \setminus P)$, $P \in U$, U open. (a) If f has pole at P, then what type of singularity does $1/f$ have at P? (b) What if f has a removable singularity at P? (c) What if f has an essential singularity at P?

Solution: Now $g := 1/f$ is holomorphic on $U \setminus S$ too, where S is a discrete set (containing P and the zeros of f) (unless f has non-isolated zeros, in which case $f \equiv 0$ (so this is possible only in case (a)), in which case we can't answer the questions, because then $1/f$ is not defined at all).

(a) $\lim_P |1/f| = 0$, hence $1/f$ has a removable singularity at P [p.106–107].

(b) Let $\hat{f} \in H(U)$ be the holomorphic extension of f. If $\hat{f}(P) = 0$, then $|\lim_P 1/f| = \infty$, hence then f has a pole at P [p.107], otherwise $1/f$ is bounded near P[1] and hence then $1/f$ has a removable singularity at P.

(c) For each n, take $P_n \neq P$ s.t. $|P_n - P| < 1/n$ and $|f(P_n)| < 1/n$ (possible, because $f[D(P,1/n) \setminus \{P\}]$ is dense for each $1/n > 0$, by Th4.1.4]). Then $|g(P_n)| \to \infty$ ($g := 1/f$), hence P is not removable. Similarly, we can also take $P_n \to P$ s.t. $|f(P_n)| > 1$, hence $|g(P_n)| < 1$, in particular, P is not a pole (because $\lim_{z \to P} |g(z)| = \infty$ is not true, because $|g(P_n)| \not\to \infty$). Thus, P must be an essential singularity point.[2]

[1] Take $r := |\hat{f}(P)| > 0$. Take $\delta > 0$ s.t. $|\hat{f}| > r/2$ on $D(P,\delta)$. Then $|1/f| = |1/\hat{f}| < 2/r$ on $D(P,\delta)$.

[2] A faster proof: For any $r > 0$, $f[D^*]$ is dense in $\mathbb{C}$, hence in $\mathbb{C} \setminus \{0\}$ too (here $D^* := D(P,r) \setminus \{P\}$). But then $g[D^*] = 1/f[D^*]$ is dense in $\mathbb{C} \setminus \{0\}$ too (because $z \mapsto 1/z$ is a homomorphism on $\mathbb{C} \setminus \{0\}$), and hence on $\mathbb{C}$; hence g has an essential singularity at P.

Exercise III.2 ([p.107] *quite easy,short*):
[Exe 4.5fd] Does f have a removable singularity, a pole or an essential singularity at $P = 0$, when (f) $f = \cos z/z$, (d) $z \cdot e^{1/z} \cdot e^{-1/z^2}$?

Solution:(f) $|\cos z/z| = |\cos z|/|z| \to 1/0 = \infty$, when $z \to 0$, hence f has a pole at P.

(d) 1^o $z_n := i/n$, $1/z_n = n/i = -in$, $1/z_n^2 = n^2/i^2 = -n^2$. Now $z_n \to 0$ and $|f(z_n)| = 1/n \cdot |e^{-in}| \cdot e^{-n^2} = e^{-n^2}/n \to 0$ (remember that $|e^{iy}| = 1$), hence 0 is not a pole.

2^o $z_n := (1+i)/n$, $z_n^2 = 2i/n^2$, $1/z_n = n/(1+i) = (1-i)n/2$, $1/z_n^2 = n^2/2i$, $|f(z_n)| = |z_n||e^{n/2}e^{-ni/2}||e^{-in^2/2}| = |\sqrt{2}/n|e^{n/2} \to \infty$, hence 0 is neither a removable singularity, hence it must be an essential singularity [p.107].

Exercise III.3 ([p.120] *easy, short*):
Prove: If $f \in H(D'(P,R))$ has an essential singularity at P, then there is a sequence $\{z_n\}$ s.t. $z_n \to P$ and $|(z_n - P)^n f(z)| > n \ \forall n$.

Solution: For each n, we have that $(z-P)^n f(z)$ is unbounded on some $D(P,r)$ [p.120], so we may choose $z_n \in D(P,1/n)$ s.t. $|(z_n - P)^n f(z)| > n$, then $z_n \to P$ etc.

Exercise III.4 ([p.139] *quite hard, not short*):
Using the calculus of residues calculate (a) [p.129] $\frac{1}{2\pi i}\oint_\gamma f(z)\,dz$ [Exe4.36d], $f(z) = e^z/[z(z+1)(z+2)]$, $\gamma :=$ the clockwise oriented triangle with vertices $1 \pm i$, -3; (b) [p.137–139] $\int_0^\infty \frac{1}{x^3+x}\,dx$ [Exe4.51].

Solution: (a) The poles of f at $0, -1, -2$ are simples poles (i.e., of order one because $(z-P_j)^1 f(z)$ is bounded but $(z-P_j)^{1-1} f(z)$ is not [p. 120]). Thus, the formula in Prop4.5.6 [p.128] simplifies to $\mathrm{Res}_f(P) = (z-P)f(z)\Big|_{z=P}$, $P = 0, -1, -2$. $\mathrm{Res}_f(0) = e^0/1 \cdot 2 = 1/2$, $\mathrm{Res}_f(-1) = e^{-1}/-1 = -1/e$, $\mathrm{Res}_f(-2) = e^{-2}/2 = 1/2e^2$. Thus, by the formula on p. 127, $\frac{1}{2\pi i}\oint_\gamma f(z)\,dz = \sum_j Ind_\gamma(P_j)\,\mathrm{Res}_f(P_j) = -1/2 + 1/e - 1/e^2$.

(b) $\int_0^\infty \frac{1}{x^3+x} \geq \int_0^1 \frac{1}{x^3+x} \geq \int_0^1 \frac{1}{x+x} = \frac{1}{2}\int_0^1 \frac{1}{x} = \infty$ (I don't want to use residies here!).

The solution is above. If you want to see, how this would be calculated, see below.

Note that if it were finite, we would try to work as in Exe4.6.5, p. 137–139, i.e., we would integrate $g(z) := \frac{\ln z}{z(z+i)(z-i)}$ around η_R (see p. 138, Fig4.7), and then deduce the result.

$\int_0^\infty \frac{1}{x^3+x} = \int_0^\infty \frac{1}{x(x+i)(x-i)}$. The poles of g at $\pm i$ are simples poles (i.e., of order one [p. 120]), so by Prop4.5.6 [p.128], we have that $\mathrm{Res}_g(i) = (z-i)g(z)\Big|_{z=i} = (\ln i)/i(2i) = -(\ln i)/2 = -(0+i\pi/2)/2$ (because $\log(re^{i\theta}) := \log r + i\theta$ [p.137]), $\mathrm{Res}_g(-i) = (z-(-i))g(z)\Big|_{z=-i} = (\ln -i)/-i(-2i) = -(\ln -i)/2 = -(i3\pi/2)/2$.

$\frac{1}{2\pi i}\oint_{\eta_R} g = 0\,\mathrm{Res}_g(0) + 1\,\mathrm{Res}_g(i) + 1\,\mathrm{Res}_g(-i) = -\pi/2$.

But this time $\oint_{\eta_R^4} \not\to 0$, so we can't deduce the integral from $(***)$. Note that our $(***)$, as the one in p. 138, would need many more details, because you can't take first the limit in the numerator, you must take it simultaneously with the denominator (or I would change the numerator and add the difference, which I'd then prove to be zero). Moreover, the infinity of the limits also cause problems (though they are not too difficult).

Exercise III.5 ([p.143] *easy, very short*):
[Exe4.18] Let f be entire and not a constant, $G(z) := f(1/z)$ $(z \neq 0)$. Prove that f is a polynomial iff G has a pole at zero.

Solution: As in Th4.7.5 (p. 142–143): $f(z) = \sum_{n=0}^\infty a_n z^n$ converges for all $z \in \mathbb{C}$, hence $G(z) := f(1/z) = \sum_{n=0}^\infty a_n z^{-n}$ for all $z \neq 0$ (and hence it must be the unique Laurent expansion of G around 0. By Def4.7.4, $f(z) = \sum_{n=0}^\infty a_n z^n$ is then the Laurent expansion of f around ∞. If is a polynomial iff this contains a finite number of positive powers iff f has a pole at ∞ (otherwise it has an essential singularity at ∞), QED.