Mat-1.151 Complex Analysis (kompleksianalyysi), 2nd exercise Wed 30.9.1998 at 12.15–14, U344 (2 sivua)

Exercise I.3 ([p.31–33] *quite easy*):
This is proved in p. 31–33: From Definitions 2.1.2 and 2.1.3 we get $(*)$, which implies 2.1.4, which implies $(**)$, which, by Cauchy–Riemann equations, implies $(***)$, i.e., Prop2.1.6, what was to be proved (by Def2.1.5).

Exercise II.1 (*quite easy, not very long*):
[Exe43&44,p.27] [Exe43] (write $f' := \frac{df}{dz}$) we have $|f|^2 = u^2 + v^2$ and $|f'|^2 = |f_x|^2$ [Prop1.4.3] $= |u_x + iv_x|^2 = u_x^2 + v_x^2 = u_x^2 + u_y^2$, because $u_x = v_y \wedge u_y = -v_x$. Thus, $\triangle(|f|^2) := [\partial_{xx} + \partial_{yy}](|f|^2) = [\partial_{xx} + \partial_{yy}](u^2 + v^2) = 2uu_{xx} + 2u_x^2 + 2uu_{yy} + 2u_y^2 = 2vv_{xx} + 2v_x^2 + 2vv_{yy} + 2v_y^2 = 4u_x^2 + 4u_y^2 + 2u(u_{xx} + u_{yy}) + 2v(v_{xx} + v_{yy}) = 4|f'|^2$, because $u_{yy} = \partial_y u_y = \partial_y - v_x = -v_{xy} = -v_{yx} = -u_{xx}$ and similarly $v_{yy} = -v_{xx}$.

[Exe44] First we note that $(\partial_x|f|^2)^2 = (2uu_x + 2vv_x)^2 = 4u^2u_x^2 + 8uvu_xv_x + 4v^2v_x^2$, hence (by Cauchy–Riemann equations $u_x = v_y \& u_y = -v_x$) we have $(\partial_x|f|^2)^2 + (\partial_y|f|^2)^2 = \ldots = 4(u^2 + v^2)(u_x^2 + u_y^2) = 4|f|^2|f'|^2$ $(*)$.

Now, $\partial_{xx}[|f|^p] = \partial_{xx}[(u^2 + v^2)^{p/2}] = \partial_x[\frac{p}{2}(u^2 + v^2)^{p/2-1}\partial_x(|f|^2)] = \frac{p}{2}(u^2 + v^2)^{p/2-1}\partial_{xx}(|f|^2) + \frac{p}{2}(p/2 - 1)(u^2 + v^2)^{p/2-2}[\partial_x(|f|^2)]^2$, so by adding a similar ∂_{yy} and using $(*)$ for the latter term we get $\triangle[|f|^p] = \frac{p}{2}(u^2+v^2)^{p/2-1}[\triangle(|f|^2)] + \frac{p}{2}(p/2-1)(u^2+v^2)^{p/2-2}4|f|^2|f'|^2 = \frac{p}{2}|f|^{p-2}[4|f'|^2] + p(p-2)|f|^{p-4}|f|^2|f'|^2 = p^2|f|^{p-2}|f'|^2$, QED.

Exercise II.2 ([p.62] *easy, quite short*):
(i) Here γ is the rectangle with vertices $\pm 3 \pm i$, counterclockwise, and[1]

$$A := \oint_\gamma \frac{\bar{z}}{8+z}\,dz = -\oint_\gamma \frac{z}{8+z}\,dz + \oint_\gamma \frac{2\,\mathrm{Re}\,z}{8+z}\,dz = -0 + \oint_\gamma \frac{2x}{8+z}\,dz$$

[Cauchy], because $f(z) := \frac{z}{8+z} \in H(D(0,8))$ and $\gamma \subset D(0,8)$. We have set $x := \mathrm{Re}\,z$, $y := \mathrm{Im}\,z$, as usually. We go on:

$$A = \oint_\gamma \frac{(2x)(8+\bar{z})}{|8+z|^2}\,dz = \oint_\gamma \frac{16x + 2x^2 - 2ixy}{(8+x)^2 + y^2}\,dz.$$

The definition is $\oint_\gamma F(z)\,dz := \int_a^b F(\gamma(t))\gamma'(t)\,dt$ [Def2.1.5,p.33], so $A = A_1 + A_2 + A_3 + A_4$, where (for A_1 we have $\gamma(x) := x - i$, $x = -3...3$, hence $\frac{\partial\gamma}{dx} = 1$; note that we use x instead of t to clarify our notation)

$$A_1 = \int_{-3}^{3} \frac{2x^2 + (16+2i)x}{(8+x)^2 + y^2}\,1dx = 0 + 2\int_0^3 \frac{2x^2}{(8+x)^2 + y^2}\,dx = -A_3,$$

because the first integrand $g(x) := \frac{(16+2i)x}{(8+x)^2+y^2}$ is antisymmetric ($g(-x) = -g(x)$) and for A_3 we have $\int_3^{-3} = -\int_{-3}^3$. For A_2 we have $x = 3$, $y = -1...1$, $\gamma(y) = 3 + yi$ and $\frac{\partial\gamma}{\partial y} = i$, hence

$$A_2 = \int_{-1}^{1} \frac{64 - 6iy}{11^2 + y^2}\,idy = 0 + 2\int_0^1 \frac{64}{11^2 + y^2}i\,dy.$$

Take $t := \overline{\mathrm{ar}}\sinh y/11$, hence $y = 11\sinh t$, $y' = 11\cosh t$ and (because $\sinh^2 t + 1 = \cosh^2 t$)

$$A_2 = 128i \int_0^{\overline{\mathrm{ar}}\sinh 1/11} \frac{1}{11^2\cosh t} 11\cosh t\,dt = i\frac{128}{11}\,\overline{\mathrm{ar}}\sinh\frac{1}{11}.$$

Similarly, for A_4 we have $x = -3$, $y = 1... -1$, $\gamma(y) = -3 + yi$ and $\frac{\partial\gamma}{\partial y} = i$, hence

$$A_4 = -\int_{-1}^{1} \frac{-30 + 6iy}{5^2 + y^2}\,idy = 0 - 2\int_0^1 \frac{-30}{5^2 + y^2}i\,dy$$

$$= 60i \int_0^{\overline{\mathrm{ar}}\sinh 1/5} \frac{1}{5^2\cosh t} 5\cosh t\,dt = 12i\,\overline{\mathrm{ar}}\sinh\frac{1}{5}.$$

Thus, $A = 0 + A_2 + A_4 = i\frac{128}{11}\,\overline{\mathrm{ar}}\sinh\frac{1}{11} + 12i\,\overline{\mathrm{ar}}\sinh\frac{1}{5}$.

[1] Note that if we had z instead of $\bar{z}$, then the integrand would be $\in H(D(0,8))$, so the integral would be 0. This exercise shows that the Cauchy Integral Theorem does not hold for (all) non-holomorphic functions.

(ii) Set $f(z) := z(z+3)/(z-8) \in H(D(2+i,4))$.
Let $\hat\gamma(t) := \gamma(a+b-t)$ (i.e., γ counterclockwise), so (this can be easily seen by a change of variable) we can integrate counterclockwise and just add a minus sign $-\oint_{\hat\gamma} \frac{f(z)}{z-(-i)}\,dz = -2\pi i f(-i)$ [Th,p.62] $= -2\pi i(-i)(-i+3)/(-i-8) = \frac{46-22i}{65}\pi$.

Exercise II.3 ([p.210] *easy, very short*):
Let u be $\mathcal{C}^1$ on $D(a,r)$ and harmonic on $D(a,r)\setminus\{a\}$. Then $\exists F \in H(D(a,r))$ s.t. $\operatorname{Re} f = u$.

Solution: (This is the proof of Lemma 7.1.2 with Th1.5.1 replaced by Th2.3.2.)

We want to find $v \in \mathcal{C}^1$ s.t. $v_x = -u_y =: f$ and $v_y = u_x =: g$ (C–R). But $= \partial_y f = -u_{yy} = u_{xx} = \partial_x g$ (outside a), hence, by Th2.3.2, such a v exists (on $D(a,r)$). Take $F := u + iv$, QED.

Exercise II.4 (*quite easy, quite short*):
Solution: (a) $h := f/g$ is entire and $|h| \le 1$, hence $h = C$ [Liouville], i.e., $f = Cg$.
(b) Let $a_1, ..., a_m$ be the zeros of g with multiplicities $n_1, ..., n_m$ (i.e., we assume that the number of zeros of g is finite; we don't study the case with an infinite number of zeros). Then $g = hG$, where $h := \prod_{k=1}^m (z-a_k)^{n_k} \in H(\mathbb{C})$ and $G \in H(\mathbb{C})$. Clearly $f = hF^2$. Now, $F = CG$, by (a), hence again $f = Cg$.

Exercise II.5 ([p.87] *easy, very short*):
If $f \in H(D(P,R))$, $0 < r < R$ and $M := M_{r,f,P} := \sup_{z\in\bar D(P,r)}$, then $|f^{(k)}(P)| \le M\frac{k!}{r^k}$ $\forall k \in \{1,2,3,...\}$, by Th3.4.1. Take $r = 1$.

[2] One way to prove it is to note that $f(a_0) = g(a_0) = 0$, $|f(z)/(z-a_0)| \le |g(z)/(z-a_0)|$, hence $f(z)/(z-a_0)$ has also a zero in the next zero (a_0, if $n_1 > 1$; otherwise a_1), because $g_1(z) := g(z)/(z-a_0)$ has; divide again by $z-$the next zero etc., use induction.