Mat-1.151 Complex Analysis (kompleksianalyysi), 1st exercise Wed 23.9.1998 at 12.15–14, U344 (1 sivua)

The book: Robert E. Greene – Steven G. Krantz: "Function theory of one complex variable", John Wiley & Sons Inc., 1997.
(The classical W. Rudin: "Real and Complex analysis", Chapters 10–20 contain approximately the same material but that book is too complex for this course.)
Lectures: Tue 14–16 U322, Thu 12–14 U322, Fri 12–14 U322, 22.9.–12.11.1998.
Exercises: Wed 12-14 U344, 23.9.–18.11.1998, no bonus (demo-like).
Half-term examinations: Mon 26.10.1998 16–19, room N & Mon 23.11.1998 16–19, room D.
How to obtain the material for the course: Exercises: take from the envelope before the room U331. Answers: as above or from U317. The book: by it or *borrow* a paper version from the folder "Mat-1.151 Kompleksianalyysi" from room U317, copy it and return it *during the same day* to the same place in order and a neat form!

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Don't hesitate to contact me, if you have any problems.

Exercise I.1 (*easy, short*):
(a) If $w = z^n$, i.e.,[1]

$$Re^{i\phi} = (re^{i\theta})^n = r^n e^{in\theta}$$

$\iff \quad r = \sqrt[n]{R} \wedge n\theta = \phi + k2\pi \quad \iff \quad r = \sqrt[n]{R} \wedge \theta = \phi/n + \frac{2k}{n}\pi$, $k = 0, ..., n-1$ (because adding $m2\pi$ to θ gives a duplicate of z).

(b) $(1+i)^{1/4} = (2^{1/2}e^{i\pi/4})^{1/4} = 2^{1/8}e^{i(\frac{1}{16}+\frac{k}{2})\pi}$, $k = 0, 1, 2, 3$, by the formula in (a).

Exercise I.2 (*easy, very short*):
$\partial z/\partial z = 1$, $\partial \bar{z}/\partial z = 0$ etc. [p. 12], so using the chain rule we get that $\frac{\partial}{\partial z} 4\bar{z}^3 = 12\bar{z}^2 \partial\bar{z}/\partial z = 0$ etc., hence:
(a) $\frac{\partial}{\partial z}(4\bar{z}^3 - z^4) = -4z^3$.

[1] We set $R := |w|$, $\phi := \operatorname{Arg} w$. Note that $(z^w)^\zeta \neq z^{\zeta w}$ is possible, when $z \in \mathbb{C} \setminus \mathbb{R}$ and $w \notin \mathbb{Z}$ (unless you define powers as sets (e.g., $1^{1/2} := \{-1, 1\}$), not single numbers), similarly, we may have $(zw)^\zeta \neq z^\zeta w^\zeta$ (but always $z^{w_1+w_2} = z^{w_1} z^{w_2}$).

(b) $\frac{\partial}{\partial \bar{z}}(\bar{z}^3 + z\bar{z}) = 3\bar{z}^2 + z$.

Exercise I.4 ([p.45]):
(For I.3 see II-paper next week.) *Let $\Omega \subset \mathbb{C}$ be an open ball or rectangle and $P \in \Omega$. Assume that $F \in \mathcal{C}(\Omega) \cap H(\Omega \setminus \{P\})$. Show that $F = H'$ for some $H \in H(\Omega)$.*

Solution: This is the proof of Th2.3.3, p. 45, and goes exactly as the proof of Th1.5.3, p. 21, except that $H, h_1, h_2 \in \mathcal{C}^1$ only, we don't know yet that $h, h_1, h_2 \in \mathcal{C}^2$, and the two references to Th1.5.1 should be to Th2.3.2.

Notes: Note that everything holds on the whole Ω, so $\frac{\partial}{\partial z}H = F$ holds everywhere on Ω and hence $H \in H(\Omega)$. We also have $F \in H$, by Cor3.1.2, p. 73. For a generalization to an arbitrary open, convex Ω, see [Rudin:R&C:Th10.14] (W. Rudin: *Real and Complex Analysis*) (which bases on 10.13).

Exercise I.5 ([I.9] *easy, not long*):
The famous *Cayley transform*:[2] $\phi(z) := i\frac{1-z}{1+z} = i\frac{(1-z)\overline{(1+z)}}{|1+z|^2} = i\frac{1+\bar{z}-z-z\bar{z}}{|1+z|^2} = i\frac{1-2i\operatorname{Im} z - |z|^2}{|1+z|^2} = \frac{2\operatorname{Im} z + i(1-|z|^2)}{|1+z|^2}$, hence $\operatorname{Im}\phi(z) = \frac{1-|z|^2}{|1+z|^2} > 0$, i.e., $\phi(z) \in U$.

One-to-one (i.e., 1–1, i.e., an injection): $\phi(z) = \phi(w) \iff \frac{1-z}{1+z} = \frac{1-w}{1+w} \iff (1-z)(1+w) = (1+z)(1-w) \iff 1 - z + w - zw = 1 + z - w - zw \iff 2z = 2w \iff z = w$.

Onto (i.e., a surjection): $w = \phi(z) \iff w + wz = i(1-z) \iff z = \frac{i-w}{i+w}$, and for this z we have $z \in \mathbb{D}$, i.e., $|z| < 1$, because $|i-w|^2 = u^2 + (w-1)^2 < u^2 + (w+1)^2 = |i+w|^2$ hence ϕ is onto, QED.[3]

[2] Extra knowledge: This is very handy with H^∞ spaces and the Fourier transform $\hat{f}(z) := \int_{\mathbb{R}} f(t)e^{itz}\,dt$. Mathematicians preferring the Laplace transform $\hat{f}(z) := \int_{\mathbb{R}} f(t)e^{-tz}\,dt$. (which is the same thing with the plane rotated by $-pi/2$) use the right half plane instead and hence use the form $\phi(z) := \frac{1-z}{1+z}$.

[3] An application for H^p spaces [p. 397]: ϕ is an isometric isomorphism between $H^\infty(\mathbb{D})$ and $H^\infty(U)$.
Note: ϕ also maps $\bar{D}$ to $\bar{U} \cup \{\infty\}$ bijectively (i.e., 1-1 and onto).