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Master in Mathematics

Dynamical Quantum Groups

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Contents

1 Introduction

The *quantum Yang-Baxter equation* (QYBE) arises naturally in the setting of statistical mechanics and quantum field theory. It was proposed by Baxter as the star-triangle relation while studying the 8-vertex model and by Yang in the study of a quantum N-body problem.
As a simple example one can consider the state models on $n \times m$ square lattices in $\mathbb{Z}^2$, then the matrix of *Boltzmann weights* satisfies the QYBE.

We can interpret the QYBE in a more algebraic setting, leading to the theory of quantum groups.
Consider V a vector space, we say that $R \in End(V \otimes V)$ satisfies the QYBE if

$$R^{12}R^{13}R^{23} = R^{23}R^{13}R^{12}$$

in $End(V \otimes V \otimes V)$. A solution of the QYBE is called R-matrix.
The classical analogue of the QYBE is the classical Yang-Baxter equation (CYBE)

$$[r_{12}, r_{23}] + [r_{13}, r_{23}] + [r_{12}, r_{13}] = 0,$$

whose solutions are called the classical r-matrices. In this case one considers the theory of Poisson-Lie groups, a geometric interpretation given by Drinfeld [Dr].
Let R be an R-matrix, we can associate to it an algebraic structure (using the RTT construction) which is exactly a Hopf algebra such that the universal element of a quasitriangular structure is the given R-matrix. We call a quasitriangular Hopf algebra *quantum group*.
In the following report we will concentrate on a generalization of the QYBE, the *quantum dynamical Yang-Baxter equation* (QDYBE). The QDYBE was introduced by G. Felder [F] and he also considered its quasiclassical limit, the CDYBE.
Let $\mathfrak{h}$ be a finite dimensional commutative Lie algebra over $\mathbb{C}$, V a semisimple finite dimensional $\mathfrak{h}$-module and $R : \mathfrak{h}^* \to End_{\mathfrak{h}}(V \otimes V)$ a meromorphic function, then the QDYBE of step γ reads:

$$R^{12}(\lambda - \gamma h^3)R^{13}(\lambda)R^{23}(\lambda - \gamma h^1) = R^{23}(\lambda)R^{13}(\lambda - \gamma h^2)R^{12}(\lambda),$$

where $\gamma \in \mathbb{C}$ (the notation is explained in section 3.1).
Similarly to the non-dynamical case, we would like to associate an algebraic structure to solutions of the QDYBE, the *dynamical quantum group*.
Our aim is to understand the solutions of the QDYBE, i.e., the dynamical R-matrices, and the dynamical quantum groups.

In the first chapter we introduce some basic notions necessary to understand the setting of quantum groups. Specifically we recall the quantum Yang-Baxter equation and the definition of R-matrix, we then introduce Hopf algebras and the concept of quantum group.
In section 2.3 we present a construction, first introduced by Faddeev, Reshetikhin and Takhtajan [FRT], that allows us to construct a quantum group starting from a given R-matrix.
In the last section we present an explicit example, the quantum group $U_q(sl_2)$.

In the second chapter we introduce the main equation, the quantum dynamical Yang-Baxter equation. Similarly to the case of the QYBE, we would like to find an algebraic structure associated to solutions

of the QDYBE, the dynamical R-matrices, that we will call dynamical quantum group.
We then introduce the dynamical quantum group A_R, a generalization of the RTT construction in the dynamical case. In section 3.2.3 we give a few results regarding the representation theory of A_R and linking it to that of R.
We then discuss a way to obtain solutions of the QDYBE using fusion and exchange operators.

In the last chapter we introduce the classical dynamical Yang-Baxter equation, i.e., the dynamical analogue of the classical Yang-Baxter equation. We give some basics notions in order to give a geometric interpretation to the CDYBE, introducing the concept of Poisson groupoid, a generalization of Drinfeld's construction for Poisson-Lie groups [Dr].

2 Basic Tools

In this chapter we introduce the Quantum Yang-Baxter equation (QYBE) and give the definition of R-matrices as solutions to the QYBE. We then introduce the algebraic structure behind R-matrices, i.e., Hopf algebras and Quantum Groups.

2.1 The Quantum Yang-Baxter equation

Consider a $\mathbb{K}$-vector space V and a linear operator $R : V \otimes_{\mathbb{K}} V \to V \otimes_{\mathbb{K}} V$, we say that R satisfies the quantum Yang-Baxter equation (QYBE) if:

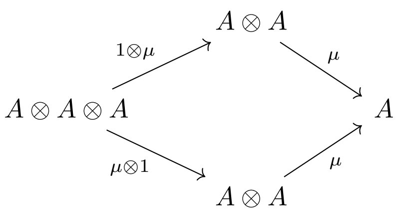
$$R^{12}R^{13}R^{23} = R^{23}R^{13}R^{12},$$

where R^{ij} is the operator acting as R on the ith and jth components in $End(V \otimes_{\mathbb{K}} V \otimes_{\mathbb{K}} V)$.

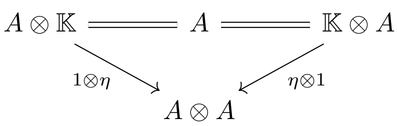
Definition 1. We define R-matrix a solution of the QYBE.

2.2 Hopf algebras

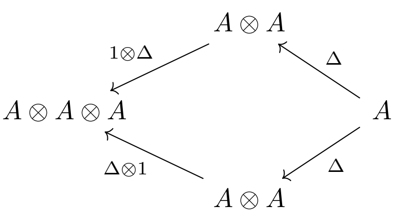
We now give the necessary definitions needed to work within the framework of Hopf algebras to better understand the QYBE.
Fix a field $\mathbb{K}$, then a $\mathbb{K}$-algebra A is a $\mathbb{K}$-vector space equipped with a bilinear product $\mu : A \otimes A \to A$. It is associative if the diagram

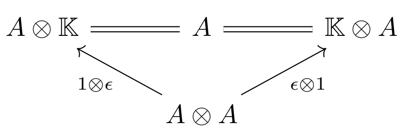


commutes.
If there is a map $\eta : \mathbb{K} \to A$ such that



commutes, then A is unital.
Similarly A is said to be coassociative with counit if there exist a coproduct $\Delta : A \to A \otimes A$ and a counit $\epsilon : A \to \mathbb{K}$ such that the diagrams

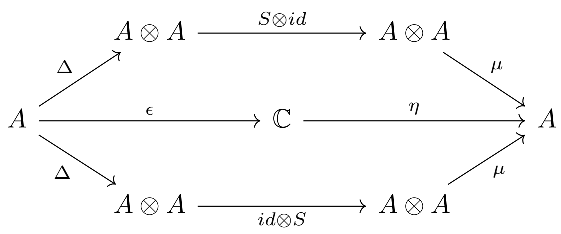




commute.

Definition 2. If $(A, \epsilon, \Delta, \mu, \eta)$ is such that (A, ϵ, Δ) is a coalgebra and (A, μ, η) is an algebra, then A is called a *bialgebra*.

Definition 3. A *Hopf algebra* is a bialgebra over $\mathbb{K}$ with a linear map $S : H \to H$, called the *antipode*, such that the following diagram commutes:



We can now introduce the general algebraic structure behind R-matrices:

Definition 4. A bialgebra, or Hopf algebra, is *quasitriangular* if there exists an element $R \in A \otimes A$, called *universal R-matrix*, such that

$$R\Delta(x)R^{-1} = \tau\Delta(x),$$

where τ is the transposition operator $\tau(a \otimes b) = b \otimes a$, and

$$(\Delta \otimes 1)R = R_{13}R_{23}, \quad (1 \otimes \Delta)R = R_{13}R_{12}.$$

Remark 5. Here the notation $\tau\Delta(x)$ is used to indicate $\tau\Delta(x)\tau = \Delta^{op}(x)$.

Given a quasitriangular Hopf algebra, the universal R-matrix satisfies the QYBE.

Definition 6. We call a quasitriangular Hopf algebra *quantum group*.

2.3 The RTT construction

Suppose R is a solution of the Yang-Baxter equation, i.e., an R-matrix, then we would like to associate to this given solution a corresponding Hopf algebra such that R determines its quasitriangular structure. To do so we follow the construction explained in detail in [FRT].

Let V be an n-dimensional $\mathbb{C}$ vector space and $R \in End(V \otimes V)$ an element satisfying the QYBE.
Define $A = A(R)$ as an associative algebra over $\mathbb{C}$ generated by $1, t_{ij}$ for $i, j = 1, ..., n$ satisying

$$RT_1T_2 = T_2T_1R\ ,$$

where $T = (t_{ij})$ and we use the notation $T_1 = T \otimes 1$ and $T_2 = 1 \otimes T$.

Proposition 7. *$A(R)$ is an Hopf algebra with coproduct $\Delta : A \to A \otimes A$ defined by $\Delta(1) = 1 \otimes 1$ and $\Delta(t_{ij}) = \sum_{k=1}^{n} t_{ik} \otimes t_{kj}$.*

Proof. We give an idea of the proof of the above result, a more general discussion regarding $A(R)$ is present in [K, VIII.6].
The above formulas define a unique algebra map $\Delta : A \to A \otimes A$ and the counit is given by $\epsilon(t_{ij}) = \delta_{ij}$.

It is sufficient to check coassociativity on the generators t_{ij} and extend by linearity.
Define I as the ideal generated by the RTT relations, we need to check that I is a coideal, i.e., that $\Delta(I) \subset I \otimes \tilde{A} + \tilde{A} \otimes I$, where $\tilde{A}$ is the free algebra without the relations on the generators.
Define $S_{ij}^{mn} := \sum_{k,l} R_{ij}^{kl} t_{km} t_{ln} - \sum_{k,l} t_{ik} t_{jl} R_{kl}^{mn}$, then we have that:

$$\begin{aligned}
\Delta(S_{ij}^{mn}) &= \sum_{k,l,p,q} R_{ij}^{kl} t_{kp} t_{lq} \otimes t_{pm} t_{qn} - \sum_{k,l,p,q} t_{ip} t_{jq} \otimes t_{pk} t_{ql} R_{kl}^{mn} = \\
&= \sum_{k,l,p,q} S_{ij}^{pq} \otimes t_{pm} t_{qn} + \sum_{k,l,p,q} t_{ik} t_{jl} R_{kl}^{pq} \otimes t_{pm} t_{qn} + \\
&+ \sum_{k,l,p,q} t_{ip} t_{jq} \otimes S_{pq}^{mn} - \sum_{k,l,p,q} t_{ip} t_{jq} \otimes R_{pq}^{kl} t_{km} t_{ln} = \\
&= \sum_{k,l,p,q} S_{ij}^{pq} \otimes t_{pm} t_{qn} + \sum_{k,l,p,q} t_{ip} t_{jq} \otimes S_{pq}^{mn}
\end{aligned}$$

□

We observe that R controls the non-commutativity of the generators t_{ij} of $A(R)$.

2.4 The quantum group $U_q(sl_2)$

In this section we compute an explicit example of quantum group and R-matrix.

Consider $\mathfrak{g} = sl_2$ and $q \in \mathbb{C}$, $q \neq 0$, such that q is not a root of unity. Define $U_q(sl_2)$ as the algebra generated by $E, F, K^{\pm}$ with the following relations:

$$\begin{aligned}
KK^{-1} = K^{-1}K = 1, \\
KEK^{-1} = q^2 E, \\
KFK^{-1} = q^{-2} F, \\
EF - FE = \frac{K - K^{-1}}{q - q^{-1}}.
\end{aligned}$$

Proposition 8. *$U = U_q(sl_2)$ is a Hopf algebra.*

Proof. Define on U the following structure:

$$\Delta(E) = E \otimes 1 + K \otimes E, \quad \Delta(F) = F \otimes K^{-1} + 1 \otimes F, \quad \Delta(K) = K \otimes K$$

$$\epsilon(E) = \epsilon(F) = 0, \quad \epsilon(K) = 1$$

$$S(E) = -K^{-1}E, \quad S(F) = -KF, \quad S(K) = K^{-1}$$

Δ, ϵ and S defined as above give U an Hopf algebra structure.
We check that $\Delta([E, F]) = [\Delta(E), \Delta(F)]$:

$$\begin{aligned}
[\Delta(E), \Delta(F)] &= [E \otimes 1 + K \otimes E, F \otimes K^{-1} + 1 \otimes F] = \\
&= [E, F] \otimes K^{-1} + K \otimes [E, F] + [K \otimes E, F \otimes K^{-1}] = \\
&= \frac{K - K^{-1}}{q - q^{-1}} \otimes K^{-1} + K \otimes \frac{K - K^{-1}}{q - q^{-1}} + KF \otimes EK^{-1} - FK \otimes K^{-1}E = \\
&= \frac{K \otimes K - K^{-1} \otimes K^{-1}}{q - q^{-1}} = \\
&= \Delta(\frac{K - K^{-1}}{q - q^{-1}}) = \Delta([E, F]).
\end{aligned}$$

So we conclude.

$\square$

We denote $\tau : U \otimes U \to U \otimes U$ to be the transposition operator $\tau(a \otimes b) = b \otimes a$. Observe that the opposite coproduct is given by

$$\Delta^{op}(E) = E \otimes K + 1 \otimes E, \quad \Delta^{op}(F) = F \otimes 1 + K^{-1} \otimes F.$$

Remark 9. One would like to recover the universal enveloping algebra $U(sl_2)$ from the quantized $U_q(sl_2)$ when $q \to 1$, to do so we need to consider the formal version of $U_q(sl_2)$ defined starting from E, F, H and relations on their brackets [K, XVII. 4]. With this definition the element K is $K = e^{qH}$ and for $q = 1$ we get $U(sl_2)$.

Let $V = \mathbb{C}^2$ and consider the tautological representation given by:

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad K = \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix}.$$

We are looking for an element $R \in U \otimes U$ such that $R\Delta^{op}(u) = \Delta(u)R$ for all $u \in U$. We take advantage of the tautological representation and look for such an R in $End(V \otimes V)$.

Consider $V \otimes V$ with basis $\{x \otimes x, x \otimes y, y \otimes x, y \otimes y\}$, then U acts on $V \otimes V$ in the following way:

$$E(v \otimes w) = \Delta(E)(v \otimes w), \quad F(v \otimes w) = \Delta(F)(v \otimes w), \quad K(v \otimes w) = \Delta(K)(v \otimes w)$$

As matrices we get:

$$E = \begin{pmatrix} 0 & q & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & q^{-1} \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad F = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ q^{-1} & 0 & 0 & 0 \\ 0 & q & 1 & 0 \end{pmatrix} \quad K = \begin{pmatrix} q^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & q^{-2} \end{pmatrix}.$$

The R-matrix preserves the eigenspaces of K so it will be of the form

$$R = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & c & 0 \\ 0 & d & e & 0 \\ 0 & 0 & 0 & f \end{pmatrix} \in End(V \otimes V).$$

We then have the following:

$$R\Delta^{op}(E) = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & c & 0 \\ 0 & d & e & 0 \\ 0 & 0 & 0 & f \end{pmatrix} \begin{pmatrix} 0 & 1 & q & 0 \\ 0 & 0 & 0 & q^{-1} \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a & qa & 0 \\ 0 & 0 & 0 & c + bq^{-1} \\ 0 & 0 & 0 & e + dq^{-1} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\Delta(E)R = \begin{pmatrix} 0 & q & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & q^{-1} \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & c & 0 \\ 0 & d & e & 0 \\ 0 & 0 & 0 & f \end{pmatrix} = \begin{pmatrix} 0 & d + qb & e + qc & 0 \\ 0 & 0 & 0 & f \\ 0 & 0 & 0 & fq^{-1} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$R\Delta^{op}(F) = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & c & 0 \\ 0 & d & e & 0 \\ 0 & 0 & 0 & f \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ q^{-1} & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & q & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ c+bq^{-1} & 0 & 0 & 0 \\ e+dq^{-1} & 0 & 0 & 0 \\ 0 & f & qf & 0 \end{pmatrix}$$

$$\Delta(F)R = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ q^{-1} & 0 & 0 & 0 \\ 0 & q & 1 & 0 \end{pmatrix} \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & c & 0 \\ 0 & d & e & 0 \\ 0 & 0 & 0 & f \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 \\ aq^{-1} & 0 & 0 & 0 \\ 0 & d+qb & e+qc & 0 \end{pmatrix}$$

Imposing the condition on R we obtain the following systems:

$$\begin{cases} a = d + qb \\ qa = e + qc \\ f = c + bq^{-1} \\ fq^{-1} = e + dq^{-1} \end{cases} \qquad \begin{cases} a = c + bq^{-1} \\ aq^{-1} = e + dq^{-1} \\ f = d + qb \\ fq = e + qc \end{cases} .$$

One then gets

$$\begin{cases} a = f \\ d = a - qb \\ c = a - bq^{-1} \\ e = q^{-1}(a-d) \\ b = q^{-1}(a-d) \end{cases} ,$$

so, up to scalars, the matrix of R is

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & b & 1-q^{-1}b & 0 \\ 0 & 1-qb & b & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} .$$

Consider $b = q$, we obtain a triangular matrix satisfying the QYBE:

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & q & 0 & 0 \\ 0 & 1-q^2 & q & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Remark 10. The R-matrix we obtained is the image in $End(V \otimes V)$ of an element in the extended $U\tilde{\otimes}U$ and not in $U \otimes U$. This implies that $U_q(sl_2)$ is not quasitriangular in a purely algebraic sense, but an R-matrix can be found in the completed tensor product.
Specifically, one finds [K, Theorem XVII 4.2]

$$R = \sum_{n=0}^{\infty} \frac{(q^{-1}-q)^n q^{-n(n-1)/2}}{[n]!} F^n \otimes E^n .$$

This infinite sum lives in $U\tilde{\otimes}U$ but for V finite dimensional representation of sl_2, since E, F act nilpotently on V, the image of R is in $End(V \otimes V)$.

Viceversa, as in [Fa], suppose given the matrix

$$R = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & q-\frac{1}{q} & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix},$$

let T be the matrix $T = \{t_{ij}\}_{i,j\in\{1,2\}}$, then the RTT relations reduce to the following 6 formulae (out of 16 only 6 are independent):

$$t_{11}t_{12} = qt_{12}t_{11}$$

$$t_{12}t_{21} = t_{21}t_{12}$$

$$t_{11}t_{21} = qt_{21}t_{11}$$

$$t_{22}t_{12} = \frac{1}{q}t_{12}t_{22}$$

$$t_{22}t_{21} = \frac{1}{q}t_{21}t_{22}$$

$$t_{11}t_{22} - t_{22}t_{11} = (q-\frac{1}{q})t_{12}t_{21}$$

Consider the *q-determinant* of T given by

$$det_q(T) = \sum_{\sigma\in S_n} (-q)^{l(\sigma)} t_{1\sigma(1)} \dots t_{n\sigma(n)} = t_{11}t_{22} - qt_{12}t_{21},$$

imposing $det_q(T) = 1$ we obtain the quantum group $SL_q(2)$, which is dual to $U_q(sl_2)$ [K, VII.5].

3 The quantum dynamical case

In this chapter we introduce the quantum dynamical Yang-Baxter equation (QDYBE), a generalization of the QYBE arising from mathematical physics, in which additional parameters appear.
Similarly to the case of quantum groups, we introduce the solutions of the QDYBE, i.e., dynamical R-matrices, and see how one can associate to such a solution an algebraic structure, which will be called a dynamical quantum group.

3.1 The quantum dynamical Yang-Baxter equation

Unlike the QYBE, the dynamical version is not an algebraic equation but a difference one, where the R-matrix is a matrix-valued function on an abelian Lie algebra instead of a matrix with scalar entries.

Let $\mathfrak{h}$ be a finite dimensional commutative Lie algebra over $\mathbb{C}$, $\gamma \in \mathbb{C}$ and V a semisimple finite dimensional $\mathfrak{h}$-module. Let $R : \mathfrak{h}^* \to End_{\mathfrak{h}}(V \otimes V)$ be a meromorphic function, then on $V \otimes V \otimes V$ the QDYBE of step γ reads:

$$R^{12}(\lambda - \gamma h^3)R^{13}(\lambda)R^{23}(\lambda - \gamma h^1) = R^{23}(\lambda)R^{13}(\lambda - \gamma h^2)R^{12}(\lambda),$$

where h^i is the *dynamical notation*, i.e., $R^{12}(\lambda - \gamma h^3)(v_1 \otimes v_2 \otimes v_3) := (R^{12}(\lambda - \gamma\mu)(v_1 \otimes v_2)) \otimes v_3$ if v_3 has weight μ, and similarly for h^1, h^2.
If $\mathfrak{h} = 0$ we obtain the usual QYBE.

A function $R_{ij} : \mathfrak{h}^* \to End(V_i \otimes V_j)$ is of zero weight if

$$[R_{ij}(\lambda), h \otimes 1 + 1 \otimes h] = 0$$

for all $h \in \mathfrak{h}$ and $\lambda \in \mathfrak{h}^*$.

Definition 11. A *Quantum dynamical R-matrix* $R : \mathfrak{h}^* \to End(V \otimes V)$ is a generically invertible solution of the QDYBE of zero weight .

3.1.1 Representation of quantum dynamical R-matrices

The following notions were introduced by Felder and Varchenko [FV], and later discussed in [EV2].

Let $M_{\mathfrak{h}^*}$ be the space of meromorphic functions on $\mathfrak{h}^*$, fix $\gamma \in \mathbb{C}$.
Denote by $V_{\mathfrak{h}}$ the category of $\mathfrak{h}$-vector spaces, with objects the diagonalizable $\mathfrak{h}$-modules and morphisms defined by $Hom_{V_{\mathfrak{h}}}(X, Y) = Hom_{\mathfrak{h}}(X, Y \otimes_{\mathbb{C}} M_{\mathfrak{h}^*})$.
Consider the bifunctor:

$$\bar{\otimes} : V_{\mathfrak{h}} \times V_{\mathfrak{h}} \to V_{\mathfrak{h}}$$

defined on objects by taking the usual tensor product and for any two morphisms $f : X \to X'$ and $g : Y \to Y'$ as

$$f\bar{\otimes}g : X \otimes Y \to X' \otimes Y'$$

$$f\bar{\otimes}g(\lambda) = f^1(\lambda - \gamma h^2)(1 \otimes g(\lambda)), \tag{1}$$

where

$$f^1(\lambda - \gamma h^2)(1 \otimes g(\lambda))(x \otimes y) = (f(\lambda - \mu)x) \otimes g(\lambda)y$$

for $g(\lambda)y$ of weight μ. The category $V_{\mathfrak{h}}$ equipped with the bifunctor $\bar{\otimes}$ is a tensor category.

Definition 12. Let $R : \mathfrak{h}^* \to End(V \otimes V)$ be a quantum dynamical R-matrix,i.e., a meromorphic function satisying QDYBE. A *representation of R* is an object $W \in V_{\mathfrak{h}}$ together with an invertible morphism $L \in End_{V_{\mathfrak{h}}}(V \bar{\otimes} W)$, called L-operator, such that

$$R^{12}(\lambda - \gamma h^3)L^{13}(\lambda)L^{23}(\lambda - \gamma h^1) = L^{23}(\lambda)L^{13}(\lambda - \gamma h^2)R^{12}(\lambda)$$

in $End_{V_{\mathfrak{h}}}(V \bar{\otimes} V \bar{\otimes} W)$.

Definition 13. Let (W, L_W) and (U, L_U) be representations of R, a morphism $A \in Hom_{V_{\mathfrak{h}}}(W, U)$ is an R-*morphism* if

$$(1 \otimes A(\lambda))L_W(\lambda) = L_U(\lambda)(1 \otimes A(\lambda - \gamma h^1)).$$

The representations of R form a category which we denote by $Rep(R)$.
The tensor product of two given representations $W, U \in Rep(R)$ is given by the pair $(W \otimes U, L_{W\otimes U})$, where $L_{W\otimes U}(\lambda) := L_W^{12}(\lambda - \gamma h^3)L_U^{13}(\lambda)$.

Proposition 14. *[EV2, Lemma 3.2] The pair $(W \otimes U, L_{W\otimes U})$ is itself a representation of R.*

Proof. We need to check that for $L_{W\otimes U}$ on $End_{V_{\mathfrak{h}}}(V \bar{\otimes} V \bar{\otimes}(W \otimes U))$ the following holds:

$$R^{12}(\lambda - \gamma h^3)L_{W\otimes U}^{13}(\lambda)L_{W\otimes U}^{23}(\lambda - \gamma h^1) = L_{W\otimes U}^{23}(\lambda)L_{W\otimes U}^{13}(\lambda - \gamma h^2)R^{12}(\lambda).$$

On the rhs we have:

$$\begin{aligned}
&R^{12}(\lambda - \gamma\tilde{h}^3)L_{W\otimes U}^{13}(\lambda)L_{W\otimes U}^{23}(\lambda - \gamma h^1) = \\
&= R^{12}(\lambda - \gamma\tilde{h}^3)L_{W\otimes U}^{13}(\lambda)L_W^{23}(\lambda - \gamma(h^1 + h^4))L_U^{24}(\lambda - \gamma h^1) = \\
&= R^{12}(\lambda - \gamma\tilde{h}^3)L_W^{13}(\lambda - \gamma h^4)L_U^{14}(\lambda)L_W^{23}(\lambda - \gamma(h^1 + h^4))L_U^{24}(\lambda - \gamma h^1).
\end{aligned}$$

Note that we write $\tilde{h}^3$ to indicate that we are using the weight of elements $z \in W \otimes U$, while we write h^3, h^4 when we consider $z \in W \otimes U$ as $z = w \otimes u$.
The lhs is:

$$\begin{aligned}
&L_{W\otimes U}^{23}(\lambda)L_{W\otimes U}^{13}(\lambda - \gamma h^2)R^{12}(\lambda) = \\
&= L_W^{23}(\lambda - \gamma h^4)L_U^{24}(\lambda)L_W^{13}(\lambda - \gamma(h^2 + h^4))L_U^{14}(\lambda - \gamma h^2)R^{12}(\lambda).
\end{aligned}$$

Using the fact that L_W and L_U satisfy the defining relation of a representation of R we conclude. □

Proposition 14 gives a structure of tensor category to $Rep(R)$.
We also introduce the notion of left and right dual representation:

Definition 15. Let $(W, L_W) \in Rep(R)$, the *right dual representation to W* is given by the pair (W^*, L_{W^*}), where W^* is the $\mathfrak{h}$-graded dual of W and

$$L_{W^*} = L_W^{-1}(\lambda + \gamma h^2)^{t_2},$$

where t_2 denotes dualization in the second component. Dually the *left dual representation of W* is given by $({}^*W, L_{{}^*W})$ with ${}^*W = W^*$ and

$$L_{{}^*W} = L_W^{t_2}(\lambda - \gamma h^2)^{-1}.$$

Note that L_{W^*} and L_{*W} are obtained by applying three different operations to L_W: inversion, shifting and dualization in the second component. To define the dual representation one must then have L_W or $L_W^{t_2}$ invertible.

3.2 h-Hopf algebroid

In this section we introduce the algebraic structures necessary to define the dynamical equivalent of quantum groups, the dynamical quantum groups, as given in [EV2]. We also introduce a construction that associates to a given meromorphic function $R : \mathfrak{h}^* \to End(V \otimes V)$ an $\mathfrak{h}$-bialgebroid called the dynamical quantum group corresponding to R.
The term algebroid comes from the fact that in the classical case one obtains, as dynamical analogues of Poisson-Lie groups, the Poisson groupoids (51). The term creates a parallelism between the classical and quantized version of the YBE and DYBE.

3.2.1 $\mathfrak{h}$-bialgebras

Let $\mathfrak{h}$ be a finite dimensional commutative Lie algebra on $\mathbb{C}$, let $M_{\mathfrak{h}^*}$ denote the field of meromorphic functions on $\mathfrak{h}^*$. Fix $\gamma \in \mathbb{C}$ with $\gamma \neq 0$.

Definition 16. An $\mathfrak{h}$*-algebra of step* γ is an associative algebra A over $\mathbb{C}$ with unit, endowed with an $\mathfrak{h}^*$-bigrading called the weight decomposition

$$A = \bigoplus_{\alpha,\beta \in \mathfrak{h}^*} A_{\alpha\beta},$$

and *left and right moment maps*, i.e., two algebra embeddings $\mu_l, \mu_r : M_{\mathfrak{h}^*} \to A_{00}$ such that $\forall a \in A_{\alpha\beta}$ and $f \in M_{\mathfrak{h}^*}$ we have

$$\mu_l(f(\lambda))(a) = a\mu_l(f(\lambda + \gamma\alpha)), \quad \mu_r(f(\lambda))(a) = a\mu_r(f(\lambda + \gamma\beta)).$$

Definition 17. A morphism between $\mathfrak{h}$-algebras is an algebra homomorphism $\varphi : A \to B$ preserving the moment maps.

Given two $\mathfrak{h}$-algebras A, B we define a third $\mathfrak{h}$-algebra given by the following operation:

Definition 18. The *matrix tensor product* of A, B is the $\mathfrak{h}$-algebra $A\tilde{\otimes}B$ where

$$(A\tilde{\otimes}B)_{\alpha\delta} = \bigoplus_{\beta} A_{\alpha\beta} \otimes_{M_{\mathfrak{h}^*}} B_{\beta\delta}.$$

Here $\otimes_{M_{\mathfrak{h}^*}}$ is the usual tensor product modulo the following relation:

$$\mu_r^A(f)a \otimes b = a \otimes \mu_l^B(f)b$$

for any $f \in M_{\mathfrak{h}^*}, a \in A, b \in B$.

On $A\tilde{\otimes}B$ define the moment maps as:

$$\mu_l^{A\tilde{\otimes}B}(f) = \mu_l^A(f) \otimes 1, \quad \mu_r^{A\tilde{\otimes}B}(f) = 1 \otimes \mu_r^B(f).$$

Definition 19. A *coproduct* on an $\mathfrak{h}$-algebra A is an homomorphism of $\mathfrak{h}$-algebras $\Delta : A \to A\tilde{\otimes}A$.

We give now a simple nontrivial example of an $\mathfrak{h}$-algebra that will be used to give a monoidal category structure to the category of $\mathfrak{h}$-algebras.

Example 20. Let $D_{\mathfrak{h}}$ be the algebra of difference operators $M_{\mathfrak{h}^*} \to M_{\mathfrak{h}^*}$, i.e., operators of the form $\sum_{i=1}^n f_i(\lambda)T_{\beta_i}$ where $f_i \in M_{\mathfrak{h}^*}$ and $\forall \beta \in \mathfrak{h}^*$ we denote T_β the field of automorphisms of $M_{\mathfrak{h}^*}$ given by $(T_\beta f)(\lambda) = f(\lambda + \gamma\beta)$.
On $D_{\mathfrak{h}}$ we define the weight decomposition as

$$D_{\mathfrak{h}} = \bigoplus (D_{\mathfrak{h}})_{\alpha\beta},$$

where $(D_{\mathfrak{h}})_{\alpha\beta} = 0$ if $\alpha \neq \beta$ and $(D_{\mathfrak{h}})_{\alpha\alpha} = \{f(\lambda)T_\alpha^{-1} : f \in M_{\mathfrak{h}^*}\}$. The moment maps are given by the tautological isomorphism

$$\mu_l = \mu_r : M_{\mathfrak{h}^*} \to (D_{\mathfrak{h}})_{00},$$

in fact $(D_{\mathfrak{h}})_{00} = \{f(\lambda)T_0^{-1} : f \in M_{\mathfrak{h}^*}\} = \{f(\lambda) : f \in M_{\mathfrak{h}^*}\} \cong M_{\mathfrak{h}^*}$.

By definition of $D_{\mathfrak{h}}$ one has $A\tilde{\otimes}D_{\mathfrak{h}}$ and $D_{\mathfrak{h}}\tilde{\otimes}A$ isomorphic to A.
The $\mathfrak{h}$-algebra $D_{\mathfrak{h}}$ is the unit object of the monoidal category of $\mathfrak{h}$-algebras.

Definition 21. A *counit* on an $\mathfrak{h}$-algebra A is a homomorphism of $\mathfrak{h}$-algebras $\epsilon : A \to D_{\mathfrak{h}}$.

Definition 22. An $\mathfrak{h}$-*bialgebroid* is an $\mathfrak{h}$-algebra A equipped with a coassociative coproduct Δ,i.e., $(\Delta \otimes Id_A) \circ \Delta = (Id_A \otimes \Delta) \circ \Delta$, and a counit ϵ such that $(\epsilon \otimes Id_A) \circ \Delta = (Id_A \otimes \epsilon) \circ \Delta = Id_A$.

Remark 23. Note that an $\mathfrak{h}$-bialgebroid is defined using the matrix tensor product $\tilde{\otimes}$, this allows multiplication by elements of $M_{\mathfrak{h}}$, i.e., meromorphic functions and not only holomorphic.

Example 24. $D_{\mathfrak{h}}$ is an $\mathfrak{h}$-bialgebroid with coproduct $\Delta : D_{\mathfrak{h}} \to D_{\mathfrak{h}}\tilde{\otimes}D_{\mathfrak{h}}$ the canonical isomorphism and counit $\epsilon = Id$.

Consider A an $\mathfrak{h}$-algebra, a linear map $S : A \to A$ is an antiautomorphism of $\mathfrak{h}$-algebras if it is an antiautomorphism of algebras and $\mu_r \circ S = \mu_l$ and $\mu_l \circ S = \mu_r$.

Definition 25. Let A be an $\mathfrak{h}$-bialgebroid, an *antipode* on A is an antiautomorphism of $\mathfrak{h}$-algebras $S : A \to A$ such that for any $a \in A$ and any presentation of $\Delta(a)$ one has

$$\sum_i a_i^1 S(a_i^2) = \mu_l(\epsilon(a)1), \quad \sum_i S(a_i^1)a_i^2 = \mu_r(\epsilon(a)1).$$

Where ϵ and Δ are the counit and coproduct on A and for $a \in A$ we have $\Delta(a) = \sum_i a_i^1 \otimes a_i^2$.

Definition 26. An $\mathfrak{h}$-bialgebroid with an antipode is called an $\mathfrak{h}$-*Hopf algebroid.*

3.2.2 The dynamical quantum group A_R

Let $\mathfrak{h}$ be a finite dimensional commutative Lie algebra, consider $V = \bigoplus_{\alpha\in\mathfrak{h}^*} V_\alpha$ a finite dimensional diagonizable $\mathfrak{h}$-module.
Let $R : \mathfrak{h}^* \to End(V\otimes V)$ be a meromorphic function such that for a generic λ we have $R(\lambda)$ invertible. Similarly to the RTT construction 2.3, we want to define an $\mathfrak{h}$-bialgebroid A_R associated to R that we will call the *dynamical quantum group corresponding to* R, i.e., a dynamical analogue of the quantum group attached to an R-matrix [EV2].

Define A_R as the quotient of the algebra $\tilde{A}$ freely generated by $M_{\mathfrak{h}^*} \otimes M_{\mathfrak{h}^*}$ and new generators which are matrix elements of the operators $L^{\pm} \in End(V) \otimes A_R$, L_{ab} and $(L^{-1})_{ab}$ for $a, b = 1, ..., dimV$.
For $f \in M_{\mathfrak{h}^*}$ we denote $f(\lambda^1)$ and $f(\lambda^2)$ the elements of respectively the first and second copy of $M_{\mathfrak{h}^*}$ in A_R. We denote the weight components of $L^{\pm}$ with respect to the natural $\mathfrak{h}$-bigrading on $End(V)$ as $(L^{\pm})_{\alpha\beta}$, so that $(L^{\pm})_{\alpha\beta} \in Hom_{\mathbb{C}}(V_\beta, V_\alpha) \otimes A_R$.
We quotient $\tilde{A}$ by the ideal defined by the following relations:

$$f(\lambda^1)L_{\alpha\beta} = L_{\alpha\beta}f(\lambda^1 + \gamma\alpha) \quad f(\lambda^2)L_{\alpha\beta} = L_{\alpha\beta}f(\lambda^2 + \gamma\beta) \quad [f(\lambda^1), g(\lambda^2)] = 0 \tag{2}$$

$$LL^{-1} = L^{-1}L = 1 \tag{3}$$

$$R^{12}(\lambda^1)L^{13}L^{23} =: L^{23}L^{13}R^{12}(\lambda^2). \tag{4}$$

The third relation is called the dynamical Yang-Baxter relation and should be read in the following way: if $\{v_\alpha\}$ is a homogeneous basis of V, and $L = \sum E_{ab} \otimes L_{ab}$, $R(\lambda)(v_a \otimes v_b) = \sum R^{ab}_{cd}(\lambda)v_c \otimes v_d$, then

$$\sum R^{xy}_{ac}(\lambda^1)L_{xb}L_{yd} = \sum R^{bd}(\lambda^2)_{xy}L_{cy}L_{ax}$$

summing over repeated indices.
To give A_R the structure of an $\mathfrak{h}$-algebra we define the moment maps as follows:

$$\mu_l(f(\lambda)) = f(\lambda^1), \quad \mu_r(f(\lambda)) = f(\lambda^2).$$

The weight decomposition is given by $f(\lambda^1), f(\lambda^2) \in (A_R)_{00}$ and $L_{\alpha\beta} \in Hom_{\mathbb{C}}(V_\beta, V_\alpha) \otimes (A_R)_{\alpha\beta}$.

We want to give A_R a $\mathfrak{h}$-bialgebroid structure, to do so we define a coproduct $\Delta : A_R \to A_R \tilde{\otimes} A_R$ as

$$\Delta(L) = L^{12}L^{13}, \;\; \Delta(L^{-1}) = (L^{-1})^{13}(L^{-1})^{12},$$

where Δ is applied to the second component of $L^{\pm}$.

Remark 27. Compare the Δ here defined to that of Proposition 7: similarly to the RTT case, the idea is to define the coproduct on the generators and to extend it. Note that here, in the Yang-Baxter relation, we are acting on $V \otimes V \otimes V$ whereas in the RTT relation we had T^1, T^2 acting on $V \otimes V$.
Similarly to Proposition 7 one must check that the given coproduct preserves the defining relations.

Proposition 28. *[EV2, Proposition 4.2]* Δ *extends to a well defined homomorphism* $A_R \to A_R \tilde{\otimes} A_R$.

Proof. By definition $\Delta(L_{\alpha\beta}) = \sum_\gamma L^{12}_{\alpha\gamma}L^{13}_{\gamma\beta}$. We need to show that Δ preserves the defining relations of A_R.
Relations 2 and 3 are invariant by definition of Δ. Consider relation 4, we have

$$\begin{aligned} R^{12}(\lambda^1_1)L^{13}L^{14}L^{23}L^{24} &= R^{12}(\lambda^1_1)L^{13}L^{23}L^{14}L^{24} \\ &=: L^{13}L^{23}R^{12}(\lambda^2_1) : L^{14}L^{24} \\ &= L^{13}L^{23}R^{12}(\lambda^1_2)L^{14}L^{24} \qquad , \\ &= L^{23}L^{13} : L^{24}L^{14}R^{12}(\lambda^2_2) : \\ &=: L^{23}L^{24}L^{13}L^{14}R^{12}(\lambda^2_2) : \end{aligned}$$

where the pedices on λ indicate that the functions are taken from the first or second component of $A_R \tilde{\otimes} A_R$. Since $A_R \tilde{\otimes} A_R$ is in the tensor product $A_R \otimes_{M_{\mathfrak{h}^*}} A_R$ we can replace λ^1_2 with λ^2_1 in the

equation.
We have then checked that

$$R^{12}(\lambda_1^1)L^{13}L^{14}L^{23}L^{24} =: L^{23}L^{24}L^{13}L^{14}R^{12}(\lambda_2^2):$$

so the proposition is proved.

□

We define the counit $\epsilon : A_R \to D_{\mathfrak{h}}$ by the formula:

$$\epsilon(L_{\alpha\beta}) = \delta_{\alpha\beta} Id_{V_\alpha} \otimes T_\alpha^{-1},$$

$$\epsilon((L^{-1})_{\alpha\beta}) = \delta_{\alpha\beta} Id_{V_\alpha} \otimes T_\alpha.$$

Similarly to the coproduct ϵ annihilates the relations 2 and 3 by definition. Relation 4 reduces to proving the following:

$$(\sum R^{12}(\lambda)(Id_{V_\alpha} \otimes Id_{V_\beta})) \otimes T_{\alpha+\beta}^{-1} = (\sum (Id_{V_\alpha} \otimes Id_{V_\beta})R^{12}(\lambda)) \otimes T_{\alpha+\beta}^{-1},$$

but R has zero weight, so the equation is satisfied.

Proposition 29. *[EV2, Proposition 4.3] The counit ϵ satisfies the counit axiom $(\epsilon \otimes Id) \circ \Delta = (Id \otimes \epsilon) \circ \Delta = Id$ for A_R.*

Combining Proposition 28 and 29 , we therefore have that A_R is an $\mathfrak{h}$-biequivariant bialgebroid,i.e., it is an $\mathfrak{h}$-bialgebroid and we have a pair of commuting actions of $\mathfrak{h}$ on A_R that behave well with the moment maps ([EV2, chapter 5]). We call it the dynamical quantum group corresponding to R.

To have an $\mathfrak{h}$-Hopf algebroid we need an antipode on A_R.

Definition 30. An invertible zero weight matrix function R is *rigid* if the element $L \in End(V) \otimes A_R$ is strongly invertible.

Consider $X \in B \otimes A$, with A, B algebras with unit and $i(X)$ the inverse, let I be the group freely generated by i, i_* with $i^2 = i_*^2 = 1$. The element X is said to be strongly invertible if $\forall g \in I$ the element $g(X)$ is well defined.
The following proposition holds:

Proposition 31. *[EV2, Proposition 4.4] R is rigid if and only if A_R admits an antipode S such that $S(L) = L^{-1}$. In this case, $S_{2n}(L) = (i^*i)^n(L)$ and $S^{2n+1}(L) = i(i^*i)^n(L)$.*
In particular, $S(L^{-1}) = i^*i(L)$.

Consequently, under the assumption of rigidity, A_R is an $\mathfrak{h}$-Hopf algebroid.
Although A_R is an $\mathfrak{h}$-Hopf algebroid, for a generic R rigid zero weight function this algebra does not have interesting dynamical representations, however in the case of R a dynamical quantum R-matrix the category $Rep(R)$ is nontrivial, and so is $Rep(A_R)$.

3.2.3 The dynamical representation

Suppose W is a diagonizable $\mathfrak{h}$-module, then we define $D^{\alpha}_{\mathfrak{h},W} \subset Hom_{\mathbb{C}}(W, W \otimes D_{\mathfrak{h}})$ to be the space of all difference operators on $\mathfrak{h}^*$ with coefficients in $End_{\mathbb{C}}(W)$ and weight α with respect to the action of $\mathfrak{h}$ in W.
Define $D_{\mathfrak{h},W} := \bigoplus_{\alpha} D^{\alpha}_{\mathfrak{h},W}$ algebra with weight decomposition $D_{\mathfrak{h},W} = \bigoplus_{\alpha,\beta}(D_{\mathfrak{h},W})_{\alpha\beta}$ defined by

$$(D_{\mathfrak{h},W})_{\alpha\beta} := \{gT_{\beta}^{-1} : g \in Hom_{\mathbb{C}}(W, W \otimes M_{\mathfrak{h}^*})\ of\ weight\ \beta - \alpha\},$$

and moment maps $\mu_r(f(\lambda)) = f(\lambda)$ and $\mu_l(f(\lambda)) = f(\lambda - \gamma h)$, where $f(\lambda - \gamma h)w = f(\lambda - \gamma\mu)w$ if $w \in W(\mu)$.

Lemma 32. *[EV2, Lemma 4.3] There is a natural embedding of $\mathfrak{h}$-algebras $D_{\mathfrak{h},W}\tilde{\otimes}D_{\mathfrak{h},U} \to D_{\mathfrak{h},W\otimes U}$, given by the formula $fT_{\beta} \otimes gT_{\delta} \to (f\bar{\otimes}g)T_{\delta}$. This embedding is an isomorphism if W, U are finite-dimensional.*

The product $f\bar{\otimes}g$ is defined in 1.

Definition 33. Let A be an $\mathfrak{h}$-algebra, a *dynamical representation of A* is a diagonalizable $\mathfrak{h}$-module W endowed with an $\mathfrak{h}$-algebra homomorphism $\pi_W : A \to D_{\mathfrak{h},W}$.
A homomorphism of dynamical representation of A is a map $\varphi \in Hom_{\mathbb{C}}(W_1, W_2 \otimes M_{\mathfrak{h}^*})$ such that $\varphi \circ \pi_{W_1}(x) = \pi_{W_2}(x) \circ \varphi$ for all $x \in A$.

We would now like to prove the following:

Proposition 34. *[EV2, Proposition 4.6] The tensor categories $Rep(A_R)$ and $Rep(R)$ are equivalent.*

To do so we introduce a few results on $Rep(A_R)$.
Let A be an $\mathfrak{h}$-Hopf algebroid, if (W, π_W) is a dynamical representation of A we denote by π^0_W the map $\pi^0_W : A \to Hom(W, W \otimes M_{\mathfrak{h}^*})$ given by $\pi^0_W(x)w = \pi_W(x)w$ for all $w \in W$, i.e., the difference operator π_W restricted to constant functions.

Definition 35. Let (W, π_W) be a dynamical representation of A, the *right dual representation to W* is (W^*, π_{W^*}), where W^* is the $\mathfrak{h}$-graded dual to W and

$$\pi^0_{W^*}(x)(\lambda) = \pi^0_W(S(x))(\lambda + \gamma h - \gamma\alpha)^t \quad \forall x \in A_{\alpha\beta}.$$

The *left dual representation to W* is the pair $({}^*W, \pi_{{}^*W})$ with ${}^*W = W^*$ and

$$\pi^0_{{}^*W}(x)(\lambda) = \pi^0_W(S^{-1}(x))(\lambda + \gamma h - \gamma\alpha)^t \quad \forall x \in A_{\alpha\beta}.$$

Proposition 36. *[EV2, Proposition 4.1] The right and left dual representation define dynamical representations of A. Moreover, if $A(\lambda) : W_1 \to W_2$ is a morphism of dynamical representations, then $A^*(\lambda) := A(\lambda + \gamma h)^t$ defines a morphism $W_2^* \to W_1^*$ and ${}^*W_2 \to^* W_1$.*

Let $R : \mathfrak{h}^* \to End(V \otimes V)$ be a meromorphic function, from the previous proposition and the results on A_R one has the following:

Lemma 37. *[EV2, Lemma 3.4, 3.5] Let W be a representation of R, the right and left dual representations of R are representations of R. If W has finite dimensional weight subspaces then ${}^*(W^*) = ({}^*W)^* = W$. If $A : W_1 \to W_2$ is a homomorphism of representations of R, then the linear map $A^*(\lambda) := A(\lambda + \gamma h^1)^t = A^t(\lambda - \gamma h^1)$ is a homomorphism of representations $W_2^* \to W_1^*$ and ${}^*W_2 \to^* W_1$ when these representations are defined.*

Consider a meromorphic function R and the $\mathfrak{h}$-algebra A_R, we can prove Proposition 34:

Proof. Define $\Gamma : Rep(A_R) \to Rep(R)$ the functor given by the identity on vector spaces and

$$L_{\Gamma(W)} = \pi^0_W(L).$$

Define the functor $\Gamma^{-1} : Rep(R) \to Rep(A_R)$ as the identity on vector spaces and

$$\pi^0_{\Gamma^{-1}(W)}(L) = L_W.$$

The two functors are inverse to each other and preserve the tensor structure.

□

3.3 Fusion and exchange construction

In this section we introduce a way to construct solutions of the QDYBE starting from classical representation theory of Lie algebras, see [ES].

3.3.1 Fusion operators

Let $\mathfrak{g}$ be a simple finite dimensional complex Lie algebra with polar decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$. Let V be a finite dimensional $\mathfrak{g}$-module with weight decomposition $V = \bigoplus_{\nu \in \mathfrak{h}^*} V[\nu]$, we denote M_λ the Verma module over $\mathfrak{g}$ with highest weight $\lambda \in \mathfrak{h}^*$, i.e., $M_\lambda = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda$ with $\mathfrak{b}$ the Borel subalgebra, x_λ its highest weight vector and x^*_λ the lowest weight vector of the dual module.
Given $\mu, \lambda \in \mathfrak{h}^*$ consider an intertwining operator

$$\Phi : M_\lambda \to M_\mu \otimes V,$$

define the *expectation value* of Φ as

$$\langle \Phi \rangle = x^*_\mu(\Phi x_\lambda) \in V[\lambda - \mu].$$

Proposition 38. *[ES, Proposition 2.2] If M_μ is irreducible the map $Hom_{\mathfrak{g}}(M_{\mu+\nu}, M_\mu \otimes V) \to V[\nu]$ given by $\Phi \to \langle \Phi \rangle$ is an isomorphism.*

This allows us to define for any $v \in V[\nu]$ the intertwining operator $\Phi^v_\lambda : M_\lambda \to M_{\lambda-\nu} \otimes V$ such that $\langle \Phi^v_\lambda \rangle = v$.
Consider now V, W finite dimensional $\mathfrak{g}$-modules, let $v \in V$ and $w \in W$ be homogeneous vectors of weight $wt(v), wt(w)$ respectively. Let $\lambda \in \mathfrak{h}^*$ then define the composition

$$\Phi^{w,v}_\lambda := (\Phi^w_{\lambda - wt(v)} \otimes 1)\Phi^v_\lambda : M_\lambda \to M_{\lambda - wt(v) - wt(w)} \otimes W \otimes V.$$

So $\Phi^{w,v}_\lambda \in Hom(M_\lambda, M_{\lambda - wt(v) - wt(w)} \otimes W \otimes V)$ and, by the previous result, there exists a unique $u \in W \otimes V[wt(v) + wt(w)]$ such that $\Phi^u_\lambda = \Phi^{w,v}_\lambda$. This defines an $\mathfrak{h}$-linear operator

$$J_{WV}(\lambda) : W \otimes V \to W \otimes V$$

given by $J_{WV}(\lambda)(w \otimes v) = \langle \Phi^{w,v}_\lambda \rangle$.

Definition 39. The operator $J^{w,v}_\lambda$ is called the *fusion operator* of V and W.

Proposition 40. *[ES, Proposition 2.3] Let V, W be finite dimensional $\mathfrak{g}$-modules, then the following hold:*

- *$J_{WV}(\lambda)$ is a rational function of λ.*
- *$J_{WV}(\lambda)$ is strictly lower triangular, i.e., $J_{WV}(\lambda) = 1+N$ with N sum of terms with strictly positive weight in the second component. In particular $J_{WV}(\lambda)$ is invertible.*
- *Let U, V, W be finite dimensional $\mathfrak{g}$-modules, the fusion operators satisfy the dynamical 2-cocycle condition:*

$$J_{U\otimes W,V}(\lambda)(J_{UW}(\lambda - h^3) \otimes 1) = J_{U,W\otimes V}(\lambda)(1 \otimes J_{WV}(\lambda)).$$

3.3.2 Exchange operators

Definition 41. The *exchange operator* on V and W, finite dimensional $\mathfrak{g}$-modules, is defined as

$$R_{VW}(\lambda) := J_{VW}(\lambda)^{-1} J^{12}_{WV}(\lambda) : V \otimes W \to V \otimes W,$$

where $J^{12} = \tau J \tau$, for $\tau(x \otimes y) = y \otimes x$.

The operator R_{VW} tells us how to exchange the intertwining operators, if $\Phi^{w,v}_\lambda = \tau \sum_i \Phi^{w_i,v_i}_\lambda$ then $R_{VW}(\lambda)(v \otimes w) = \sum_i v_i \otimes w_i$.

Proposition 42. *For U, V, W finite dimensional $\mathfrak{g}$-modules the exchange matrices satisfy:*

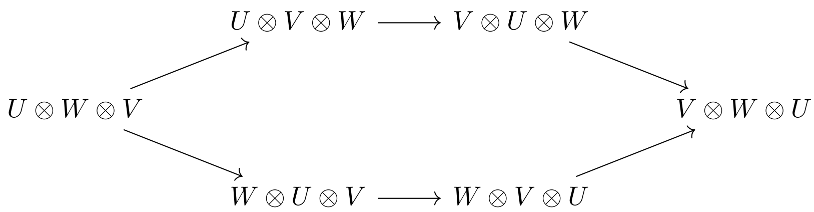
$$R^{VW}(\lambda - h^3)R^{VU}(\lambda)R^{WU}(\lambda - h^1) = R^{WU}(\lambda)R^{VU}(\lambda - h^2)R^{VW}(\lambda).$$

In particular R_{VV} is a solution of the QDYBE.

Proof. Let $u \in U, v \in V, w \in W$ be homogeneous elements and

$$\Phi^{u,v,w}_\lambda = \Phi^u_{\lambda - wt(v) - wt(w)} \circ \Phi^w_{\lambda - wt(v)} \circ \Phi^v_\lambda.$$

Define $\sigma : U \otimes W \otimes V \to V \otimes W \otimes U$ as $\sigma(x \otimes y \otimes z) = z \otimes y \otimes x$, then we can write $\Phi^{u,v,w}_\lambda = \sum_i \sigma \Phi^{v_i,w_i,u_i}_\lambda$ in two different ways as given by the following diagram:



Using the two ways we obtain the relation on the exchange matrices.

□

Example 43. [ES, Example 1] Consider $\mathfrak{g} = sl_2$ with generators e, f, h and $V = \mathbb{C}^2$ with basis $\{x, y\}$. The action of $\mathfrak{g}$ on V is given by taking:

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Using the triangularity property on $J_{V,V}(\lambda)$ one gets that

$$J_{V,V}(\lambda)(x \otimes x) = x \otimes x, \quad J_{V,V}(\lambda)(y \otimes y) = y \otimes y, \quad J_{V,V}(\lambda)(y \otimes x) = y \otimes x.$$

We need to compute $J_{V,V}(\lambda)(x \otimes y)$.
Consider $\Phi^x_{\lambda+1}(x_{\lambda+1}) = x_\lambda \otimes x$ and $\Phi^y_\lambda(x_\lambda) = x_{\lambda+1} \otimes y + g(\lambda) f x_{\lambda+1} \otimes x$, we use the intertwining property to determine $g(\lambda)$:

$$\begin{aligned} 0 = \Phi^y_\lambda(e x_\lambda) &= (e \otimes 1 + 1 \otimes e)\Phi^y_\lambda(x_\lambda) = \\ &= x_{\lambda+1} \otimes x + g(\lambda) e f x_{\lambda+1} \otimes x = \\ &= x_{\lambda+1} \otimes x + (1+\lambda) g(\lambda) x_{\lambda+1} \otimes x \ , \end{aligned}$$

so $g(\lambda) = -\frac{1}{1+\lambda}$. Then

$$\Phi^{x,y}_\lambda(x_\lambda) = x_\lambda \otimes (x \otimes y - \frac{1}{\lambda+1}) y \otimes x.$$

So we get

$$J_{V,V}(\lambda) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -\frac{1}{\lambda+1} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$R_{V,V}(\lambda) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\frac{1}{\lambda+1} & 0 \\ 0 & \frac{1}{\lambda+1} & 1 - \frac{1}{(\lambda+1)^2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We can check that $R_{V,V}$ is a solution of the QDYBE

$$R^{12}(\lambda - h^3) R^{13}(\lambda) R^{23}(\lambda - h^1) = R^{23}(\lambda) R^{13}(\lambda - h^2) R^{12}(\lambda).$$

It is enough to check that the relation holds on the basis of $V \otimes V \otimes V$, we start with the easy case:

$$\begin{aligned} &R^{12}(\lambda - h^3) R^{13}(\lambda) R^{23}(\lambda - h^1)(x \otimes x \otimes x) = \\ &= x \otimes x \otimes x = \\ &= R^{23}(\lambda) R^{13}(\lambda - h^2) R^{12}(\lambda)(x \otimes x \otimes x) \end{aligned}$$

and

$$\begin{aligned} &R^{12}(\lambda - h^3) R^{13}(\lambda) R^{23}(\lambda - h^1)(y \otimes y \otimes y) = \\ &= y \otimes y \otimes y = \\ &= R^{23}(\lambda) R^{13}(\lambda - h^2) R^{12}(\lambda)(y \otimes y \otimes y) \end{aligned}$$

Consider $x \otimes x \otimes y$, remember that x and y have weight $1, -1$, so we have:

$$\begin{aligned}
&R^{12}(\lambda - h^3)R^{13}(\lambda)R^{23}(\lambda - h^1)(x \otimes x \otimes y) = \\
&= R^{12}(\lambda - h^3)R^{13}(\lambda)(x \otimes x \otimes y + \frac{1}{\lambda} x \otimes y \otimes x) = \\
&= R^{12}(\lambda - h^3)[x \otimes x \otimes y + \frac{1}{\lambda+1} y \otimes x \otimes x + \frac{1}{\lambda} x \otimes y \otimes x] = \\
&= x \otimes x \otimes y - \frac{1}{\lambda(\lambda+1)} x \otimes y \otimes x + \frac{1}{\lambda+1}(1 - \frac{1}{\lambda^2}) y \otimes x \otimes x + \\
&+ \frac{1}{\lambda} x \otimes y \otimes x + \frac{1}{\lambda^2} y \otimes x \otimes x = \\
&= x \otimes x \otimes y + \frac{1}{\lambda+1} x \otimes y \otimes x + \frac{1}{\lambda} y \otimes x \otimes x
\end{aligned}$$

and

$$\begin{aligned}
&R^{23}(\lambda)R^{13}(\lambda - h^2)R^{12}(\lambda)(x \otimes x \otimes y) = \\
&= R^{23}(\lambda)R^{13}(\lambda - h^2)(x \otimes x \otimes y) = \\
&= R^{23}(\lambda)(x \otimes x \otimes y + \frac{1}{\lambda} y \otimes x \otimes x) = \\
&= x \otimes x \otimes y + \frac{1}{\lambda+1} x \otimes y \otimes x + \frac{1}{\lambda} y \otimes x \otimes x
\end{aligned} .$$

Similarly for any element of the basis of $V \otimes V \otimes V$ the relation holds, i.e., R_{VV} is a dynamical R-matrix.

4 The classical dynamical case

Quantization of the classical Yang-Baxter equation (CYBE) led to the theory of quantum groups, its dynamical analogue is the classical dynamical Yang-Baxter equation (CDYBE).
In this section we follow [S] and [EV1] to introduce some basic notions linked to the CDYBE and give a geometric meaning to its solutions, the dynamical r-matrices, using the notion of dynamical Poisson groupoid, the dynamical analogue of Poisson-Lie groups.
Similarly to the QYBE, let V be a finite dimensional semisimple $\mathfrak{h}$-module and $R : \mathfrak{h}^* \to End(V \otimes V)$ of the form $R = 1 - \gamma r + O(\gamma^2)$ a solution of the QDYBE of step γ, then $r : \mathfrak{h}^* \to End(V \otimes V)$ satisfies the CDYBE.
The function r is the *classical limit* of R, and R is a *quantization* of r.

4.1 The classical dynamical Yang-Baxter equation

Let $\mathfrak{g}$ be a Lie algebra and $r \in \mathfrak{g} \otimes \mathfrak{g}$, then:

Definition 44. The *classical Yang-Baxter equation* (CYBE) is given by

$$[r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{13}, r^{23}] = 0.$$

Solutions of the CYBE are called r-matrices.

Let $\mathfrak{h} \subset \mathfrak{g}$ be a subalgebra, an element $x \in \mathfrak{g} \otimes \mathfrak{g}$ is said to be $\mathfrak{h}$-invariant if

$$[k \otimes 1 + 1 \otimes k, x] = 0 \quad \forall k \in \mathfrak{h}.$$

For $x \in \mathfrak{g}^3$ define

$$Alt(x) = x^{123} + x^{231} + x^{312}.$$

Definition 45. The *classical dynamical Yang-Baxter equation* (CDYBE) is the differential equation for an $\mathfrak{h}$-invariant holomorphic function $r : D \to \mathfrak{g} \otimes \mathfrak{g}$, with $D \subset \mathfrak{h}^*$, given by

$$Alt(dr) + [r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{13}, r^{23}] = 0.$$

Here $dr : D \to \mathfrak{g}^3$ is considered as a holomorphic function $dr(\lambda) = \sum_i x_i \otimes \frac{\partial r^{23}}{\partial x_i}(\lambda)$, for any basis (x_i) of $\mathfrak{h}$.

Explicitly we have

$$Alt(dr) = \sum_i x_i^1 \frac{\partial r^{23}}{\partial x_i} + \sum_i x_i^2 \frac{\partial r^{31}}{\partial x_i} + \sum_i x_i^3 \frac{\partial r^{12}}{\partial x_i}.$$

Definition 46. A function $r : D \to \mathfrak{g} \otimes \mathfrak{g}$ satisfying the CDYBE is called *dynamical r-matrix.*

In [E] we find the following conjecture:

Any classical dynamical r-matrix can be quantized.

This has been proved in the non-dynamical case by Etingof P. and Kazhdan D., and in the dynamical case for skew-symmetric solutions with some additional assumptions by Xu P..

4.2 Dynamical Poisson groupoids

We now introduce the classical limit of the notion of dynamical quantum groups, i.e., Poisson groupoids. A more detailed analysis is given in [EV1] and [W].

Remark 47. We will only consider groupoids built on small categories, so objects and morphisms form a set.

Definition 48. A *groupoid* is a small category in which every morphism is invertible.

We will consider groupoids given by:

- a set X (denoting the groupoid itself) of arrows
- a set P of objects
- two surjective maps $s, t : X \to P$ called source and target
- a composition map $m : \{(a, b) \in X \times X : s(a) = t(b)\} \to X$
- an injective map $E : P \to X$ called identity map

satisfying various conditions. In particular there exists an involution $i : X \to X$ such that $s(i(x)) = t(x)$, $s(x) = t(i(x))$, $m(x, i(x)) = id_{t(x)}$ and $m(i(x), x) = id_{s(x)}$.
The notion of groupoid generalizes that of group, in particular a groupoid with only one object is a group.

Definition 49. A *Lie groupoid* is a groupoid equipped with a smooth structure, i.e., the set of objects and morphisms are both smooth manifolds and the structure maps are smooth.

Definition 50. [W][EV1] A *Poisson groupoid* is a Lie groupoid X endowed with a poisson bracket such that the graph of the composition map is a coisotropic submanifold of $X \times X \times \bar{X}$, i.e., the smooth functions vanishing on it are closed under Poisson bracket.
Here $\bar{X}$ indicates the opposite Poisson manifold to X.

We can now introduce a special class of Poisson groupoids which we will call dynamical Poisson groupoids.
Let G be a Lie group and $\mathfrak{g}$ its Lie algebra, $H \subset G$ a connected Lie subgroup with Lie algebra $\mathfrak{h}$. Define the coadjoint action as for any $h \in H$, $x \in \mathfrak{h}$ and $\mu \in \mathfrak{h}^*$:

$$Ad^* : H \to Aut(\mathfrak{h}^*)$$

$$Ad^*(h)(\mu)(x) := \mu(Ad(h^{-1})(x)).$$

Let $U \subset \mathfrak{h}^*$ be an open subset invariant under the coadjoint action.
Consider the manifold $X(G, H, U) := U \times G \times U$, it has a natural structure of Lie groupoid given by taking $X = X(G, H, U)$, $P = U$, $s(u_1, g, u_2) = u_2$, $t(u_1, g, u_2) = u_1$, $E(u) = (u, 1, u)$ and $m((u_1, f, u_2), (u_2, g, u_3)) = (u_1, fg, u_3)$, while the inversion is defined by using inversion on G as $i(u_1, g, u_2) = (u_2, g^{-1}, u_1)$. This groupoid is the direct product of the trivial groupoid with base U and the group G.
On X we consider a left and right commuting actions of H defined as:

$$l(h)(u_1, g, u_2) = (Ad^*(h)(u), hg, u_2),$$

$$r(h)(u_1, g, u_2) = (u_1, gh, Ad^*(h^{-1})(u_2)).$$

We can also define the diagonal action of H on $X \times X$ as $\Delta(h)(x, y) = (r(h)^{-1}x, l(h)y)$, which preserves the composition map.
For any $a \in \mathfrak{h}$ consider the functions on X defined by $a_1(u_1, g, u_2) = a(u_1)$ and $a_2(u_1, g, u_2) = a(u_2)$. Recall that given (M, ω) a symplectic manifold and H a Lie group, then a symplectic action of H on M is said to be a *Hamiltonian action* if there exists a moment map $\mu : M \to \mathfrak{h}^*$.

Definition 51. The pair $(X, \{\})$, where $\{,\}$ is a Poisson bracket on X, is a *dynamical Poisson groupoid* if the following holds:

- the actions l, r are Hamiltonian, with t, s being their moment maps, and for any $a, b \in \mathfrak{h}$ one has $\{a_1, b_2\} = 0$.
- Let $X \bullet X := X \times X//\Delta(H)$ be the Hamiltonian reduction of $X \times X$ by the diagonal actions and $\bar{m} : X \bullet X \to X$ the reduction of the composition map by H. Then $\bar{m}$ is a Poisson map.

Remark 52. If $H = 1$ a dynamical Poisson groupoid is a Poisson-Lie group.

References

[B] D. Bump, Course notes on Quantum Groups, Stanford, spring 2019

[Dr] V. G. Drinfeld, *Quantum groups*, Proc. Int. Congr. Math (Berkley,1986) 1, 798-820.

[E] P. Etingof, *On the Dynamical Yang-Baxter equation*, 2000.

[ES] P. Etingof, O. Schiffman, *Lectures on the dynamical Yang-Baxter equation*, 1999.

[EV1] P. Etingof, A. Varchenko, *Geometry and classification of solutions of the Classical Dynamical Yang-Baxter equation*, 1997.

[EV2] P. Etingof, A. Varchenko, *Solution of the Quantum Dynamical Yang-Baxter equation and Dynamical Quantum Groups*, 1997.

[EV3] P. Etingof, A. Varchenko, *Exchange dynamical quantum groups*, 1998.

[Fa] L. D. Faddeev, *Quantum groups*, Bol. Soc. Bras. Mat 20, 47–54 (1989)

[F] G. Felder, *Conformal field theory and integrable systems associated to elliptic curves, 1994*

[FV] G. Felder, A. Varchenko, *On representation of the elliptic quantum group* $E_{\tau,\eta(sl_2)}$, Commun. Math. Phys. 181(1996), 746-762

[L] I. Losev, Course notes on Representation theory, Yale, 2015

[K] C. Kassel, *Quantum groups*, Vol. 155 Springer, 2012

[Ki] O. Kivinen, Notes on quantum groups

[S] O. Schiffmann, *On classification of dynamical r-matrices*, 2018

[W] A. Weinstein, *Coisotropic calculus and Poisson groupoids*, J.Math.Soc.Japan 40 (1998), 705-727.

[FRT] L. D. Faddeev, N. Y. Reshetikhin, L. A. Takhtajan, *Quantization of Lie groups and Lie algebras*, Leningrad Math, 1989