$\mathbb{Z}$-algebras from Coulomb branches

Geometric Representation Theory, Perimeter Institute

(Report on work in progress)

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Outline

- Review of Coulomb branches and line defects in $3d\mathcal{N} = 4$ theories
- $\mathbb{Z}$-algebras
- Examples
- Generalized affine Springer theory
- Directions

BFN Coulomb branches and "line defects"

- I'll concentrate on the math side of the story. Fix $G \supset B \supset T, \mathcal{O} = \mathbb{C}[[t]], \mathcal{K} = \mathbb{C}((t)), N \in \mathrm{Rep}(G)$.
- Recall the BFN space of triples

$$\mathcal{R}_{G,N} := \{(g,s) \in \mathrm{Gr}_G \times N_{\mathcal{O}} | gs \in N_{\mathcal{O}}\}.$$

  The BM homology $H_*^{G_{\mathcal{O}} \rtimes \mathbb{C}^\times}(\mathcal{R}_{G,N})$ with convolution product is an associative algebra $\mathcal{A}^{\hbar}_{G,N}$, filtered quantization of a commutative algebra $\mathcal{A}_{G,N}$=ring of functions on "the" Coulomb branch.
- Flavor deformation: $1 \to G \to \widetilde{G} \to G_F \to 1$ gives $\widetilde{\mathcal{A}}^{\hbar}_{G,N} = H_*^{\widetilde{G}_{\mathcal{O}} \rtimes \mathbb{C}^\times}(\mathcal{R}_{G,N})$. Will always take G_F to be a torus.
- Heuristically, these are endomorphisms of objects in a "category of line operators" (Dimofte-Garner-Geracie-Hilburn, Webster, Weekes, etc.) which is something like $G_{\mathcal{K}}$-equivariant D-modules on $N_{\mathcal{K}}$.

Convolution

- This category should contain objects coming from $\eta = (U, \mathbf{P})$, where $U \subset N_{\mathcal{K}}$ is a $\mathbf{P}$-stable lattice and $\mathbf{P}$ is a parahoric subgroup of $G_{\mathcal{K}}$. The above people simply define $\mathrm{Hom}(\eta, \eta') = H_*^{\mathbf{P}' \rtimes \mathbb{C}^\times}({}_{\eta}\mathcal{R}_{\eta'})$, where

$$ {}_{\eta}\mathcal{R}_{\eta'} = \{[g, s] \in G_{\mathcal{K}} \times^{\mathbf{P}'} U' | gs \in U\}. $$

- There is an associative multiplication $\mathrm{Hom}(\eta, \eta') \otimes_{\mathbb{C}} \mathrm{Hom}(\eta', \eta'') \to \mathrm{Hom}(\eta, \eta'')$ via the following modification of the BFN convolution product.

$$
\begin{array}{ccccc}
{}_{\eta}\mathcal{R}_{\eta'} \times {}_{\eta'}\mathcal{R}_{\eta''} & \xleftarrow{p} & p^{-1}({}_{\eta}\mathcal{R}_{\eta'} \times {}_{\eta'}\mathcal{R}_{\eta''}) & \xrightarrow{q} & q(p^{-1}({}_{\eta}\mathcal{R}_{\eta'} \times {}_{\eta'}\mathcal{R}_{\eta''})) \\
\Big\downarrow{\scriptstyle i} & & \Big\downarrow{\scriptstyle j} & & \Big\downarrow{\scriptstyle m} \\
\mathcal{T}_{\eta'} \times {}_{\eta'}\mathcal{R}_{\eta''} & \xleftarrow[p]{} & G_{\mathcal{K}} \times {}_{\eta'}\mathcal{R}_{\eta''} & & {}_{\eta}\mathcal{R}_{\eta''}
\end{array}
$$

$\mathbb{Z}$-algebras from Coulomb branches

- Taking a sequence of $\eta_0, \eta_1, \ldots$ one can produce new associative algebras.
- Just to give you an idea, if $\chi : G_F^\vee \to \mathbb{C}^\times$ is a cocharacter of the flavor group (i.e. a stability condition on the Higgs side) one can take $\mathbf{P} = G_\mathcal{O}$ and $U = (t^\chi)^i N_\mathcal{O}$ for some i. Actually, you can just think of $\widetilde{G} = G \times \mathbb{C}^\times_{dil}$ and $\chi, \eta_i \in \mathbb{Z}$. We'll write ${}_i\mathcal{R}_j, {}_i\mathcal{A}_j^\hbar$ and so forth for the resulting spaces and Homs.
- This yields a $\mathbb{Z}$-category, i.e. one whose objects are $\mathbb{Z}$ and Homs given as above. Taking direct sums gives a "$\mathbb{Z}$-algebra". Of course, you can do much wilder things too by picking any sequence of η_i for various parahorics, or maybe even taking $\mathbb{Z}^m$-categories and algebras for some m.

$\mathbb{Z}$-algebras

Definition

A $\mathbb{Z}$-algebra is a nonunital associative algebra

$$B = \bigoplus_{i \geq j \geq 0} B_{ij}$$

(These would probably better be called ”lower triangular $\mathbb{N}$-algebras".) We require also diagonal idempotents and that multiplication happens in a matrix-graded way.

- You should think of these as corresponding to a noncommutative Proj-construction, i.e. the modules one usually cares about are graded ("column vector" modules), and we discard torsion modules.
- In particular, BFN's "Line bundles on Coulomb branches" yields the commutative case. There $B = \mathrm{End}(\bigoplus_{k \geq 0} \mathcal{O}(k))$

Localization

- Each ${}_{\eta'}\mathcal{R}_{\eta''}$ is of finite codimension inside the vector bundle $\mathcal{T}_{\eta'} \to G_{\mathcal{K}}/\mathbf{P}$. When $\mathbf{P} = G_{\mathcal{O}}$, the latter space is the affine Grassmannian, and there is a natural inclusion Gr_T " = " $\mathrm{Gr}_G^T \hookrightarrow \mathrm{Gr}_G$.
- As in BFN, since $\mathbf{P} = G_{\mathcal{O}}$ this gives an embedding of ${}_{\eta}\mathcal{A}_{\eta'}$ to the ring of $\hbar-$difference operators on $\mathrm{Lie}(T)$, which is the Coulomb branch for $(G, N) = (T, 0)$.
- In particular, one recovers the ${}_{\eta}\mathcal{A}_{\eta} \otimes {}_{\eta'}\mathcal{A}_{\eta'}$-bimodules ${}_{\eta}\mathcal{A}_{\eta'}$ *embedded into* $\mathcal{A}^{\hbar}_{T,0}$.
- Moreover, there are explicit localization formulas for the classes over minuscule orbits (similar to ones in BFN), with appropriate changes in the Euler classes.

Example: The Gordon-Stafford construction

- In the case $G = \mathrm{GL}_n, N = \mathrm{Ad} \oplus V$ we get by Kodera-Nakajima that ${}_i\mathcal{A}_i^\hbar$ is the rational Cherednik algebra for parameters $t, i\hbar$ (aligning with Kodera-Nakajima's conventions).
- The bimodules ${}_i\mathcal{A}_j^{\hbar=0}$ agree with $\delta^{i-j}A^{i-j}$ (where A=diagonally alternating polynomials) as computed in BFN's line bundles paper. In particular, in the commutative case we get the same $\mathbb{Z}$-algebra

$$\bigoplus_{i \geq j \geq 0} \delta^{i-j}A^{i-j}.$$

- Turning on loop rotation, results of Losev imply that these must quantize to the Gordon-Stafford/Heckman-Opdam bimodules. (This gives some hints to the actual equivalences of categories, but those are of course harder.)
- For direct computation, there is a caveat coming from the embedding to difference (and not differential) operators.

Other examples

- Trigonometric case (for any G): Work in progress with E. Gorsky and A. Oblomkov. The GL_n case is again similar to the Dunkl embedding but since we're working with difference operators, it's not obvious that we get the same shift bimodules as in e.g. Bellamy-Ginzburg. The commutative degeneration is $\mathrm{Hilb}^n(\mathbb{C}^\times \times \mathbb{C})$.
- The case $N = \mathrm{Ad} + V^\ell$ for $\ell > 1$ should also work in parallel to above, but now we also have more flavor deformation directions. A $\mathbb{Z}$-algebra construction in this vein has been done by Gordon algebraically, and it would be interesting to compare the resulting constructions.

Other examples

- One can also upgrade the above (in type A, say) to K-theory, in which case one gets $\mathrm{Hilb}^n(\mathbb{C}^\times \times \mathbb{C}^\times)$ from the DAHA (Oblomkov) and in the cyclotomic cases expects related multiplicative quiver varieties (Braverman-Etingof-Finkelberg, Chalykh).
- The elliptic DAHA cases involve the right sort of embedding to elliptic difference operators, but it's not completely clear how to get enough parameters, in particular for the $C^\vee C$ (which is a starting point for Rains' theory).
- Webster's tilting bundle constructions also yield different $\mathbb{Z}$-algebras, and of course there is a whole picture coming from symplectic duality (Bezrukavnikov-Okounkov, BLPW, Losev, etc.)

Springer theory

- Recall generalized affine Springer fibers: $a : G_{\mathcal{K}} \times N_{\mathbf{P}} \to N_{\mathcal{K}}$, $M_\gamma := a^{-1}(\gamma)$.
- If we replace G by $\widetilde{G}$ when $\widetilde{G} = G \times \mathbb{C}^\times_{dil}$, denote the resulting GASF $\widetilde{M}_\gamma$. The fibers of $\mathrm{Gr}_{\widetilde{G}} \to \mathrm{Gr}_{G_F}$ are isomorphic to Gr_G and the preimage of $[t^i]$ intersected with $\widetilde{M}_\gamma$ is $M_{t^i\gamma}$.
- A straightforward modification of the convolution from Hilburn-Kamnitzer-Weekes and Garner-K. gives an action

$$ {}_j\mathcal{A}_i^\hbar \otimes H_*^{Stab_{G_{\mathcal{K}}}(\gamma)}(M_{t^i\gamma}) \to H_*^{Stab_{G_{\mathcal{K}}}(\gamma)}(M_{t^j\gamma}) $$

(with some assumptions on the stabilizer). In particular, a module for our $\mathbb{Z}$-algebra.
- Now one can ask things about good filtrations and such, but I won't do that. Instead, kill the loop rotation and all resulting equivariance on the $\widetilde{M}_\gamma$ side. This results in a column vector module over $\bigoplus_{i\geq j\geq 0} {}_j\mathcal{A}_i$, in particular a quasicoherent sheaf on $\mathrm{Proj} \bigoplus_{i\geq 0} A^{i-j}$.

More on Springer theory

- For the $N = \mathrm{Ad}$ case get some sheaves on a partial resolution of the commuting variety from classical ASF (joint work in progress w/ Gorsky and Oblomkov). Finite generation results (similar to Yun, Bouthier-Kazhdan-Varshavsky) would imply coherence (conjecture); size of $Stab_{G_{\mathcal{K}}}(\gamma)$ tells us about the support of the resulting sheaves. For GL_n get sheaves on $\mathrm{Hilb}^n(\mathbb{C}^\times \times \mathbb{C})$.
- For the $N = \mathrm{Ad} + V$ case, get sheaves on $\mathrm{Hilb}^n(\mathbb{C}^2)$ starting from plane curve singularities after Garner-K. (and other GASF or sheaves on $N_{\mathcal{K}}$). Also expect direct relations to knot homology (ORS/OR/GNR conjectures).

Directions

- The Procesi bundle in the above cases is related to a certain (generalized) affine Springer fiber, but I don't have a good explanation other than on the level of formulas of why things work out. The Iwahori version (in progress w/ Losev and Boixeda-Alvarez) might give more clues to this.
- As stated earlier, more careful investigation of the possible flavor deformations/parameters should give information about stability conditions on the (multiplicative) quiver varieties in these stories, as explored by Gordon for the cyclotomic case. Dually, it would be interesting to consider quantized Gieseker varieties (see José's poster).
- Etc.

Thank you!