SHALIKA GERMS FOR TAMELY RAMIFIED ELEMENTS IN GL_n

OSCAR KIVINEN AND CHENG-CHIANG TSAI

Abstract. Degenerating the action of the elliptic Hall algebra on the Fock space, we give a combinatorial formula for the Shalika germs of tamely ramified regular semisimple elements γ of GL_n over a nonarchimedean local field. As a byproduct, we compute the weight polynomials of affine Springer fibers in type A and standard orbital integrals of tamely ramified regular semisimple elements.

As further corollaries, we show that point-counts of compactified Jacobians of planar curves are given by non-negative integral polynomials, and prove a virtual version of the Cherednik-Danilenko conjecture on the Betti numbers of compactified Jacobians. Our results also provide further evidence for the Oblomkov-Rasmussen-Shende conjecture relating compactified Jacobians and HOMFLY-PT invariants of algebraic knots.

Following known connections between our methods and the Hilbert scheme of points on $\mathbb{A}^2$, we conjecture that the Shalika germs of γ correspond to residues of torus localization weights of a certain (quasi-)coherent sheaf $\mathcal{F}_\gamma$ on $\mathrm{Hilb}^n(\mathbb{A}^2)$, thereby finding a geometric interpretation for the germs.

Contents

1. Introduction

Shalika germs are a family of functions on a neighborhood of the identity in an algebraic group G defined over a nonarchimedean local field F. They were introduced in [71] and further studied in for example [19,47,67,78]. We refer to [45] for a survey. In particular, given $f \in C_c^\infty(G(F))$ an Iwahori bi-invariant function on G, the Shalika germ expansion of orbital integrals [19,71] states that

Proposition 1.1.
$$I_\gamma(f) = \sum_{\mathbb{O}\in G(0)} \Gamma_{\mathbb{O}}(\gamma) I_{\mathbb{O}}(f)$$
for any $\gamma \in U$, a neighborhood of the identity in $G(F)$. Here $G(0)$ is the set of unipotent orbits in $G(F)$, $I_\gamma, I_{\mathbb{O}}$ are the corresponding orbital integrals and $\Gamma_{\mathbb{O}} : U \to \mathbb{C}$ are functions called the Shalika germs.

See Corollary 2.7 for a more precise version of the Shalika germ expansion we will use.

Computing the Shalika germs for a given group G is an important, in general open problem, which can be used for example to deduce formulas for regular semisimple orbital integrals (see Section 2). When $G = GL_n$ and γ is tamely ramified, these orbital integrals include the weight polynomials of corresponding affine Springer fibers for type A and compactified Jacobians (see Theorem 1.10). Understanding the Shalika germs for general γ is also essential to the determination of precise character values of $G(F)$, see for example [57] for GL_n and [74] for general G.

In the present paper, we give an explicitly computable algorithm to compute the Shalika germs of tamely ramified elements in $GL_n(F)$, using the representation theory of the *elliptic Hall algebra* introduced in [11]. This is an algebra whose definition was motivated by automorphic forms over function fields, although its appearance in the present paper is far from obvious. Surprisingly, it follows from the relation of the elliptic Hall algebra to various knot invariants (recalled in Section 5) that the final closed-form answer for the Shalika germs in the totally ramified case is given by certain HOMFLY-type invariants of algebraic knots. For more general tamely ramified elements, one gets certain linear combinations of these invariants.

This seems to be a completely new connection between harmonic analysis on p-adic groups and knot invariants, which we hope to explore further.

Our main results are stated in Subsection 1.1 below. The impatient reader may now skip there, but we take the liberty of providing some context for our results before proceeding. The starting point for our method is as follows: In [79], Waldspurger gives a rather complicated inductive formula for what we call *Steinberg germs* (see Theorem 2.19 and Section 6) of γ in this case, built inductively from elements in $GL_{n'}$, $n'|n$. These germs are related to the original Shalika germs by an a priori complicated change of basis, so in principle the algorithm of *loc. cit.* gives formulas for the Shalika germs as well.

The algorithm in [79] rests on a clever choice of test functions and a version of "Kazhdan's lemma", using which one can bootstrap computation of the germs to what is essentially just linear algebra. A similar idea is used again by Waldspurger in [80] and also by the second author in [76] based on a lemma of Kim-Murnaghan [43] to obtain less precise results for general groups. Waldspurger mentions similar strategies due to Kazhdan, Henniart, and others. The issue is that the resulting linear algebra is usually cumbersome to carry out and many steps of the algorithm have no obvious conceptual meaning. As Waldspurger writes in [79]:

> *L'auteur est convaincu qu'il existe une bonne combinatoire, moins naïve que celle utilisée ici, qui devrait permettre de calculer les germes.* - J.-L. Waldspurger [79]

Our method of computation of the germs will use Waldspurger's techniques from [78, 79]. Notably we replace, or rather extend, the PSH-algebra calculations in *loc. cit.* by the elliptic Hall algebra, the combinatorics of which provide the "bonne combinatoire" sought after by Waldspurger. This greatly clarifies the resulting computations, giving a formula which is essentially computable by hand [1].

More precisely, much of this paper is concerned with a symmetric function in infinitely many variables which we denote by $\mathbf{f}_\gamma$, attached to any tamely ramified regular semisimple element $\gamma \in \mathfrak{gl}_n(F)$. The expansion of $\mathbf{f}_\gamma$ in different bases of the ring of symmetric functions encodes both the Shalika and Steinberg germ expansions of γ, giving a simple explanation for the change of basis from above (see Theorem 1.6 and the discussion below). The $\mathbf{f}_\gamma$ for any tamely ramified γ as before can be constructed using the shuffle algebra action of the elliptic Hall algebra on the *Fock space* of symmetric functions depending on two parameters, constructed in [26, 69]. Details are explained in Section 6.

We have called $\mathbf{f}_\gamma$ "the master symmetric function" for lack of a better name. When γ is totally ramified, one of our main results identifies $\mathbf{f}_\gamma$, appropriately normalized, with another symmetric function which we denote $\mathbf{f}_{\vec{p},\vec{q}}$. The symmetric function $\mathbf{f}_{\vec{p},\vec{q}}$ will be introduced in Section 5. A closely related version of $\mathbf{f}_{\vec{p},\vec{q}}$ has appeared before in the knot theory literature; $\mathbf{f}_{\vec{p},\vec{q}}$ is the $t \to 1$ limit of a symmetric function implicitly appearing in the definition of the *superpolynomial* for an iterated torus knot in the work of Cherednik–Danilenko [13,33]. A precise definition is given in Sections 4 and 5.

In the papers [78, 79], Waldspurger already implicitly introduces the master symmetric function $\mathbf{f}_\gamma$, but mostly as a bookkeeping tool which turns out to be helpful

[1] Sage code with several examples is available at `http://math.aalto.fi/~kivineo3/Shalika.zip` and in the arXiv submission.

because of the relation to calculations in the Hall algebra of $GL_n(\mathbb{F}_q)$, under its incarnation as Zelevinsky's PSH-algebra [81]. Apparently, the relationship of Shalika germs to this Hall algebra was first suggested to Waldspurger by B. Srinivasan. We note that the papers [78, 79] a priori employ the group $GL_n(F)$ instead of the Lie algebra $\mathfrak{gl}_n(F)$. However, their connection is straightforward and is explained in Section 2. We make the case that the master symmetric function, in its "degenerate" and "deformed" incarnations (cf. Definitions 5.3, 5.10) is a completely natural object and arises as a vector in the Fock space representation of the elliptic Hall algebra.

As we explain in Section 8, expressing the Shalika germs using $\mathbf{f}_\gamma$ has strong implications for integrality and positivity properties of the germs as well as the orbital integrals of the elements in question, as has been expected by various authors. Through known connections between the elliptic Hall algebra, Hilbert schemes of points, and HOMFLY-type knot invariants (see e.g. [32] for a survey) we also find a new conjectural incarnation of the Shalika germs of GL_n in terms of the Hilbert scheme of points on $\mathrm{Hilb}^n(\mathbb{A}^2)$, which we hope will help us understand the structural properties of the germs in general.

1.1. **Main results.** Let us now state our main results in some detail. Let $G = GL_n$, $\mathfrak{g} = \mathrm{Lie}(G) = \mathfrak{gl}_n$. Let F be a complete discrete valuation field, with $\mathcal{O}$ its ring of integers, $\mathfrak{m}_F$ the maximal ideal, and $k := \mathcal{O}/\mathfrak{m}_F$ the residue field. We endow F and any of its finite extensions with the standard valuation. Let $\gamma \in \mathfrak{g}(\mathcal{O})$ be a regular semisimple element that lives in a maximal torus that splits over a tamely ramified extension. Let $F(\gamma)$ be the commutative algebra generated by γ in the algebra of matrices. For the sake of simplicity, in the introduction we will assume that γ is inertially elliptic, meaning that $F(\gamma)/F$ is a totally ramified degree n extension. We refer to Remark 1.7 for the general case.

Let u be a uniformizer of $F(\gamma)$ such that $u^n \in F$ [46, Proposition II.5.12]. It might be inspiring to think of the example when $F = k((t))$ and u is of the form

$$u = \begin{pmatrix} 0 & 0 & \cdots & 0 & t \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \ddots & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix} \tag{1.1}$$

so that $u^n = t$. In general, that $F(\gamma)$ is a complete DVF guarantees that we can always write

$$\gamma = \sum_{r \in \frac{1}{n}\mathbb{Z}_{\geq 0}} a_r u^{nr}, \tag{1.2}$$

where $a_i \in \mathcal{O}^\times$. Consider the set of *root valuations*

$$RV(\gamma) := \left\{ r \in \frac{1}{n}\mathbb{Z}_{\geq 0} \mid r \notin \mathrm{span}_{\mathbb{Z}}\{1, r' \in \frac{1}{n}\mathbb{Z}_{\geq 0} \mid r' < r,\ a_{r'} \neq 0\} \right\}.$$

The set $RV(\gamma)$ is finite; in fact one can check that

$$RV(\gamma) = \{\mathrm{val}_F(\sigma(\gamma) - \gamma) \mid \sigma \in \mathrm{Gal}(F^{sep}/F),\ \sigma(\gamma) \neq \gamma\}.$$

We also define two other sets of invariants of γ as follows:

Definition 1.2. Write $RV(\gamma) = \{r_1, r_2, \cdots, r_k\}$, where $r_1 > \cdots > r_k$. Let $r_k = m_k/n_k$ be the reduced expression, and inductively write $r_i = m_i/n_i n_{i+1} \dots n_k$, where $m_i \in \mathbb{Z}_{>0}$, $n_i \in \mathbb{Z}_{\geq 0}$ are coprime. We call $(m_1, n_1), \dots, (m_k, n_k)$ the *Puiseux pairs* of γ.

Define another sequence of pairs of integers $(p_1, q_1), \dots, (p_k, q_k)$ related to the previous one by $p_i = n_i$, $q_i = m_i - m_{i+1} n_i$. We call $(p_1, q_1), \dots, (p_k, q_k)$ the *Newton pairs* of γ. We will write $(\vec{p}, \vec{q})$ for this sequence.

Remark 1.3. If we adapt the slight abuse of notation that $t^{m/n} := u^m$, then a typical example of γ with given Newton pairs (or root valuations) is given by:

$$\gamma = t^{q_k/p_k}(a'_k + t^{q_{k-1}/p_k p_{k-1}}(a'_{k-1} + \cdots (a'_2 + a'_1 t^{q_1/p_1 \cdots p_k}))\cdots)$$

where $a'_i \in \mathcal{O}^\times$ are arbitrary.

Remark 1.4. Suppose that $F = \mathbb{C}((t))$, and that (1.2) defines a convergent Taylor series over $\mathbb{C}$. Then the Puiseux (equivalently, Newton) pairs determine the *topological type* of the singularity $\{\mathrm{char}(\gamma) = 0\}$ cut out by the characteristic polynomial of γ. Recall that this is by definition the knot in S^3 determined by intersecting $\{\mathrm{char}(\gamma) = 0\} \subset \mathbb{C}^2$ with a small three-sphere centered at the origin. Conversely, if we impose the condition $q_k \geq p_k$, the topological type uniquely determines the Puiseux pairs. We note that our conventions for relating Puiseux data and the topological type are opposite to the conventions used in e.g. [23].

Example 1.5. Let $F = k((t))$, $n = 4$ and

$$\gamma = u^6 + u^7 = \begin{pmatrix} 0 & t^2 & t^2 & 0 \\ 0 & 0 & t^2 & t^2 \\ t & 0 & 0 & t^2 \\ t & t & 0 & 0 \end{pmatrix}$$

as in (1.1). Then $r_1 = 7/4, r_2 = 3/2$ and

$$\mathrm{char}(\gamma) = x^4 - 2t^3 x^2 - 4t^5 x - t^7 + t^6$$

The Puiseux pairs are $(m_1, n_1) = (7, 2), (m_2, n_2) = (3, 2)$ and the Newton pairs are $(p_1, q_1) = (2, 1), (p_2, q_2) = (2, 3)$. The link is the "$(2, 13)$-cable of the trefoil". This example features also for example in [23, p. 58].

As explained in Section 4, for every pair of coprime natural numbers (e, d) one may define a homomorphism from the ring of symmetric functions over $\mathbb{Q}(q)$ to itself $\varphi_{d/e} : \mathrm{Sym}_q \to \mathrm{Sym}_q$ by letting

$$\varphi_{d/e}(e_k) = E_{d,e,k} := \sum_{\pi \in \mathbb{D}_{kd,ke}} q^{\mathrm{area}(\pi)} e_\pi \tag{1.3}$$

where $\mathbb{D}_{kd,ke}$ is the set of Dyck paths in $(kd \times ke)$ rectangle below the diagonal and e_k are the elementary symmetric functions; see Proposition 4.12 and Definition 3.6 for the details.

Starting from the Newton pairs of γ, we define the symmetric function $\mathbf{f}_{\vec{p},\vec{q}}$ for $(\vec{p}, \vec{q})$ as follows:

$$\mathbf{f}_{\vec{p},\vec{q}} := \varphi_{q_d/p_d}(\cdots \varphi_{q_1/p_1}(e_1))\cdots)$$

Denote by $\mathbf{f}_{\vec{p},\vec{q}}|_{q \mapsto q^{-1}}$ the image of $\mathbf{f}_{\vec{p},\vec{q}}$ under the involution on $\mathbb{Q}(q)$ sending q to q^{-1}. This simply means changing q^{area} to $q^{-\mathrm{area}}$ in Eq. (1.3). Denote by ω the involution on Sym_q that switches e_λ and the homogeneous symmetric functions h_λ (see §3.2). Our first main theorem is the following.

Theorem 1.6. *Let k be a finite field so that F is a non-archimedean local field. Let $\gamma \in \mathfrak{g}(\mathcal{O})$ be inertially elliptic as before. For any partition $\lambda \vdash n$ let n_λ (resp. u_λ be the nilpotent (resp. unipotent) orbit of Jordan type λ, and denote by $\Gamma_\lambda(\gamma) := \Gamma_{n_\lambda}(\gamma)$ the Shalika germ for $\gamma \in \mathfrak{g}(F)$ (resp. $\Gamma_\lambda(1+\gamma) := \Gamma_{u_\lambda}(1+\gamma)$ for $1+\gamma \in G(F)$) in Proposition 1.1. We have*

$$\sum_{\lambda \vdash n} \Gamma_{\lambda^t}(\gamma)\widetilde{h}_\lambda = \sum_{\lambda \vdash n} \Gamma_{\lambda^t}(1+\gamma)\widetilde{h}_\lambda = q^{\Xi(\gamma)}\omega \mathbf{f}_{\vec{p},\vec{q}}|_{q\mapsto q^{-1}} \tag{1.4}$$

where $\widetilde{h}_\lambda \in \mathrm{Sym}_q$ are defined after Corollary 3.9 and $\Xi(\gamma) \in \mathbb{Z}$ is as in Definition 6.12. Note also the transposition in λ. In particular, the Shalika germs of γ only depend on its Newton pairs.

The Theorem above is restated and generalized to general tamely ramified γ in Section 6. That is, one can essentially take the left-hand side of Eq. (1.4) to be the definition of $\mathbf{f}_\gamma$, see Definition 6.6. Similarly, one gets what we call *Steinberg germs* from Theorem 2.19 by expanding $\mathbf{f}_\gamma$ in the elementary symmetric functions. That is,

$$\mathbf{f}_\gamma := \sum_{\lambda \vdash n} \Gamma_{\lambda^t}(\gamma)\widetilde{h}_\lambda = \sum_{\lambda \vdash n} \Gamma^{St}_\lambda(\gamma) e_\lambda$$

and the content of Theorem 1.6 is then that this master symmetric function $\mathbf{f}_\gamma$ equals $\mathbf{f}_{\vec{p},\vec{q}}$ as defined above up to a simple change of variables, see Corollary 6.22 for details.

Remark 1.7. So far, we have been restricting to the case where $F(\gamma)/F$ is totally tamely ramified and γ is elliptic. In the case where our extension is tamely ramified but not necessarily totally ramified, the above theorem also works with appropriate modifications, as explained in Theorem 6.27. Of course, in this case one needs to modify the statement of Theorem 1.6 to account for intermediate unramified extensions.

Finally, if γ is not elliptic but still regular semisimple and belongs to a Levi $L(\lambda)$ in which it is elliptic, $\mathbf{f}_\gamma$ is proportional to the product of the master symmetric functions of the factors, see Corollary 6.47. When $F = k((t))$ and $F(\gamma)$ is a product of totally tamely ramified extensions, $\mathbf{f}_\gamma$ is uniquely determined by the Puiseux series of the branches of the spectral curve, which correspond to the different blocks of γ.

On the harmonic analysis side, the recursive definition of $\mathbf{f}_\gamma$ essentially boils down to writing

$$\gamma^{\leq} := \sum_{r \leq r_k} a_r u^{nr}, \ \gamma^{>} := \sum_{r > r_k} a_r u^{n(r-r_k)}$$

in terms of (1.2), so that we have $\gamma = \gamma^{\leq} + u^{nr_k}\gamma^{>}$. The centralizer of $\gamma^{\leq}$ in $GL_n(F)$ is isomorphic to some $GL_{n'}(F')$ for a tamely and totally ramified extension F'/F of degree $e = p_k$ with $n' := n/e$; in fact p_k is the (minimal) denominator of r_k and $F' = F(u^{nr_k})$. The Shalika germs for γ are then computed from those of $\gamma^{>}$, viewing $\gamma^{>}$ as an element in $\mathfrak{gl}_{n'}(F')$. In detail, this is slightly more involved as we need to pass between the group GL_n and the Lie algebra $\mathfrak{gl}_n$ to use the results of [79]. Details are explained in Section 6.1.

As we show in Theorem 6.35, the Dyck path recursion for $\mathbf{f}_\gamma$ can be translated to the following fairly complicated recursive formula for the Shalika germs using compositions, see Section 6.3 for details.

Theorem 1.8. *The transition matrix between the Shalika germs for γ and $\gamma^>$, indexed by $\lambda \vdash n$ and $\lambda' \vdash n/e$ respectively, is given by*

$$M^{d/e}_{\lambda,\lambda'} = \left(c_{\lambda'} q^s \sum_{\mu \vdash n/e} \frac{|S_{\lambda'} \cap C_\mu|}{b_\mu \lambda'!} \prod_{i=1}^{\ell(\mu)} \Big(\sum_{\alpha \vDash e\mu_i} \mathrm{wt}_{d/e}(\alpha)_{q \mapsto q^{-1}} q^{-n(\alpha^t)} \widetilde{h}_\alpha \Big) \right) \Bigg|_{\widetilde{h}_\lambda}$$

where $e := p_k$, $d := q_k = e \cdot r_k$, and $|_{\widetilde{h}_\lambda}$ means we pick the coefficient of $\widetilde{h}_\lambda$. Here $\mathrm{wt}(\alpha)_{d/e}$ is as in Eq. (5.11), and $|S_{\lambda'} \cap C_\mu|$ is the number of elements in the symmetric group $S_{n'}$ which belong to both the Young subgroup $S_{\lambda'}$ and have cycle type μ. We also use the normalizing constants

$$c_{\lambda'} = (1-q)^{n'} [\lambda']_q!, \; b_\mu = \prod_{i=1}^{\ell(\mu)} (1 - q^{\mu_i})$$

defined in Section 3 and $s := n(\lambda'^t) + \frac{(dn'-1)(en'-1)+n'-1}{2}$.

Remark 1.9. We do not have a good conceptual explanation for the entries of the transition matrix $M^{d/e}$, but give some combinatorial stipulations about their structure in Section 6.3.

Finally, we explain how our results recover combinatorial formulas for many basic orbital integrals. For any partition $\lambda \vdash n$, let $\mathbf{P}_\lambda \subset G(\mathcal{O})$ be the standard parahoric subgroup consisting of elements whose reduction in $G(k)$ is block-wise lower triangular with block sizes λ. In particular $\mathbf{I} := \mathbf{P}_{(1^n)}$ is the standard Iwahori subgroup. Let $\mathbf{1}_\lambda \in C^\infty(\mathfrak{g}(\mathcal{O}))$ be the characteristic function of $\mathrm{Lie}\,\mathbf{P}_\lambda$, the associated parahoric subalgebra, divided by the measure of $\mathrm{Lie}\,\mathbf{P}_\lambda$. With this normalization and the normalization of measures introduced in Section 2, the integrals of $\mathbf{1}_\lambda$ over the orbit of γ give exactly the point counts of affine Springer fibers in the case $F = \mathbb{F}_q((t))$, or their weight polynomials when we are coming from characteristic zero via reduction.

The values of these integrals are obtained easily from the master symmetric function by pairing it with the complete homogeneous symmetric functions:

Theorem 1.10. *Let $\lambda \vdash n$ be a partition. Then*

$$I_\gamma(\mathbf{1}_\lambda) = \langle \mathbf{f}_\gamma, h_\lambda \rangle = |\mathrm{Sp}^\lambda_\gamma(k)/\Lambda_\gamma(k)|$$

where we pair using the usual Hall inner product and $\mathrm{Sp}^\lambda_\gamma$ is the affine Springer fiber of γ in the partial affine flag variety $G(F)/\mathbf{P}_\lambda$, see Section 8. See also Section 2 for the normalizations of measures we are using.

The following is more or less obvious from Theorem 1.10 and has been a folklore conjecture for quite long. We give details in Section 8.2.

Corollary 1.11. *The point-counts of (local) compactified Jacobians of plane curves over a finite field are polynomials in q and only depend on the Newton-Puiseux pairs of γ. In addition, they are polynomials with nonnegative integer coefficients. Similarly, the weight polynomials of local compactified Jacobians over $\mathbb{C}$ are nonnegative polynomials in q.*

Proof. Theorem 1.10 combined with Proposition 8.10 implies that

$$\langle \mathbf{f}_\gamma, h_\lambda \rangle$$

is the number of points of the projective variety $X_\gamma = \mathrm{Sp}_\gamma^\lambda/\Lambda$. In the case over $\mathbb{C}$, we reduce to this case after spreading out and modding out by q outside a finite set of primes.

It is well known that $\langle h_\mu, h_\lambda\rangle$ counts the number of certain nonnegative integer matrices with row sums μ and column sums λ and in particular this count is a nonnegative integer. Since $\mathbf{f}_\gamma = \sum \sigma_\lambda(\gamma) h_\lambda$ with $\sigma_\lambda(\gamma) \in \mathbb{N}[q]$ by Corollary 6.22, we get

$$|X_\gamma(\mathbb{F}_q)| \in \mathbb{N}[q]$$

as desired. □

We note that the element γ does not have to be elliptic for this to hold. Indeed, as explained in Section 2 the point-counts on regular semisimple affine Springer fibers can be reduced to those of elliptic elements. The above is also in line with the expectation that all local compactified Jacobians (for elliptic γ, say) are paved by affines. We also note that combining Theorem 1.6 with Theorem 1.10 gives strong evidence for the expectations of [64, Section 5] relating compactified Jacobians with the HOMFLY homology of algebraic links. More details are explained in Section 5.

Our final main result follows from our method using the elliptic Hall algebra. The detailed statement is formulated and proved in Section 6.2. In Section 9 we connect this Theorem to the geometry of Hilbert schemes of points on $\mathbb{A}^2$.

Theorem 1.12. *Let γ be elliptic and tamely ramified. The master symmetric function $\mathbf{f}_\gamma$ admits a canonical t-deformation $\widehat{\mathbf{f}}_\gamma$ which admits a t-deformed version of the Shalika germ expansion:*

$$\widehat{\mathbf{f}}_\gamma = \sum_{\lambda\vdash n} \widetilde{\Gamma}_{\lambda^t}(\gamma)\widetilde{H}_\lambda$$

where $\widetilde{H}_\lambda$ are the modified Macdonald polynomials.

1.2. **Further directions.** We now remark on further generalizations of our results. To reiterate, throughout the paper, $G = GL_n$ and γ is tamely ramified, meaning the algebra $F(\gamma) := F[x]/f_\gamma$ where f_γ is the minimal polynomial of γ, is a product of tamely ramified extensions of the field F. It is unlikely the methods in this paper will yield results for other groups, in that the elliptic Hall algebra seems to be confined to work with $G = GL_n$ only. There are also a number of geometric simplifications in the $G = GL_n$ case for the affine Springer fibers.

It is however interesting to ponder what part of the theory carries through to other G. For example, the knot invariants of Section 5 could potentially be replaced by Hochschild homology of Rouquier complexes of Soergel bimodules for other groups, though the representation-theoretic superpolynomials have not been defined in this generality. In a different direction, our main results have direct implications for inner forms of GL_n, namely $GL_m(D)$ for a division algebra of dimension d^2 over F, with $md = n$, as follows from the generalized Jacquet-Langlands correspondence of [20].

Another direction of generalization is to understand similar formulas for the wildly ramified elements in $GL_n(F)$. In particular, it would be interesting to know whether there is a finite algorithm that determines the Shalika germs when $F(\gamma)$ is arbitrarily ramified.

It also seems likely our methods can be used to glean information about the geometry of the mixed characteristic affine Springer fibers. More precisely, while the comparison to plane curves fails in mixed characteristic, the geometric interpretation of orbital integrals from [29, 42, 49] still goes through but the geometry is now replaced by the Witt vector affine flag varieties of [83] and the affine Springer fibers therein. For example, Proposition 8.2 still goes through in mixed characteristic. This is presumably the case for the rest of Section 8 as well, but some technical groundwork seems to be needed.

Finally, we note that by the results of [6, 70] the elliptic Hall algebra is the decategorification of a form of induction-restriction functors for a coherent realization of affine character sheaves, and one expects this to be reflected on the constructible side of Langlands duality. Indeed, in an appropriate sense, which we will not make precise, each γ gives rise to $G(F)$-equivariant constructible sheaf on $\mathfrak{g}(F)$ by taking the extension by zero of the constant sheaf on the conjugacy class. One may view the induction provided by the plethysms $\widehat{\varphi}_{m/n}$ from Theorem 4.5 below as some shadow of yet-to-be-defined induction-restriction functors for affine character sheaves of this sort (meaning that we impose no singular support condition). While we only work with the elements γ themselves, it would be compelling to understand the a categorified version of this induction on the level of affine character sheaves.

1.3. **Outline of the paper.** In Section 2, we review some general theory of orbital integrals and various versions of the Shalika germ expansion. In Section 3 we introduce background on symmetric functions, and in Section 4 we define and study a degenerate version of the Elliptic Hall Algebra. Section 5 is devoted to making the connection of our results to HOMFLY type knot invariants precise. It appears before the technical heart of the paper, Section 6, because results of the latter are strongly guided by the computation of the knot superpolynomials. Section 7 is devoted solely to example computations to illustrate our method. Finally, we discuss some applications in Section 8 and the relationship of our results to the Hilbert scheme of points on $\mathbb{A}^2$ in Section 9.

Acknowledgments. The authors thank Francois Bergeron, Pierre-Henri Chaudouard, Stephen DeBacker, Eugene Gorsky, Thomas Hales, Bertrand Lemaire, Yen-Chi Roger Lin, Anton Mellit, Fiona Murnaghan, Andrei Neguț, Alexei Oblomkov, Koji Shimizu, Yan Soibelman, Loren Spice, Minh-Tam Trinh, Jean-Loup Waldspurger, Zhiwei Yun, and Wei Zhang for interesting conversations.

2. Orbital integrals

In this section, we fix k a finite field and let F be a non-archimedean local field with residue field k and $\mathcal{O}$ its ring of integers. The group G will be GL_n or a product $\prod GL_{n_i}$ for which one can reduce the discussion to the former case. Recall that $\mathbf{I} \subset G(\mathcal{O})$ is the Iwahori subgroup consisting of elements whose reduction is upper triangular. For any $\gamma \in G(F)$ and $f \in C_c^\infty(G(F))$ (henceforth complex-valued), the orbital integral is

$$I_\gamma(f) := \int_{g \in C_{G(F)}(\gamma)\backslash G(F)} f(g^{-1}\gamma g)dg$$

Definition 2.1. For γ semisimple, the measure dg is defined as follows: On $G(F)$ we have the up to a scalar unique Haar measure, which we will normalize so that $G(\mathcal{O})$ has measure 1. $C_{G(F)}(\gamma)$ is a product of general linear groups over extensions F' of F, and we use the same normalization. The same is done for $\gamma \in \mathfrak{g}(F)$ semisimple.

For general $\gamma \in G(F)$ or $\gamma \in \mathfrak{g}(F)$, the orbit of γ is locally closed and the centralizer unimodular, therefore the orbit of γ under the adjoint action admits an invariant measure. A way to see this is to identify the orbit with the orbit inside $\mathfrak{gl}_n(F)^*$ via the natural equivariant embedding of varieties $GL_n \hookrightarrow \mathfrak{gl}_n$ and the Killing form. The coadjoint orbit admits a natural symplectic structure (even in this non-archimedean setting), whose top wedge is an invariant volume form.

We will not attempt to fix a normalization for arbitrary γ. However when $\gamma \in G(F)$ is unipotent, we will normalize the measure as follows, following [40, 78]. Let $\lambda \vdash n$ and consider the standard (Richardson) parabolic $P = P(\lambda^t)$ for the unipotent orbit associated to λ. Let N be its unipotent radical and M its Levi factor. Let I_λ^G be the orbital integral on the unipotent orbit with Jordan type λ. For $f \in C_c^\infty(G(F))$ let $f^P \in C_c^\infty(M(F))$ be defined by

$$f^P(m) = \delta_P(m)^{1/2} \int_{G(\mathcal{O})} \int_{N(F)} f(k^{-1}mnk)dndk \tag{2.1}$$

where δ_P is the modular function for P. By [40, Proposition 5], the linear forms given by the unipotent orbital integrals $I_\lambda^G(-)$ for $\lambda \vdash n$ are proportional to $f \mapsto f^P(1)$. We normalize the measure on the unipotent orbit so that $I_\lambda^G(f) = f^P(1)$, where on $N(F)$ we take the Haar measure such that $N(\mathcal{O})$ has measure 1. This discussion applies verbatim with G replaced by $\mathfrak{g}$, etc., with the resulting formula being

$$f^{\mathfrak{p}}(X) = \int_{G(\mathcal{O})} \int_{\mathfrak{n}(F)} f(\mathrm{ad}(k^{-1})(X+Y))dY dk \tag{2.2}$$

and we normalize the nilpotent orbital integral $I_\lambda^{\mathfrak{g}}(f) = f^{\mathfrak{p}}(0)$, again with the Haar measure on $\mathfrak{n}(F)$ such that $\mathfrak{n}(\mathcal{O})$ has measure 1. We will drop the superscripts $G, \mathfrak{g}$ when understood from the context. This is even more merited in view of

Lemma 2.2. *Let $f \in C^\infty(\mathfrak{g}(\mathcal{O})/\mathrm{Lie}(\mathbf{I}))$ be the characteristic function of a standard parahoric. Then the restriction of f to $G(\mathcal{O}) \subset \mathfrak{g}(\mathcal{O})$ equals the characteristic function of the corresponding standard parabolic subgroup, and*

$$f^P(1) = f^{\mathfrak{p}}(0).$$

See Lemma 2.37 where the content here is discussed in the more general setting in §2.4 in terms of parabolic induction/restriction.

2.1. **Shalika germs.** For any subset $\Omega \subset G(F)$ we denote by $J(\Omega)$ the space of invariant distributions on $G(F)$ supported on elements of the form $g^{-1}\gamma g$ with $g \in G(F)$, $\gamma \in \Omega$. The famous Howe conjecture states that

Theorem 2.3 ([40], [36], [17], [5]). *For any compact subset $\Omega \subset G(F)$ and an open subgroup $K \subset G(F)$, the restriction of $J(\Omega)$ to $C_c(G(F)/K)$ is finite-dimensional.*

The same is true when we replace Ω by a compact subset in $\mathfrak{g}(F)$, K by an open sub-$\mathcal{O}$-module in $\mathfrak{g}(F)$ and $C_c(G(F)/\Omega)$ by $C_c(\mathfrak{g}(F)/K)$. In this article we will make use of precise versions of the above finiteness. A particularly important one is the following theorem, proved by Hales [34, Thm. 1] for the span of regular semisimple orbital integrals, and which follows in general from Proposition 2.29 (see also Courtés [18, Thm 1.10]):

Theorem 2.4. *Recall that $G = GL_n$. Let $\mathcal{U} \subset G(F)$ be the (F-points of the) unipotent variety and $\mathbf{I} \subset G(F)$ be an Iwahori subgroup. Then the restriction of*

$J(G(\mathcal{O}))$ to $C_c(G(F)/\mathbf{I})$ is equal to that of $J(\mathcal{U})$ to $C_c(G(F)/\mathbf{I})$. Both restrictions have a basis given by unipotent orbital integrals.

Remark 2.5. Hales [34] works over characteristic zero F, as does Waldspurger [78, 79]. However, the most essential ingredient for Hales is the original Shalika expansion [71], which also works in positive characteristic assuming finiteness of unipotent orbits and convergence of the unipotent orbital integrals. This was proved in [54] when the characteristic is *good* for G. In particular, for GL_n it holds in arbitrary characteristic. Moreover, the more general Proposition 2.29 works in arbitrary characteristic, so we do not have to worry about this issue.

Remark 2.6. As far as invariant distributions are concerned, any test function in $C_c(G(F)/K)$ can be averaged by K-conjugation into $C_c(K\backslash G(F)/K)$. Likewise in Theorem 2.4 we can replace $C_c(G(F)/\mathbf{I})$ by $C_c(\mathbf{I}\backslash G(F)/\mathbf{I})$.

The above theorem is a variant of the so-called Shalika germ expansion, reinterpreted as:

Corollary 2.7. *For any $\gamma \in G(\mathcal{O})$, there exist constants $\Gamma_\lambda(\gamma)$ where λ runs over unipotent orbits of $G(F)$ (i.e. partitions of n) such that for any $f \in C_c(G(F)/\mathbf{I})$, we have*

$$I_\gamma(f) = \sum_\lambda \Gamma_\lambda(\gamma) I_\lambda(f). \tag{2.3}$$

For the Lie algebra case one has the following which works also for arbitrary connected reductive group G provided that $\operatorname{char} k \gg \operatorname{rank} G$.

Theorem 2.8. *[Thm. 2.1.5., [19]] Let $\mathcal{N} \subset \mathfrak{g}(F)$ be the (F-points of the) nilpotent cone and $\operatorname{Lie}\mathbf{I} \subset \mathfrak{g}(F)$ be an Iwahori subalgebra. Then the restriction of $J(\mathfrak{g}(\mathcal{O}))$ to $C_c(\mathfrak{g}(F)/\operatorname{Lie}\mathbf{I})$ is equal to that of $J(\mathcal{N})$ to $C_c(\mathfrak{g}(F)/\operatorname{Lie}\mathbf{I})$. Both restrictions have a basis given by nilpotent orbital integrals.*

In particular, the given bound when $G = GL_n$ is $\operatorname{char} k > 2n$. Whenever the theorem works, it asserts the existence of unique functions $\Gamma_\lambda(\gamma)$ such that (2.3) holds for $f \in C_c(\mathfrak{g}(F)/\operatorname{Lie}\mathbf{I})$ where $I_\lambda(f)$ is the integral on nilpotent orbits this time; by abuse of language we will denote the germs again by $\Gamma_\lambda(\gamma)$, thanks to the following proposition:

Proposition 2.9. *Let $\gamma \in \mathfrak{g}(\mathcal{O})$. Fix a nilpotent orbit λ in $\mathfrak{g}(F)$ – it corresponds to a unipotent orbit in $G(F)$ under $x \mapsto 1 + x$. Under this matching, the following are equal:*

1. *The Lie algebra Shalika germ $\Gamma_\lambda(\gamma)$ from Theorem 2.8.*
2. *The Lie algebra Shalika germ $\Gamma_\lambda(c + \gamma)$ from Theorem 2.8, for any $c \in \mathcal{O}$.*
3. *The Lie group Shalika germ $\Gamma_\lambda(c+\gamma)$ from Theorem 2.4 and Corollary 2.7, for any $c \in \mathcal{O}$ such that $c + \gamma \in G(\mathcal{O})$.*

Proof. (1) equals (2) since the central translation doesn't affect orbital integrals.

Just like the unipotent orbital integrals on $G(F)$ may be computed using Eq. (2.1), so may the nilpotent ones on $\mathfrak{g}(F)$ using Eq. (2.2). By Lemma 2.2 the restriction to $G(\mathcal{O})$ of the characteristic function of any standard parahoric subalgebra of $\mathfrak{g}(F)$ is the the characteristic function of the corresponding standard parahoric subgroup and the nilpotent orbital integral equals the unipotent orbital integral of the restriction. By [78, Corollaire 4.4.], the nilpotent and unipotent orbital integrals are determined as distributions on Iwahori-(bi-)invariant functions by their values on $\mathbf{1}_{\mathrm{Lie}(\mathbf{P}_\lambda)} \in C_c(\mathfrak{g}(\mathcal{O})/\operatorname{Lie}\mathbf{I})$ where $\lambda \vdash n$. Therefore (2) and (3) have the same Shalika germs, either for the group or the algebra. Therefore (2) equals (3). □

Remark 2.10. Theorem 2.8 is expected to hold even if $\operatorname{char} k \leq 2n$. For example, in [51, §5] it is shown that (2.3) holds for any fixed function and for fixed γ which is *quasi-regular* (see the abstract to [51], and note in particular that when $\operatorname{char} k > n$ quasi-regularity is equivalent to regular semisimplicity) and in a small enough neighborhood $V_f \ni \gamma$. Consequently, (2.3) holds for quasi-regular elements in a small enough neighborhood and for all functions in $C_c(\mathfrak{g}(\mathcal{O})/\operatorname{Lie}\mathbf{I})$, since the latter is finite-dimensional.

This is what we used in the proof of Proposition 2.9, and thus our Shalika germ results for tamely ramified regular semisimple γ will also work for Lemaire's germs. Even more generally, in [51, §5.2, pp. 505] Lemaire defines normalized Shalika germs $\widetilde{b}_i$ defined on the set of all quasi-regular elements using homogeneity. Remark 6.31 together with propositions 2.11 and 2.9 show that such $\widetilde{b}_i$ also agree with the Lie algebra/group Shalika germs and in particular the above results, as well as the computations in Section 6 extend to this case as well given $f \in C_c(\mathfrak{g}(\mathcal{O})/\operatorname{Lie}\mathbf{I})$.

In the Lie algebra setup, for a nilpotent orbit $\mathbb{O} \subset \mathfrak{g}(F)$ the Shalika germs enjoy the following homogeneity property.

Proposition 2.11. *For $G = GL_n$, we have*

$$\Gamma_{\mathbb{O}}(t\gamma) = |t|^{-\frac{1}{2}\dim\mathbb{O}}\Gamma_{\mathbb{O}}(\gamma).$$

Remark 2.12. For general reductive group when $\operatorname{char} k$ is very good we have $\Gamma_{\mathbb{O}}(t^2\gamma) = |t|^{-\dim\mathbb{O}}\Gamma_{t^{-2}\mathbb{O}}(\gamma) = |t|^{-\dim\mathbb{O}}\Gamma_{\mathbb{O}}(\gamma)$ since $\mathbb{O}$ and $t^{-2}\mathbb{O}$ are the same orbit in $\mathfrak{g}(F)$. However $\mathbb{O}$ and $t^{-1}\mathbb{O}$ are typically different orbits and the identity in Proposition 2.11 does not hold in general.

Remark 2.13. In terms of partitions, if $\mathbb{O} \leftrightarrow \lambda \vdash n$,

$$\frac{1}{2}\dim\mathbb{O} = \sum_{i=1}^{\ell(\lambda^t)}(i-1)\lambda_i^t =: n(\lambda^t)$$

2.2. **Steinberg germs.** Another way to make use of Theorem 2.4, analogous to that of Corollary 2.7, is proposed by Waldspurger [79, Prop. 2.4.]. In fact, Waldspurger goes further to study the space of distributions $J(G(F)_c)$, where $G(F)_c$ is defined as follows.

Definition 2.14. Let us say an element $g \in G(F) = GL_n(F)$ is *compact mod center* if all its eigenvalues have the same valuation; we will suppress the "mod center" and just call them *compact* when no confusion should arise. Let $G(F)_c \subset G(F)$ be the subset of all compact elements.

Let us also call $\gamma \in \mathfrak{g}(F)$ compact if it belongs to some parahoric subalgebra, or equivalently that it is conjugate to an element in $\mathfrak{g}(\mathcal{O})$.

Remark 2.15. Note that in the group case, not all compact-modulo-center elements are literally central translations of compact elements. Instead, they become compact in the usual sense under the map to PGL_n.

We will also need the following alternative characterization of the compact modulo center elements in GL_n:

Lemma 2.16. *The set $G(F)_c$ coincides with the union of conjugates of normalizers of standard parahorics. In particular, for GL_n we have*

$$G(F)_c = \mathrm{Ad}(G(F))\left(\bigcup_{\substack{e|n,\alpha\vdash n/e,\\ d\in\mathbb{Z}}} u^{nd/e}\mathbf{P}_{\alpha^e}\right)$$

where u is the matrix from the introduction and $\mathbf{P}_{\alpha^e}$ is a standard parahoric subgroup.

Remark 2.17. Note that $u^{nd/e}, d \in \mathbb{Z}$, normalizes $\mathbf{P}_{\alpha^e}$. Compare also to the beginning of [79, Section 4].

Definition 2.18. Let St_n be the Steinberg representation of $G(F) = GL_n(F)$. More generally, for $\lambda \vdash n$ (Definition 3.1) let $P(\lambda) \subset GL_n(F)$ be the corresponding parabolic and $L(\lambda)$ its Levi subgroup. Let St_λ be the parabolic induction[2] of the Steinberg representation of $L(\lambda)$ to $G(F)$

We will henceforth identify St_λ with its character, an invariant distribution on $G(F)$. We denote by $\mathrm{St}_{\lambda,c}$ the restriction of St_λ to $G(F)_c$, i.e. the distribution which is truncated to be 0 outside $G(F)_c$. Denote by $J_{\mathrm{St},c} \subset J(G(F)_c)$ the subspace spanned by $\mathrm{St}_{\lambda,c}$. Write $C_c(\mathbf{I}\backslash G(F)/\mathbf{I}) = \bigoplus_{k\in\mathbb{Z}} C_c(\mathbf{I}\backslash G(F)^{\mathrm{val}=k}/\mathbf{I})$ where $G(F)^{\mathrm{val}=k}$ is the subset of elements whose determinant has valuation k. In [78, Proposition 2.4.] and [79, Proposition III 4.], when $\mathrm{char}\, F = 0$, Waldspurger proved:

Theorem 2.19. *For $k \in \mathbb{Z}$ the restriction of $\mathrm{St}_{\lambda,c}$ to $C_c(\mathbf{I}\backslash G(F)^{\mathrm{val}=k}/\mathbf{I})$ is non-zero iff λ is divisible by $n/\gcd(k,n)$. The restriction of $J_{\mathrm{St},c}$ to $C_c(\mathbf{I}\backslash G(F)^{\mathrm{val}=k}/\mathbf{I})$ has a basis given by the restrictions of these $\mathrm{St}_{\lambda,c}$.*

Theorem 2.20. *For any regular semisimple $\gamma \in G(F)^{\mathrm{val}=k}$ that is compact mod center, there exist unique constants $\Gamma_\lambda^{St}(\gamma)$ indexed by $\lambda \vdash \gcd(k,n)$, so that for any $f \in C_c(\mathbf{I}\backslash G(F)^{\mathrm{val}=k}/\mathbf{I})$ we have*

$$I_\gamma(f) = \sum_\lambda \Gamma_\lambda^{St}(\gamma)\mathrm{St}_{n'\lambda,c}(f) \tag{2.4}$$

where $n' := n/\gcd(k,n)$. We shall call the constants $\Gamma_\lambda^{St}(\gamma)$ the Steinberg germs *of γ.*

Remark 2.21. A few remarks are in order. The most important one is that in [78, 79] there is an assumption on $\mathrm{char}\, F = 0$. However, it is easy to see that the proof of [78, Proposition 2.4.] only uses characteristic-independent facts about the representation theory of $G(F)$. In the next section, Section 2.3, we will in particular construct unique distributions $\mathrm{St}_{\lambda,c}$ satisfying Theorems 2.19, 2.20 and [79, V 11, V12]. *If* one were able to carry out Clozel's work in [17] in positive characteristic, then these $\mathrm{St}_{\lambda,c}$ could safely be identified with the truncated characters of parabolic inductions of Steinberg representations. While it is somewhat awkward we cannot do this right now, it will not affect the computation of the Shalika germs themselves.

Further, we note that the "*St*"-superscript stands for Steinberg, and should not be confused with the notion of "stability" in the automorphic forms literature.

[2] Usually parabolic inductions are normalized by the modulus character. In our case, the modulus character is trivial on compact (mod center) elements because it is a homomorphism to $(\mathbb{R}^+, \times)$ that is trivial on center. Since we will immediately restrict to compact elements, the normalization will have no effect on what follows.

2.3. **Study of germs via rigid cocenters of affine Hecke algebras.** In this subsection we digress to mention an approach to the Howe conjecture using the cocenter of the (extended) affine Hecke algebra

$$\mathcal{H} := C_c(\mathbf{I}\backslash G(F)/\mathbf{I})$$

following [16, 37].

Note that as an abstract algebra, $\mathcal{H}$ only depends on the *residue field* of F (more precisely its size), through for example the well-known description by generators and relations. In particular, the results in this subsection apply in all characteristics and can be viewed as alternative proofs for some results in the previous two subsections, as well as strengthening those in [37, Section 5]. If the reader is more geometrically inclined, it does no harm to skip this subsection and black-box the transfer of characteristic zero results from the previous chapter to (possibly very large) positive characteristic using e.g. the theory of "nearby fields" (corps proches) or the model-theoretic apparatus.

Consider the space $J(G(F)_c)$ of invariant distributions supported on the compact mod center elements of $G(F)$. There is a natural map

$$J(G(F)_c) \to \mathcal{H}^*$$

to the linear dual of the AHA given by evaluating the distributions on the functions. The $G(F)$-invariance of the distributions amounts to this map factoring through the *cocenter*

$$J(G(F)_c) \to (\mathcal{H}/[\mathcal{H},\mathcal{H}])^* \to \mathcal{H}^* \tag{2.5}$$

We will soon see that the map in Eq. (2.5) can be further shown to factor through the dual of the *rigid* cocenter introduced in [16]. First, note that $\mathcal{H}$ and $\mathrm{tr}(\mathcal{H}) := \mathcal{H}/[\mathcal{H},\mathcal{H}]$ are graded by the valuation of the determinant, i.e. as before, we have the decomposition

$$\mathcal{H} = \bigoplus_{k\in\mathbb{Z}} C_c(\mathbf{I}\backslash G(F)^{\mathrm{val}=k}/\mathbf{I})$$

and similarly for the cocenter. Let us denote $\mathcal{H}^{\mathrm{val}=k}$, $\mathrm{tr}(\mathcal{H})^{\mathrm{val}=k} := \mathrm{tr}(\mathcal{H}^{\mathrm{val}=k})$ the corresponding subspaces.

This is further refined by the *Newton decomposition* of the group $G(F)$ as well as $\mathcal{H}$ from [37]. From the Cartan decomposition, we have $\mathbf{I}\backslash G(F)/\mathbf{I} \cong \widetilde{W}$, where $\widetilde{W} \cong \mathbb{Z}^n \rtimes S_n$ is the extended affine Weyl group associated to $G = GL_n$. We also write $W^{\mathrm{fin}} := S_n$ in the above. We will identify $X_*(T) \cong \mathbb{Z}^n$ in a standard way. We have a decomposition $\widetilde{W} \cong \Omega \ltimes W^{\mathrm{aff}}$ where W^{aff} is the affine Weyl group and $\Omega \cong \mathbb{Z}$.

Definition 2.22. The *Kottwitz map* is the projection $\kappa : \widetilde{W} \twoheadrightarrow \Omega$. The *Newton map* $\nu' : \widetilde{W} \to \frac{1}{n}\mathbb{Z}^n$ is defined as follows. If $w^k.x = \lambda + x$ we let $\nu'(w) = \lambda/k$. By sorting, we may take this to be the unique dominant element in the W^{fin}–orbit of $\nu'(w)$, which gives another map $\nu^+ : \widetilde{W} \to (\frac{1}{n}\mathbb{Z}^n)_{\geq 0}$.

Together, we get a map

$$\pi := (\kappa, \nu^+) : \widetilde{W} \to \Omega \times (\frac{1}{n}\mathbb{Z}^n)_{\geq 0} =: \aleph$$

Note that $\pi(w) = \pi(w')$ whenever w and w' are conjugate.

Definition 2.23. For $\nu \in \aleph$, the *Newton stratum* of $G(F)$ is

$$G(\nu) := \bigcup_{w\in\widetilde{W} \text{ minimal length, } \pi(w)=\nu} \mathrm{Ad}(G(F))\mathbf{I}w\mathbf{I}$$

Theorem 2.24 (Theorem A, [37]). *We have the* Newton decomposition of $G(F)$ *is*

$$G(F) = \bigsqcup_{\nu\in\aleph} G(\nu)$$

i.e. the group decomposes as a disjoint union of Newton strata.

For the algebraic/combinatorial point of view, we have the following theorem of He and Nie [38, Thm. 6.7] and He [37, Thm. 11]:

Theorem 2.25. *We have a Newton decomposition for the cocenter* $\mathrm{tr}(\mathcal{H})$ *of the Iwahori-Hecke algebra* $\mathcal{H}$

$$\mathrm{tr}(\mathcal{H}) = \bigoplus_{\nu\in\aleph} \mathrm{tr}(\mathcal{H})_\nu \tag{2.6}$$

where $\mathrm{tr}(\mathcal{H})_\nu$ *is spanned by the images of the Iwahori-Matsumoto generators* T_w *with* $\pi(w) = \nu \in \aleph$ *and* w *is of minimal length. Moreover, the image of* T_w *for minimal length* w *depends only on its conjugacy class. These* T_w*, one for each conjugacy class with* $\pi(w) = \nu$*, form a basis of* $\mathrm{tr}(\mathcal{H})_\nu$.

We are ready to define the *rigid cocenter* following [37] and its relation to $J(G(F)_c)$. Let us call $\nu^+ \in (\frac{1}{n}\mathbb{Z}^n)_{\geq 0}$ central if it lives in the diagonal $\frac{1}{n}\mathbb{Z}_{\geq 0}$.

Definition 2.26. The *rigid cocenter* of $\mathcal{H}$ is

$$\mathrm{tr}(\mathcal{H})^{rig} := \bigoplus_{\substack{\nu=(\kappa,\nu^+)\in\aleph,\\ \nu^+ \text{ central}}} \mathrm{tr}(\mathcal{H})_\nu$$

By [37, Prop. 21], $\mathrm{tr}(\mathcal{H})^{rig}$ is exactly the image of the subspace of $\mathbf{I}$-bi-invariant C^∞-functions on $G(F)$ represented by functions supported on the compact-mod-center elements. More precisely, we have

Proposition 2.27. *The set* $G(F)_c \subset G(F)$ *of compact-mod-center elements in* $G(F)$ *is exactly*

$$G(F)_c = \bigsqcup_{\substack{\nu=(\kappa,\nu^+)\in\aleph,\\ \nu^+ \text{ central}}} G(F)_\nu$$

Corollary 2.28. *Identify* $\mathrm{tr}(\mathcal{H})^{rig}$ *as a direct summand of* $\mathrm{tr}(\mathcal{H})$ *(as vector spaces) using* (2.6)*. Then the map* (2.5) *factors as*

$$J(G(F)_c) \to (\mathrm{tr}(\mathcal{H})^{rig})^* \hookrightarrow (\mathrm{tr}(\mathcal{H}))^*$$

In particular, the image of a distribution in $J(G(F)_c)$ *under the map* (2.5) *is determined by its image in* $(\mathrm{tr}(\mathcal{H})^{rig})^*$.

Combining Theorem 2.25 and Corollary 2.28, we get

Proposition 2.29. *The image of any* $D \in J(G(F)_c)$ *under* (2.5) *is determined by* $D([\mathbf{I}w\mathbf{I}])$ *where* w *runs over a set of minimal length representatives for conjugacy classes in* $\widetilde{W}$ *s.t.* $\nu^+(w)$ *is central.*

We remark that by definition $\nu^+(w)$ is central iff $\nu^+(w^m) = m\nu^+(w)$ is central. Note further that $\mathrm{tr}(\mathcal{H})^{rig}$ is still graded by $k \in \Omega$ given by $\kappa : \widetilde{W} \to \Omega$. In fact, composing with the Cartan decomposition $G(F) = \sqcup_{w\in\widetilde{W}} IwI$ we have a map $G(F) \twoheadrightarrow \Omega$, which is a homomorphism and, abusing notation slightly, the so-called Kottwitz map κ on $G(F)$. This map is just

$$\kappa := \mathrm{val} \circ \det : G(F) \to \mathbb{Z}$$

by identifying $\Omega \cong \mathbb{Z}$. We will be interested in understanding the restrictions of the unipotent orbital integrals $I_\lambda(-)$ and the truncated Steinberg characters $\mathrm{St}_{\lambda,c}$ to $\mathrm{tr}(\mathcal{H}^{rig})$. Further, we want to understand the truncations of the latter for a fixed $k \in \Omega$.

Imitating [79, IV 1.] we define

Definition 2.30. Suppose $\lambda \in P(n/e)$. Let $f_\alpha^{d,e} \in \mathcal{H}^{\mathrm{val}=\frac{nd}{e}}$ be the characteristic function of $u^{nd/e}\mathbf{P}_{\alpha^e}$ where $\mathbf{P}_{\alpha^e}$ is the standard parahoric associated to α^e.

Here

$$u = \begin{pmatrix} 0 & 0 & \cdots & 0 & t \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \ddots & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix}$$

is the matrix in Eq. (1.1) except that we can take t to be any uniformizer for F. We have that u normalizes $\mathbf{I}$, and $u^{n/e}$ normalizes $\mathbf{P}_{\alpha^e}$.

Note that by Lemma 2.16, $f_\alpha^{d,e}$ is supported on the compact-mod-center elements and therefore its image in the cocenter lies in the rigid part. Let us now fix $k \in \mathbb{Z} \cong \Omega$ and only look at the span $[\mathbf{I}w\mathbf{I}]$ for $w \in W^{\mathrm{aff}} \times \{k\}$ that are compact, i.e. restrict to $\mathrm{tr}(\mathcal{H})^{rig,\mathrm{val}=k}$. Let $e = n/\gcd(n,k)$ and $d = ke/n$, so that $k = nd/e$. Note that as $\Omega \subset \widetilde{W}$ we can also view k as a length zero element in $\widetilde{W}$. From Proposition 2.27 it is then not hard to see the following:

Proposition 2.31. *There is a basis of* $\mathrm{tr}(\mathcal{H})^{rig,\mathrm{val}=k}$ *given by (images in the cocenter of) characteristic functions* $[\mathbf{I}w_i\mathbf{I}]$ *where* $w_i = (\sigma_i, k) \in W^{\mathrm{aff}} \rtimes \Omega$ *with* $\sigma_i \in S_{n/e}$ *a set of minimal length representatives for conjugacy classes in* $S_{n/e}$*. Here we embed (as sets)*

$$S_{n/e} \hookrightarrow S_n \hookrightarrow W^{\mathrm{aff}} \times \{k\} \subset \widetilde{W}$$

where the first inclusion is given by permutation of the first n/e *elements. In particular, this space has dimension the number of partitions of* n/e*.*

Proof. By construction, each w_i is compact, of minimal length, and the elements are in distinct conjugacy classes. For each $k \in \Omega$ we would like to know the number of compact conjugacy classes in $W^{\mathrm{aff}} \times \{k\}$. If it is the number of partitions of n/e we are done.

Note that the map $G(F) \twoheadrightarrow \Omega$ is given by $g \mapsto \mathrm{val}(\det(g)) \in \mathbb{Z} \cong \Omega$, and there is a section $\widetilde{W} \hookrightarrow GL_n(F)$ with image generated by permutation matrices and diagonal matrices with diagonal entries in $t^{\mathbb{Z}}$, where $t \in F$ is a fixed uniformizer. It's easy to see that an element $w \in \widetilde{W}$ has $\nu^+(w)$ central iff its image in $GL_n(F)$ under this section is compact (e.g., one can verify both properties by replacing w with some power of w that lives in the lattice part $\mathbb{Z}^n$), which is the case iff all eigenvalues for a $g \in GL_n(F)$ have the same valuation.

When $\mathrm{val}(\det(g)) = k$, that g is compact is equivalent to that all eigenvalues have valuations $k/n = d/e$ (the latter is the reduced expression). For compact g in the image of the section $\widetilde{W} \subset GL_n(F)$, we need each cycle of the permutation to have length divisible by $e = n/\gcd(k,n)$. Conversely, for every partition of n for which all parts are divisible by e, we have a unique compact conjugacy class in $W^{\mathrm{aff}} \times \{k\}$ mapping to k. Hence the space has dimension equal to the number of partitions of n/e. ∎

Example 2.32. If $n = 4$, $e = 2$, $d = 1$, $k = 2$, then $S_{n/e} = S_2$ which is abelian. The elements $(1,2), (s,2) \in W^{\text{aff}} \rtimes \Omega$ send $(a,b,c,d) \in \mathbb{R}^4$ to $(c+1, d+1, a, b)$ and $(d+1, c+1, a, b)$ respectively. As elements of $\widetilde{W} = S_n \ltimes \mathbb{Z}^n$ we have $w_1 = ((13)(24), (0,0,1,1))$ and $w_2 = ((1324), (0,0,1,1))$.

Corollary 2.33. *The images of the functions $f_\alpha^{d,e}$ from Proposition 3.14 for $\alpha \vdash n/e$ also give a basis of $\operatorname{tr}(\mathcal{H})^{rig,\text{val}=k}$ indexed by partitions of n/e.*

Proof. The linear independence is clear by imitating e.g. [78, Corollaire 4.4.] again. By Proposition 2.31 the dimension is the number of partitions of n/e. □

By the previous Corollary, for any $w \in W^{\text{aff}} \times \{k\} \subset \widetilde{W}$ we can write

$$[\mathbf{I}w\mathbf{I}] = \sum_{\alpha \vdash \frac{n}{e}} c(w,\alpha) f_\alpha^{d,e}$$

for some constants $c(w,\alpha)$, and for chosen w_i as above, the matrix with entries $c(w_i,\alpha)$ is a change-of-basis matrix. By the well-known generators-and-relations description of the Iwahori-Hecke algebra $\mathcal{H}$, where $[\mathbf{I}w\mathbf{I}]$ corresponds to the "standard basis" T_w, together with the Cartan decomposition of $G(F)$, we see that $c(w,\alpha)$ are rational functions in q that depend only on n, k and α, but *not* on the local field F. We forego the explicit computation of these rational functions, although it should be an interesting exercise.

By Lemma 2.16 combined with Definition 2.30 and Proposition 2.27 we also have that the image of a distribution $D \in J(G(F)_c)$ in $(\operatorname{tr}(\mathcal{H})^{rig,\text{val}=k})^*$ is also determined by $D(f_\alpha^{d,e})$ where $d,e,k \in \mathbb{Z}$ are as before, and $\alpha \vdash n/e$ is a partition.

Consider now the unipotent orbital integrals I_λ and let $k = 0$. Let $f_\alpha = f_\alpha^{0,1}$ be the characteristic function of the standard parahoric $\mathbf{P}_\alpha$. Pairing with f_α for varying α, λ, Proposition 3.13 (which is basically computed in [78, Proposition 4.2] whose proof is characteristic-independent) shows that we get an invertible matrix, in particular the I_λ give a basis of $J(G(\mathcal{O})) = J(G(F)_c)^{\text{val}=0}$. Combined with Eq. (2.5) and Proposition 2.31, we get a new proof of Theorem 2.4. We also have

Theorem 2.34. *Fix n as before and let $d,e \in \mathbb{Z}$, with $e|n$ be arbitrary. There exist unique elements $\widetilde{\operatorname{St}}_{\lambda,c,k} \in (\operatorname{tr}(\mathcal{H})^{rig,\text{val}=k})^*$ whose pairing with the family of test functions $f_\alpha^{d,e}$ is given by the right-hand-side of Proposition 3.14, that is*

$$\widetilde{\operatorname{St}}_{\lambda,c,k}(f_\alpha^{d,e}) = (-1)^{nd-k}\langle e_\alpha, h_\lambda\rangle$$

When $\operatorname{char} F = 0$ the elements $\widetilde{\operatorname{St}}_{\lambda,c} = \sum_k \widetilde{\operatorname{St}}_{\lambda,c,k}$ coincide with the image of the truncated Steinberg characters $\operatorname{St}_{\lambda,c}$ under (2.5). Moreover, these $\widetilde{\operatorname{St}}_{\lambda,c,k}$ for $\lambda \vdash n/e$ form a basis of $(\operatorname{tr}(\mathcal{H})^{rig,\text{val}=k})^*$.

Proof. Theorem 2.19 combined with Proposition 3.14 shows the first assertion. The second follows from the invertibility of the square matrix given by the pairings $\widetilde{\operatorname{St}}_{\lambda,c,k}(f_\alpha^{d,e}) = (-1)^{nd-k}\langle e_\alpha, h_\lambda\rangle$ for varying α, λ. □

Combining the results for all $k \in \mathbb{Z}$, we get an element $\widetilde{\operatorname{St}}_{\lambda,c} \in (\operatorname{tr}(\mathcal{H})^{rig})^*$ that serves as the image of this version of $\operatorname{St}_{\lambda,c}$ under (2.5) satisfying Theorems 2.19 and 2.20.

Finally, let us note that given these $\widetilde{\operatorname{St}}_{\lambda,c}$, the essentially combinatorial, characteristic-independent proof of [79, Lemme V 12] goes through with the truncated Steinberg characters replaced by these $\widetilde{\operatorname{St}}_{\lambda,c}$. In particular, Proposition 6.17 goes through in arbitrary characteristic for tamely ramified γ.

2.4. **Parabolic induction.**

Definition 2.35. Suppose $M \subset P = MN \subset G$ are compatible Levi subgroup and parabolic subgroup defined over $\mathcal{O}$. For $f \in C_c^\infty(G(F))$, we define its parabolic restriction (also called parabolic descent, or constant term) $\mathrm{Res}_M^G(f) \in C_c^\infty(M(F))$ as in (2.1):

$$\mathrm{Res}_M^G(f)(m) \coloneqq \int_{G(\mathcal{O})} \int_{N(F)} f(gmng^{-1}) dndg$$

where the measure is normalized so that $G(\mathcal{O})$ and $N(\mathcal{O})$ have measure 1.

Definition 2.36. Let M, N, P be as above and $\mathfrak{m} \coloneqq \mathrm{Lie}\, M$. For $f \in C_c^\infty(\mathfrak{g}(F))$, we define its parabolic restriction $\mathrm{Res}_{\mathfrak{m}}^{\mathfrak{g}}(f) \in C_c^\infty(\mathfrak{m}(F))$ as in (2.2):

$$\mathrm{Res}_{\mathfrak{m}}^{\mathfrak{g}}(f)(X) \coloneqq \int_{G(\mathcal{O})} \int_{\mathrm{Lie}\, N(F)} f(\mathrm{Ad}(g)(X+Y)) dY dg$$

where the measure is normalized so that $G(\mathcal{O})$ and $\mathrm{Lie}\, N(\mathcal{O})$ have measure 1.

Recall that $G(F)_c^{\mathrm{val}=0} \subset G(F)$ is the subset of elements whose eigenvalues all have valuation 0. As $G = GL_n$, we realize $G(F)_c^{\mathrm{val}=0}$ also as a subset of $\mathfrak{g}(F)$.

Lemma 2.37. *If $f \in C_c^\infty(G(F)_c^{\mathrm{val}=0})$, then Definition 2.35 and 2.36 agree; we have $\mathrm{Res}_M^G(f) = \mathrm{Res}_{\mathfrak{m}}^{\mathfrak{g}}(f)$.*

Proof. In both definitions, the resulting $\mathrm{Res}_M^G(f)$ and $\mathrm{Res}_{\mathfrak{m}}^{\mathfrak{g}}(f)$ is evidently supported on $M(F)_c^{\mathrm{val}=0}$. Here one may view M as a product of general linear groups and $M(F)_c^{\mathrm{val}=0}$ is again the subset of elements whose all eigenvalues have valuation 0. For $m \in M(F)_c^{\mathrm{val}=0}$, $\mathrm{Ad}(m) : N(F) \to N(F)$ preserves the Haar measure on $N(F)$. This shows

$$\int_{N(F)} f(gmng^{-1}) dn = \int_{\mathfrak{n}(F)} f(g(m+n)g^{-1}) dn$$

and thus the two definitions agree. □

Example 2.38. One has obviously that $\mathrm{Res}_M^G(\mathbf{1}_{G(\mathcal{O})}) = \mathbf{1}_{M(\mathcal{O})}$ from Definition 2.35, and hence also $\mathrm{Res}_{\mathfrak{m}}^{\mathfrak{g}}(\mathbf{1}_{G(\mathcal{O})}) = \mathbf{1}_{M(\mathcal{O})}$.

Proposition 2.39. *Suppose $\gamma \in M(F)$ is G-regular, meaning γ is regular when viewed as an element of G. Then*

$$I_\gamma^G(f) = \left|\det(\mathrm{Ad}(\gamma)_{\mathfrak{g}/\mathfrak{m}} - \mathrm{id}_{\mathfrak{g}/\mathfrak{m}})\right|^{-1/2} \cdot I_\gamma^M(\mathrm{Res}_M^G(f))$$

Proposition 2.40. *Suppose $\gamma \in \mathfrak{m}(F)$ is G-regular, meaning γ is regular when viewed as an element of $\mathfrak{g}$. Then*

$$I_\gamma^G(f) = \left|\det(\mathrm{ad}(\gamma)_{\mathfrak{g}/\mathfrak{m}})\right|^{-1/2} \cdot I_\gamma^M(\mathrm{Res}_{\mathfrak{m}}^{\mathfrak{g}}(f))$$

Definition 2.41. We define $\mathrm{Ind}_{\mathfrak{m}}^{\mathfrak{g}} : C_c^\infty(\mathfrak{m}(F))^* \to C_c^\infty(\mathfrak{g}(F))^*$ to be the adjoint of $\mathrm{Res}_{\mathfrak{m}}^{\mathfrak{g}}$. It is called *parabolic induction.* Same for $\mathrm{Ind}_M^G : C_c^\infty(M(F))^* \to C_c^\infty(G(F))^*$

In particular, Proposition 2.40 effectively says that we have the equality of invariant distributions

$$\left|\det(\mathrm{ad}(\gamma)_{\mathfrak{g}/\mathfrak{m}})\right|^{-1/2} \cdot \mathrm{Ind}_M^G I_\gamma^M(-) = I_\gamma^G(-)$$

More generally, an important property is that $\mathrm{Res}_{\mathfrak{m}}^{\mathfrak{g}}$ and Res_M^G send $G(F)$-coinvariant to $M(F)$-coinvariant and equivalently $\mathrm{Ind}_{\mathfrak{m}}^{\mathfrak{g}}$ and Ind_M^G send $M(F)$-invariant to $G(F)$-invariant [45, Lemma 13.1].

Proposition 2.42. *Suppose* $\mathbb{O}$ *is a nilpotent orbit of* $\mathfrak{m}(F)$ *and* $\widetilde{\mathbb{O}}$ *the induced orbit in the sense of Lusztig-Spaltenstein, i.e.* $\widetilde{\mathbb{O}}$ *contains an open dense subset of* $\mathbb{O}+\operatorname{Lie}N(F)$. *Then* $\operatorname{Ind}_{\mathfrak{m}}^{\mathfrak{g}} I_{\mathbb{O}}^{M} = I_{\widetilde{\mathbb{O}}}^{G}$.

Remark 2.43. In terms of partitions (in the sense of Definition 3.1), if $M = GL_{n_1} \times \cdots \times GL_{n_r}$, and $\mathbb{O}$ is a unipotent orbit corresponding to a sequence of partitions

$$\lambda^{(1)} \vdash n_1, \ \ldots, \ \lambda^{(r)} \vdash n_r$$

the induced orbit is

$$\widetilde{\mathbb{O}} \leftrightarrow (\lambda_1^{(1)} + \cdots + \lambda_1^{(r)}, \ldots, \lambda_k^{(1)} + \cdots + \lambda_k^{(r)})$$

where k is the length of the longest $\lambda^{(i)}$. For example, when $M = T$, the zero orbit in T induces to the principal one in GL_n.

Corollary 2.44. *For* $\gamma \in \mathfrak{m}(F)$ which is G-regular*, we have*

$$\Gamma_{\widetilde{\mathbb{O}}}^{G}(\gamma) = \begin{cases} 0 & \text{if } \widetilde{\mathbb{O}} \text{ is not induced from } M, \\ \left|\det(\operatorname{ad}(\gamma)|_{\mathfrak{g}/\mathfrak{m}})\right|^{-1/2} \cdot \Gamma_{\mathbb{O}}^{M}(\gamma) & \text{if } \widetilde{\mathbb{O}} \text{ is induced from } \mathbb{O} \subset \mathfrak{m}(F). \end{cases}$$

where $\Gamma_{\widetilde{\mathbb{O}}}^{G}(\gamma)$ *and* $\Gamma_{\mathbb{O}}^{M}(\gamma)$ *are defined as in Theorem 2.8 for* G *and* M*, respectively.*

3. Symmetric functions and combinatorics

In this section, we review some theory of symmetric functions relevant to the computation of Shalika germs. The theory is very well covered in many sources, see for example [35, Section 3].

3.1. Combinatorics.

We begin with two combinatorial definitions.

Definition 3.1. A *partition* of an integer $n > 0$, written $\lambda \vdash n$ or $\lambda \in P(n)$ is a nonincreasing sequence of positive integers

$$\lambda_1 \geq \ldots \geq \lambda_k > 0, \ \ \sum_i \lambda_i = n$$

and a *composition* of n, written $\alpha \vDash n$ is an ordered collection $(\alpha_1, \ldots, \alpha_k)$ of positive integers such that $\sum_i \alpha_i = n$. In both cases, we write $\ell(\lambda) = \ell(\alpha) = k$ for the length of the composition or partition and denote by λ^t, α^t the conjugate partition (resp. composition).

Recall that partitions $\lambda \vdash n$ index conjugacy classes in the symmetric group S_n. Similarly, to any composition $\alpha \vDash n$ we can associate the *Young subgroup* $S_{\alpha_1} \times \cdots \times S_{\alpha_k} \subseteq S_n$, whose conjugacy class only depends on $\operatorname{sort}(\alpha)$, the partition obtained by sorting α. We will draw the Young/Ferrers diagrams of partitions in French notation. We think of them as lying in $\mathbb{Z}_{\geq 0}^2$ with the first box always at $(0,0)$. For a box $\square \in \lambda$ with coordinates (i,j) we denote

$$a(\square) = \lambda_i - i - 1, \ l(\square) = \lambda_j^t - j - 1, \ a'(\square) = i, \ l'(\square) = j \tag{3.1}$$

the arm, leg, coarm, and coleg lengths of the box. The *q,t-content* of a box is defined to be $q^{a'(\square)} t^{l'(\square)}$. Finally, we have

Definition 3.2. For two partitions (or compositions) λ, μ define

$$\mathbf{M}(\lambda, \mu)$$

to be the set of nonnegative integer matrices (of size $\ell(\lambda) \times \ell(\mu)$) whose rows sum to λ and columns sum to μ.

Definition 3.3. A *standard Young tableau* is a filling of the Ferrers diagram of $\lambda \vdash n$ with the letters $1, \ldots, n$ such that the letters increase in columns and rows.

Given a Young tableau and a box $\square_i$ labeled i, we define the arm length as $a(\square_i)$ and so on. We let z_i be the $q,t-$content of the box $\square_i$.

We will also need the following Lemma in Sections 5, 6.

Lemma 3.4. *To each composition $\alpha \vDash n$ is associated a unique Young tableau $T(\alpha)$ defined as follows. To each α_i we assign the sequence of numbers $\sum_{j=1}^{i-1} \alpha_j + 1, \sum_{j=1}^{i-1} \alpha_j + 2, \ldots \sum_{j=1}^{i} \alpha_j$ and form a tableau by taking one-row diagrams with these fillings, and then dropping them on top of each other, with the rule that gravity brings boxes as low as possible. In particular, the tableau decomposes as a sequence of horizontal α_i-strips.*

Example 3.5. To the compositions $4 = 2 + 2$, $4 = 1 + 2 + 1$ and $4 = 1 + 3$ we assign the tableaux

$$\begin{array}{|c|c|} \hline 3 & 4 \\ \hline 1 & 2 \\ \hline \end{array} \qquad \begin{array}{|c|c} \cline{1-1} 4 & \\ \cline{1-1} 2 & \\ \hline 1 & \multicolumn{1}{c|}{3} \\ \hline \end{array} \qquad \begin{array}{|c|c|c|} \cline{1-1} 2 & \multicolumn{2}{c}{} \\ \hline 1 & 3 & 4 \\ \hline \end{array}$$

The final combinatorial gadget we will need are Dyck paths.

Definition 3.6. Let $m, n, k \geq 1, (m, n) = 1$. Then the set

$$\mathbb{D}_{km,kn}$$

will be the collection of lattice paths in a $kn \times km$-rectangle in $\mathbb{Z}^2_{\geq 0}$, fitting under the diagonal (which has slope m/n).

The *area* of a Dyck path $D \in \mathbb{D}_{km,kn}$ is defined to be the number of full squares between the path and the diagonal. Similarly, the *coarea* of D is defined as $(m-1)(n-1)/2 - \mathrm{area}(D)$, which is the number of squares below the path.

3.2. **The ring of symmetric functions.** Let $\mathrm{Sym}_{q,t}$ be the ring of symmetric functions over $\mathbb{Q}(q,t)$ in the alphabet $\{X_1, \ldots, X_n, \ldots\}$ and denote the five usual bases of monomial, homogeneous, elementary, Schur, and power sum symmetric functions by

$$\{m_\lambda\}, \{h_\lambda\}, \{e_\lambda\}, \{s_\lambda\}, \{p_\lambda\}$$

Here λ is a partition in the sense of Definition 3.1. Note that the first four are also bases of $\mathrm{Sym} = \mathrm{Sym}_{\mathbb{Z}}$ while the last one needs a ring containing $\mathbb{Q}$.

Recall that the *modified Macdonald polynomials* $\widetilde{H}_\lambda[X; q, t], \lambda \vdash n$ are the unique symmetric functions with the properties

$$\widetilde{H}_\mu[X(1-q); q, t] \in \mathbb{Q}(q,t)\{s_\lambda | \lambda \geq \mu\} \tag{3.2}$$

$$\widetilde{H}_\mu[X(1-t); q, t] \in \mathbb{Q}(q,t)\{s_\lambda | \lambda \geq \mu^t\} \tag{3.3}$$

$$\langle \widetilde{H}_\mu[X; q, t], s_{(n)} \rangle = 1 \tag{3.4}$$

Here the last pairing is the Hall inner product, defined in Definition 3.10.

We do not require much of the advanced theory of Macdonald polynomials, but let us note down the following definition as well as some specializations.

Definition 3.7. The operator ∇ of Bergeron and Garsia scales by definition each $\widetilde{H}_\lambda$ by $q^{n(\lambda)} t^{n(\lambda^t)}$ where $n(\lambda) = \sum_{i=1}^{\ell(\lambda)} (i-1)\lambda_i$.

From [35, Proposition 3.5.8.] we have

Lemma 3.8 (The limit of $\widetilde{H}_\lambda$ as $q \to 1$)**.** *The modified Macdonald symmetric function $\widetilde{H}_\lambda$ at $q = 1$ is given by*

$$\widetilde{H}_\lambda[X;1,t] = (1-t)^{|\lambda|}[\lambda^t]_t! h_{\lambda^t}[X/(1-t)] =: \widetilde{h}_{\lambda^t}[X;t] \tag{3.5}$$

in other words a plethystically transformed homogeneous symmetric function, up to normalization.

From the q,t-symmetry $\widetilde{H}_\lambda[X;t,q] = \widetilde{H}_{\lambda^t}[X;q,t]$ we immediately have

Corollary 3.9 (The limit as $t \to 1$)**.**

$$\widetilde{H}_\lambda[X;q,1] = \widetilde{h}_\lambda[X;q]$$

In addition to the q,t-symmetry, we have the symmetry under inverting q and t [35, Proposition 3.5.12.]

$$t^{-n(\mu)}q^{-n(\mu^t)}\omega\widetilde{H}_\mu[X;q,t] = \widetilde{H}_\mu[X;q^{-1},t^{-1}] \tag{3.6}$$

so that in particular $\lim_{t\to 1}\widetilde{H}_\mu[X;q^{-1},t^{-1}] = q^{n(\mu^t)}\omega\widetilde{h}_\mu[X;q^{-1}]$. Note that this implies

$$\omega\widetilde{h}_\mu[X;q] = q^{n(\mu^t)}\widetilde{h}_\mu[X;q^{-1}] \tag{3.7}$$

We will denote $\widetilde{h}_\lambda := \widetilde{H}_\lambda[X;q,1]$ and call these the *specialized Macdonald symmetric functions* or the *plethystically transformed homogeneous symmetric functions*. Later on, we will also need the prefactor

$$c_\lambda(q) := (1-q)^{|\lambda|}[\lambda]_q! = \prod_{i=1}^{\ell(\lambda)}\prod_{j=1}^{\lambda_i}(1-q^j) \tag{3.8}$$

in Eq. (3.5). Note that

$$\widetilde{h}_\lambda = c_\lambda^{Wal}(q)e_\lambda\left[\frac{X}{q-1}\right] = (-1)^n c_\lambda^{Wal}\omega e_\lambda\left[\frac{X}{1-q}\right]$$

where $c_\lambda^{Wal} := (-1)^n c_\lambda = \prod_{i=1}^{\ell(\lambda)}\prod_{j=1}^{\lambda_i}(q^j-1)$ is exactly the prefactor defined on [78, pp. 201].

Note that compared to the Macdonald polynomials, $\widetilde{h}_\lambda$ are much simpler in behaviour. For instance, they are multiplicative:

$$\widetilde{h}_\lambda\widetilde{h}_\mu = \widetilde{h}_{\lambda+\mu}$$

and one can deduce combinatorial expansions for them in terms of the other standard bases via known relations between h_λ and these bases. This will turn out to be important in the proof of Theorem 6.35. Further, the ∇-operator from Definition 3.7 becomes a ring homomorphism on symmetric functions in this limit.

We will also need a few different inner products on the ring of symmetric functions, the interplay of whom turns out to play a key role. We remark that by "inner product" we simply mean a symmetric bilinear form valued in $\mathbb{Q}(q,t)$.

Definition 3.10. (1) The *Hall inner product* is the inner product on $\mathrm{Sym}_{q,t}$ defined by

$$\langle s_\lambda, s_\mu\rangle = \delta_{\lambda\mu}$$

(2) The *q-inner product* is

$$\langle f,g\rangle_q := \langle f, g\left[\frac{X}{(1-q)}\right]\rangle$$

(3) The *q,t-inner product* is

$$\langle f,g\rangle_{q,t} := \langle f, g\left[\frac{(1-q)}{(1-t)}X\right]\rangle$$

(4) The *geometric inner product* is

$$(f,g) = -q^{\deg f}\left\langle (\nabla^{-1}(f))[X(1-t^{-1})], g[X(1-t^{-1})]\right\rangle_{q,t^{-1}}$$

Remark 3.11. There is a natural Frobenius characteristic map from the direct sum of the representation rings of symmetric groups

$$\bigoplus_{n\geq 0}\mathrm{Rep}(S_n)\to \mathrm{Sym}_{\mathbb{Z}}$$

Endowing the source with the natural inner product on characters, and the target with the Hall inner product ((1) in the above Definition), this map is an isomorphism of Hopf algebras and an isometry.

Similarly, if we let

$$\mathrm{Hall}(GL(\mathbb{F}_q))$$

be the Hall algebra of the general linear groups as n ranges from 0 to ∞ there is a natural inner product on this space, again coming from the convolution product on characters, and a map

$$\mathrm{Hall}(GL(\mathbb{F}_q))\to \mathrm{Sym}_q$$

which is an isomorphism of Hopf algebras and an isometry with respect to the inner product $\langle f,g\rangle_{Zel} := \langle f, g\left[\frac{X}{q-1}\right]\rangle$. Note that this inner product differs from (2) by a plethysm $X\to -X$. For this perspective, we refer to [81, 10.2.]

Remark 3.12. The last inner product will be used in Section 9 and is the one naturally arising from the geometry of Hilbert schemes of points. It can be more easily characterized as the unique inner product satisfying

$$(\widetilde{H}_\lambda, \widetilde{H}_\mu) = \delta_{\lambda\mu}g_\lambda$$

where

$$g_\lambda = \prod_{\square\in\lambda}(1-q^{a(\square)}t^{-l(\square)-1})(1-q^{-a(\square)-1}t^{l(\square)}) \tag{3.9}$$

3.3. **Orbital integrals and symmetric functions.** Let us review how symmetric functions arise in the theory of orbital integrals on GL_n. This essentially goes back to [78, 79]. First, we describe how to interpret unipotent/nilpotent orbital integrals of characteristic functions of standard parahorics using the theory of symmetric functions.

Proposition 3.13. *Let $\mathbf{1}_\lambda$ be the characteristic function of the standard parahoric subgroup $\mathbf{P}_\lambda\subset GL_n(F)$ corresponding to the partition λ,* divided by the measure of *$\mathbf{P}_\lambda$. Let $I_{\mu^t}(-)$ be the orbital integral over the unipotent class of type μ^t. Then we have*

$$I_{\mu^t}(\mathbf{1}_\lambda) = \langle h_\lambda, \widetilde{h}_\mu\rangle \tag{3.10}$$

where $\langle -,-\rangle$ is usual Hall inner product from Definition 3.10.

Proof. Let ψ_λ be the characteristic function of $\mathbf{P}_\lambda$, i.e. $\psi_\lambda = c_\lambda(q)c_n(q)^{-1}\mathbf{1}_\lambda$[3]. By [78, Prop. 4.2.] we get that in the normalizations we have chosen,

$$I_{\mu^t}(\psi_\lambda) = c_\lambda^{Wal}(q)c_\mu^{Wal}(q)c_n^{Wal}(q)^{-1}\langle x_\lambda, y_\mu\rangle_{Zel}$$

where x_λ, y_μ in the notations of [78,81] correspond to our h_λ and e_μ by [81, 10.2.].

Since $e_\mu\left[\frac{X}{q-1}\right] = (-1)^n h_\mu\left[\frac{X}{1-q}\right]$ we get

$$I_{\mu^t}(\psi_\lambda) = (-1)^{4n}c_\lambda c_\mu c_n^{-1}\langle h_\lambda, h_\mu\left[\frac{X}{1-q}\right]\rangle$$

Absorbing c_μ to $\widetilde{h}_\mu$ and taking into account the difference between ψ_λ and $\mathbf{1}_\lambda$ we get

$$I_{\mu^t}(\mathbf{1}_\lambda) = c_\lambda^{-1}c_n I_{\mu^t}(\psi_\lambda) = \langle h_\lambda, \widetilde{h}_\mu\rangle$$

□

A similar result holds for the truncated Steinberg distributions $\mathrm{St}_{\lambda,c,k}$ from Theorems 2.19, 2.20.

Proposition 3.14 (Lemme V 11. [79])**.** *Suppose $e \geq 1, (d,e) = 1$ and $\mu \in P(n/e)$. Denote also $k = nd/e$, and let $f_\mu^{d,e}$ be the characteristic function of $u^{nd/e}\mathbf{P}_{\mu^e}$ where $\mathbf{P}_{\mu^e}$ is the parahoric associated to μ^e, again divided by the measure of the subset as in the previous Proposition. Note Here u is the matrix in Eq.* (1.1)*, or any suitable lift of the generator of $\Omega \subset \widetilde{W}$ to the group $G(F)$. Note that $f_\mu^{0,e} = \mathbf{1}_\mu$.*

Then

$$\mathrm{St}_{e\lambda,c,k}(f_\mu^{d,e}) = (-1)^{nd-nd/e}\langle h_\mu, e_\lambda\rangle$$

where the pairing is the Hall inner product and $\mathrm{St}_{e\lambda,c,k}(-)$ is the distribution from Theorem 2.34.

In particular, when $d = 0$, we have $\mathrm{St}_{e\lambda,c,k}(f_\mu^{0,e}) = \langle h_\mu, e_\lambda\rangle$.

Proof. This is a direct translation of [79, V. II], with the identifications $x_\mu = h_\mu, y_\lambda = e_\lambda$ as above. Note that we could use the isometry property of ω to switch the roles of e_μ, h_λ. □

4. The elliptic Hall algebra

In this section, we define the elliptic Hall algebra (EHA) and recall some necessary facts about it. Apart from Theorem 4.9 results in this section are contained in [25,33,59–62,68–70,75]. For the basic theory, our main references are [25,59,75] and for the results on symmetric functions, one may refer to [59–62,69].

For most of the paper, in particular for the application in the proofs of our main results in Section 2, we want to understand the $t \to 1$ degeneration of the Fock space representation of the EHA.

Definition 4.1. The *elliptic Hall algebra* (quantum toroidal $\mathfrak{gl}_1$) is the $\mathbb{C}$-algebra $\mathcal{E} = \mathcal{E}_{q_1,q_2,q_3}$ depending on $q_1, q_2, q_3 \in \mathbb{C}^\times, q_1q_2q_3 = 1$, generated by elements

$$P_{m,n},\ (m,n) \in \mathbb{Z}^2\backslash(0,0)$$

and satisfying the relations

$$[P_{m_1,n_1}, P_{m_2,n_2}] = 0$$

[3]Warning: Waldspurger uses the notation φ_λ for our ψ_λ. For him, ψ_λ denotes a different function. We reserve the notation φ for the plethysms introduced in the next sections.

if $(m_1,n_1),(m_2,n_2)$ lie on the same line through the origin, and

$$[P_{m_1,n_1},P_{m_2,n_2}] = \frac{\theta_{m_1+m_2,n_1+n_2}}{\alpha_1}$$

if $(m_1,n_1),(m_2,n_2),(m_1+m_2,n_1+n_2)$ form a quasi-empty triangle. Here

$$\exp(\sum_{k=1}^{\infty} P_{km,kn}\alpha_k x^k) = \sum_{\ell=1}^{\infty}\theta_{\ell m,\ell n}x^\ell$$

for $(m,n)=1$ and

$$\alpha_k = \frac{(q_1^k-1)(q_2^k-1)(q_3^k-1)}{k}$$

Proposition 4.2 (Triangular decomposition)**.** *Let $\mathcal{E}^>$ be the subalgebra generated by the $P_{1,n}, n\in\mathbb{Z}$, $\mathcal{E}^<$ be the subalgebra generated by the $P_{-1,n}, n\in\mathbb{Z}$, and $\mathcal{E}^0$ be the subalgebra generated by $P_{0,\pm k}, k\in\mathbb{Z}_{>0}$. The multiplication map gives a $\mathbb{C}$-linear isomorphism*

$$\mathcal{E}^<\otimes\mathcal{E}^0\otimes\mathcal{E}^>\to\mathcal{E}$$

For the rest of this paper, we may as well restrict our attention to the positive part $\mathcal{E}^>$ of the EHA, or rather the nonnegative part $\mathcal{E}^\geq$ which is by definition generated by $P_{1,n}, n\in\mathbb{Z}, P_{0,k}, k>0$. Further, we wish to study the $q_3\to 1$ limit of this algebra and the Fock space representation. The relationship of the parameters q_1,q_2,q_3 to the Macdonald theory parameters is $q_1=q, q_3=t^{-1}$ so that this limit amounts to setting $t=1$. We will use these identifications freely.

Note that since the definition of $\mathcal{E}$ is symmetric in the q_i, this choice is immaterial for many things. Importantly, it does matter for the Fock space representation (to be introduced soon), whose definition is not symmetric in the q_i.

Let us now remark on the structure of $\mathcal{E}$ in the limit $t\to 1$ as an abstract algebra, although this will not be important for us. Consider the quantum torus in one variable, or in other words the algebra of q-difference operators on $\mathbb{C}^\times$. It is the $\mathbb{C}[q^\pm]$-algebra

$$\mathfrak{D} := \mathbb{C}[q^\pm]\langle X^\pm, D^\pm\rangle / DX - qXD.$$

Considering this associative algebra as a Lie algebra we get a 2-dimensional central extension $\mathfrak{D}_{c_1,c_2}$ [25] with central charges $c_1,c_2\in\mathbb{C}$ defined as

$$[X^{i_1}D^{j_1}, X^{i_1}D^{j_2}] = (q^{j_1i_2}-q^{j_2i_1})X^{i_1+i_2}D^{j_1+j_2} - \delta_{(i_1,j_1),(-i_2,-j_2)}q^{i_1j_1}(i_1c_1+j_1c_2).$$

By [25] we may view $\mathcal{E}$ as a quantization of the universal enveloping algebra of $\mathfrak{D}_{c_1,c_2}$, and taking the $q_3\to 1$ limit recovers just this universal enveloping algebra, at least up to a completion.

For example for the limit $q_3\to 1$, we have (see [75, Proposition 5.6.]) that

$$(1-q)P_{1,m} = D^mX, (q^{-1}-1)P_{-1,m} = X^{-1}D^m$$

and

$$(1-q^{-m})P_{0,m} = D^m.$$

Remark 4.3. It is not possible to directly set $q_3\to 1$ in the defining relations of the EHA as given above. A way to circumvent this is to redefine:

$$\exp((1-q_3)^{-1}\sum_{k=1}^{\infty}P_{km,kn}\alpha_k x^k) = \sum_{\ell=1}^{\infty}\theta_{\ell m,\ell n}x^\ell$$

or alternatively to rescale the generators of $\mathcal{E}^<$ by $1-q_2$ and those of $\mathcal{E}^>$ by $1-q_1$. Effectively, this gives an integral form of $\mathcal{E}$ in the sense of Lusztig. See e.g. [75, Section 5.4.] and [63] for details.

As the name suggests, $\mathcal{E}$ specializes to the Hall algebra of coherent sheaves on an elliptic curve over a finite field (when q_1 is the Frobenius eigenvalue on H^1 and q_2 its conjugate). In that setting, the slope of vector bundles gives rise to natural Hall subalgebras. These lift to $\mathcal{E}$, and are by definition the commutative subalgebras "living on lines through the origin".

Definition 4.4. Let $m, n \in \mathbb{Z}^2_{\geq 0}$, $(m,n) = 1$. The *slope* $\frac{m}{n}$*-subalgebra* of $\mathcal{E}^{\geq 0}$ is the subalgebra $\mathcal{E}^{m/n}$ generated by $P_{km,kn}, k \geq 0$

Theorem 4.5 ([59]). *Let* $\mathrm{Sym}_{q,t}$ *be the algebra of symmetric functions over* $\mathbb{C}(q,t)$ *as introduced in Section 3. There is an algebra isomorphism*

$$\widehat{\varphi}_{m/n} : \mathrm{Sym}_{q,t} \to \mathcal{E}^{m/n}$$

sending $p_k \mapsto P_{km,kn}$.

We will call this homomorphism the *slope* m/n *plethysm.*

4.1. **The Fock space.**

Definition 4.6. The *Fock space* is the $\mathbb{C}(q,t)$-vector space $\mathcal{F}$ spanned by the basis

$$\{|\lambda\rangle\}_{\lambda\vdash n, n\geq 0}$$

Recall that $\mathcal{F}$ appears naturally from the Hilbert scheme of points on $\mathbb{A}^2$ or symmetric functions over $\mathbb{C}(q,t)$. We will freely identify $\mathcal{F}$ with the space of symmetric functions $\mathrm{Sym}_{q,t}$ (see Section 3) so that the basis $|\lambda\rangle$ corresponds to the Macdonald basis $\widetilde{H}_\lambda$. The reason for our usage of the Fock space as opposed to just $\mathrm{Sym}_{q,t}$ will become clear in Section 9.

Theorem 4.7 ([26, 69]). *There is an action of* $\mathcal{E}_{q_1,q_2,q_3}$ *on* $\mathcal{F}$ *by so called shuffle algebra operations.*

We will be interested in the action of the operators $P_{km,kn} \in \mathcal{E}^{\geq 0}$ and more generally the slope m/n subalgebras $\mathcal{E}^{m/n}$ in the Fock space, especially in the $t \to 1$ limit. For example, the operators $P_{0,m}$ act as multiplication by the symmetric functions p_m, and the operator $P_{1,0}$ is a so called Macdonald eigenoperator.

In [60] the matrix coefficients of the operators $P_{km,kn}$ in the basis $|\lambda\rangle$ are computed (see also [26]). Below the orthogonalizing inner product $\langle\lambda|\mu\rangle = \delta_{\lambda\mu} g_\lambda$ corresponds to the geometric inner product $(-,-)$ on symmetric functions, see Definition 3.10.

Theorem 4.8 ([60], see Eq. (37) in [33]). *We have*

$$\langle\lambda|P_{km,kn}|\mu\rangle = \frac{\gamma^{kn}}{[k]_q}\cdot\frac{g_\lambda}{g_\mu}\sum_{\mu=\lambda+\square_1+\ldots+\square_{kn}}^{\mathrm{SYT}}\left[\sum_{j=0}^{k-1}(qt)^j\frac{z_{n(k-1)+1}z_{n(k-2)+1}\cdots z_{n(k-j)+1}}{z_{n(k-1)}z_{n(k-2)}\cdots z_{n(k-j)}}\right]\cdot$$

$$\cdot\frac{\prod_{i=1}^{kn} z_i^{S'_{m/n}(i)}(qt\chi_i - 1)}{\left(1-qt\frac{z_2}{z_1}\right)\cdots\left(1-qt\frac{z_{kn}}{z_{kn-1}}\right)}\prod_{1\leq i<j\leq kn}\omega'^{\,-1}\left(\frac{z_j}{z_i}\right)\prod_{1\leq i\leq kn}^{\square\in\lambda}\omega'^{\,-1}\left(\frac{z(\square)}{z_i}\right)$$

where

$$\omega'(x) = \frac{(x-1)(x-qt)}{(x-q)(x-t)},\ \gamma = \frac{(q-1)(t-1)}{qt(qt-1)}$$

and

$$S_{m/n}(i)' := \lfloor \frac{im}{n} \rfloor - \lfloor \frac{(i-1)m}{n} \rfloor$$

Although we do not need the full strength of the formula in Theorem 4.8, it is recorded here for our computations in Section 5 and possible generalizations. The $t \to 1$ limit of this formula for $\mu = \varnothing$ is studied in Proposition 5.20.

We will now begin to study the degeneration of the representation on $\mathcal{F}$ as $t \to 1$. The most important fact about the $t = 1$ limit is the following.

Proposition 4.9. *In the Fock representation at $t = 1$, the positive half $\mathcal{E}^{\geq}_{q,1/q,1}$ acts by multiplication operators.*

Proof. As shown in [63], the operators $P_{m,n}$ for $(m,n) \in \mathbb{Z} \times \mathbb{N}$ generate $\mathcal{E}^{\geq}$ over $\mathbb{Z}[q_1^{\pm}, q_2^{\pm}]$, as do the operators $H_{m,n}$ which are defined by the identity

$$1 + \sum_{s=1}^{\infty} \frac{H_{sm,sn}}{x^s} = \exp\left(\sum_{s=1}^{\infty} \frac{P_{sm,sn}}{sx^s}\right)$$

By [62, Theorem 2.15.] one can write the action of either $H_{m,n}$ or $P_{m,n}$ as a contour integral, for example:

$$H_{m,n} \cdot f[X] = \int_{0<X<|z_n|<\cdots<|z_1|<\infty} \frac{z_i^{S'_{m/n}(i)}}{\prod_{i=1}^{n-1}(1 - qt\frac{z_{i+1}}{z_i}) \prod_{i<j} \omega'(z_j/z_i)}$$

$$\wedge^\bullet\left(-\frac{X}{z_1}\right) \cdots \wedge^\bullet\left(-\frac{X}{z_n}\right) \cdot f\left[X - (1-q)(1-t)\sum_{i=1}^{n} z_i\right] \prod_{a=1}^{n} \frac{dz_a}{2\pi i z_a}$$

Where $\bigwedge^\bullet(-\frac{X}{z}) = \sum_{k=0}^{\infty} \frac{h_k}{z^k}$. Here the contours are concentric circles in the prescribed order and are contained between the poles $0, x_1, \ldots, \infty$, see e.g. [60, 62] for details.

Now the plethystic operator

$$f[X] \mapsto f[X \pm (1-q)(1-t)z] = \exp\left[\pm \sum_{k=1}^{\infty} \frac{p_k^{\dagger} z^k}{k}\right] \cdot f[X]$$

at $t = 1$ becomes just the identity, so that this is a multiplication operator. □

Remark 4.10. We note that this Proposition is conjectured in [8, 9].

Remark 4.11. The operators $P_{1,n}, n \in \mathbb{N}$ can be described as follows, see e.g. [9]. In plethystic notation, their action on the Fock space is given by

$$P_{1,n} \cdot f[X] = f[X + \frac{(1-t)(1-q)}{z}] \sum_{i \geq 0} (-z)^i e_i[X] \Bigg|_{z^n}$$

where by $|_{z^i}$ we mean extracting the coefficient of z_i in this series. At $t = 1$ this becomes just multiplication by e_n. In general when $n \in \mathbb{Z}$, the limits of $P_{1,n}$ are still multiplication operators by above. It is however not true that the algebra generated by these operators over $\mathbb{Z}[q_1^{\pm}, q_2^{\pm}]$ is all of $\mathcal{E}^{\geq}$ anymore.

In addition to the $P_{m,n}$ we want to understand the elements $E_{km,kn} := \varphi_{m/n}(e_k)$ from [61, 68, 69] in the limit $t \to 1$.

Proposition 4.12. *Suppose that $m,n>0$ and $gcd(m,n)=1$. At $t=1$ the operator $\varphi_{m/n}(e_k)|_{t=1}$ becomes a multiplication operator by the symmetric function:*

$$E_{m,n,k} := \sum_{D\in\mathbb{D}_{km,kn}} q^{\mathrm{area}(D)} e_D.$$

Here D is a Dyck path in $(km\times kn)$ rectangle below the diagonal, area(D) *is the area between D and the diagonal, and $e_D := \prod_{horizontal\ steps\ h_i(D)\ of\ D} e_{h_i(D)}$.*

Proof. Given Proposition 4.9, this is [9, Eq. (4.5.4)] (see also [7]). □

Remark 4.13. In fact, according to [25] while the construction of the limit $t=q_3\to 1$ of the algebra $\mathcal{E}$ is independent of our choice in q_1,q_2,q_3, the construction of the Fock representation $\mathcal{F}$ naturally breaks the symmetry (in physics, this is related to the threefold symmetry of the refined topological vertex). The action of the skein algebra of the torus on that of the solid torus made explicit in [56] corresponds to the $q_2=(qt)^{-1}\to 1$ limit, and can be thought of as a "rotation" of our representation by 120 degrees.

4.2. **Double affine Hecke algebras.** In order to define the superpolynomials in the next section, it will be relevant for us to treat $\mathcal{E}$ as the limit of the spherical double affine Hecke algebras as $n\to\infty$, and the Fock space representation as a limit of the polynomial representations of the spherical DAHA. This point of view is adopted in e.g. [68].

Definition 4.14. The *double affine Hecke algebra* (DAHA) $\mathbb{H}_n$ is the $\mathbb{Q}(q,t)$-algebra generated by

$$X_1^{\pm},\dots,X_n^{\pm},Y_1^{\pm},\dots,Y_n^{\pm},T_1,\dots,T_n$$

with the relations

$$[X_i,X_j]=0 \qquad [Y_i,Y_j]=0 \tag{4.1}$$

$$(T_i-t)(T_i+t^{-1})=0 \qquad [T_i,T_j]=0,\ T_iT_{i+1}T_i=T_{i+1}T_iT_{i+1} \tag{4.2}$$

$$T_iX_j=X_jT_i \qquad T_iY_j=Y_jT_i \tag{4.3}$$

$$T_iX_iT_i=X_{i+1} \qquad T_i^{-1}Y_iT_i^{-1}=Y_{i+1} \tag{4.4}$$

$$Y_1X_1\cdots X_n=qX_1\cdots X_nY_1 \qquad Y_2^{-1}X_1Y_2X_1^{-1}=T_1^2 \tag{4.5}$$

where $|i-j|>1$.

The *spherical DAHA* is the subalgebra

$$\mathbb{SH}_n := \mathbf{e}\,\mathbb{H}_n\,\mathbf{e}$$

where $\mathbf{e} := \frac{1}{[n]_t!}\sum_{w\in S_n} t^{\ell(w)}T_w$ is the symmetrizing idempotent for the finite Hecke algebra. Note also that $\mathbb{H}_n$ contains two affine Hecke algebras of S_n as subalgebras, namely one generated by the T_i,X_i and another one generated by the T_i,Y_i. We will denote these by $\mathcal{H}_n^{aff,X},\mathcal{H}_n^{aff,Y}$.

The following is proved in [13,33,68] and will be essential for our computations:

Lemma 4.15. *There is an action of the braid group $B_3=\widehat{SL_2(\mathbb{Z})}$ on $\mathbb{H}_{q,t}$ by algebra automorphisms.*

Proof. See [13, Section 1.3.]. □

The generators of this action are

$$\tau_+ := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},\ \tau_- = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

and they act by

$$\tau_+ : T_i \mapsto T_i,\ X_i \mapsto X_i,\ Y_i \mapsto Y_i X_i (T_1 \cdots T_{i-1})^{-1} (T_{i-1} \cdots T_1)^{-1}$$
$$\tau_- : T_i \mapsto T_i,\ X_i \mapsto X_i Y_i (T_{i-1} \cdots T_i)(T_1 \cdots T_{i-1}),\ Y_i \mapsto Y_i$$

Next, let

$$P^{(n)}_{0,k} = \mathbf{e}(\sum_{i=1}^n Y_i^k)\,\mathbf{e} \in \mathbb{SH}_n \tag{4.6}$$

For arbitrary integers $(a,b) \in \mathbb{Z}^2 \backslash 0$ we have the following.

Proposition 4.16 (Section 2.2., [68])**.** *Let $k = \gcd(a,b)$ and $\gamma_{a/k,b/k}$ be any matrix of the form*

$$\gamma_{a/k,b/k} = \begin{pmatrix} * & a/k \\ * & b/k \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$$

Then the elements

$$P^{(n)}_{(a,b)} := \gamma_{a/k,b/k}(P^{(n)}_{0,k}) \tag{4.7}$$

are well-defined, i.e. do not depend on the chosen matrix.

Proposition 4.17. *The elements $P^{(n)}_{a,b}$ generate $\mathbb{SH}_n$ as an algebra.*

Further, one can show the $P^{(n)}_{a,b} \in \mathbb{SH}_n$ satisfy relations similar to those of $P_{a,b} \in \mathcal{E}$. In fact, by [68, Theorem 4.6.], we have

Proposition 4.18. *There is a surjective algebra homomorphism*

$$\mathcal{E} \twoheadrightarrow \mathbb{SH}_n \tag{4.8}$$

for all n, sending

$$P_{a,b} \mapsto P^{(n)}_{a,b}$$

This map restricts to a surjection

$$\mathcal{E}^> \twoheadrightarrow \mathbb{SH}^+_n$$

where $\mathbb{SH}^+_n$ is generated by $P^{(n)}_{a,b}$ with $a > 0$ or $a = 0, b > 0$.

It remains to connect these facts to the Fock space. Recall the following

Definition 4.19. The *polynomial representation* of $\mathbb{H}_n$ is

$$\mathrm{Ind}^{\mathbb{H}_n}_{\mathcal{H}^{aff,Y},n} 1 \cong \mathbb{C}(q,t)[X_1^\pm, \dots, X_n^\pm]$$

The polynomial representation of $\mathbb{SH}_n$ is $\mathbf{e}\,\mathbb{C}(q,t)[X_1^\pm, \dots, X_n^\pm]$.

It is clear from the above that by restricting the action of $\mathbb{SH}_n$ on the polynomial representation to the positive part $\mathbb{SH}^+_n$ we get an action on symmetric polynomials in n variables $\mathbb{C}(q,t)[X_1, \dots, X_n]^{S_n}$.

Theorem 4.20 (Section 5.1, [68])**.** *This action together with the Fock representation of $\mathcal{E}$ intertwine the surjections $\mathrm{Sym}_{q,t} \twoheadrightarrow \mathbb{C}(q,t)[X_1, \dots, X_n]^{S_n}$ and $\mathcal{E}^> \twoheadrightarrow \mathbb{SH}^+_n$.*

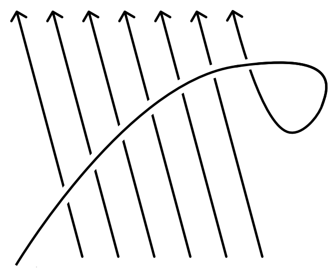


FIGURE 1. The Coxeter braid $\mathbf{cox}_7$.

5. Knot invariants

5.1. **Algebraic knots.** Recall from Definition 1.2 and Remark 1.4 that to any (reduced) germ of a plane curve $\{f = 0\} \subset \mathbb{C}^2$ we may associate both a Puiseux expansion and the link $\mathrm{Link}_0(f) \subset S^3$. To simplify the discussion, we let f be irreducible, although appropriately interpreted all our results hold for any f. These correspond to each other as follows. For a single Newton pair, we have the torus knot $T(p,q)$. It is the braid closure of the q :th power of the Coxeter braid $\mathbf{cox}_p$ (see Fig. 1).

Next, for knots L_1, L_2 in the solid torus, or more precisely elements in the skein algebra of the annulus, we define the *satellite* of L_1 by L_2, denoted $L_1 * L_2$ by thickening L_1 to an annulus and placing the diagram of L_2 inside this annulus. Note that this operation is "acting on the right". Denote T_p^q the annular closure of the diagram of $\mathbf{cox}_p^q$ shown in Fig. 1 (in the blackboard framing). Finally, for a given sequence $(\vec{p}, \vec{q})$ define the iterated torus knot

$$T(\vec{p},\vec{q}) \coloneqq T_{p_d}^{q_d} * (T_{p_{d-1}}^{q_{d-1}} * (\cdots * (T_{p_1}^{q_1})\cdots) \tag{5.1}$$

where we think of these as links in S^3 by filling the core of the thickened annulus.

Remark 5.1. The sequence, or pair of sequences $(\vec{p}, \vec{q})$ is denoted $(\vec{\mathbf{r}}, \vec{\mathbf{s}})$ in [13]. Note that it can be *any* sequence of (coprime) integers, in which generality we obtain *iterated torus knots*. However, the Newton pairs are always positive and eventually have $p_k = 1$.

An alternative way to produce the iterated torus link is by *cabling* (see [23, Appendix A]), for which we need yet another sequence $(\vec{p}, \vec{a})$ where $a_d = q_d, a_i \coloneqq a_{i+1}p_{i+1}p_i + q_i, 1 \le i < d$. For a pair of coprime integers (p,a) we let the (p,a)-cable of a link $L \subset S^3$ be the link $\mathrm{Cab}(p,a)(L)$ formed by thickening L to a small solid torus and placing the torus knot $T(p,a)$ inside it. Note that this operation is "acting on the left". Then it is an instructive exercise to check that

$$T(\vec{p},\vec{q}) = \mathrm{Cab}(p_1,a_1)\cdots(Cab(p_d,a_d)(\bigcirc)\cdots)$$

Remark 5.2. We are again opposite to the conventions in [13, 23, 52]. Note that in [13] the notation $(\vec{a}, \vec{p})$ is used instead. For the satellite construction, we refer to [52, Section 4].

5.2. **Superpolynomials.** The *superpolynomial* for links in S^3 has been proposed as a three-variable polynomial specializing to the HOMFLY-PT polynomial [21]. There are two main definitions for it:

1) as the Poincaré polynomial

$$P_L(a,q,t) := \sum_{i,j,k} q^i t^j a^k \dim \mathrm{HHH}^{i,j,k}(L)$$

of the *triply graded Khovanov-Rozansky* homology (or HOMFLY homology) $\mathrm{HHH}(L)$. This is a homology theory for knots and links in S^3 defined using a braid presentation of L and Soergel bimodules. This is the invariant that has been compared with Hilbert schemes of points on plane curve singularities, as well as their compactified Jacobians, in [64]. For more details, we refer to the survey [32].

2) For iterated torus links, a definition of $P_L(a,q,t)$ was given by Cherednik-Danilenko [13] using double affine Hecke algebras; see also [4, 33] and will be repeated in Definition 5.5.

The first and second definitions are known to agree for torus knots and conjectured to agree in general, but this has not been proved at the time of writing. We will use the second definition in this paper, but this also comes with a caveat. Namely, the polynomial is defined using a cabling presentation as in Eq. 5.1 and the topological invariance is not clear.

More precisely, it is not known whether there exist two distinct presentations of some iterated torus knot (/link) L as iterated cables, so that the resulting polynomials are different (see e.g. [56, p. 6]). In other words, this second version of the superpolynomial is not immediately a topological invariant of L. On the other hand, in this paper we only care about algebraic knots, where any ambiguity in the resulting isotopy type of the link is fixed by setting $q_1 > p_1$ (this is reflected in the choice of a coordinate in the Puiseux expansion). In other words, we may speak of $P_L(a,q,t)$ as an *invariant of the algebraic knot*. See also [14, Theorem 4.3.(ii)].

Next, we will recall the approaches of Cherednik-Danilenko and Gorsky-Negut [13, 33] to superpolynomials of iterated torus knots and how they degenerate at $t = 1$. In fact [33] only work out the torus knot case, while [13] do not use the elliptic Hall algebra, so one should regard what is below as a mixture of the two.

In [33], the approach is as follows. For a sequence of pairs of coprime integers $(p_1,q_1),\dots,(p_d,q_d)$ we have an iterated torus knot $T(\vec{p},\vec{q}) = T^{p_d}_{q_d} * \cdots * T^{p_1}_{q_1}$ as above. By Theorem 4.5, we have also have algebra homomorphisms $\mathrm{Sym}_{q,t} \to \mathcal{E}^{q_i/p_i}, i = 1,\dots,d$ sending $p_k \mapsto P_{kq_i,kp_i}$. By Theorem 4.7 the algebra $\mathcal{E}$ acts on the Fock space $\mathcal{F} \cong \mathrm{Sym}_{q,t}$ by shuffle algebra operations. we denote the action of $E \in \mathcal{E}$ on $f \in \mathrm{Sym}_{q,t}$ by $E \cdot f$.

Definition 5.3. The *full,* or *deformed master symmetric function* associated to $(\vec{p},\vec{q})$ is

$$\widehat{\mathbf{f}}_{\vec{p},\vec{q}} = \widehat{\varphi}_{q_d/p_d}(\cdots(\widehat{\varphi}_{q_2/p_2}(P_{q_1,p_1}\cdot 1)\cdot 1)\cdots)\cdot 1 \tag{5.2}$$

A recursive description is thus as follows. Set $f_{(p_1,q_1)} = P_{q_1,p_1}\cdot 1$, and for $j = 2,\dots d$ define $f_{(p_1,q_1),\dots,(p_j,q_j)}$ as follows. First, expand $f_{(p_1,q_1),\dots,(p_{j-1},q_{j-1})}$ in terms of the power sum symmetric functions p_k and replace all p_k in the resulting expression by the operators P_{q_ik,p_ik}, then act on $1 \in$ the Fock representation. The result is a symmetric function.

Define the *evaluation vector* from [33, Eq. (39)]:

$$\mathfrak{v}(a) = \sum_{\mu\vdash n} \frac{\widetilde{H}_\mu}{g_\mu} \prod_{\square\in\mu} (1 - aq^{a'(\square)}t^{l'(\square)}) \tag{5.3}$$

where

$$g_\mu := \prod_{\square\in\mu} (1 - q^{a(\square)}t^{-l(\square)-1}) \prod_{\square\in\mu} (1 - q^{-a(\square)-1}t^{l(\square)})$$

and $a'(\square), l'(\square)$ denote the coarm and coleg of a box in the Ferrers diagram.

Remark 5.4. The factor g_μ is the product of the weights of the $\mathbb{G}_m^2$-representation $\Lambda^\bullet T_\mu^\vee \operatorname{Hilb}^n(\mathbb{A}^2)$, where the variable a encodes the exterior degree.

Definition 5.5. Let $L = T(\vec{p}, \vec{q})$ be an iterated torus link. The *superpolynomial* of L is defined to be

$$\mathbf{P}_L(a, q, t) := (\widehat{\mathbf{f}}_{\vec{p},\vec{q}}, \mathfrak{v}(a))$$

where $\mathfrak{v}(a)$ is defined in Eq. (5.3). Note that we are using the geometric inner product.

Remark 5.6. We make three remarks on the above definitions.

(1) Note that the full master symmetric function and the superpolynomial depend on three variables q, t, a. At $a = 0$ the evaluation vector simplifies to

$$\mathfrak{v}(0) = \sum_{\mu\vdash n} \frac{\widetilde{H}_\mu}{g_\mu}$$

and it is indeed the quantity

$$(\mathbf{f}_{\vec{p},\vec{q}}, \mathfrak{v}(0)) = \langle \mathbf{f}_{\vec{p},\vec{q}}, e_n \rangle$$

at $t = 1$ that gives the spherical orbital integrals in Section 2.

(2) The homogeneity property of Shalika germs from Proposition 2.11 is reflected in the $\widetilde{H}_\lambda$-expansion of $\widehat{\mathbf{f}}_{\vec{p},\vec{q}}$. Namely, the operator ∇ of Bergeron and Garsia scales each $\widetilde{H}_\lambda$ by $q^{n(\lambda)}t^{n(\lambda^t)}$ where $n(\lambda) = \sum_{i=1}^{\ell(\lambda)}(i-1)\lambda_i$, and in the $t = 1$ limit this will turn into scaling the $\widetilde{h}_\lambda$ in the expansion of $\mathbf{f}_\gamma$ by $q^{n(\lambda)}$, as we will prove in the next section (see Remark 6.31). Note that on the level of the link of the singularity, this scaling $\gamma \mapsto t\gamma$ corresponds to adding a full twist.

(3) Note also that the full master symmetric function could be decorated by a partition or even a sequence of partitions, by replacing the first vacuum state "1" in the Fock space by a modified Macdonald polynomial $\widetilde{H}_\lambda$ (resp. replacing all of the vacuum states by $\widetilde{H}_{\lambda_d}, \dots, \widetilde{H}_{\lambda_1}$). In principle, our methods give formulas for these cases as well at $t = 1, a = 0$.

Let us now discuss how the above connects to the approach in [13]. In *loc. cit.* the following "evaluation homomorphism" or "coinvariant" $\{-\}_{ev} : \mathbb{H}_n \to \mathbb{C}(q,t)$ on the DAHA is defined:

$$\{-\}_{ev} : X_a \mapsto q^{-(\rho,a)},\ Y_b \mapsto q^{-(\rho,b)},\ T_i \mapsto t$$

Recall also from the discussion around Definition 4.19 that the DAHA acts on its polynomial representation

$$\mathbb{C}(q,t)[X_1^\pm, \dots, X_n^\pm]$$

and this restricts to an action of $\mathbb{SH}_n^+$ on symmetric polynomials in n variables. Recall also the $\widehat{\mathrm{SL}_2(\mathbb{Z})}$-action and the elements $P_{a,b}^{(n)} \in \mathbb{SH}_n$. The *DAHA-Jones* polynomial of Cherednik-Danilenko is defined in [13, Eq. (4.26)] (with slightly different notation) as

$$JD_{\vec{p},\vec{q}}^{(n)}(q,t) = \{\gamma_{q_1/p_1}(\cdots(\gamma_{q_{d-1}/p_{d-1}}(P_{q_d,p_d}^{(n)}\cdot 1)\cdot 1)\cdots)\cdot 1)\}_{ev} \tag{5.4}$$

This is related to the superpolynomial P_L from above by

Theorem 5.7 (Cherednik's stabilization conjecture, Section 3.4., [33])**.** *We have*

$$JD_{\vec{p},\vec{q}}^{(n)}(q,t) = P_L(t^n,q,t)$$

Remark 5.8. When $m = 0$, the superpolynomials coincide with the Poincaré polynomials of triply graded Khovanov-Rozansky homologies of L by the explicit computations in [24,39,53]. It is an important open problem to verify this for algebraic links and more general iterated torus links.

We now come to explicit combinatorial formulas for the superpolynomials and the master symmetric functions. We will first recall the full torus knot case as in [33], and then work out the general formula at $a = 0, t = 1$.

The following formula for the full master symmetric function of a torus knot is given by [33, Theorem 1.1]. Let T be a standard Young tableau on n letters. For the box labeled i in its diagram, denote by z_i the q,t-content of the box. Recall also that

$$\omega'(x) = \frac{(x-1)(x-qt)}{(x-q)(x-t)}$$

and

$$S_i' := S_{q/p}(i)' := \lfloor \frac{iq}{p} \rfloor - \lfloor \frac{(i-1)q}{p} \rfloor$$

and define $\nu := \frac{(1-q)(1-t)}{(1-qt)}$. Then we have the following.

Theorem 5.9 (Theorem 1.1. and Eq. (37), [33])**.**

$$\widehat{\mathbf{f}}_{m,n} = \sum_{\lambda\vdash n} \nu^n \frac{\widetilde{H}_\lambda}{g_\lambda} \sum_{\mathrm{SYT}(\lambda)} \frac{\prod_{i=1}^n z_i^{S'_{m/n}(i)}(qtz_i-1)}{\left(1-qt\frac{z_2}{z_1}\right)\cdots\left(1-qt\frac{z_n}{z_{n-1}}\right)} \prod_{i<j} \omega'\left(\frac{z_j}{z_i}\right)^{-1} \tag{5.5}$$

Note that to pass from this description to the case of iterated cables is rather cumbersome. Namely, one should expand each $\widetilde{H}_\lambda$ in the power sum symmetric functions (or perhaps the elementary ones [61]), replace each p_k by operators of the form $P_{kq,kp}$, use the formula [33, Eq. (37)], and proceed, but it is not obvious if this gives rise to any simple combinatorial formula.

However, at $t = 1$ the above formula massively simplifies and we may write down the general result. To do this, let us introduce some notation.

Definition 5.10. The *degenerate master symmetric function* (or just *master symmetric function* in the rest of the text) of an iterated torus knot $T(\vec{p},\vec{q})$, is the $t = 1$ specialization of $\widehat{\mathbf{f}}_{\vec{p},\vec{q}}$:

$$\mathbf{f}_{\vec{p},\vec{q}} := \widehat{\mathbf{f}}_{\vec{p},\vec{q}}|_{t=1}$$

Remark 5.11. This definition stems from a somewhat unfortunate notation clash between knot homology, symmetric functions and point-counting on affine Springer fibers. Since the t-variable is naturally inverted from the point of view of Macdonald theory, it could be more appropriate to define $\mathbf{f}_{\vec{p},\vec{q}} = \widehat{\mathbf{f}}_{\vec{p},\vec{q}}|_{q=1}$ and then replace t by q everywhere. Since the q,t-formulas are transposition-symmetric under switching q,t, this will affect our formulas by a $q \to q^{-1}$ in the Shalika germ expansion as well as the orbital integrals, see Theorem 6.27.

Given Definition 5.10, we can now write down the resulting symmetric function $\mathbf{f}_{\vec{p},\vec{q}}$ given a sequence of pairs of coprime integers $(\vec{p},\vec{q})$.

Definition 5.12. Let $\lambda \vdash n$ and $d,e \geq 1$ with $(d,e) = 1$. Define

$$E_{d,e,\lambda} = \prod_{j=1}^{\ell(\lambda)} E_{d,e,\lambda_j}$$

where $E_{d,e,\lambda_j} = \sum_{D \in \mathbb{D}_{d\lambda_j, e\lambda_j}} q^{\mathrm{area}(D)} e_D$ is as in Proposition 4.12.

With this notation, the slope p/q plethysm from Theorem 4.5 in the limit $t = 1$ can be expressed using the following definition.

Definition 5.13. The *slope q/p-plethysm in the $t = 1$ limit* is the homomorphism

$$\varphi_{q/p} : \mathrm{Sym}_{q,t} \to \mathrm{Sym}_{q,t}$$

defined by letting

$$\varphi_{q/p}(e_k) = E_{q,p,k}$$

Combining the definition of $\mathbf{f}_{\vec{p},\vec{q}}$ as the limit at $t = 1$ of $\widehat{\mathbf{f}}_{\vec{p},\vec{q}}$ with the result of Proposition 4.9, the degenerate master symmetric function $\mathbf{f}_{\vec{p},\vec{q}}$ can be inductively computed by expanding $\mathbf{f}_{(p_{i-1},q_{i-1})}, i = 1,\ldots,d$ in the elementary symmetric functions $e_\lambda = \prod_{j=1}^{\ell(\lambda)} e_{\lambda_j}$ and by replacing each e_{λ_j} by the operator E_{q_i,p_i,λ_j} defined in Proposition 4.12. In other words,

Theorem 5.14. *The degenerate master symmetric function $\mathbf{f}_{\vec{p},\vec{q}}$ can be computed as*

$$\mathbf{f}_{\vec{p},\vec{q}} = \varphi_{q_d/p_d}(\cdots\varphi_{q_1/p_1}(e_1)\cdots)$$

As we will also be interested in the expansion of $\mathbf{f}_{\vec{p},\vec{q}}$ in terms of the $\widetilde{h}_\lambda$, let us now study the limit of the formula in Eq. (5.5) as $t \to 1$.

Definition 5.15. We call the coefficient of $\widetilde{H}_\lambda$ in Eq. (5.5) for a fixed $T \in \mathrm{SYT}(\lambda)$ the (q,t)-*weight* of the SYT T. We will denote it by $\widehat{\mathrm{wt}}_{m/n}(T)$. Note that the weight depends on m/n.

Lemma 5.16. *By comparison to Eq.* (5.5)*, a convenient formula for the weight is given by*

$$\widehat{wt}_{m/n}(T) = \frac{\prod_{i=1}^n z_i^{S_{m/n}(i)-1}}{\prod_{i=2}^n (1 - \frac{1}{z_i})(1 - qt\frac{z_{i-1}}{z_i})} \prod_{i<j} \omega\left(\frac{z_i}{z_j}\right) \tag{5.6}$$

where

$$\omega(x) = \frac{(1-x)(1-qtx)}{(1-qx)(1-tx)}$$

and

$$S_{m/n}(i) = \lceil \frac{im}{n} \rceil - \lceil \frac{(i-1)m}{n} \rceil$$

Proof. First, note that $S_{m/n}(n-i) = S'_{m/n}(i)$ and $\omega'(x^{-1}) = \omega(x)$. In particular, reversing the labeling on the z_i we see

$$\mathrm{wt}_{m/n}(T) = \frac{\nu^n}{g_\lambda} \frac{z_i^{S_{m/n}(i)} \prod_i (qtz_i - 1)}{\prod_{i=2}^n (1 - qt\frac{z_{i-1}}{z_i})} \prod_{i>j} \omega\left(\frac{z_i}{z_j}\right)^{-1}$$

On the other hand, one can check that

$$g_\lambda = z_1 \cdots z_n \frac{\nu^n \prod_{i=1}^n (1 - z_i^{-1})(1 - qtz_i)}{\prod_{i<j} \omega\left(\frac{z_i}{z_j}\right) \omega\left(\frac{z_j}{z_i}\right)}$$

Plugging the latter equation into the former one, we are done. □

Let us now study the limit as $t \to 1$.

Proposition 5.17.

(1) *Let $T \in \mathrm{SYT}(\lambda)$. then the order of vanishing of the weight* $\mathrm{wt}(T)_{m/n}$ *at $t = 1$ equals*

$$v(T) = |\lambda| - \ell(\lambda) - \pi(T) \tag{5.7}$$

where $\pi(T)$ is the number of pairs of consecutive boxes $\square_i, \square_{i+1}$ in T s.t. they lie in consecutive columns. Note that this number is always ≥ 0 and independent of m/n.

(2) *Suppose that $v(T) = 0$, so that the weight does* not *vanish at $t = 1$. Then it is equal to*

$$\frac{\prod_{i=1}^n z_i^{S_{m/n}(i)-1}}{\prod_{i=2}^n (1 - z_i^{-1})(1 - q\frac{z_{i-1}}{z_i})}$$

Here z_i are (q,t)-contents of boxes in T now specialized at *$t = 1$ and as in [33], we simply ignore $\ell(\lambda) - 1 + n(T)$ zero factors in the denominator.*

Proof. From the formula for the weight in Eq. (5.6) independence of the subscript m/n is clear. Looking at the denominator, we have the claimed factor and the factors

$$\omega\left(\frac{z_i}{z_j}\right) = \frac{(1 - \frac{z_i}{z_j})(1 - qt\frac{z_i}{z_j})}{(1 - q\frac{z_i}{z_j})(1 - t\frac{z_i}{z_j})},\ i < j$$

At a first glance it looks like this factor is always just 1 at $t = 1$. However, this only holds if $\frac{z_i}{z_j} \notin \{\frac{1}{q}, \frac{1}{t}, \frac{1}{qt}\}$ (wheel conditions). For example, if $\frac{z_i}{z_j} = 1/q$, then we get

$$\omega\left(\frac{z_i}{z_j}\right) = \frac{(1 - 1/q)(1 - t)}{(1 - t/q)}$$

which gives a zero at $t = 1$ or order 1. Similarly, for $\frac{z_i}{z_j} = 1/t$ we get a zero of order 1 from $\frac{(1-1/t)(1-q)}{(1-q/t)}$, and at $\frac{z_i}{z_j} = 1/qt$ we get a pole of order 1 from $\frac{(1-1/qt)}{(1-1/t)(1-1/q)}$. Since each box which has a box to the right of it contributes a zero, each box with a box above it contributes a zero, and a box with a box diagonally above it contributes a pole, we see that this gives a total number of $|\lambda| - 1$ zeroes.

The factors $(1 - z_i^{-1})$ in the denominator vanish at $t = 1$ iff $z_i = t^k$ for some $k \geq 1$, i.e. $\square_i$ is at the beginning of a row and we ignore the first factor. Therefore, they contribute a pole of order $\ell(\lambda) - 1$.

Finally, the factors $(1-qtz_{i-1}/z_i)$ contribute a pole of order $\pi(T)$ at $t=1$, since this factor vanishes at $t=1$ iff $z_i = qz_{i-1}$. The result follows. □

Remark 5.18. The first author thanks Eugene Gorsky for explanations related to the combinatorics of this Proposition.

Recall from Lemma 3.4 that to each composition $\alpha \vDash n$ there is associated an unique Young tableau.

Lemma 5.19. *The weight* $\mathrm{wt}(T)_{m/n}$ *does not vanish iff* T *comes from a composition.*

Proof. We need to show that only the tableaux coming from compositions have $v(T) := |\lambda| - \ell(\lambda) - \pi(T) = 0$, cf. Eq. (5.7). Note that $v(T) = 0$ iff $\pi(T) = |\lambda| - \ell(\lambda)$. Additionally, the latter is an upper bound (i.e. the condition is satisfied for every box except the ends of the rows, which is obviously the maximum number of boxes), so we are looking to maximize the number of consecutive pairs of boxes in consecutive columns.

For a tableau of shape λ, coming from a composition $\alpha = \alpha_1 + \cdots + \alpha_r$ we have exactly

$$\pi(T) = (\alpha_1 - 1) + \cdots + (\alpha_r - 1) = |\alpha| - \ell(\alpha)$$

by construction. Conversely, if $\pi(T) = |\lambda| - \ell(\lambda)$, the horizontal strip coming from the top boxes in the diagram must have consecutive labels. Stripping it away gives α_r, and we continue inductively to build $\alpha_1 + \cdots + \alpha_r$. □

At $t = 1$ we then finally have

Proposition 5.20.

$$P_{m,n} \cdot 1 = \sum_{\alpha \vDash n} \mathrm{wt}(\alpha)_{m/n} \widetilde{h}_\alpha = \sum_{\alpha \vDash n} \frac{(-1)^{n-\ell(\alpha)} z_1^{S_1} \cdots z_n^{S_n}}{c_{\alpha-1}(q)} \widetilde{h}_\alpha \tag{5.8}$$

where $S_i = \lceil \frac{im}{n} \rceil - \lceil \frac{(i-1)m}{n} \rceil$. *Here* $(m,n) = 1$.

Proof. From the second part of Proposition 5.17 we have that the denominator of $\mathrm{wt}(\alpha)_{m/n}$ is

$$\prod_{i=2}^{n} \frac{1}{(1 - z_i^{-1})(1 - qz_{i-1}/z_i)}$$

with the convention that zero factors are ignored.

Arranging the q,t-contents at $t = 1$ into a vector by reading the tableau box by box, we write

$$z(\alpha) = (1, q, \ldots, q^{\alpha_1 - 1}, 1, q, \ldots, q^{\alpha_2 - 1}, \ldots, 1, \ldots, q^{\alpha_r - 1})$$

and by definition $z(\alpha)_i = z_i$.

It is easy to see that the $1 - qz_i/z_{i+1}$ factors are only nonvanishing at the ends of the parts of α so this becomes

$$\prod_{j=1}^{r} \frac{1}{(1-q^{-1}) \cdots (1 - q^{-\alpha_j - 1})} \prod_{j=1}^{r-1} \frac{1}{1 - q^{\alpha_j}} = \frac{(-1)^{n-\ell(\alpha)} z_1 \cdots z_n}{(1-q)^{n-1} [\alpha_1]! \cdots [\alpha_{r-1}]! [\alpha_r - 1]!} = \frac{(-1)^{n-\ell(\alpha)} z_1 \cdots z_n}{c_{\alpha-1}(q)}$$

where $[m]! = \prod_{i=1}^{m} (1 - q^m)/(1-q)$ and $\alpha - 1$ is the composition where we remove 1 from the last part.

Since the numerator was $z_1^{S_{m/n}(1)-1}\cdots z_n^{S_{m/n}(n)-1}$ we get the result. For further reference, we will denote the coefficient for a fixed α the *weight of* α:

$$\mathrm{wt}(\alpha)_{m/n} = \frac{(-1)^{n-\ell(\alpha)} z_1^{S_{m/n}(1)}\cdots z_n^{S_{m/n}(n)}}{c_{\alpha-1}(q)} \tag{5.9}$$

□

In order to write down the transition matrix of Shalika germs, we will also need the case when m,n are not coprime. This is the $t \to 1$ limit of the formula in Theorem 4.8 at $\mu = \varnothing$.

Proposition 5.21.

$$P_{km,kn} \cdot 1 = \sum_{\alpha \vDash kn} \mathrm{wt}(\alpha)_{m/n} \widetilde{h}_\alpha \tag{5.10}$$

where for $\alpha \vDash kn$

$$\mathrm{wt}(\alpha)_{m/n} = \left(1 + \sum_{j=1}^{k-1} q^j \frac{z_{(k-j)n}\cdots z_{(k-1)n}}{z_{(k-j)n+1}\cdots z_{(k-1)n+1}}\right) \frac{(-1)^{n-\ell(\alpha)} z_1^{S_{m/n}(1)}\cdots z_n^{S_{m/n}(n)}}{c_{\alpha-1}(q)} \tag{5.11}$$

Proof. Comparing to Theorem 4.8 and Proposition 5.20 this is proved exactly in the same way, but we also have the coefficient

$$\sum_{j=0}^{k-1} q^j \frac{z_{n(k-1)+1}\cdots z_{n(k-j)+1}}{z_{n(k-1)}\cdots z_{n(k-j)}}$$

where the first summand is to be just read as 1. □

6. The combinatorial formulas

In this section, we state and prove the inductive combinatorial formula for the Shalika germs and the Steinberg germs, as well as the orbital integrals themselves. Our method on the harmonic analysis side heavily based on results of [79], in particular the combinatorial result Lemme V 12. therein. Currently, it can be regarded as the most technical part of our computations, but we also hope the results in this section give insight to the rather brute-force approach in [79].

Definition 6.1. Let $m \geq n \geq 0$, let $\lambda \vdash m$ have n parts and $\mu \vdash n$ and consider the set $\Upsilon_\lambda^\mu \subset \mathbb{Z}_{\geq 0}^n$ defined by

$$\Upsilon_\lambda^\mu := \left\{ (d_1,\ldots,d_n) \in \mathbb{Z}_{\geq 0}^n \mid (\sum_{i=1}^k d_i)(\sum_{i=k+1}^n \lambda_i) - (\sum_{i=1}^k \lambda_i)(\sum_{i=k+1}^n d_i) \right.$$
$$\left. \begin{cases} = 0, & \text{if there is a } j \text{ so that } k = \sum_{i=1}^j \mu_i \\ > 0, & \text{otherwise} \end{cases}, \ \forall k = 1,\ldots,n-1 \right\}$$

Example 6.2. Let $m = n$, $\mu = (n)$ and $\lambda = (1^n)$. Then

$$\Upsilon_{(1^n)}^{(n)} = \left\{ \vec{d} \mid (n-k)\cdot \sum_{i=1}^k d_i - k \cdot \sum_{i=k+1}^n d_i > 0, \forall k = 1,\ldots,n-1 \right\}$$

Proposition 6.3. *The subset* $D_{d,n} \subseteq \Upsilon_{(1^n)}^{(n)}$ *given by* $\vec{d}$ *with* $\sum_i d_i = d$ *is in bijection with the slope* d/n *rational Dyck paths* $\mathbb{D}_{d,n}^<$ strictly under the diagonal.

Proof. Giving a Dyck path is the same as giving the sequence of its horizontal steps. If we are looking at d/n-Dyck paths, these steps have to sum to d and there are at most n steps, which gives us a sequence $(d_1, \dots, d_n)$. Since such a Dyck path lies above the line with slope d/n, we must have

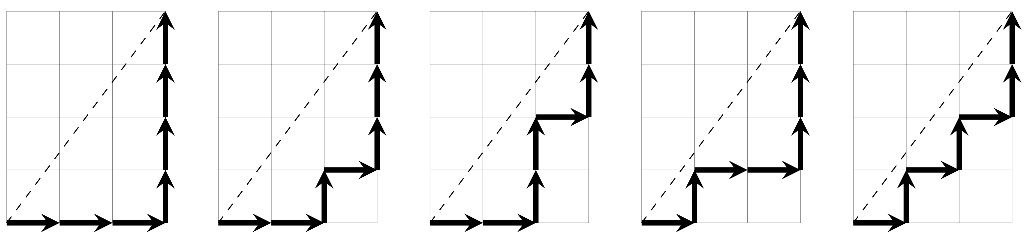
$$(n-k)(\sum_{i=1}^{k} d_i) - k(d - \sum_{i=1}^{k} d_i) = n \sum_{i=1}^{k} d_i - kd > 0$$

for all k (if we allowed all Dyck paths, for noncoprime d, n equality could also hold). The converse is clear. □

Example 6.4. Let $n = 4$, $d = 3$. The allowed sequences are

$$(3,0,0,0), (2,1,0,0), (2,0,1,0), (1,2,0,0), (1,1,1,0)$$

and these correspond to the Dyck paths



For any $P = \sum_{\vec{d}} a_{\vec{d}} X^{\vec{d}} \in \mathbb{Z}[[X_1, \dots, X_n]]$ we define the truncation

$$\Upsilon_{\lambda}^{\mu}(P) \coloneqq \sum_{\vec{d} \in \Upsilon_{\lambda}^{\mu}} a_{\vec{d}} X^{\vec{d}} \tag{6.1}$$

Remark 6.5. Our Υ_{λ}^{μ} is equal to the "polynomial" component of the set $\Gamma_{\lambda,\mathbb{Z}}^{\mu}$ defined in [79, I 10]. It appears that this polynomial component constitutes the primary (if not the exclusive) usage of $\Gamma_{\lambda,\mathbb{Z}}^{\mu}$ in [79]. We have avoided this notation in order not to get it confused with the Shalika germs $\Gamma_{\lambda}(-)$. Note also that on pages 856 and 878 of *loc. cit.* the confusing notation $\Gamma_{\lambda}^{\mu_P}$ is used, but this seems to be a misprinted $\Gamma_{\lambda}^{\mu} P$.

In general, one should think of the elements of $\Gamma_{\lambda,\mathbb{Z}}^{\mu}$ and Υ_{λ}^{μ} Lie-theoretically as follows. $\mathbb{Z}^n$ is the weight lattice of GL_n, and each collection of n integers $\lambda_1, \dots, \lambda_n$ gives a linear form on $\mathbb{Z}^n$ as defined above. On the other hand, μ gives a parabolic subgroup of GL_n, and the (in)equalities above decide that this linear form should (not) vanish on the relative root subspaces of the corresponding Levi subgroup. This cuts out a cone in the apartment of T. Fixing the coordinatewise sum is intersecting this cone with an affine hyperplane. Further restricting to the nonnegative part is intersecting with yet another cone, i.e. the positive orthant. We remark that the definition of Υ_{λ}^{μ} is related to the definition of "Hecke-regular functions" in [3, Section 4].

6.1. **Comparison to Waldspurger's recursion.** The goal of this subsection is to recall the recursive computation of the Steinberg germs from [79] and in the inertially elliptic case to compare it to the recursive definition of $\mathbf{f}_{\vec{p},\vec{q}}$ from Section

5. We also extend the construction to apply to general tamely ramified elements, even when there is no knot in the picture.

To start with, if we suppose $\gamma \in GL_n(F)$ is elliptic, tamely ramified and compact, the recursion of [79] for determining the Steinberg germs $\Gamma_\lambda^{St}(\gamma)$ proceeds in two steps, as explained in VII 7. of *loc. cit.* We now recall this process in our notation. As our eventual goal is to compute the Steinberg and Shalika germs of regular semisimple tamely ramified $\gamma \in \mathfrak{gl}_n$ or $\gamma \in GL_n$ by a combination of homogeneity and the computation for compact elements in the group done below, we set up some definitions. Their utility will immediately become clear.

Definition 6.6. We now define the *master symmetric function of* γ, $\mathbf{f}_\gamma \in \mathrm{Sym}$ for

$$\gamma \in GL_n(\mathcal{O}), GL_n(F)_c, \mathfrak{gl}_n(F)$$

respectively.

(1) Let $\gamma \in GL_n(\mathcal{O})$ be regular semisimple. By Corollary 2.7 we have the group Shalika germs $\Gamma_\lambda(\gamma)$ for γ and the Steinberg germs $\Gamma_\lambda^{St}(\gamma)$. Let the master symmetric function of γ be

$$\mathbf{f}_\gamma := \sum_\lambda \Gamma_{\lambda^t}(\gamma)\widetilde{h}_\lambda$$

or equivalently by Propositions 3.13, 3.14, $\mathbf{f}_\gamma = \sum_\lambda \Gamma_\lambda^{St}(\gamma)e_\lambda$. Note the transposition in the Shalika expansion, forced upon us by Proposition 3.13.

(2) For general compact $\gamma \in GL_n(F)_c$, not necessarily with $\mathrm{val}(\det(\gamma)) \neq 0$, we define

$$\mathbf{f}_\gamma := \sum_\lambda \Gamma_\lambda^{St}(\gamma)e_\lambda$$

(3) Finally, if $\gamma \in \mathfrak{gl}_n(F)$, define

$$\mathbf{f}_\gamma := \sum_\lambda \Gamma_{\lambda^t}(\gamma)\widetilde{h}_\lambda$$

where $\Gamma_\lambda(\gamma)$ are the Lie algebra Shalika germs from Theorem 2.8.

Remark 6.7. These symmetric functions are elements of Sym, the ring of symmetric functions over $\mathbb{C}$. As follows from Theorem 6.27, they only depend on certain discrete invariants attached to γ in the tamely ramified case. Moreover, the coefficients in any of the above expansions are rational functions of q, the size of the residue field of F, which only have poles at roots of unity. It therefore does no harm to consider $\mathbf{f}_\gamma$ as an element of Sym_q, the ring of symmetric functions over $\mathbb{C}(q)$.

Remark 6.8. If γ is topologically unipotent, i.e. $\gamma = 1+\gamma'$ where γ' is topologically nilpotent, then it follows from Proposition 2.9 that the Lie algebra version of $\mathbf{f}_\gamma$ coincides with the group version $\mathbf{f}_\gamma$, relating definitions (1) and (3). In this case we also have $\mathbf{f}_\gamma = \mathbf{f}_{\gamma'}$ on the Lie algebra. Below, we tacitly avoid keeping track of which $\mathbf{f}_\gamma$ we work with, as it should be clear from the context. Note that in case (2) there is no obvious analog of the Shalika germs. Of course, one may define these via a change of basis a posteriori, but the harmonic analysis meaning is unclear.

Definition 6.9. Let γ be as in any of the cases (1)–(3) above. The coefficients of the expansion of $\mathbf{f}_\gamma$ in the complete homogeneous symmetric functions are called the *Dyck germs* of γ. In the group case, one can think of these as the expansions of the corresponding orbital integrals as linear combinations of Fourier transforms of nilpotent orbital integrals truncated to compact elements, although we do not use this fact.

Let us recall the setup of [79, Séction VI–VII], in slightly simplified form (the simplification being that for us $E = F$ and $r = 1$ in the notation of *loc. cit.*). Now F is a nonarchimedean local field (of arbitrary characteristic, see Remark 2.5), $\gamma \in GL_n(F)$ is elliptic and tamely ramified so that $F(\gamma)/F$ is a tamely ramified extension of F. Here for any semisimple $\gamma \in GL_n(F)$, we can realize $F(\gamma)$ as the commutative subalgebra of $\mathfrak{gl}_n(F)$ given as the center of the centralizer of γ in $\mathfrak{gl}_n(F)$, and $F(\gamma)^\times$ is the corresponding torus in $GL_n(F)$.

Definition 6.10. Let F'/F be a tamely ramified extension with ramification index $e = e_{F'/F}$. We call $\delta \in (F')^\times$ cuspidal for F'/F if $(\mathrm{val}_{F'}(\delta), e) = 1$ and if the reduction of $t^{-\mathrm{val}_{F'}(\delta)}\delta^e$ in the residue field of $\mathcal{O}_{F'}$ generates the residue field over that of $\mathcal{O}_F$ for any uniformizer $t \in F$.

We will also need the following [79, VII 4.]

Lemma 6.11. *Let E/F be a non-trivial tamely ramified extension and $\gamma \in 1 + \mathfrak{m}_E$ be such that $F(\gamma) = E$. Then we can always write $\gamma = \eta(1 + \delta\gamma')$ with $\eta \in 1 + \mathfrak{m}_F$, $\gamma' \in 1 + \mathfrak{m}_E$ and $\delta \in E$, such that δ is cuspidal for $F' := F(\delta)/F$ with F'/F a non-trivial extension.*

Note that in both lemmas, while γ', δ are not uniquely determined by γ, the integers $e = e_{F'/F}$, $f = [\mathcal{O}_{F'}/\mathfrak{m}_{F'} : \mathcal{O}_F/\mathfrak{m}_F]$, $d = \mathrm{val}_{F'}(\delta)$ and the extension F' are all uniquely determined by γ. In fact, more numerical invariants are preserved. Let us define

Definition 6.12. Let E/F be a tamely ramified extension of degree n and $\gamma \in E$ be such that $E = F(\gamma)$. Take F^s any separable closure of F and let $\gamma_1, ..., \gamma_n$ be all conjugates of γ in F^s. We define $RV_F(\gamma)$ to be the *multiset* $RV_F(\gamma) := \{\mathrm{val}_F(\gamma_i - \gamma_j) \mid 1 \leq i < j \leq n\}$. We also define

$$\Xi_F(\gamma) = \left(\sum_{r \in RV_F(\gamma)} r\right) - (n - [E^{ur} : F])/2$$

where E^{ur} is the maximal unramified subextension of E/F.

Remark 6.13. The underlying set of our multiset $RV_F(\gamma)$ already appeared as the set $RV(\gamma)$ in Definition 1.2. In particular, $RV_F(\gamma)$ determines the *Puiseux pairs* and *Newton pairs* of γ as in Definition 1.2.

When $F = k((t))$ is a function field of large enough characteristic and $\gamma \in \mathfrak{gl}_n(\mathcal{O}_F)$, we note that $\Xi(\gamma)$ equals the dimension of the affine Springer fiber Sp_γ. Presumably this holds for all characteristics and even for the mixed-characteristic version, but the authors are not aware of a proof in the literature.

Given a multiset S of rational numbers and $\alpha \in \mathbb{Q}$, we denote by αS and $S + \alpha$ the multisets of the same cardinality as S, given by multiplying each element by α, resp. adding α to each element.

Lemma 6.14. *Let γ, δ and γ' be as in Lemma 6.11. Let e and f be the ramification index and residual degree of F'/F in the corresponding notations. Write $n' := n/ef$.*

(1) *We have $RV_F(\gamma) = (\frac{1}{e}RV_{F'}(\gamma')^{ef} \sqcup \{0\}^{n(n-n')/2}) + \mathrm{val}_F(\delta)$.*

(2) *We have* $\Xi_F(\gamma) = f \cdot \Xi_{F'}(\gamma') + \frac{n(n-1)}{2}\mathrm{val}_F(\delta) - \frac{n-n/e}{2}$.
(3) *When* $E = F(\gamma) \supsetneq F$ *and* E/F *is totally ramified, the sequence of Newton pairs of* γ' *is given by deleting the last Newton pair* (p_k, q_k) *of* γ*, where* $q_k/p_k = \mathrm{val}_F(\delta)$*, or more precisely* $p_k = e$ *and* $q_k = e \cdot \mathrm{val}_F(\delta)$.

To carry out the recursive computation of $\mathbf{f}_\gamma$, we first assume our elliptic, tamely ramified group element γ is also of the form $\gamma \equiv 1 \mod \mathfrak{m}_{F(\gamma)}$, following Lemma 6.11. We also define $\gamma'' = \delta\gamma'$, similar to [79, VII 4.].

Example 6.15. In the totally ramified case, assuming $\gamma = 1 + a_d u^{nr_k} + \cdots + a_1 u^{nr_1}$ with $r_k < \cdots < r_1$ (compare to Eq. (1.2)), we can write $\gamma = \eta(1 + \delta\gamma')$ with η, δ, γ' satisfying the above conditions, in the following way: If $r_d \notin \mathbb{Z}$, then we simply take $\eta = 1$, $\delta = a_d u^{nr_k}$, and $\gamma' = 1 + \frac{a_{k-1}}{a_k} u^{n(r_{k-1}-r_k)} + \cdots$ In general, η takes care of possible central translations of γ, which don't affect the computation by Theorem 2.8. For example, if $\gamma = 1 + u^4 + u^6 + u^7$ in GL_4, we can take $\eta = 1 + u^4$, in which case

$$\gamma\eta^{-1} = 1 + u^6 + u^7 - u^{10} - u^{11} + \cdots = 1 + u^6(1 + u - u^4 - u^5 + \cdots)$$

so that we can take $\delta = u^6$ and $\gamma' = 1 + u - u^4 - u^5 + \cdots$

The first step of the recursion is essentially [79, Prop. VII 5.], recalled in Proposition 6.17 below. To state it, we need some notation.

Definition 6.16. Let $X(T) = \sum_{i\geq 0} h_i T^i \in \mathrm{Sym}[[T]]$. Fix $\lambda' \vdash n'$ and number the boxes in its diagram $1, \ldots, n'$, going from left to right and bottom to top. Let

$$P = \prod_{k=1}^{n'} X(X_k)$$

and let $x_n(d\lambda', q) \in \mathrm{Sym}[q^{\pm 1}]$ be the coefficient of T^n in the series

$$\Upsilon^{\lambda'}_{(1^{n'})}(P)$$

evaluated at $X_\square = q^{a'(\square) - (\lambda'_{l'(\square)+1} - 1)/2} T$, where $\Upsilon^{\lambda'}_{(1^{n'})}$ is as defined in Eq. (6.1), $a'(\square), l'(\square)$ are the coarm and coleg lengths as in Eq. (3.1) and $\square$ runs over the boxes in λ'. See also [79, Section I 10., V 12.].

Proposition 6.17. *Write* $\gamma = 1 + \delta\gamma'$ *and* $\gamma'' := \delta\gamma'$ *as in Lemma 6.11. Let* $n' = n/e$*, where e is as after Lemma 6.11. Evaluating at* $q = |k|$ *we have the following equality of symmetric functions in* Sym*:*

$$\sum_{\lambda \vdash n} \Gamma^{St}_\lambda(\gamma) e_\lambda = (-1)^{n-n'} q^{(n'nd-n)/2} \sum_{\lambda' \vdash n'} (-1)^{n'-\ell(\lambda')} \Gamma^{St}_{\lambda'}(\gamma'') x_n(d\lambda', q) \tag{6.2}$$

where $x_n(d\lambda', q)$ *is as defined in Definition 6.16 and the Steinberg germs are as in Theorem 2.19.*

We refer the reader to [79, Lemme VII 5., Lemme V 12.] for details. We warn the interested reader that there are some printing errors in the latter lemma, e.g. the second displayed equation on p. 880 should have a subscripted X_2, a sentence after it there seems to be an extra "A" in front of λ, and on line 7 of p. 881 another subscript seems to have gone astray. Also in the statement the "q" should not be a subscript of $t(\lambda)$ but on the same line as (-1). Also in the former Lemma, the exponent of q should have a $-n/2$ instead of $-n/e$.

Proposition 6.17 does not quite yet cover the induction for general tamely ramified compact $\gamma \in GL_n(F)$ as outlined in [79, Section VII 7.], as the right hand side of (6.2) has γ'' which is not congruent to 1 for an essential reason; $\mathrm{val}_F(\gamma'')$ is in general not an integer.

Write $\gamma'' := \delta\gamma'$. and let d, f be as in Lemma 6.14 so that $\mathrm{val}_F(\det_G(\delta)) = nd/e$. By Theorem 2.19 we have the Steinberg germs $\Gamma^{St}_\lambda(\gamma'')$ for each $\lambda \vdash n/e$. Note that even though $\gamma'' \in GL_n(F)$, these are partitions of n/e. In contrast, we have germs $\Gamma^{St}_{\lambda'}(\gamma')$ for $\lambda' \vdash n/ef$ where γ' is now thought of as an element of $G' := Z_{GL_n}(\delta) \cong GL_{n/ef}(F(\delta))$. The associated master symmetric functions are

$$\mathbf{f}_{\gamma''} = \sum_{\lambda \vdash n/e} \Gamma^{St}_\lambda(\gamma'') e_\lambda \text{ for } \gamma'' \in GL_n(F)$$

and similarly

$$\mathbf{f}_{\gamma'} = \sum_{\lambda' \vdash n/ef} \Gamma^{St}_{\lambda'}(\gamma') e_{\lambda'} \text{ for } \gamma' \in GL_{n/ef}(F')$$

We have

Proposition 6.18. *Let γ'' and γ' be as above. The master symmetric functions of γ'' and γ' are related by*

$$\mathbf{f}_{\gamma''} = (-1)^{n-n/e} \tau_f(\mathbf{f}_{\gamma'})$$

where τ_f is the plethysm/Adams operation defined on symmetric functions by $\tau_f : p_r \mapsto p_{rf}$ for all $r \in \mathbb{Z}_{\geq 0}$. Note that when $f = 1$ this allows us to compare the Steinberg germs of γ'' and γ' directly up to a sign.

Proof. This is a direct translation of the $r = 1$ case of [79, Proposition VII 2.], where the sign is $(-1)^{nd-nd/e}$. But $nd - nd/e \equiv n - n/e \bmod 2$ whenever $\gcd(d, e) = 1$. Note that our γ'' was denoted by γ *loc. cit.*. □

Let us now compare the construction of $\mathbf{f}_{\vec{p},\vec{q}}$ from Section 5 to $\mathbf{f}_\gamma$ introduced above, and extend the construction to apply to general tamely ramified extensions of local fields, even when there is no knot in the picture. We start with a Lemma.

Lemma 6.19. *For any $d \in \mathbb{Z}_{\geq 0}$, $n, e \in \mathbb{Z}_{\geq 1}$, with $e|n$ and $\lambda' \vdash n/e$, we have*

$$x_n(d\lambda', q) = q^{-(ed\sum_{i=1}^{\ell(\lambda')}(\lambda'_i)^2 - en')/2} \prod_{i=1}^{\ell(\lambda')} \left(\sum_{\pi \in \mathbb{D}^<_{e\lambda'_i, d\lambda'_i}} q^{\mathrm{coarea}(\pi)} h_\pi \right)$$

where we only sum over Dyck paths strictly under the diagonal. Here $n' = n/e$ as before. (Note that the n' in [79] is n/ef.)

Proof. Since

$$x_n(d\lambda', q) = \prod_{i=1}^{\ell(\lambda')} x_{e\lambda'_i}(d\lambda'_i, q)$$

(see e.g. the second displayed equation of p. 883 in [79]), we may restrict to a single factor in the product.

Proposition 6.3 tells us that the terms in $\Upsilon^{d\lambda'_i}_{1^{n'}}(P)$ which contribute to the coefficient $x_{e\lambda'_i}(d\lambda'_i, q)$ of $T^{e\lambda'_i}$ of the evaluation of this series at $T_\Box = q^{a'(\Box)-(d\lambda'_i-1)/2}T$ are in bijection with $(d\lambda'_i, e\lambda'_i)$-Dyck paths strictly under the diagonal.

Collecting these terms, we evaluate $\Upsilon^{\lambda'}_{1^{n'}}(P)$ from Definition 6.16 at

$$T_\Box = q^{a'(\Box)-(\lambda_{l'(\Box)+1}-1)/2} = q^{a'(\Box)-(d\lambda'_i-1)/2}T$$

since $l'(\square)+1=1$ for all boxes and $\lambda = d\lambda'_i$. (Note that in the definition of P there is a $+1/2$ instead of a $-1/2$ as in [79, V 12], because our $a'(\square)$ equals $i-1$ from *loc. cit.*.) We can thus factor out $q^{-(d\lambda'_i-1)/2}$. Since we are interested in the coefficient at $e\lambda'_i$, we get an overall factor of $q^{-e\lambda'_i(d\lambda'_i-1)/2}$. Taking the product over i gives $q^{-(ed\sum_{i=1}^{\ell(\lambda')}(\lambda'_i)^2-en')/2}$ as desired.

It is not hard to see that the coarea of the Dyck path corresponding to a term $h_\pi T_\pi$ contributing to $x_{e\lambda'_i}(d\lambda'_i,q)$ where $T_\pi = T_1^{h(\pi)_1}\cdots T_{d\lambda'_i}^{h(\pi)_{d\lambda'_i}}$ is exactly $\mathrm{coarea}(\pi) = \sum_k (k-1)h(\pi)_k$. Here $h(\pi)_k$ denote the horizontal steps of π and h_π is the homogeneous symmetric function associated to the partition of $e\lambda'_i$ given by the horizontal steps of π.

□

Recall the operators $E_{d,e,\lambda'}$ from Definition 5.12. Similarly, we define

$$H_{d,e,\lambda'} = \prod_i H_{d,e,\lambda'_i} = \prod_i \left(\sum_{\pi\in\mathbb{D}_{e\lambda'_i,d\lambda'_i}} q^{\mathrm{coarea}(\pi)} h_\pi \right) \tag{6.3}$$

Motivated by Lemma 6.19, we also define

$$H^<_{d,e,\lambda'} := \prod_{i=1}^{\ell(\lambda')} \left(\sum_{\pi\in\mathbb{D}^<_{e\lambda'_i,d\lambda'_i}} q^{\mathrm{coarea}(\pi)} h_\pi \right)$$

We want to relate these functions to the slope d/e-plethysms $\varphi_{d/e}$ on symmetric functions, as defined in Section 5. Note that these operators involve the area statistic on Dyck paths, rather than the coarea.

Lemma 6.20. *We have*

$$q^{n'nd/2-n/2} x_n(d\lambda',q) = q^{\frac{(dn'-1)(en'-1)+n'-1}{2}} \prod_{i=1}^{\ell(\lambda'_i)} \left(\sum_{\pi\in\mathbb{D}^<_{e\lambda'_i,d\lambda'_i}} q^{-\mathrm{area}(\pi)} h_\pi \right)$$

Proof. We have

$$H^<_{d,e,\lambda'} = q^{\sum \delta_{\lambda'_i}} \prod_i \left(\sum_{\pi\in\mathbb{D}^<_{e\lambda'_i,d\lambda'_i}} q^{-\mathrm{area}(\pi)} h_\pi \right)$$

where $\delta_{\lambda'_i} = \frac{(d\lambda'_i-1)(e\lambda'_i-1)+\lambda'_i-1}{2}$. Then

$$\sum_i \delta_{\lambda'_i} = (ed\sum_i (\lambda'_i)^2 + \sum_i (-d-e+1)\lambda'_i)/2$$

and therefore by Lemma 6.19

$$x_n(d\lambda',q) = q^{-(ed\sum_{i=1}^{\ell(\lambda')}(\lambda'_i)^2-en')/2} H^<_{d,e,\lambda'} = q^{(1-d)n'/2} \prod_i \left(\sum_{\pi\in\mathbb{D}^<_{e\lambda'_i,d\lambda'_i}} q^{-\mathrm{area}(\pi)} h_\pi \right)$$

Further multiplying this by $q^{n(n'd-1)/2}$ we get $q^{\frac{nn'd-n-dn'+n'}{2}} = q^{\frac{(n-1)(dn'-1)+n'-1}{2}}$ in front.

□

For convenience, let us denote

$$H^{-}_{d,e,\lambda'} = \prod_i \left(\sum_{\pi \in \mathbb{D}_{e\lambda'_i, d\lambda'_i}} q^{-\operatorname{area}(\pi)} h_\pi \right)$$

and

$$H^{-,<}_{d,e,\lambda'} = \prod_i \left(\sum_{\pi \in \mathbb{D}^{<}_{e\lambda'_i, d\lambda'_i}} q^{-\operatorname{area}(\pi)} h_\pi \right)$$

Consider now the operator on symmetric functions which takes each $h_{\lambda'}$ and replaces it by $H^{-}_{d,e,\lambda'}$. We can think of this as the conjugation of $\varphi_{d/e}$ by the involution ω, together with negating all the powers of q that appear in the definition.

More precisely, take an elliptic tamely ramified $\gamma \in G(\mathcal{O})$ as above and let γ', γ'' be as in Proposition 6.17. By the process just described, we compute

$$q^{\frac{(dn'-1)(en'-1)+n'-1}{2}} \omega \varphi_{d/e}|_{q \mapsto q^{-1}} \omega^{-1}(\mathbf{f}_{\gamma''}) = q^{\frac{(dn'-1)(en'-1)+n'-1}{2}} \sum_{\lambda' \vdash n/e} \sigma_{\lambda'}(\gamma'') H^{-}_{d,e,\lambda'} \tag{6.4}$$

Where $\sigma_{\lambda'}$ are the "Dyck germs" of Definition 6.9 that appear in the $h_{\lambda'}$-expansion of $\mathbf{f}_{\gamma''}$ and the relevance of the q-power will become clear soon. Our main technical result is

Theorem 6.21. *The right-hand sides of Eqs.* (6.2) *and* (6.4) *are equal up to a sign. More precisely,*

$$q^{n'nd/2-n/2}(-1)^{n-n'} \sum_{\lambda' \vdash n'} (-1)^{n'-\ell(\lambda')} \Gamma^{St}_{\lambda'}(\gamma'') x_n(d\lambda', q) = (-1)^{n-n'} q^{\delta_{n'}} \sum_{\lambda' \vdash n'} \sigma_{\lambda'}(\gamma'') H^{-}_{e,d,\lambda'}$$

where $\delta_{n'} := \frac{(dn'-1)(en'-1)+n'-1}{2}$ *In particular, the left-hand sides are also equal up to the same sign.*

Proof. Dividing out the sign and using Lemma 6.20, the LHS reads

$$q^{\delta_{n'}} \sum_{\lambda' \vdash n'} (-1)^{\frac{n}{e}-\ell(\lambda')} \Gamma^{St}_{\lambda'}(\gamma'') \prod_{i=1}^{\ell(\lambda')} \left(\sum_{\pi \in \mathbb{D}^{<}_{e\lambda'_i, d\lambda'_i}} q^{-\operatorname{area}(\pi)} h_\pi \right)$$

and we can divide out $q^{\delta_{n'}}$ on both sides.

On the other hand, by definition we have

$$H^{-}_{e,d,\lambda'_i} = \sum_{\alpha \vDash \lambda'_i} \sum_{\substack{\pi \in \mathbb{D}_{e\lambda'_i, d\lambda'_i} \\ \operatorname{touch} \pi = \alpha}} q^{-\operatorname{area}(\pi)} h_\pi$$

where $\operatorname{touch}(\pi) = \alpha$ specifies that π touches the diagonal at α.

Given two arbitrary compositions, denote by $\alpha + \beta$ their concatenation. By sorting, this gives a partition of $|\alpha| + |\beta|$ of length $\ell(\alpha) + \ell(\beta)$. Given a collection $\vec{\alpha}$ of compositions

$$\alpha^{(1)} \vDash \lambda'_1, \ldots, \alpha^{\ell(\lambda')} \vDash \lambda'_{\ell(\lambda')}$$

refining the parts of λ', we write $\vec{\alpha} \leftrightarrow \lambda'$.

We now further expand the LHS and collect terms as follows. Note that from the equation $\sum_{k=1}^{n} (-1)^k h_{n-k} e_k = 0$ it follows that

$$e_n = \sum_{\alpha \vDash n} (-1)^{\ell(\alpha)} h_\alpha$$

where we sum over all compositions of n. Writing $\lambda' = (\lambda'_1, \dots, \lambda'_{\ell(\lambda')})$ we then have

$$e_{\lambda'} = \prod_{i=1}^{\ell(\lambda')} \Big(\sum_{\alpha \vDash \lambda'_i} (-1)^{\ell(\alpha)} h_\alpha \Big) \tag{6.5}$$

By definition of the Dyck germs of $\mathbf{f}_{\gamma''}$,

$$\sum_{\lambda' \vdash n/e} \Gamma^{St}_{\lambda'}(\gamma'') e_{\lambda'} = \sum_{\lambda' \vdash n/e} \sigma_{\lambda'}(\gamma'') h_{\lambda'}$$

Fix $\mu' \vdash n/e$. Plugging Eq. (6.5) in the expression

$$\sum_{\lambda' \vdash n/e} \Gamma^{St}_{\lambda'}(\gamma'') e_{\lambda'}$$

and collecting all collections of compositions whose sum has associated partition μ' we see that

$$\sigma_{\mu'}(\gamma'') = \sum_{\substack{\vec{\alpha} \leftrightarrow \lambda' \vdash n/e \\ \mathrm{sort}(\alpha^{(1)} + \dots + \alpha^{(k)}) = \mu'}} (-1)^{\frac{n}{e} - \ell(\lambda')} \Gamma^{St}_{\lambda'}(\gamma'')$$

where the sum runs over all $\lambda' \vdash n/e$ and all collections of compositions $\vec{\alpha}$ refining the parts of λ', such that the partition given by adding the compositions and sorting is exactly μ'. It now remains to replace $H^-_{d,e,\lambda'}$ by a similar expansion. By definition we have

$$H^-_{d,e,\lambda'} = \prod_{i=1}^{\ell(\lambda')} \Big(\sum_{\alpha \vDash \lambda'_i} \sum_{\substack{\pi \in \mathbb{D}_{e\lambda'_i, d\lambda'_i} \\ \mathrm{touch}\, \pi = \alpha}} q^{-\mathrm{area}(\pi)} h_\pi \Big)$$

Picking one of the summands over α, we notice that

$$\sum_{\substack{\pi \in \mathbb{D}_{e\lambda'_i, d\lambda'_i} \\ \mathrm{touch}\, \pi = \alpha}} q^{-\mathrm{area}(\pi)} h_\pi = \prod_{i=1}^{\ell(\alpha)} H^{-,<}_{d,e,\alpha_i}$$

where we use the fact that the area statistic is additive on concatenation of Dyck paths (but note the coarea is not). Here $H^{-,<}_{d,e,\alpha_i}$ is defined as above. Again collecting all $\vec{\alpha} \leftrightarrow \mu'$ we see that the terms contributing to $H^-_{d,e,\mu'}$ on the right are of the form

$$(-1)^{\frac{n}{e} - \ell(\lambda')} \Gamma^{St}_{\lambda'}(\gamma'') \prod_{k=1}^{\ell(\lambda')} \prod_{i=1}^{\ell(\alpha^{(k)})} H^{-,<}_{d,e,\alpha_i^{(k)}}$$

where $\vec{\alpha}$ refines parts of λ' and sums and sorts to μ'. Summing over all such collections we get the desired result. $\square$

We have the following corollary, which is a form of Theorem 1.6 from the introduction.

Corollary 6.22. *Let $\gamma \in GL_n(\mathcal{O})$ be inertially elliptic and tamely ramified, with Newton pairs $(\vec{p}, \vec{q})$. Consider the master symmetric function $\mathbf{f}_\gamma$ for γ as defined in Definition 6.6 and the (degenerate) master symmetric function $\mathbf{f}_{\vec{p},\vec{q}}$ introduced in Definition 5.10. Then $\mathbf{f}_\gamma = q^{\Xi(\gamma)} \omega \mathbf{f}_{\vec{p},\vec{q}}|_{q \mapsto q^{-1}}$. Here $\Xi(\gamma)$ is as defined in Definition 6.12.*

Proof. By assumption γ lies in the ring of integers of the totally and tamely ramified extension $F(\gamma)/F$. Dividing γ by an element in $\mathcal{O}_F^\times$, we may assume γ is topologically unipotent, i.e. $\gamma \equiv 1 (\operatorname{mod} \mathfrak{m}_{F(\gamma)})$. The resulting γ is of the type Lemma 6.11 and Proposition 6.17 apply to.

Write $\gamma = \eta(1+\delta\gamma')$ as in Lemma 6.11. By Theorem 6.21 combined with the $f = 1$ case of Proposition 6.18 we see that the master symmetric functions of $\gamma \in G(F)$ and $\gamma' \in Z_{G(F)}(\delta)$ are related by the equation

$$\mathbf{f}_\gamma = q^{\frac{(dn'-1)(en'-1)+n'-1}{2}}(-1)^{n-n'}\omega\varphi_{d/e}|_{q\mapsto q^{-1}}\omega^{-1}((-1)^{n-n'}\tau_1(\mathbf{f}_{\gamma'}))$$

where d, e are as defined after Lemma 6.11.

First, we note the overall sign cancels. Second, to see that the power of q adds up to $\Xi(\gamma)$, note that the exponent for each step is $\frac{1}{2}((dn'-1)(n-1)+n'-1) = \frac{1}{2}(dn'(n-1)+n'-n)$. On the other hand, as both γ and γ' are inertially elliptic, by Lemma 6.14(2), $\Xi(\gamma)-\Xi(\gamma')$ is equal to $\frac{1}{2}(\frac{d}{e}n(n-1)-(n-n')) = \frac{1}{2}(dn'(n-1)+n'-n)$. By Lemma 6.14(3), this proves the corollary inductively. □

We note that in Theorem 1.6 we worked with topologically nilpotent elements, which can be taken to be just $\gamma - 1$ where γ is in Corollary 6.22. The Newton pairs associated to γ (see Definition 1.2) is obviously equal to the Newton pairs associated to $\gamma - 1$.

Remark 6.23. Lemma 6.11 could have worked without η but with the price that F'/F might be a trivial extension. In fact, one can run the recursion without choosing an η at each step and this still gives us a sequence of pairs of integers $(\vec{p}, \vec{q})$ and hence an associated master symmetric function $\mathbf{f}_{\vec{p},\vec{q}}$. See e.g. Example 6.24 below for an example. That the result is independent of which way we proceed is clear from that $\Gamma_\lambda(\gamma) = \Gamma_\lambda(\gamma/\eta)$ for $\eta \in \mathcal{O}_F^\times$. One may ask whether this is clear from the "combinatorial" setting of Section 5 where one starts simply with a sequence $(\vec{p}, \vec{q})$. Indeed, the relevant symmetry can be deduced for the full master symmetric functions $\widehat{\mathbf{f}}_{\vec{p},\vec{q}}$ and hence for the $\mathbf{f}_{\vec{p},\vec{q}}$ as well, using Cherednik-Danilenko's "reduction cases" in [14, (4.4)–(4.5)].

Example 6.24. As a continuation of Example 6.15 and the above Remark, we write the recursion of Corollary 6.22 in two different ways. Let $\gamma = 1 + u^4 + u^6 + u^7$. Proceeding as in the Corollary, we get

$$\gamma = 1 + u^4 + u^6 + u^7 = (1 + u^4)(1 + u^6(1 + u - u^4 - u^5 + \cdots))$$

and further $\gamma' = 1 + u - u^4 - u^5 + \cdots = 1 + u(1 - u^3 - u^4 + \cdots)$. This shows that the master symmetric function is

$$\mathbf{f}_\gamma = q^8\omega\varphi_{3/2}(\varphi_{1/2}(e_1))|_{q\mapsto q^{-1}}$$

which is also written out in Example 7.1. If we were to use Lemma 6.11 but without η, we would get $\gamma = 1 + u^4(1 + u^2 + u^3)$ and $\gamma' = 1 + u^2 + u^3 = 1 + u^2(1 + u)$. This gives

$$\mathbf{f}_\gamma = q^8\omega\varphi_{1/1}(\varphi_{1/2}(\varphi_{1/2}(e_1)))|_{q\mapsto q^{-1}}$$

We leave it to the reader to verify these two are the same symmetric function.

When γ is not inertially elliptic, we still have a finite algorithm to compute the master symmetric function $\mathbf{f}_\gamma$. From Lemma 6.11 we have the following.

Proposition 6.25. *Take a topologically unipotent elliptic tamely ramified $\gamma \in GL_n(F)$. There exist*

(1) *A sequence of triples of integers* $(f_i, q_i, p_i), i = 1, \dots, k$ *with* $(q_i, p_i) = 1$,
(2) *A sequence of fields* $F = F_k \subsetneq F_{k-1} \subsetneq \cdots \subsetneq F_1 \subsetneq F_0 = F(\gamma)$, *and*
(3) *A sequence of elements* $\gamma_1, \dots, \gamma_k = \gamma \in F_0^\times$

such that for $i = 1, ..., k$ *we have*

(a) *Each* γ_i *is topologically unipotent, i.e.* $\gamma_i \in 1 + \mathfrak{m}_{F(\gamma_i)}$.
(b) *There exists* $\eta_i \in 1 + \mathfrak{m}_{F_i}$ *and* $\delta_i \in F_{i-1}$ *such that* $\gamma_i = \eta_i(1 + \delta_i \gamma_{i-1})$ *and that* δ_i *is* F_{i-1}/F_i*-cuspidal. In particular* $F_{i-1} = F_i(\delta_i)$.
(c) f_i *(resp.* p_i*) is the residue degree (resp. ramification degree) of* F_{i-1}/F_i, *and* $q_i = \mathrm{val}_{F_{i-1}}(\delta_i)$.

Moreover, the sequence of triples (f_i, q_i, p_i) *is uniquely determined by* γ.

We also get the following generalization of Lemma 6.14(3).

Lemma 6.26. *Let* γ, δ *and* γ' *be as in Lemma 6.11. Let* e *and* f *be the ramification index and residual degree of* F'/F *in the corresponding notations. Write* $n' := n/ef$. *Then the sequence of triples in Proposition 6.25 associated to* γ' *is given by deleting the last triple* (f_k, q_k, p_k) *is the sequence of triples associated to* γ. *We have* $q_k/p_k = \mathrm{val}_F(\delta)$, $e = p_k$ *and* $f = f_k$.

Denote the residue field of F_i in Proposition 6.25 by k_i. Combining the Proposition 6.17, Proposition 6.18, Proposition 6.25 and Lemma 6.26 we get the following Theorem, which is the most general expression for the master symmetric function of a compact, elliptic and tamely ramified $\gamma \in \mathrm{GL}_n(F)_c$.

Theorem 6.27. *Let* $\gamma \in GL_n(F)$ *be topologically unipotent, elliptic and tamely ramified with discrete invariants* $(\vec{f}, \vec{q}, \vec{p})$. *Then*

$$\mathbf{f}_{\gamma_i} = |k_i|^{\frac{(dn/e-1)(n-1)+n/e-1}{2}} \omega \varphi_{q_i/p_i}|_{q \mapsto |k_i|^{-1}} \omega^{-1}(\tau_{f_i}(\mathbf{f}_{\gamma_{i-1}}))$$

For $b \geq 1$ *let* $\tau'_b : \mathrm{Sym}_q \to \mathrm{Sym}_q$ *be the operator sending* $p_k \mapsto p_{bk}$ *for all* $k \geq 1$ *and* $q \mapsto q^b$. *In particular, we may write*

$$\mathbf{f}_\gamma = q^{\Xi(\gamma)} \omega \varphi_{q_k/p_k}(\tau'_{f_k}(\varphi_{q_{k-1}/p_{k-1}}(\tau'_{f_{k-1}}(\cdots \varphi_{q_1/p_1}(\tau'_{f_1}(e_1))\cdots)))|_{q \mapsto q^{-1}}$$

Proof. The proof is essentially the same as for Corollary 6.22, with the difference that now one also applies the plethysm τ_b with $b > 1$. As the cardinality of the residue field in Proposition 6.17 is that of the base field, and we are applying the proposition recursively, we need to raise the variable q to the residue degree at each step. For any symmetric function g of degree a/b with $a, b \in \mathbb{Z}_{\geq 1}$ and $b|a$, we have

$$\langle \tau_b(g), e_\lambda \rangle = \begin{cases} 0, & \lambda \neq b\lambda' \text{ for any } \lambda' \vdash a/b \\ (-1)^{a-a/b} \langle g, e_{\lambda'} \rangle, & \lambda = b\lambda' \text{ for some } \lambda' \vdash a/b \end{cases}$$

and

$$\langle \tau_b(g), h_\lambda \rangle = \begin{cases} 0, & \lambda \neq b\lambda' \text{ for any } \lambda' \vdash a/b \\ \langle g, h_{\lambda'} \rangle, & \lambda = b\lambda' \text{ for some } \lambda' \vdash a/b \end{cases}$$

In particular, any sequence of operators of the form $\omega \tau'_b \omega$ in the given expression may be replaced by $(-1)^{a-a/b} \tau'_b$ where a/b is the degree of the symmetric function these operators are being applied to. This introduces the overall sign $(-1)^{\sum_{i=1}^k (\prod_{j=1}^i p_j f_j) - f_i^{-1}(\prod_{j=1}^i p_j f_j)}$. However, applying Proposition 6.18 repeatedly gives the same overall sign, and together they cancel. Lastly, the power $q^{\Xi(\gamma)}$ is computed in the same way as in Corollary 6.22, with Lemma 6.26 replacing Lemma 6.14(c). ☐

Just as in the discussion following Corollary 6.22, note that the first case considered in Theorem 6.27 applies verbatim to $\gamma+1$ where $\gamma \in \mathfrak{gl}_n(F)$ is topologically nilpotent, elliptic, and tamely ramified. In particular, the theorem gives a formula for the Lie algebra $\mathbf{f}_\gamma$ in Definition 6.6 (3). Note also that more generally than in the Theorem, if $\gamma \in \mathrm{GL}_n(F)$ is compact of valuation $\neq 0$, it is still possible to compute $\mathbf{f}_\gamma$ using Proposition 6.18 combined with the Theorem. These cases cover all of the three possibilities in Definition 6.6.

Motivated by the above and Definition 5.10, we also define the "combinatorial" counterpart of $\mathbf{f}_\gamma$ as above. Namely, given *any* sequence of triples of positive integers $(\vec{f}, \vec{q}, \vec{p})$ we define

$$\mathbf{f}_{(\vec{f},\vec{q},\vec{p})} := \varphi_{q_k/p_k}(\tau'_{f_k}(\varphi_{q_{k-1}/p_{k-1}}(\tau'_{f_{k-1}}(\cdots\varphi_{q_1/p_1}(\tau'_{f_1}(e_1))\cdots)))) \tag{6.6}$$

Let us finally note that Theorem 6.28 specializes to [78, Théoréme 1.3.] which addresses the following situation: Suppose F'/F is unramified of degree f, $X \in \mathcal{O}_{F'}$ generates the residue field k' of F' and $Y \in \mathfrak{gl}_{n/f}(\mathcal{O}_{F'})$ is such that $F'(Y)$ is an unramified extension of degree n/f. Let $0 \leq a < b$ be integers and $\gamma := 1 + t^a X + t^b Y$, $\gamma' := 1 + t^{b-a} Y$. We have

Theorem 6.28.

$$\mathbf{f}_\gamma = |k|^{a(n^2-n)/2} \omega(\nabla|^a_{t=1,q=1/|k|} \omega \tau_f(\mathbf{f}_{\gamma'}))$$

where ∇ is the Macdonald eigenoperator from Definition 3.7.

Proof. This follows from the fact that $\nabla|^a_{t=1} = \varphi_{a/1}$ as an operator on symmetric functions and Theorem 6.27 with $f_1 = f, q_1 = a, p_1 = 1$. □

Remark 6.29. In order to compare the Shalika germs $s_\lambda(\gamma)$ in [78] to ours, we notice there is a factor of $c^{Wal}_\lambda(q)$ and another of $c^{Wal}_{\lambda'}(q^f)$ inside the plethysm used in *loc. cit.*. This is explained by the fact that there is a mismatch between [78, 79], namely our master symmetric function $\mathbf{f}_\gamma$ is defined to cohere with the latter paper [79], whereas in [78] paper the Shalika germs are collected into a generating function

$$\sum_\lambda s_{\lambda^t}(\gamma, q) c^{Wal}_\lambda(q) h_\lambda$$

instead of $\mathbf{f}_\gamma = \sum_\lambda \Gamma_{\lambda^t}(\gamma) \widetilde{h}_\lambda$ which has an additional plethysm $X \mapsto X/(q-1)$. Composing this with τ_f explains the power $q \mapsto q^f$.

Finally, on the LHS of [78, Théorème 1.3.] we have factors of the form $q^{-an(\lambda^t)}$ which are exactly the ones coming from homogeneity of Shalika germs as observed in Remark 5.6. This matches the appearance of $\nabla|^a_{t=1}$ above.

Remark 6.30. In [79] an unramified character and some "twisted" Steinberg germs appear. While these are not studied in the present paper, they also have nice expressions and combinatorics in terms of symmetric functions. For example, the fundamental lemma proved in [79] can be given meaning in this language. We will do this elsewhere.

Remark 6.31. As can be seen in Theorem 6.28 or by changing the first Newton pair from (p, q) to $(p, q+p)$, the ∇−operator at $t = 1$ (note the overloaded notation: this t is the parameter in $\mathrm{Sym}_{q,t}$, not our chosen uniformizer in F) corresponds to multiplying the element γ in the Lie algebra or taking $1 + \gamma \to 1 + t\gamma$ in the group. Comparing to our formulas, this actually yields homogeneity of Shalika germs for the tamely ramified elements (compare to the proof of [78, Lemme 1.2.]).

More concisely, for any tamely ramified compact $\gamma \in \mathfrak{gl}_n(F)$, the following equation holds:

$$\nabla|_{t=1}\mathbf{f}_\gamma = \mathbf{f}_{t\gamma} \tag{6.7}$$

6.2. **A canonical t-deformation.** Let us shortly discuss a canonical t-deformation of $\mathbf{f}_\gamma$ as defined above for $\gamma \in G(F)$ or $\gamma \in \mathfrak{g}(F)$ that are tamely ramified and elliptic.

Note that by induction, as explained in Theorem 6.27, $\mathbf{f}_\gamma$ is constructed using the steps in Theorem 6.21 as well as Proposition 6.18 (or Theorem 6.28), which are operations on symmetric functions, namely compositions of slope m/n plethysms $\varphi_{m/n} : \mathrm{Sym}_q \to \mathrm{Sym}_q$, the specialized nabla operator $\nabla|_{t=1}$, scalar multiplication, and the Adams operations τ_f. Promoting $\nabla|_{t=1}$ to $\nabla : \mathrm{Sym}_{q,t} \to \mathrm{Sym}_{q,t}$, the slope m/n plethysms to a family of operators coming from the elliptic Hall algebra via $\widehat{\varphi}_{m/n} : \mathrm{Sym}_{q,t} \to \mathcal{E}^{m/n}$, and keeping the τ_f as they are, we may run the similar recursion which only depends on the datum of γ. In particular, we define

Definition 6.32. Let $\gamma \in \mathfrak{gl}_n$ be tamely ramified, topologically nilpotent and elliptic. Let $(\vec{f}, \vec{p}, \vec{q})$ be the discrete invariants attached to γ by Proposition 6.25.

Eq. (6.6) deforms to involve a t as explained above, and with this motivation we define the *full master symmetric function* of $(\vec{f}, \vec{p}, \vec{q})$ as

$$\widehat{\mathbf{f}}_{(\vec{f},\vec{p},\vec{q})} := \widehat{\varphi}_{q_d/p_d}(\tau'_{f_d}(\cdots\tau'_{f_3}(\widehat{\varphi}_{q_2/p_2}(\tau'_{f_2}(\widehat{\varphi}_{q_1/p_1}(\tau'_{f_1}(e_1))\cdot 1))\cdot 1))\cdots)\cdot 1$$

Note that in the totally ramified case we recover the deformed master symmetric function $\widehat{\mathbf{f}}_{\vec{p},\vec{q}}$ from Section 5. This symmetric function, while carrying all the information and behaving nicely with respect to combinatorics, is again not the direct generalization of the functions $\mathbf{f}_\gamma$, as we need to twist by ω. Motivated by this, we define

$$\widehat{\mathbf{f}}_\gamma := (qt)^{\Xi(\gamma)}\omega\widehat{\mathbf{f}}_{(\vec{f},\vec{p},\vec{q})}|_{q\mapsto q^{-1},t\mapsto t^{-1}} \tag{6.8}$$

Example 6.33. With these conventions, we for example have

$$\widehat{\mathbf{f}}_\gamma = qt\omega\nabla e_2$$

in the case $\gamma = u^3 \in \mathfrak{gl}_2(F)$.

In general we have proved Theorem 1.12 from the introduction, namely

Theorem 6.34. *Let $\gamma \in \mathfrak{g}(F)$ be compact, tamely ramified and elliptic. Then the master symmetric function admits a canonical t-deformation, namely*

$$\widehat{\mathbf{f}}_\gamma = \sum_\lambda \widetilde{\Gamma}_{\lambda^t}(\gamma)\widetilde{H}_\lambda$$

where $\widetilde{H}_\lambda$ are the modified Macdonald polynomials. In particular, the Shalika germs $\Gamma_\lambda(\gamma)$ admit a canonical t-deformation.

6.3. **The formula for Shalika germs.** In this section, we will state and prove the main combinatorial formula for Shalika germs. Let

$$\mathbf{f}_\gamma = \sum_{\lambda\vdash n} \Gamma_{\lambda^t}(\gamma)\widetilde{h}_\lambda$$

be the Shalika expansion of the master symmetric function for $\gamma \in \mathfrak{gl}_n(F)$ that is elliptic, topologically nilpotent and tamely ramified.

In the notation of Theorem 6.27, the "cabling process" passing from $\mathbf{f}_{\gamma_{i-1}}$ to $\mathbf{f}_{\gamma_i}$ with new Newton exponents (p_i, q_i) essentially expands $\mathbf{f}_{\gamma_{i-1}}$ in the $\{h_\lambda\}$, replacing each h_λ by $H^-_{q_i,p_i,\lambda}$. If $f_i > 1$, we also need to precompose with τ_{f_i}, which is essentially the "slope $0/f_i$" case of this process.

On the level of the Shalika expansions themselves, we treat these two cases separately, similar as they are, so that each step of the algorithm in Theorem 6.27 is broken into two steps. It is clear from Definition 6.6 that the transition matrix for Shalika germs is exactly the matrix for the operator $\omega\varphi_{q_i/p_i}|_{q\mapsto q^{-1}}\omega$ or τ_{f_i} written in the bases $\{\widetilde{h}_\lambda\}_{\lambda\vdash n}$ and $\{\widetilde{h}_{\lambda'}\}_{\lambda'\vdash n/e}$. In either case, we denote this matrix by

$$M^{d/e} = \{M^{d/e}_{\lambda,\lambda'}\} \tag{6.9}$$

with the convention that $M^{0/e}$ corresponds to the operator τ_e.

Theorem 6.35. *Let $e \geq 1$ and $d = 0$ or $d \geq 1$ with $(e,d) = 1$. Considering the symmetric group of permutations on n/e letters, denote by $|S_{\lambda'} \cap C_\mu|$ is the number of permutations simultaneously lying in the Young/parabolic subgroup $S_{\lambda'} := S_{\lambda'_1} \times \cdots \times S_{\lambda'_\ell}$ and the conjugacy class C_μ of permutations with cycle type μ.*

When $d \geq 1$, the transition matrix $M^{d/e}$ of Shalika germs has a combinatorial description as follows:

$$M^{d/e}_{\lambda,\lambda'} = \left(c_{\lambda'} q^s \sum_{\mu\vdash n/e} \frac{|S_{\lambda'} \cap C_\mu|}{b_\mu \lambda'!} \prod_{i=1}^{\ell(\mu)} \Big(\sum_{\alpha \vDash e\mu_i} \mathrm{wt}_{d/e}(\alpha)_{q\mapsto q^{-1}} q^{-n(\alpha^t)} \widetilde{h}_\alpha \Big) \right) \Bigg|_{\widetilde{h}_\lambda} \tag{6.10}$$

where $c_{\lambda'}$ is as in Eq. (3.8), $b_\mu := \prod_i (1-q^{\mu_i})$, $s := n({\lambda'}^t) + \frac{(dn'-1)(en'-1)+n'-1}{2}$ and $\mathrm{wt}(\alpha)_{d/e}$ is defined in Eq. (5.11). When $d = 0$ the transition matrix is given by

$$M^{0/e}_{\lambda,\lambda'} = \left(c_{\lambda'} \sum_{\mu\vdash n/e} \frac{|S_{\lambda'} \cap C_\mu|}{b_\mu \lambda'!} \prod_{i=1}^{\ell(\mu)} \Big(\sum_{\alpha \vDash e\mu_i} \mathrm{wt}_{0/e}(\alpha) \widetilde{h}_\alpha \Big) \right) \Bigg|_{\widetilde{h}_\lambda} \tag{6.11}$$

where $\mathrm{wt}_{0/e}(\alpha)$ is still defined by Eq. (5.11) but with $S_{0/e}(i) := 0$ for all i when $d = 0$.

Proof. Essentially, we need to compute the slope "d/e plethysm" of the functions $\widetilde{h}_{\lambda'}$, i.e. expand them in the p_μ and replace each p_k by $\varphi_{d/e}(p_k) = P_{dk,ek}$ and bring the result back to the basis $\widetilde{h}_\lambda$. In order to conform to the recipe in Theorem 6.27 we also need to conjugate the plethysm by ω and invert q, as well as multiply the result by $q^{\frac{(dn'-1)(en'-1)+n'-1}{2}}$. In the case $d = 0$ there is no conjugation by ω and we simply have $\varphi_{0/e}(p_k) := \tau_e(p_k) = p_{ek}$.

We first notice that by [22] the *untransformed* complete homogeneous symmetric functions satisfy

$$h_{\lambda'} = \sum_\mu \frac{|S_{\lambda'} \cap C_\mu|}{\lambda'!} p_\mu$$

where $|S_{\lambda'} \cap C_\mu|$ is as defined above. Since

$$\widetilde{h}_{\lambda'} = c_{\lambda'} h_{\lambda'} \left[\frac{X}{1-q} \right],$$

we get

$$\widetilde{h}_{\lambda'} = c_{\lambda'} \sum_{\mu\vdash n'} \frac{|S_{\lambda'} \cap C_\mu|}{\lambda'! b_\mu} p_\mu$$

where $b_\mu = \prod_i (1-q^{\mu_i})$ is the reciprocal of the principal specialization of p_μ and $c_{\lambda'}$ is as before. Let now $d > 0$.

We first note that the matrix elements of $\omega\varphi_{d/e}|_{q\mapsto q^{-1}}\omega$ in the $\widetilde{h}_\lambda$-basis are the same as those of $\varphi_{d/e}$ in the $\omega\widetilde{h}_\lambda$-basis. On the other hand, $\omega\widetilde{h}_\lambda = q^{n(\lambda^t)}\widetilde{h}_\lambda[X;q^{-1}]$ by Eq. (3.7). Applying $\varphi_{d/e}|_{q\mapsto q^{-1}}$ to $\widetilde{h}_\lambda[X;q^{-1}]$ is clearly the same as applying $\varphi_{d/e}$ to $\widetilde{h}_\lambda$ and then inverting q. By Proposition 5.21 we get

$$\varphi_{d/e}(\widetilde{h}_{\lambda'}) = \left(c_{\lambda'}\sum_{\mu\vdash n/e}\frac{|S_{\lambda'}\cap C_\mu|}{b_\mu\lambda'!}\prod_{i=1}^{\ell(\mu)}(\sum_{\alpha\vDash e\mu_i}\mathrm{wt}_{d/e}(\alpha)\widetilde{h}_\alpha)\right)$$

and by the above argument applying ω and $\varphi_{d/e}|_{q\mapsto q^{-1}}$ gives

$$\varphi_{d/e}|_{q\mapsto q^{-1}}(\omega\widetilde{h}_{\lambda'}) = \left(c_{\lambda'}q^{n(\lambda'^t)}\sum_{\mu\vdash n/e}\frac{|S_{\lambda'}\cap C_\mu|}{b_\mu\lambda'!}\prod_{i=1}^{\ell(\mu)}(\sum_{\alpha\vDash e\mu_i}\mathrm{wt}_{d/e}(\alpha)|_{q\mapsto q^{-1}}\widetilde{h}_\alpha[X;q^{-1}])\right)$$

Applying ω once more gives

$$\omega\varphi_{d/e}|_{q\mapsto q^{-1}}(\omega\widetilde{h}_{\lambda'}) = \left(c_{\lambda'}q^{n(\lambda'^t)}\sum_{\mu\vdash n/e}\frac{|S_{\lambda'}\cap C_\mu|}{b_\mu\lambda'!}\prod_{i=1}^{\ell(\mu)}(\sum_{\alpha\vDash e\mu_i}\mathrm{wt}_{d/e}(\alpha)|_{q\mapsto q^{-1}}q^{-n(\alpha^t)}\widetilde{h}_\alpha)\right)$$

and multiplying by the prefactor $q^{\frac{(dn'-1)(en'-1)+n'-1}{2}}$ gives the result.

The proof for the $d = 0$ case is similar, except we do not need to apply ω nor invert q. Note that Eq. (5.11) still holds in this case by [61] and is closely related to the classical Pieri rule for Macdonald polynomials.

□

Remark 6.36. One may view Theorem 6.35 as giving a combinatorial expression for the "λ'-colored" master symmetric functions of torus knots at $a = 0, t = 1$.

Remark 6.37. In the "slope zero" case of this Theorem, where F'/F is an unramified extension of residue degree f, this matrix was essentially computed in [78].

6.3.1. *Integrality properties.* In this section, we conjecture a different combinatorial approach to the (renormalized) Shalika germs. While it may not seem obvious from the previous formulas, we have the following.

Proposition 6.38. *The symmetric functions h_λ expand with $\mathbb{Z}[q]$-coefficients in the basis $h_\lambda\left[\frac{X}{1-q}\right]$ and the symmetric functions $\widetilde{h}_\lambda$ expand with $\mathbb{Z}[q]$-coefficients in the basis h_λ.*

Proof. The bases $\{m_\lambda\}, \{h_\lambda\}$ are dual with respect to the Hall inner product, so that $m_\lambda[X(1-q)]$ is the basis dual to $h_\lambda\left[\frac{X}{1-q}\right]$ by standard properties of plethysm. Therefore, we need to check that

$$\langle h_\lambda, m_\mu[X(1-q)]\rangle = \langle h_\lambda[X(1-q)], m_\mu\rangle$$

is in $\mathbb{Z}[q]$. The inner product is nonzero only when there exists an integer matrix with row sums λ and column sums μ, with only a single nonzero entry in each row (see e.g. [81, 3.17.]). It is also integral by e.g. [35, p. 52]. The second statement follows from a similar argument. □

From the above and the fact that the transition matrix $M^{d/e}(h)$ in the h_λ-basis is integral (we use the notation $M(h)$ to denote the conjugation to the appropriate basis, similar to [22]) by Theorem 6.21, we get that the coefficients of $\mathbf{f}_\gamma$ in the basis

$h_\lambda\left[\frac{X}{1-q}\right]$ are integral. On the other hand, since $\widetilde{h}_\lambda = c_\lambda h_\lambda\left[\frac{X}{1-q}\right]$ these coefficients are exactly $c_\lambda \Gamma_{\lambda^t}(\gamma)$. Motivated by this, we have the following.

Definition-Proposition 6.39. The *renormalized* Shalika germs

$$\Gamma_\lambda^{ren}(\gamma) := c_{\lambda^t}\Gamma_\lambda(\gamma)$$

are integral, i.e. $\Gamma_\lambda^{ren}(\gamma) \in \mathbb{Z}[q]$.

Proof. We have $\widetilde{h}_1 = h_1$. By Theorem 6.27 we can get $\mathbf{f}_\gamma$ up to a sign by multiplying h_1 by various $M^{d/e}$ together. This results in an obviously integral expression for $\mathbf{f}_\gamma$ in the basis h_λ. By Proposition 6.38 we get that the expansion in the $h_\lambda\left[\frac{X}{1-q}\right]$-basis is also integral. □

Remark 6.40. The renormalized Shalika germs are not in $\mathbb{N}[q]$ in general, even up to an overall sign. In particular it is easy to find examples for which $c_\lambda\Gamma_\lambda(\gamma)$ has both positive and negative integer coefficients.

One might also renormalize the Shalika germs in a representation-theoretic way so that one further divides out by $1-q^f$ as well as $q^{\Xi(\gamma)+n-1}$, where f is the degree of the maximal unramified subextension of $F(\gamma)/F$. More precisely, by [77, Eq. (3.6)] one has

$$(1-q^f)^{-1}q^{-\Xi(\gamma)-n+1}c_{(1^n)}\Gamma_{(n)}(\gamma) = 1.$$

For the regular unipotent orbit, this normalization coincides with the normalization used by Shelstad in [72]. In general, we set

$$\Gamma_\lambda^{dW}(\gamma) := (1-q^f)^{-1}q^{-\Xi(\gamma)-n+1}c_{\lambda^t}\Gamma_\lambda(\gamma)$$

We call this the *degenerate Whittaker normalization*, following [55, 72].

Example 6.41. Let $\gamma = u^3$ in $\mathfrak{gl}_2$. Then

	(11)	(2)
Ordinary	$\frac{-1}{q-1}$	$\frac{q^2}{q-1}$
Renormalized	$1-q^2$	q^3-q^2
Degenerate Whittaker	$-q^{-1}-q^{-2}$	1

Similarly, if $\gamma = u^4 \in \mathfrak{gl}_3$ we have

	(111)	(21)	(3)
Ordinary	$\frac{-1}{-q^3+q^2+q-1}$	$\frac{2q^4+q^3}{-q^3+q^2+q-1}$	$\frac{-q^5}{-q^2+2q-1}$
Renormalized	q^3-1	$-2q^5+q^4+q^3$	q^6-q^5
Degenerate Whittaker	$q^{-3}+q^{-4}+q^{-5}$	$2q^{-1}-q^{-2}$	1

Note that in the latter example, one can directly compare the subregular germ to the formula in [67, (10.3)]. The extra discriminant factors there stem from a normalization difference, just like in the comparison between the regular germs in [72] and [66].

We now discuss some conjectures regarding the behavior of the functions $\Gamma_\lambda^{dW}, \Gamma_\lambda^{ren}$. Computations suggest the following conjecture.

Conjecture 6.42. *The $\Gamma_\lambda^{dW}(\gamma)$ are integral polynomials in q^{-1}, i.e. lie in $\mathbb{Z}[q^{-1}]$.*

Next, we try to give a combinatorial interpretation for the polynomials Γ_λ^{ren}. Let $\lambda, \mu \vdash n \geq 1$. Let $\mathcal{G}(\lambda)$ be the set of directed graphs (loops allowed) with vertex set the boxes of the Ferrers diagram of λ, labeled with $\{1, \dots, n\}$ and edges only between boxes in the same row. Further, require each vertex to have in- and outdegree 1. Let $\mathcal{G}(\lambda, \mu) \subset \mathcal{G}(\lambda)$ be the subset of graphs whose connected components sort to give the partition μ. Note that μ is necessarily a *refinement* of λ.

Lemma 6.43. *There is a natural bijection $S_\lambda \cap C_\mu \leftrightarrow \mathcal{G}(\lambda, \mu)$.*

Proof. Writing a cycle decomposition for elements on the left gives rise to a graph by drawing the boxes labeled $1, \dots, n$ and adding edges $a_i \to a_{i+1}$ for each cycle $(a_1 \cdots a_k)$. The converse is clear. □

Next, we note that by deleting at least one edge from each cycle of a graph $\mathtt{G} \in \mathcal{G}(\lambda)$, we get a composition of n, by remembering the ordering on the original boxes of λ. This composition naturally refines λ. Accordingly for $\mathtt{G} \in \mathcal{G}(\lambda)$, we say $\alpha \vDash n$ refines $\mathtt{G}$ if we can obtain the composition α by deleting edges from $\mathtt{G}$. Finally, for $e \geq 1$, let $e\mathtt{G}$ be the graph obtained by e-dilating each cycle in $\mathtt{G}$.

In order to only have one kind of combinatorial object, we may further associate to each $\mathtt{G}' \in \mathcal{G}(\lambda', \mu)$ and a composition $\alpha \leq e\mathtt{G}'$ exactly $\prod_{i=1}^{\ell(\mu)} \mu_i$ different graphs by cyclic permutation of vertices in $\mathtt{G}'$. It is easy to see these graphs $\mathtt{G} \leq m\mathtt{G}'$ are the ones coming exactly from $m\mathtt{G}'$ by removal of one or more edges so that the resulting composition is α.

Next, define the weight of a graph to be

$$\mathrm{wt}(\mathtt{G})_{d/e} = q^{-\sum_{v \in \mathtt{G}} \mathrm{coarm}(v) S_{d/e}(v)}$$

where coarm is the i-coordinate of the vertex minus 1, counting from the start of the chain v belongs to.

Conjecture 6.44. *In the "renormalized" basis $h_\lambda[\frac{X}{1-q}] = \widetilde{h}_\lambda / c_\lambda$, the transition matrix of Shalika germs is given by*

$$M^{d/e}_{\lambda,\lambda'} = \frac{c_\lambda}{c_{\lambda'}} M_{\lambda,\lambda'} = (-1)^{n-\ell(\lambda)} \frac{1}{\lambda'!} \sum_{\mathtt{G}' \in \mathcal{G}(\lambda)} (-1)^{n/e - \ell(\alpha(\mathtt{G}'))} \sum_{\substack{\mathtt{G} \leq e\mathtt{G}' \\ sort(\mathtt{G}) = \lambda}} \mathrm{wt}(\mathtt{G}_{d/e}) \tag{6.12}$$

This is a purely combinatorial conjecture, which we expect to be verifiable by direct comparison of Eqs. (6.10) (6.12).

In effect, Eq. (6.12) gives a conjectural combinatorial interpretation for the renormalized Shalika germs, which are integral polynomials in q^{-1}.

Remark 6.45. Eq. (6.12) was conjectured in a slightly different form by the second author in 2018, based on extensive computer experiments, a slightly different algorithm based on [76], and an expectation for (6.12) when $q \to 1$.

6.4. **The formulas for orbital integrals.** In this section, we give a combinatorial reformulation of the orbital integrals of characteristic functions of standard parahorics, in particular we prove Theorem 1.10 from the introduction.

Theorem 6.46. *Let $\gamma \in \mathfrak{g}(F)$ be compact, tamely ramified and regular semisimple, and let $\mathbf{1}_\lambda$ be the characteristic function of the standard parahoric $\mathbf{P}_\lambda$ associated to $\lambda \vdash n$, divided out by its measure (with the normalization of measures as before). Then*

$$I_\gamma(\mathbf{1}_\lambda) = \langle \mathbf{f}_\gamma, h_\lambda \rangle$$

where we pair using the Hall inner product and $\mathbf{f}_\gamma$ *is as above.*

Proof. From Theorem 2.19, we have

$$I_\gamma(\mathbf{1}_\lambda) = \sum_\mu \Gamma^{St}_\mu(\gamma)\mathrm{St}_{\mu,c}(\mathbf{1}_\lambda)$$

and by Theorem 6.21 plus Definition 6.6 we have

$$\sum_\mu \Gamma^{St}_\mu(\gamma)e_\mu = \sum_\mu \Gamma_\mu(\gamma)\widetilde{h}_\mu = \sum_\mu \sigma_\mu(\gamma)h_\mu = \mathbf{f}_\gamma$$

The result then follows from Propositions 3.13, 3.14 and Proposition 2.9. □

As a corollary of the proof, we get

Corollary 6.47. *For* $\gamma \in \mathfrak{m}(F) \subset \mathfrak{g}(F)$ *as in Theorem 6.46, where* $\mathfrak{m}$ *is the Lie algebra of a Levi subgroup conjugate to* $L(\mu)$*, let* $\gamma_1, \dots, \gamma_{\ell(\mu)}$ *be the "diagonal blocks" of* γ*. Then we have the following identity of master symmetric functions:*

$$\mathbf{f}_\gamma = \left|\det(\mathrm{ad}(\gamma)_{\mathfrak{g}/\mathfrak{m}})\right|^{-1/2} \prod_{i=1}^{\ell(\mu)} \mathbf{f}_{\gamma_i}$$

where $\mathbf{f}_{\gamma_i}$ *is the master symmetric function of* γ_i *as defined in Definition 6.6.*

Proof. As in the statement, suppose γ belongs to a Levi of type μ, WLOG to the standard one and has blocks $\gamma_1, \dots, \gamma_{\ell(\mu)}$. Then by Proposition 2.40

$$I^G_\gamma(\mathbf{1}_\lambda) = \left|\det(\mathrm{ad}(\gamma)_{\mathfrak{g}/\mathfrak{m}})\right|^{-1/2} I^M_\gamma(\mathrm{Res}^{\mathfrak{g}}_{\mathfrak{m}}(\mathbf{1}_\lambda))$$

Let us write $\mathrm{Res}_\mu = \mathrm{Res}^{\mathfrak{g}}_{\mathfrak{m}}$. By [79, Lemme IV 3.] and Lemma 2.37, we get

$$\mathrm{Res}_\mu(\mathbf{1}_\lambda) = \sum_{m \in \mathbf{M}(\lambda,\mu)} \otimes_{j=1}^{\ell(\mu)} \mathbf{1}_{m_{\cdot,j}}$$

where $\mathbf{1}_{m_{\cdot,j}}$ is the characteristic function of the corresponding standard parahoric normalized by its measure and $\mathbf{M}(\lambda,\mu)$ is as in Definition 3.2. It is clear that this implies

$$I^M_\gamma(\mathrm{Res}_\mu(\mathbf{1}_\lambda)) = \sum_{m \in \mathbf{M}(\lambda,\mu)} \prod_{j=1}^{\ell(\mu)} \langle h_{m_{\cdot,j}}, \mathbf{f}_{\gamma_j}\rangle$$

On the other hand, the first displayed equation on [79, p. 883] implies that we may write the RHS of the above equation as

$$\langle h_\lambda, \prod \mathbf{f}_{\gamma_i}\rangle$$

Multiplying by $\left|\det(\mathrm{ad}(\gamma)_{\mathfrak{g}/\mathfrak{m}})\right|^{-1/2}$, we are done. □

7. Examples

Example 7.1. Let $\gamma = u^7 + u^6 \in \mathfrak{gl}_4(F)$ and $\mathrm{char}(k) \neq 2$, following Examples 1.5, 6.15 and 6.24. Let us write down the master symmetric function. On the knot theory/combinatorial side it is

$$\mathbf{f}_{(1,2),(3,2)} = \varphi_{3/2}(\varphi_{1/2}(e_1)) = \varphi_{3/2}(e_2) =$$
$$= \left(q^2 + q + 1\right) e_{1,1,1,1} + \left(q^5 + 2q^4 + 4q^3 + 2q^2 + 2q\right) e_{2,1,1} +$$
$$\left(q^6 + q^4 + q^2\right) e_{2,2} + \left(q^7 + q^6 + 2q^5 + q^4\right) e_{3,1} + q^8 e_4$$

Indeed, there are 23 Dyck paths in a 6×4 rectangle with these horizontal steps and area statistics. We recover $\mathbf{f}_\gamma$ from Corollary 6.22, namely

$$\mathbf{f}_\gamma = q^8\omega\mathbf{f}_{(1,2),(3,2)}|_{q\mapsto q^{-1}} = \left(q^8+q^7+q^6\right)h_{1,1,1,1} + \left(2q^7+2q^6+4q^5+2q^4+q^3\right)h_{2,1,1} + \left(q^6+q^4+q^2\right)h_{2,2} + \left(q^4+2q^3+q^2+q\right)h_{3,1} + h_4$$

The weight polynomial of the spherical affine Springer fiber is

$$q^{\dim \mathrm{Sp}_\gamma}\langle \mathbf{f}_{(1,2),(3,2)}, e_4\rangle|_{q\mapsto q^{-1}} = 1+q+2q^2+3q^3+4q^4+4q^5+4q^6+3q^7+q^8$$

and that of the Iwahori affine Springer fiber is

$$q^{\dim \mathrm{Sp}_\gamma}\langle \mathbf{f}_{(1,2),(3,2)}, e_{1111}\rangle|_{q\mapsto q^{-1}} = 1+4q+10q^2+20q^3+34q^4+48q^5+54q^6+48q^7+24q^8$$

Note that the first one is just the sum of the coefficients of the various h_λ in $\mathbf{f}_\gamma$. It agrees up to $q\mapsto q^{-1}$ with the computation in [13, Eq. (3.1)] – it seems that there is a typo in that paper, repeating one from Piontkowski's work [65].

Finally, we illustrate Theorem 1.8, i.e. the combinatorial formula for Shalika germs. On the second step of our induction $n=4, n'=2$. Suppose we want to compute the entry $M^{3/2}_{211,2}$ of our transition matrix in the $\widetilde{h}_\lambda$-basis. We have

$$c_2 = (1-q)(1-q^2),\ b_2 = (1-q^2),\ b_{11} = (1-q)^2, \lambda'! = 2$$

and therefore $\widetilde{h}_2 = \frac{q+1}{2}p_{11} + \frac{1-q}{2}p_2$. Now since $(p_2,q_2) = (3,2)$ we must apply the slope $3/2$ plethysm and replace $p_{11}\mapsto P^2_{3,2}, p_{(2)}\mapsto P_{6,4}$, up to conjugation by ω.

Now by formula (5.8) we have $P_{3,2} = \frac{1}{1-q}\widetilde{h}_{11} - \frac{q}{1-q}\widetilde{h}_2$ so

$$P^2_{3,2} = \frac{1}{(1-q)^2}\widetilde{h}_{1111} - \frac{2q}{(1-q)^2}\widetilde{h}_{211} + \frac{q^2}{(1-q)^2}\widetilde{h}_{22}$$

There are 8 compositions of 4, and we compute

$$S_{3/2}(1) = 2, S_{3/2}(2) = 1, S_{3/2}(3) = 2, S_{3/2}(4) = 1$$

Plugging this in to Eq. (5.11) gives

$$\mathrm{wt}(2+1+1)_{3/2} = \frac{-q(1+q^2)}{(1-q)^2(1-q^2)},\ \mathrm{wt}(1+2+1)_{3/2} = \frac{-2q^2}{(1-q)^2(1-q^2)},$$
$$\mathrm{wt}(1+1+2)_{3/2} = \frac{-q(1+q)}{(1-q)^3},$$

so that the coefficient of $\widetilde{h}_{211}$ in $P_{6,4}$ is

$$\frac{-q(1+q^2)}{(1-q)^2(1-q^2)} + \frac{-2q^2}{(1-q)^2(1-q^2)} + \frac{-q(1+q)}{(1-q)^3} = \frac{-2q^2-2q}{(q-1)^3}$$

Taken together, we get

$$\frac{-2q(q+1)}{2(1-q)^2} + \frac{(1-q)(-2q^2-2q)}{2(q-1)^3} = 0$$

One verifies in Sage that the slope $3/2$ plethysm of $\widetilde{h}_2$ has vanishing coefficient for $\widetilde{h}_{211}$. More generally, one can compute that

$$M^{3/2} = \begin{pmatrix} \frac{1}{q^5-q^3-q^2+1} & \frac{-q^6-q^5}{q^4-q^3-q+1} & \frac{q^7}{q^3-q^2-q+1} & 0 & 0 \\ 0 & 0 & \frac{q^6}{q^2-2q+1} & \frac{-2q^8}{q^2-2q+1} & \frac{q^{10}}{q^2-2q+1} \end{pmatrix}$$

where the rows are indexed by the partitions $(2), (11)$ and the columns are indexed by $(4), (31), (22), (211), (1111)$.

Example 7.2. The simplest elliptic case with three Puiseux pairs appears in [13, Eq. (3.8.)] as well as in [65] as an example where previous methods fail. This example corresponds to the plane curve singularity $\mathbb{C}[[t^8, t^{12} + t^{14} + t^{15}]]$, so we have $(p_1, q_1) = (p_2, q_2) = (2, 1), (p_3, q_3) = (2, 3)$. The dimension of the ASF is 42 in this case. Using Sage, we compute

$$\begin{gathered}\langle \mathbf{f}_\gamma, h_8 \rangle = \\ q^{42} + 7q^{41} + 24q^{40} + 56q^{39} + 104q^{38} + 166q^{37} + 236q^{36} + 306q^{35} + 370q^{34} + \\ 424q^{33} + 465q^{32} + 492q^{31} + 507q^{30} + 510q^{29} + 504q^{28} + 488q^{27} + 466q^{26} + \\ 437q^{25} + 406q^{24} + 370q^{23} + 335q^{22} + 298q^{21} + 264q^{20} + 230q^{19} + 199q^{18} + \\ 168q^{17} + 143q^{16} + 118q^{15} + 97q^{14} + 78q^{13} + 63q^{12} + 48q^{11} + 38q^{10} + \\ 28q^9 + 21q^8 + 15q^7 + 11q^6 + 7q^5 + 5q^4 + 3q^3 + 2q^2 + q + 1\end{gathered}$$

which by Theorem 1.10 is the weight polynomial of the compactified Jacobian in this case. We refer the reader to the attached computer program for computing the Shalika germs and other data in this case.

Example 7.3. Let $G = GL_4$ and $\gamma = u^6$. This is an element whose characteristic polynomial is $x^4 - t^6$, so that the link is a $(6, 4)$-torus link. The element γ is conjugate to one in a Levi isomorphic to $GL_2 \times GL_2$, and on each of the blocks we have an equivalued element of valuation $3/2$. We compute the master symmetric function to be the product of $\left|\det(\mathrm{ad}(\gamma)_{\mathfrak{g}/\mathfrak{m}})\right|^{-1/2} = q^6$ and the two factors in this case, namely $\mathbf{f}_\gamma = (qh_{11} + h_2)^2$. The Shalika expansion of $\mathbf{f}_\gamma$ reads

$$\mathbf{f}_\gamma = \left(\frac{q^{10}}{q^2 - 2q + 1}\right)\widetilde{h}_{1111} + \left(\frac{-2q^8}{q^2 - 2q + 1}\right)\widetilde{h}_{211} + \left(\frac{q^6}{q^2 - 2q + 1}\right)\widetilde{h}_{22}$$

Theorem 6.46 gives that $I_\gamma(\mathbf{1}_{(4)}) = q^8 + 2q^7 + q^6$ and $I_\gamma(\mathbf{1}_{(1^4)}) = 24q^8 + 24q^7 + 6q^6$. Note that up to $q \leftrightarrow t$, the first result agrees with the numerator of [39, Example 1.3.] at $a = 0, q = 1$.

Example 7.4. Let us work out an unramified example. Suppose $k = \mathbb{F}_q, p \neq 2$, and $a \in \mathbb{F}_q^\times - (\mathbb{F}_q^\times)^2$. Let

$$\gamma = \begin{pmatrix} 0 & at \\ t & 0 \end{pmatrix}$$

Then γ splits over a degree two unramified extension of F. By Hilbert's Theorem 90, stable conjugacy in GL_n is rational conjugacy, so by [82, (3.5.4)] we should have

$$I_\gamma^{GL_n}(\mathbf{1}_{\mathfrak{g}(\mathcal{O})}) = \mathrm{SO}_\gamma(\mathbf{1}_{\mathfrak{sl}_n(\mathcal{O})}) = q + 2$$

where SO_γ is the stable orbital integral in SL_n, defined as in [82, Section 3.5.3.].

Indeed, we are in a situation where $1 + \gamma \sim 1 + tX$ with X generating $\mathbb{F}_{q^2}$ over $\mathbb{F}_q$, and by Theorem 6.28 and Proposition 2.9

$$\mathbf{f}_\gamma = q\omega\nabla_{t=1}\omega\tau_2(h_1)|_{q\mapsto q^{-1}} = qh_{11} + 2h_2$$

By Theorem 6.46 we get

$$I_\gamma(\mathbf{1}_{(2)}) = \langle \mathbf{f}_\gamma, h_2 \rangle = q + 2$$

as desired.

Example 7.5. Next, we discuss the simplest "mixed" example. Let a be as above, and

$$\gamma = \begin{pmatrix} 0 & 0 & 0 & at^2 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

This element is elliptic and splits over a degree 4 extension, with a maximal unramified subextension of degree 2. The discrete invariants from Proposition 6.25 are $(f, q, p) = (2, 1, 2)$, so

$$\mathbf{f}_\gamma = q^2 \omega \varphi_{1/2}|_{q \mapsto q^{-1}} \omega(\tau_2(e_1)) = q^2 h_{22} + 2q h_{31} + 2h_4$$

and for example $I_\gamma(\mathbf{1}_{(4)}) = \langle \mathbf{f}_\gamma, h_4 \rangle = q^2 + 2q + 2$. We note that $I_\gamma(\mathbf{1}_{(11111)}) = 6q^2 + 8q + 2$. Note that the Iwahori affine Springer fiber in this case has six components (cf. Theorem 8.4).

Example 7.6. Finally, we consider an example of the form considered in [78, 1.5.]. Let $f_1 = f_2 = 2$ and $F = F_2 \subset F_1 \subset F_0$ be a tower of unramified extensions, both of degree 2. Suppose $X_i \in \mathcal{O}_{F_i}$ are so that their reduction in the residue field generates the residue field over that of F_{i+1}. Consider $\gamma = 1 + tX_1 + t^2 X_0$. The discrete invariants from Theorem 6.27 are $(2, 1, 1), (2, 1, 1)$ and the master symmetric function is

$$\mathbf{f}_\gamma = (q^8 + 2q^6) h_{1111} + (4q^7 + 8q^5 + 4q^4 + 4q^3) h_{211} + (4q^6 + 6q^4 + 4q^2) h_{22} + (4q^3 + 4q^2 + 4q) h_{31} + 4h_4$$

Note that on the second step of the recursion we apply τ_2 to a degree two symmetric function. To compare with the transition matrix in Example 7.1 we compute

$$M^{0/2} = \begin{pmatrix} \frac{2}{q^5 - q^3 - q^2 + 1} & \frac{-2q^2 - 2}{q^4 - q^3 - q + 1} & \frac{q^2 + 4q + 1}{q^3 - q^2 - q + 1} & \frac{-4q}{q^2 - 2q + 1} & \frac{q^2 + q}{q^2 - 2q + 1} \\ 0 & 0 & \frac{4}{q^2 - 2q + 1} & \frac{-4q - 4}{q^2 - 2q + 1} & \frac{q^2 + 2q + 1}{q^2 - 2q + 1} \end{pmatrix}$$

Note that the $(2), (211)$-entry is *not* zero in this example. This is because the coefficient of $\widetilde{h}_{211}$ in $P_{0,4}$ is $\frac{-4q^2 - 4}{q^3 - 3q^2 + 3q - 1}$ and in $P^2_{0,2}$ it is $\frac{-4q - 4}{q^2 - 2q + 1}$, which gives

$$\frac{1}{2} \left(\frac{-4q^2 - 4}{q^3 - 3q^2 + 3q - 1} (1 - q) + \frac{-4q - 4}{q^2 - 2q + 1} (1 + q) \right) = \frac{-4q}{q^2 - 2q + 1}$$

8. Applications

8.1. **Affine Springer fibers.** Let $G = GL_n/F$ where $F = k((t))$ with $k = \mathbb{F}_q$. Appropriately modifying the definition of Sp_γ below to account for mixed characteristic F, we get similar results but leave these for the interested reader. Suppose $\mathbf{P} \subset G(F)$ is a parahoric subgroup. Let $\mathrm{Fl}_\mathbf{P} = G(F)/\mathbf{P}$ be the corresponding partial affine flag variety.

Definition 8.1. Let $\gamma \in \mathfrak{g}(F)^{rs}$. The affine Springer fiber is the reduced ind-subscheme of $\mathrm{Fl}_\mathbf{P}$ defined by

$$\mathrm{Sp}^\mathbf{P}_\gamma(k) = \{ g\mathbf{P} | \mathrm{Ad}(g^{-1})\gamma \in \mathrm{Lie}(\mathbf{P}) \}$$

Let T_γ be the centralizer of γ. Then it acts naturally on $\mathrm{Sp}^\mathbf{P}_\gamma$. Let S_γ be the maximal unramified subtorus of T_γ, and $X_*(S_\gamma) =: \Lambda_\gamma$ its cocharacter lattice. As in [29, Section 15], the centralizer action gives rise to an action of Λ_γ on $\mathrm{Sp}^\mathbf{P}_\gamma$.

Recall that $1_{\mathbf{P}}$ is the characteristic function of $\mathbf{P}$ divided by the measure of $\mathbf{P}$. Unraveling the definitions, it is not hard to prove (see for example [29, Theorem 15.8.])

Proposition 8.2. *We have*

$$|\mathrm{Sp}_\gamma^{\mathbf{P}}(k)/\Lambda_\gamma(k)| = I_\gamma(\mathbf{1_P}).$$

See Definition 2.1 for the normalization in I_γ.

Recall that Theorem 6.46 says that if $\mathbf{P}$ is of type λ and γ is tamely ramified and regular semisimple, then

$$I_\gamma(\mathbf{1_P}) = \langle \mathbf{f}_\gamma, h_\lambda\rangle$$

which is a polynomial with nonnegative coefficients. This result, combined with Proposition 8.2 implies

Corollary 8.3. *When γ is tamely ramified, the number of points $|\mathrm{Sp}_\gamma^{\mathbf{P}}(k)/\Lambda_\gamma(k)|$ is a polynomial in q with nonnegative coefficients.*

Let us also note the following application to components of affine Springer fibers.

Theorem 8.4. *Let $\gamma \in \mathfrak{g}(F)$ be compact and regular semisimple. Then the number of geometric components of $\mathrm{Sp}_\gamma^{\mathbf{I}}/\Lambda_\gamma$ stable under $Gal(\overline{k}/k)$ is always a divisor of $|W| = n!$.*

Proof. By [77, Eq. (4.5)], the number of Frobenius-stable geometric components is the coefficient of the leading term in q of the integral of $\mathbf{1}_{11\cdots 1}$ along the orbit of γ. By Theorem 6.46, this orbital integral can be computed using $\mathbf{f}_\gamma$ by pairing it with $h_{11\cdots 1}$. On the other hand, $\mathbf{f}_\gamma$ is formed by multiplying the master symmetric functions for the blocks of γ. Suppose for a moment γ is totally ramified. By Lemma 8.5 the smallest power in the Dyck germs is 1, it appears with coefficient one, and it appears for the least dominant partition. Since pairing with h_λ does not introduce powers of q, the highest power of q appearing in

$$\langle \mathbf{f}_\gamma, h_{11\cdots 1}\rangle$$

is $q^{\dim \mathrm{Sp}_\gamma}$ and it appears with coefficient $\langle h_\lambda, h_{11\cdots 1}\rangle = \frac{|S_n|}{|S_\lambda|}$ where λ is the smallest partition in the dominance order appearing in the Dyck expansion of $\mathbf{f}_\gamma$. This proves the claim in the totally ramified case.

For the general case where the construction $\mathbf{f}_\gamma$ involves the operator τ_f, cf. Proposition 6.18, note that the plethysm τ_f is designed so that for any homogeneous symmetric function $\mathbf{f}$ of degree $n' = n/f$, $\lambda \vdash n$, we have

$$\langle h_\lambda, \tau_f(\mathbf{f})\rangle = \begin{cases} \langle h_{\lambda/f}, \mathbf{f}\rangle, & \text{if } \lambda \text{ is divisible by } f \\ 0, & \text{if } \lambda \text{ is not divisible by } f \end{cases}$$

In particular, we may reduce these cases to the computation above. □

Lemma 8.5. *Let γ be inertially elliptic. Then the smallest power of q appearing in the coefficients of $\mathbf{f}_{\vec{p},\vec{q}} = \sum_\lambda \sigma_\lambda(\gamma)e_\lambda$ is $1 = q^0$ and it only appears in front of the smallest partition in dominance order for which $\sigma_\lambda(\gamma) \neq 0$. In addition, it appears with coefficient 1.*

Proof. We will prove this by induction. For one Puiseux pair it is clear, as there is always a Dyck path with area 0 (for example in Example 6.4 it is the rightmost Dyck path). Suppose the statement holds for Newton pairs $(p_1, q_1), \dots, (p_{i-1}, q_{i-1})$. Since

$$\varphi_{q_i/p_i}(e_\lambda) = \prod_{j=1}^{\ell(\lambda)} \left(\sum_{\pi \in \mathbb{D}_{\lambda_j q_i, \lambda_j p_i}} q^{area(\pi)} e_\pi \right)$$

the Dyck paths appearing when applying φ_{q_i/p_i} to e_λ can be thought of as concatenations of $(p_i\lambda_j, q_i\lambda_j)$-Dyck paths where $j = 1, \dots, \ell(\lambda)$. The "least dominant" horizontal steps, i.e. the paths with the smallest area, appearing in all the possible concatenations of Dyck paths under the application of φ_{q_i/p_i} can appear from any e_λ. However, by induction only the term of $\mathbf{f}_{(p_1,q_1),\dots,(p_{i-1},q_{i-1})}$ having the smallest power of q in the e_λ–expansion contributes a term of the form e_ν where ν is the unique $(p_i|\lambda|, q_i|\lambda|)$-Dyck path with area 0 and the coefficient is 1. □

Remark 8.6. Let m/n be the minimal root valuation of γ. When γ is inertially elliptic, the above shows that the minimal partition appearing in the e_λ-expansion is formed from the horizontal steps of the maximal staircase partition fitting under a line of slope m/n, i.e. the one with parts $\lfloor \frac{(m-k)n}{m} \rfloor, k = 1, \dots m$. For example, when $m/n = 3/7$ this gives the partition $4 + 2(+0)$, and the corresponding Dyck path in the 3×7 rectangle has horizontal steps $3, 2, 2$. In particular when $m/n \geq 1$ the horizontal steps give the one-column partition. It is easy to extend this to γ non-elliptic by multiplying the corresponding e_λ together. This gives another (slightly more general) proof of a Theorem of Z. Yun in type A, which states that the *minimal reduction type* of γ determines the number of components in the Iwahori affine Springer fiber.

Remark 8.7. Theorem 8.4 proves [77, Conjecture 8.7.] in type A. From the main result of [77], there are always exactly $n!$ components when the depth is > 1. In fact the last statement is true for depth ≥ 1 because any depth-1 element either differs from a depth > 1 element by a central element, or is contained in a Levi subalgebra in which case we can reduce the assertion to the Levi case as in the proof of Theorem 6.46.

Theorem 8.4 has the following interesting corollary about the W-representation given by $H^*(\mathrm{Sp}^{\mathbf{I}}_\gamma/\Lambda_\gamma)$. Let us assume that γ is tamely ramified, and note that the top degree part of the cohomology is always pure. A well-known argument using finite Springer theory tells us there is a graded isomorphism of vector spaces:

$$H^*(\mathrm{Sp}^{\mathbf{P}_\lambda}_\gamma/\Lambda_\gamma) \cong H^*(\mathrm{Sp}^{\mathbf{I}}_\gamma/\Lambda_\gamma)^{W_\lambda}$$

In particular, knowing the dimensions of the top degree cohomologies of each $\mathrm{Sp}^{\mathbf{P}_\lambda}_\gamma/\Lambda$ tells us exactly all the dimensions of the W_λ-invariants of the representation on top degree cohomology of $\mathrm{Sp}^{\mathbf{I}}_\gamma/\Lambda_\gamma$. Recall that using the Hall inner product and Frobenius reciprocity, this is the same as knowing the inner products of the Frobenius character with h_λ. Since the h_λ are a basis of the ring of symmetric functions, this uniquely determines the representation. A similar argument shows that assuming purity, $\mathbf{f}_\gamma$ actually determines the Frobenius character of $H^*(\mathrm{Sp}^{\mathbf{I}}_\gamma/\Lambda_\gamma)$ in the elliptic case. In fact, the Frobenius character will simply be $\mathbf{f}_\gamma$ if these assumptions are satisfied.

More precisely, we get

Theorem 8.8. *Let γ be tamely ramified. The $W = S_n$-representation on $H^{top}(\mathrm{Sp}_\gamma^{\mathbf{I}}/\Lambda_\gamma)$ has Frobenius character h_ν, where ν is the smallest partition in dominance order appearing in the h_λ−expansion of $\mathbf{f}_\gamma$. In particular, when γ has depth ≥ 1, this is the regular representation by above.*

If we assume the purity conjecture and that γ is elliptic, $\mathbf{f}_\gamma$ is the Frobenius character of $H^*(\mathrm{Sp}_\gamma^{\mathbf{I}}/\Lambda_\gamma)$.

Proof. For the first statement, let C_γ be the number of components in the Iwahori affine Springer fiber of γ, defined as in Theorem 8.4. As in the proof of Theorem 8.4, the master symmetric function has the form

$$\mathbf{f}_\gamma = C_\gamma q^{\Xi(\gamma)} h_\nu + O(q^{\Xi(\gamma)-1})$$

Pairing this with h_λ for varying $\lambda \vdash n$ and taking the leading term in q gives the trace of Frobenius on $H^{top}(\mathrm{Sp}_\gamma^{\mathbf{P}_\lambda}/\Lambda_\gamma)$ for varying λ. But since the top cohomology is always pure, the coefficient of $q^{\Xi(\gamma)}$ in $\langle \mathbf{f}_\gamma, h_\lambda\rangle$ in fact equals the top Betti number of $\mathrm{Sp}_\gamma^{\mathbf{P}_\lambda}/\Lambda_\gamma$. Since an S_n-representation is uniquely determined by the dimensions of its invariants under Young subgroups and these are given exactly by $\langle h_\nu, h_\lambda\rangle$ in this case, we are done.

For the second statement, if purity holds we see $\langle \mathbf{f}_\gamma, h_\lambda\rangle$ is the ordinary Poincaré polynomial of $\mathrm{Sp}_\gamma^{\mathbf{P}_\lambda}/\Lambda_\gamma$, in particular the graded dimension of the space of S_λ-invariants of $H^*(\mathrm{Sp}_\gamma^{\mathbf{I}}/\Lambda_\gamma)$. Since these pairings determine $\mathbf{f}_\gamma$ uniquely, the graded Frobenius character of $H^*(\mathrm{Sp}_\gamma^{\mathbf{I}}/\Lambda_\gamma)$ equals $\sum_\lambda \langle \mathbf{f}_\gamma, h_\lambda\rangle m_\lambda = \mathbf{f}_\gamma$. □

Remark 8.9. Note that this proves [31, Conjecture 7.17.] in type A.

Suppose for a moment γ is a split element, i.e. lies in some split maximal torus. In [12], Zongbin Chen proves that the generating function (summing over elements of varying root valuation data) for the number of points on a so called *fundamental domain* of Sp_γ is rational, and that the number of points only depends on the root valuation datum. This is further related to the "weighted" Shalika expansion of Arthur, indeed the rationality is proved using homogeneity properties of these functions. See [12] for more details. We have not compared our techniques with the weighted Arthur-Shalika expansion, but it would be interesting to see how Chen's results could be combined with ours.

8.2. **Compactified Jacobians.** In this section, we apply Theorem 1.10 to show that the point-counts of compactified Jacobians of rational, unibranch plane curves are polynomials in q.

Let us recall some relevant material from [49]. Let C be a reduced, projective and geometrically connected curve over the residue field k, with only planar singularities. Suppose for simplicity that the normalization of C is rational. Let $\overline{\mathrm{Pic}}(C)$ be the compactified Picard scheme of C. It is the moduli space whose closed points parametrize torsion-free rank one sheaves on C. For each $c \in \mathrm{Sing}(C)$ fix an isomorphism $\widehat{\mathcal{O}}_{C,c} \cong k[[x,y]]/f$ and let Sp_c be the affine Springer fiber associated to $\gamma_c := \gamma_f \in \mathfrak{gl}_{\deg_x f}$ where γ_f is the companion matrix of f. Let Λ_c be the lattice part of the centralizer of γ_f and $\Lambda = \mathrm{Pic}(C)/\mathrm{Jac}(C)$. Fix a section $\Lambda \to \mathrm{Pic}(C)$ of the quotient map.

From [49, Proposition 2.3.1.] we have

Proposition 8.10. *There is a universal homeomorphism*

$$\prod_{c \in Sing(C)} \mathrm{Sp}_c/\Lambda_c \to \overline{\mathrm{Pic}}(C)/\Lambda$$

If k is a finite field, we have

Corollary 8.11. *Let k'/k be a finite extension. Then*

$$\left| \prod_{c \in Sing(C)} \mathrm{Sp}_c(k')/\Lambda_c \right| = \left| \overline{\mathrm{Pic}}(C)(k')/\Lambda \right|$$

Combined with Corollary 8.3, we have

Theorem 8.12. *The number of points on $\overline{\mathrm{Pic}}(C)$ is a polynomial in $q = |k|$. In addition, it is a polynomial with nonnegative integer coefficients.*

A standard spreading-out argument, combined with [41, Theorem 1] and the previous Theorem gives

Corollary 8.13. *Let $k = \mathbb{C}$. Then $X = \overline{\mathrm{Pic}}(C)$ is strongly polynomial-count in the sense of Katz [41], and the E-polynomial*

$$E_X(x,y) := \sum_{p,q} e_{p,q} x^p y^q$$

is given by the weight polynomial of $\overline{\mathrm{Pic}}(C)$ as $E_X(x,y) = P_X(xy)$, defined by

$$P_X(q) = \sum_{i,j} (-1)^i q^j \dim \mathrm{gr}^j_W H^i(\overline{\mathrm{Pic}}(C))$$

Yet another corollary of Corollary 8.3 together with Corollary 8.11 and Definition 5.5 is a virtual version of [13, Conjecture 2.4.(iii)], which compares Betti numbers of Jacobian factors with superpolynomials at $q = 1$ (i.e. $t = 1$ in our notation). Using Definition 5.5, the more precise statement is that

Proposition 8.14. *For unibranch $C = \{f(x,y) = 0\} \subset \mathbb{C}^2$, the weight polynomial of $\overline{\mathrm{Jac}}(C)$ is given by the superpolynomial at $a = 0, t = 1$, with q replaced by $1/q$, up to multiplying by $q^{\dim \overline{\mathrm{Jac}}(C)}$. That is,*

$$P_{\overline{\mathrm{Jac}}(C)}(q) = q^{\dim \overline{\mathrm{Jac}}(C)} \mathbf{P}_{Link(C)}(a = 0, q = 1/q, t = 1)$$

8.3. **Orbital integrals.** Let us finally comment on possible other applications of our results, as the explicit computation of orbital integrals bears on many problems in number theory and automorphic forms.

For example, in [73] Shin and Templier prove an equidistribution theorem for "families" of automorphic L-functions (for any G). Their main result [73, Theorem 1.3.] rests on an explicit, residue-characteristic independent bound for the size of orbital integrals derived by Kottwitz from the Shalika germ expansion. For $G = GL_n$, our methods should be applicable to give sharper bounds and as they remark, possible improvements on their analytic results. It would be interesting to see more analytic applications of our results.

In his Beyond Endoscopy -proposal [48], Langlands computes global orbital integrals for $GL(2)$ using "elementary" methods. In the thesis of Espinosa Lara [50], which builds on work of Altug [1], the corresponding local orbital integrals are computed and compared via a product formula to Langlands' results. In Altug's work analysis of orbital integrals is used to "isolate" the contribution of the trivial representation to a certain trace formula Langlands introduces.

A priori, as suspected by Arthur in [2], it should be possible to use an explicit computation of the local orbital integrals (which is where our results come in) to have similar results for GL_n. It would be interesting to see how the possible application to Beyond Endoscopy plays out.

9. Hilbert schemes of points

In this section we give a, frankly tentative, conjectural geometric expression for the Shalika germs of $\gamma \in \mathfrak{g}(F)$ in terms of the Hilbert scheme of points on $\mathbb{A}^2_K$. We consider these Hilbert schemes over a field K, which is algebraically closed of characteristic zero. We also take K as the field of coefficients for the Borel-Moore homology of the (generalized) affine Springer fibers we will be considering.

There is another field we need to keep track of, namely the field F over which we consider GL_n. The reader may want to restrict to the case $F = k((t))$, i.e. a function field over a finite field, but we believe that with appropriate definitions, everything in this section can be made to work over a mixed-characteristic F. In either case, the BM homology of Sp_γ considered in the previous section should be interpreted as the étale cohomology of the Verdier dualizing complex placed in negative degrees, and we use $K = \overline{\mathbb{Q}}_\ell$-coefficients.

To conform with much of the affine Springer fiber and Hilbert scheme literature, the reader may also work with the discretely valued field $F = \mathbb{C}((t))$ and work in singular BM homology for the associated analytic space of Sp_γ, where γ is taken to be regular semisimple and we use $K = \mathbb{C}$-coefficients in homology. Note that the Shalika germs of γ cannot be defined in terms of harmonic analysis in this case, as F is not locally compact. Be that as it may, via Theorem 6.27 the germs can still be defined, up to appropriate normalization, as the coefficients in the expansion of the combinatorially defined master symmetric function $\mathbf{f}_{(\vec{f},\vec{p},\vec{q})} = \mathbf{f}_{\vec{p},\vec{q}}$ from Eq. (6.6).

9.1. **The Fock space and Hilbert schemes on $\mathbb{A}^2$.** Let $\mathrm{Hilb}^n(\mathbb{A}^2)$ be the Hilbert scheme of n points on $\mathbb{A}^2$, see e.g. [35]. There is a natural action of $\mathbb{G}_m^2$ on it given by scaling the coordinates on $\mathbb{A}^2$. The following theorem is by now classical, and should admit an obvious generalization to other K than $K = \mathbb{C}$.

Proposition 9.1 ([35])**.** *When $K = \mathbb{C}$, the direct sum of the equivariant K-theory groups of $\mathrm{Hilb}^n(\mathbb{A}^2), n \geq 0$ is upon localization naturally isomorphic to $\mathcal{F}$:*

$$K(\mathrm{Hilb}^\bullet) := \left(\bigoplus_{n\geq 0} K^{\mathbb{G}_m^2}(\mathrm{Hilb}^n(\mathbb{A}^2))\right) \otimes_{\mathbb{C}[q^\pm,t^\pm]} \mathbb{C}(q,t) \cong \mathcal{F} \cong \mathrm{Sym}_{q,t}$$

The fixed point basis on the left corresponds to the basis $|\lambda\rangle = \widetilde{H}_\lambda$ on the right.

Keeping the above assumptions, from [26, 69] we have

Proposition 9.2. *Under the isomorphism of Proposition 9.1, the action of the elliptic Hall algebra $\mathcal{E}$ on the Fock space $\mathcal{F}$ from Theorem 4.7 is realized on $K(\mathrm{Hilb}^\bullet)$ by certain geometric correspondences.*

We recall the following Theorem from [31, Theorem 1.1.].

Theorem 9.3. *Let $F = \mathbb{C}((t))$. To each conjugacy class of (regular) semisimple $\gamma \in \mathfrak{gl}_n(F)$ and a compact subgroup $L_\gamma \subseteq Z_{G(F)}(\gamma)$, we may associate a quasi-coherent sheaf*

$$\mathcal{F}_\gamma \in \mathrm{QCoh}^{\mathbb{G}_m}(\mathrm{Hilb}^n(T^*\mathbb{G}_m))$$

with the property that

$$\Gamma(\mathrm{Hilb}^n(T^*\mathbb{G}_m), \mathcal{F}_\gamma(m)) = H_*^{L_\gamma}(\mathrm{Sp}_{t^m\gamma})$$

Moreover, the homological grading on the affine Springer fiber side can be recovered from the $\mathbb{G}_m$-action dilating cotangent fibers on $\mathrm{Hilb}^n(T^*\mathbb{G}_m)$.

This theorem is proved using a Coulomb branch $\mathbb{Z}$-algebra construction, whose details will not be explained here. Roughly speaking, $\mathcal{F}_\gamma$ is first constructed as a $\mathbb{Z}_{\geq 0}$-graded module over the homogeneous coordinate ring of $\mathrm{Hilb}^n(T^*\mathbb{G}_m))$, which can be obtained by a certain blow-up as explained in *op. cit.* The k-th graded piece of the module giving rise to $\mathcal{F}_\gamma$ is defined as $H_*^{L_\gamma}(\mathrm{Sp}_{t^k\gamma})$ and the action of the homogeneous coordinate ring on $\bigoplus_{k\geq 0} H_*^{L_\gamma}(\mathrm{Sp}_{t^k\gamma})$ is defined using a convolution product similar to [27].

Remark 9.4. The proof of [31, Theorem 1.1.] works verbatim in the case $F = k((t))$ with étale cohomology replacing singular cohomology for Sp_γ but for example the convolution product has not been defined yet in the case when F is of mixed characteristic.

9.2. **Hilbert schemes on spectral curves.** We now sketch an extension of Theorem 9.3 for L_γ the identity subgroup along the lines of [27,31], to produce from a regular semisimple $\gamma \in \mathfrak{g}(F)$ a sheaf

$$\mathcal{F}_\gamma \in \mathrm{QCoh}^{\mathbb{G}_m\times\mathbb{G}_m}(\mathrm{Hilb}(\mathbb{A}^2))$$

The other $\mathbb{G}_m$-action now present records the "number of points" grading on the homology of the Hilbert scheme of points $\mathrm{Hilb}^\bullet(C)$ for the spectral curve $C = \{\mathrm{char}(\gamma) = 0\}$, or equivalently the "connected component" grading on the homology of a *generalized* $(GL_n, \mathrm{Ad}+V)$*-affine Springer fiber* associated to the companion matrix of γ as in [27]. Note that unlike for ordinary affine Springer fibers, we need to choose a representative in the conjugacy class of γ.

As shown in *loc. cit.*, the GASF is just the intersection of the positive part of the affine Grassmannian with the ordinary affine Springer fiber for γ conjugated to its companion matrix. To construct the module appearing in Theorem 9.3, we need an analogous notion to the dilation $\gamma \mapsto t^k\gamma$ appearing in the statement of the Theorem. First of all, let γ be the companion matrix of a polynomial $f = \sum_{i=0}^n a_i x^i \in F[x]$ and let $\chi(t) = \mathrm{diag}(t^{n-1}, t^{n-2}, \ldots, t, 1)$ where t is a uniformizer of F. Then we have

Lemma 9.5. *For any k, the matrix*

$$\chi^k t^k \gamma \chi^{-k}$$

is the companion matrix of $f_k := \sum_{i=0}^n t^{(n-i)k} a_i x^i$.

We denote by C_k the germ of the plane curve singularity (recall $F \in \{k((t)), \mathbb{C}((t))\}$)

$$C_k := \{\mathrm{char}(t^k\chi^k\gamma\chi^{-k}) = f_k = 0\}$$

Example 9.6. Let $\gamma = \begin{pmatrix} 0 & t^3 \\ 1 & 0 \end{pmatrix}$ be the companion matrix of $x^2 - t^3 \in F[x]$. Then $\chi^k t^k \gamma \chi^{-k} = \begin{pmatrix} 0 & t^{3+2k} \\ 1 & 0 \end{pmatrix}$ is the companion matrix of $x^2 - t^{3+2k}$.

Now we will use the $\mathbb{Z}$-algebra construction in [31, Section 5] with "flavor symmetry". Namely, in the notation of *loc. cit.*, we let η_k be the cocharacter of $\widetilde{GL}_n(F) = GL_n(F) \times F^\times$ sending $t \mapsto (\chi^k, t^{-(n-1)k})$. The corresponding one-parameter subgroup acts on $\mathfrak{gl}_n(F) + F^n$ by $\gamma \mapsto t^k \chi^k \gamma \chi^{-k}$ and $v \mapsto t^{-(n-1)k}\chi^k v = v$, so that $(1,0,\dots,0)^t \mapsto (1,\dots,0)^t$. As explained in [31, Section 5.3.], the sequence $\{\eta_k\}_{k=0}^\infty$ of cocharacters defines a Coulomb branch $\mathbb{Z}$-algebra

$$ {}_\bullet\mathcal{A}_\bullet = \bigoplus_{i<j} {}_{\eta_i}\mathcal{A}_{\eta_j} \tag{9.1} $$

depending on two parameters $(c, \hbar)$.

Since on the Ad-factor η_k is just a twisted form of the usual dilation cocharacter $\gamma \mapsto t^k \gamma$, which corresponds to the shift functor in the trigonometric Cherednik algebra case studied in [31], we make the following conjecture.

Conjecture 9.7. *The $\mathbb{Z}$-algebra of Eq.* (9.1) *with parameters specialized to $c = -\nu\hbar, \hbar = 1$ can be identified with the $\mathbb{Z}$-algebra constructed by Gordon and Stafford in [28] for the rational Cherednik algebra of $\mathfrak{gl}_n$. Here $\nu \in \mathbb{C} - (\mathbb{Z} + \frac{1}{2})$.*

It would follow from this conjecture that at $\hbar = 0$, the above $\mathbb{Z}$-algebra degenerates to "the" homogeneous coordinate ring of $\mathrm{Hilb}^n(\mathbb{A}^2)$ as explained e.g. in [28, Proposition 1.7.].

Remark 9.8. For the "standard" dilation cocharacter η_i' acting by $(\gamma, v) \mapsto (t^i\gamma, t^i v)$, the degeneration result, i.e. that ${}_{\eta_i'}\mathcal{A}_{\eta_j'}$ at $\hbar = 0$ equals the global sections of $\mathcal{O}(i-j)$ on $\mathrm{Hilb}^n(\mathbb{A}^2)$, follows from [10, Section 3].

Assuming the above degeneration for ${}_\bullet\mathcal{A}_\bullet$, the construction of [31, Section 7] yields

Proposition 9.9. *There exits a quasi-coherent sheaf $\mathcal{F}_\gamma$ on $\mathrm{Hilb}^n(\mathbb{A}^2)$ such that by the main theorem of [27] the global sections of $\mathcal{O}(k) \otimes \mathcal{F}_\gamma$ are given by the Borel-Moore homologies of Hilbert schemes of points on the curves C_k:*

$$ H^0(\mathcal{O}(k) \otimes \mathcal{F}_\gamma) = H_*(\mathrm{Hilb}^\bullet(C_k)) $$

Finally, one hopes to compare the results of [69] and [26] on the K-theory of $\mathrm{Hilb}^n(\mathbb{A}^2)$ to the construction of $\mathcal{F}_\gamma$ as follows. Recall the convolution action of the EHA on the K–theory $K(\mathrm{Hilb}^\bullet)$ from Proposition 9.2. Parallel to the construction of the full master symmetric function $\widehat{\mathbf{f}}_\gamma$ of Definition 6.32, one can use this action to recursively construct a K-class $[\mathcal{G}_\gamma] \in K(\mathrm{Hilb}^\bullet)$ from the datum of γ as in Eq. (6.8). Via the identification of $K(\mathrm{Hilb}^\bullet)$ and Sym, we have a "deformed Shalika germ expansion"

$$ [\mathcal{G}_\gamma] = \widehat{\mathbf{f}}_\gamma = \sum_{\lambda \vdash n} \widetilde{\Gamma}_{\lambda^t}(\gamma) \widetilde{H}_\lambda $$

Remark 9.10. In the recursive construction of $\widehat{\mathbf{f}}_\gamma$, which is arguably closer to the "constructible" than the "coherent" realization of affine character sheaves, one may think of the passage from $\gamma^>$ to γ as an action by the EHA on the level of the K–group of the "rigid part" of the direct sum of derived categories of $GL_n(F)$-equivariant constructible sheaves on $\mathfrak{gl}_n(F)$, but this seems difficult to make precise.

Note that the sheaf $\mathcal{F}_\gamma$ from Proposition 9.9 is $\mathbb{G}_m^2$-equivariant. Conjecturally it is also coherent, in which case we may write its class in localized equivariant K-theory as a linear combination of fixed point classes. Recall from Proposition 9.1 that the fixed points are indexed by $\lambda \vdash n$ and correspond to $\widetilde{H}_\lambda$ in the Fock space. Now writing

$$[\mathcal{F}_\gamma] = \sum_{\lambda \vdash n} \widehat{\Gamma}_{\lambda^t}(\gamma)\widetilde{H}_\lambda$$

inside $K(\mathrm{Hilb}^\bullet)$ gives us coefficients $\widehat{\Gamma}_\lambda(\gamma) \in \mathbb{Q}(q,t)$.

It seems natural to conjecture an equality of K-theory classes $[\mathcal{F}_\gamma] = [\mathcal{G}_\gamma]$, or equivalently whether $\widehat{\Gamma}_\lambda(\gamma) = \widetilde{\Gamma}_\lambda(\gamma)$, where the LHS and RHS are as defined above. This would in particular imply the following conjecture.

Conjecture 9.11. *The coefficients $\widehat{\Gamma}_\lambda(\gamma)$ limit to the Shalika germs $\Gamma_\lambda(\gamma)$ of γ as $t \to 1$. In particular, they can be thought of as a natural t-deformation of the Shalika germs of γ and*

$$[\mathcal{F}_\gamma] \xrightarrow{t\to 1} \mathbf{f}_\gamma$$

Besides looking at K-theory classes, similar to [31, Conjecture 1.9.] one can even make the stronger conjecture that we have an equality $\mathcal{F}_\gamma = \mathcal{G}_\gamma$ for an actual (class of a) complex $\mathcal{G}_\gamma \in D^b\,\mathrm{Coh}^{\mathbb{G}_m^2}(\mathrm{Hilb}^n(\mathbb{A}^2))$ constructed using e.g. the functors in [60]. This seems too much to hope for, as the construction of $\mathcal{G}_\gamma$ involves derived pushforwards and is in some examples a genuine complex in $D^b\,\mathrm{Coh}^{\mathbb{G}_m^2}(\mathrm{Hilb}^n\,\mathbb{A}^2)$, whereas $\mathcal{F}_\gamma$ is always concentrated in a single degree by construction. It is for example possible that $\mathcal{G}_\gamma$ is the image of $\mathcal{F}_\gamma$ by a perverse autoequivalence on the Hilbert scheme, but we have no evidence to support such a guess.

Remark 9.12. When $F = \mathbb{C}((t))$ and γ is homogeneous and elliptic i.e. its characteristic polynomial is quasi-homogeneous with a single Puiseux pair (m,n) the coefficients $\widehat{\Gamma}_\lambda(\gamma)$ appear, up to multiplication by a combinatorial factor, at the end of [64, Section 5] under the name $g_{m/n}$ and some values for them are computed using explicit combinatorics of the Hilbert schemes on the spectral curves. One can check that these coefficients limit to the Shalika germs as $t \to 1$.

Conjecture 9.11 applies to *any* compact regular semisimple element, including homogeneous non-elliptic elements. For example, when γ is a split equivalued element of valuation 1, the master symmetric function $\mathbf{f}_\gamma = q^{\binom{n}{2}}p_1^n$ only has a non-vanishing leading Shalika germ, see Corollaries 6.47 and 2.44. On the other hand, [31, Proposition 9.11.] in the trigonometric case suggests that

$$[\mathcal{F}_\gamma] = \nabla p_1^n,$$

whose expansion in the modified Macdonald polynomials is quite nontrivial. However, up to a power of q, this expansion limits as $t \to 1$ to the Shalika expansion $\mathbf{f}_\gamma$, as is easy to see from the multiplicativity of ∇ at $t = 1$.

From the point of view of harmonic analysis, finding a good interpretation for the categorification and t–deformation of Shalika and other germs seems fascinating. We leave these explorations for future work.

References

[1] Altug, Salim Ali. Beyond Endoscopy via the trace formula. Diss. Princeton University, 2013.

[2] Arthur, James. "A stratification related to characteristic polynomials." Advances in Mathematics 327 (2018): 425-469.
[3] Arthur, James, and Laurent Clozel. Simple algebras, base change, and the advanced theory of the trace formula. No. 120. Princeton University Press, 1989.
[4] Aganagic, Mina, and Shamil Shakirov. "Knot homology and refined Chern–Simons index." Communications in Mathematical Physics 333.1 (2015): 187-228.
[5] Barbasch, Dan, and Allen Moy. "A new proof of the Howe conjecture." Journal of the American Mathematical Society 13.3 (2000): 639-650.
[6] Ben-Zvi, David, David Nadler, and Anatoly Preygel. "A spectral incarnation of affine character sheaves." Compositio Mathematica 153.9 (2017): 1908-1944.
[7] Bergeron, Francois, et al. "Compositional (km, kn)-shuffle conjectures." International Mathematics Research Notices 2016.14 (2016): 4229-4270.
[8] Bergeron, François. "Open questions for operators related to rectangular Catalan combinatorics." arXiv preprint arXiv:1603.04476 (2016).
[9] Bergeron, François. "Symmetric Functions and Rectangular Catalan Combinatorics." arXiv preprint arXiv:2112.09799 (2021). Journal of Combinatorics Vol. 8, No. 4 (2017), 673–703.
[10] Braverman, Alexander, Michael Finkelberg, and Hiraku Nakajima. "Line bundles on Coulomb branches." Adv. Theor. Math. Phys. Volume 25, Number 4, 957–993, 2021
[11] Burban, Igor, and Olivier Schiffmann. "On the Hall algebra of an elliptic curve, I." Duke Mathematical Journal 161.7 (2012): 1171-1231.
[12] Chen, Zongbin. "On the local behavior of weighted orbital integrals and the affine Springer fibers." arXiv preprint arXiv:2103.15091 (2021).
[13] Cherednik, Ivan, and Ivan Danilenko. "DAHA and iterated torus knots." Algebraic & Geometric Topology 16.2 (2016): 843-898.
[14] Cherednik, Ivan, and Ivan Danilenko. "DAHA approach to iterated torus links." Categorification in geometry, topology, and physics 684 (2017): 159-267.
[15] Chevalley, Claude. Introduction to the theory of algebraic functions of one variable. No. 6. American Mathematical Soc., 1951.
[16] Ciubotaru, Dan, and Xuhua He. "Cocenters and representations of affine Hecke algebras." J. Eur. Math. Soc. (JEMS) 19 (2017): 3143–3177.
[17] Clozel, Laurent. "Orbital integrals on p-adic groups: a proof of the Howe conjecture." Annals of Mathematics 129.2 (1989): 237-251.
[18] F. Courtès, Distributions invariantes sur les groupes réductifs quasi-déployés, Can. J. Math. 2006.
[19] DeBacker, Stephen. "Homogeneity results for invariant distributions of a reductive p-adic group." Ann. Sci. Éc. Norm. Supér (2002): 391-422.
[20] Deligne, Pierre, D. Kazhdan and M.-F. Vigneras, Representations des algebres centrales simples p-adiques, in Representations des groups reductifs sur un corps local, Hermann, Paris, (1984), 33-117
[21] Dunfield, Nathan M., Sergei Gukov, and Jacob Rasmussen. "The superpolynomial for knot homologies." Experimental Mathematics 15.2 (2006): 129-159.
[22] Egecioglu, Ömer, and Jeffrey B. Remmel. "Brick tabloids and the connection matrices between bases of symmetric functions." Discrete Applied Mathematics 34.1-3 (1991): 107-120.
[23] Eisenbud, David, and Walter D. Neumann. Three-dimensional link theory and invariants of plane curve singularities. No. 110. Princeton University Press, 1985.
[24] Elias, Ben, and Matthew Hogancamp. "On the computation of torus link homology." Compositio Mathematica 155.1 (2019): 164-205.
[25] Feigin, Boris, et al. "Quantum toroidal $\mathfrak{gl}_1$-algebra: Plane partitions." Kyoto Journal of Mathematics 52.3 (2012): 621-659.
[26] Feigin, B. L., and A. I. Tsymbaliuk. "Equivariant K-theory of Hilbert schemes via shuffle algebra. Kyoto J. Math. 51 (4), 831–854 (2011)." arXiv preprint arXiv:0904.1679: 21562261-1424875.
[27] Garner, Niklas, and Oscar Kivinen. "Generalized Affine Springer Theory and Hilbert Schemes on Planar Curves." International Mathematics Research Notices (2022).
[28] Gordon, Iain, and J. Toby Stafford. Rational Cherednik algebras and Hilbert schemes. Advances in Mathematics 198.1 (2005): 222–274.
[29] Goresky, Mark, Robert Kottwitz, and Robert Macpherson. "Homology of affine Springer fibers in the unramified case." Duke Mathematical Journal 121.3 (2004): 509-561.

[30] Goresky, Mark, Robert Kottwitz, and Robert MacPherson. "Codimensions of Root Valuation Strata." Pure and Applied Mathematics Quarterly 5.4 (2009): 1253-1310.

[31] Gorsky, Eugene, Oscar Kivinen, and Alexei Oblomkov. "The affine Springer fiber-sheaf correspondence." arXiv preprint arXiv:2204.00303 (2022).

[32] Gorsky, Eugene, Oscar Kivinen, and José Simental. "Algebra and geometry of link homology Lecture notes from the IHES 2021 Summer School." (2021).

[33] Gorsky, Eugene, and Andrei Neguţ. "Refined knot invariants and Hilbert schemes." Journal de mathématiques pures et appliquées 104.3 (2015): 403-435.

[34] Hales, Thomas C. "Unipotent Representations and Unipotent Classes $SL(N)$." American Journal of Mathematics 115.6 (1993): 1347-1383.

[35] Haiman, Mark. "Combinatorics, symmetric functions and Hilbert schemes." Current developments in mathematics 2002.1 (2002): 39-111.

[36] Harish-Chandra, Admissible invariant distributions on reductive p-adic groups, Preface and notes by S. DeBacker and P. Sally, University Lecture Series, vol. 16, American Mathematical Society, Providence, RI, 1999. CMP 99:16

[37] He, Xuhua. "Cocenters of-adic groups, I: Newton decomposition." Forum of Mathematics, Pi. Vol. 6. Cambridge University Press, 2018.

[38] He, Xuhua, and Sian Nie. "Minimal length elements of extended affine Weyl groups." Compositio Mathematica 150.11 (2014): 1903-1927.

[39] Hogancamp, Matthew, and Anton Mellit. "Torus link homology." arXiv preprint arXiv:1909.00418 (2019).

[40] Howe, Roger. "The Fourier transform and germs of characters (case of GL_n over a p-adic field)." Mathematische Annalen 208.4 (1974): 305-322.

[41] Katz, Nicholas. Appendix to Hausel, Tamás, and Fernando Rodriguez-Villegas. "Mixed Hodge polynomials of character varieties." Inventiones mathematicae 174.3 (2008): 555-624.

[42] Kazhdan, David, and George Lusztig. "Fixed point varieties on affine flag manifolds." Israel Journal of Mathematics 62.2 (1988): 129-168.

[43] Kim, Ju-Lee, and Fiona Murnaghan. "Character expansions and unrefined minimal K-types." American journal of mathematics 125.6 (2003): 1199-1234.

[44] Khovanov, Mikhail. "Triply-graded link homology and Hochschild homology of Soergel bimodules." International journal of mathematics 18.08 (2007): 869-885.

[45] Kottwitz, Robert E. "Harmonic analysis on reductive p-adic groups and Lie algebras." Harmonic analysis, the trace formula, and Shimura varieties. Vol. 4. Amer. Math. Soc. Providence, RI, 2005. 393-522.

[46] Lang, Serge. Algebraic number theory. Vol. 110. Springer Science & Business Media, 2013.

[47] Langlands, Robert P. "Orbital integrals on forms of $SL(3)$, I." American Journal of Mathematics 105.2 (1983): 465-506.

[48] Langlands, Robert P. "Beyond endoscopy." Contributions to automorphic forms, geometry, and number theory 611 (2004): 697.

[49] Laumon, G. "Fibres de Springer et jacobiennes compactifiées, Algebraic geometry and number theory, 515–563." Progr. Math 253.

[50] Lara, Malors Emilio Espinosa. Explorations on Beyond Endoscopy. Diss. University of Toronto, 2022.

[51] Lemaire, Bertrand. "Intégrales orbitales sur $GL(N, \mathbb{F}_q((t)))$". Journal of the Institute of Mathematics of Jussieu, 20(2), 423-515. doi:10.1017/S1474748019000227 (2021)

[52] Maulik, Davesh. "Stable pairs and the HOMFLY polynomial." Inventiones mathematicae 204.3 (2016): 787-831.

[53] Mellit, Anton. "Homology of torus knots." arXiv preprint arXiv:1704.07630 (2017).

[54] McNinch, George J. "Nilpotent orbits over ground fields of good characteristic." Mathematische Annalen 329.1 (2004): 49-85.

[55] Mœglin, Colette, and Jean-Loup Waldspurger. "Modeles de Whittaker dégénérés pour des groupes p-adiques." Mathematische Zeitschrift 196 (1987): 427-452.

[56] Morton, Hugh, and Peter Samuelson. "The HOMFLYPT skein algebra of the torus and the elliptic Hall algebra." Duke Mathematical Journal 166.5 (2017): 801-854.

[57] Murnaghan, Fiona. "Local character expansions and Shalika germs for $GL(n)$." Mathematische Annalen 304.1 (1996): 423-455.

[58] Ngô, Bao Châu. "Le lemme fondamental pour les algebres de Lie." Publications Mathématiques de l'IHÉS 111.1 (2010): 1-169.

[59] Neguţ, Andrei. "The shuffle algebra revisited." International Mathematics Research Notices 2014.22 (2014): 6242-6275.

[60] Neguţ, Andrei "Moduli of flags of sheaves and their K-theory." Algebraic Geometry (2015).

[61] Neguţ, Andrei. "The m/n Pieri rule." International Mathematics Research Notices 2016.1 (2016): 219-257.

[62] Neguţ, Andrei. "Operators on symmetric polynomials." arXiv preprint arXiv:1310.3515 (2013).

[63] Neguţ, Andrei. "An integral form of quantum toroidal $\mathfrak{gl}_1$." arXiv preprint arXiv:2209.04852 (2022).

[64] Oblomkov, Alexei, Jacob Rasmussen, and Vivek Shende. "The Hilbert scheme of a plane curve singularity and the HOMFLY homology of its link." Geometry & Topology 22.2 (2018): 645-691.

[65] Piontkowski, Jens. "Topology of the compactified Jacobians of singular curves." Mathematische Zeitschrift 255.1 (2007): 195-226.

[66] Repka, Joe. "Shalika's germs for p-adic GL (n). I. The leading term." Pacific Journal of Mathematics 113.1 (1984): 165-172.

[67] Repka, Joe. "Shalika's germs for p-adic GL (n). II. The subregular term." Pacific Journal of Mathematics 113.1 (1984): 173-182.

[68] Schiffmann, Olivier, and Eric Vasserot. "Cherednik algebras, W-algebras and the equivariant cohomology of the moduli space of instantons on $\mathbb{A}^2$." Publications mathématiques de l'IHÉS 118.1 (2013): 213-342.

[69] Schiffmann, Olivier, and Eric Vasserot. "The elliptic Hall algebra and the K-theory of the Hilbert scheme of $\mathbb{A}^2$." Duke Mathematical Journal 162.2 (2013): 279-366.

[70] Schiffmann, Olivier, and Eric Vasserot. "Hall algebras of curves, commuting varieties and Langlands duality." Mathematische Annalen 353.4 (2012): 1399-1451.

[71] Shalika, Joseph A. "A theorem on semi-simple p-adic groups." Annals of Mathematics 95.2 (1972): 226-242.

[72] Shelstad, Diana. "A formula for regular unipotent germs." Astérisque 171.172 (1989): 275-277.

[73] Shin, Sug Woo, and Nicolas Templier. "Sato–Tate theorem for families and low-lying zeros of automorphic L L-functions: With appendices by Robert Kottwitz [A] and by Raf Cluckers, Julia Gordon, and Immanuel Halupczok [B]." Inventiones mathematicae 203 (2016): 1-177.

[74] Loren Spice, "Explicit asymptotic expansions in p-adic harmonic analysis ii", arXiv preprint arXiv:2108.12935 (2021).

[75] Tsymbaliuk, Alexander. "The affine Yangian of $\mathfrak{gl}_1$ revisited." Advances in Mathematics 304 (2017): 583-645.

[76] Tsai, Cheng-Chiang. "Inductive structure of Shalika germs and affine Springer fibers." arXiv preprint arXiv:1512.00445 (2015).

[77] Tsai, Cheng-Chiang. "Components of affine Springer fibers." International Mathematics Research Notices (2020): 1882-1919.

[78] Waldspurger, Jean-Loup. "Sur les germes de Shalika pour les groupes linéaires." Mathematische Annalen 284.2 (1989): 199-221.

[79] Waldspurger, Jean-Loup. "Sur les intégrales orbitales tordues pour les groupes linéaires: un lemme fondamental." Canadian Journal of Mathematics 43.4 (1991): 852-896.

[80] Waldspurger, Jean-Loup. "Endoscopie et changement de caractéristique." Journal of the Inst. of Math. Jussieu (2006): 423-525.

[81] Zelevinsky, Andrey V. Representations of finite classical groups: a Hopf algebra approach. Vol. 869. Springer, 2006.

[82] Yun, Zhiwei. "Lectures on Springer theories and orbital integrals." In: Geometry of Moduli Spaces and Representation Theory. IAS/PCMI Series Vol. 24. American Mathematical Society (2017)

[83] Zhu, Xinwen. "Affine Grassmannians and the geometric Satake in mixed characteristic." Annals of Mathematics 185.2 (2017): 403-492.

Aalto University

Academia Sinica and NSYSU