Gromov-Witten Theory and Virtual Localization

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Abstract

Gromov-Witten theory is a curve counting theory in modern enumerative geometry, developed rigorously in the 1990s. The theory is based on the notion of a stable map from a curve to an ambient space and counts of such maps are obtained via intersection theory in the space parametrizing such maps. Many technical tools have been developed for computations in Gromov-Witten theory and perhaps the most important of these is the virtual localization formula. The goal of this thesis is to present the statement, proof and applications of this result.

In order to apply the localization formula in Gromov-Witten theory, it must be proved for geometric objects called stacks. The theorem concerns the so-called virtual classes in the Chow group of a stack and thus in order to give a rigorous statement, one has to define three things: stacks, their Chow groups and virtual classes. The definitions are somewhat involved, mostly scattered in several original papers and often ignored in introductory material. This thesis presents the essential constructions and results of the original papers. The proof of the localization theorem is then based on the definition of virtual classes and properties of Chern classes on stacks.

The rest of the thesis focuses on applications of the localization theorem in Gromov-Witten theory. Before the applications, a general introduction to Gromov-Witten theory is presented. More specifically, the stack of stable maps is constructed and some important geometric properties of stable maps are proved. The virtual localization formula is then applied to the Gromov-Witten theory of the projective line. In particular, using an explicit formula for the localization, Hurwitz numbers are expressed as integrals of tautological classes in the moduli space of curves. As another application, the explicit formula is used to evaluate so-called Hodge integrals. These examples illustrate the interesting and non-trivial consequences of the localization theorem.

Keywords Gromov-Witten theory, stacks, Chow group, virtual fundamental class, virtual localization, Hurwitz numbers, Hodge integrals

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Tiivistelmä

Gromovin–Wittenin teoria on modernin enumeratiivisen geometrian osa-alue, jossa pyritään laskemaan käyrien lukumääriä jossain ympäröivässä avaruudessa. Teorian matemaattisesti täsmällinen perusta muotoiltiin tarkasti 1990-luvulla ja pohjautuu niin sanotun stabiilin kuvauksen käsitteeseen. Gromovin–Wittenin teoriassa on käytössä monia teknisiä työkaluja, joista mahdollisesti tärkein on virtuaaliluokan lokalisaatiolause. Tämän työn tavoitteena on esittää lauseen täsmällinen muotoilu ja todistus sekä sovelluksia Gromovin–Wittenin teoriassa.

Jotta lokalisaatiolausetta voidaan soveltaa Gromovin–Wittenin teoriassa, se täytyy muotoilla skeemoja yleisemmille geometrisille objekteille, joita kutsutaan pinoiksi. Lokalisaatiolause koskee pinojen virtuaaliluokkia, jotka ovat luokkia pinojen Chow-ryhmässä. Lauseen täsmällistä muotoilua varten tulee siis määritellä kolme asiaa: pinot, niiden Chow-ryhmät ja virtuaaliluokat. Näiden objektien tarkat määritelmät ovat suhteellisen teknisiä ja hajallaan useissa alkuperäisissä artikkeleissa ja tässä työssä on tavoitteena antaa tiivis, mutta täsmällinen yhteenveto näiden artikkeleiden tärkeimmistä tuloksista. Lokalisaatiolauseen todistus perustuu virtuaaliluokan määritelmään ja Chern-luokkien ominaisuuksiin pinoilla.

Lokalisaatiolauseen todistuksen jälkeen työssä esitellään lokalisaatiolauseen sovelluksia Gromovin–Wittenin teoriassa. Gromovin–Wittenin teorian perusteiden esittelyn jälkeen keskitytään projektiviisen suoran tapaukseen ja sovelletaan lokalisaatiolausetta kahdessa esimerkissä. Ensimmäiseksi näytetään, miten Hurwitz-luvut voidaan esittää niin sanottujen tautologisten luokkien integraaleina, ja toiseksi johdetaan suljettu muoto niin sanotuille Hodge-integraaleille. Molemmat esimerkit havainnollistavat lokalisaatiolauseen mielenkiintoisia ja epätriviaaleja seurauksia.

Avainsanat Gromovin–Wittenin teoria, pinot, Chow-ryhmä, virtuaalinen perusluokka, virtuaalilokalisaatio, Hurwitz-luvut, Hodge-integraalit

Contents

1 Introduction

Given a smooth projective variety X and a homology class $\beta \in H_2(X)$, we can construct the moduli stack $\overline{M}_{g,n}(X,\beta)$ of stable maps from a genus g curve with n marked points to X representing the class β. In Gromov-Witten theory, we are interested in this stack and its intersection theory. For example, Gromov-Witten invariants are obtained by intersecting certain cycles in the Chow group of $\overline{M}_{g,n}(X,\beta)$ with the so-called virtual fundamental class. Concrete computations in the theory are often difficult and it is very convenient to have powerful general tools at one's disposal. In case X is equipped with a torus action, there is an important tool called the virtual localization formula. The action on X induces an action on the stack $\overline{M}_{g,n}(X,\beta)$ and the localization formula allows one to reduce computations on $\overline{M}_{g,n}(X,\beta)$ to computations on the fixed stack. This tool has been used for example in Gromov-Witten theory of $\mathbb{P}^1$ in [24] and [9] and in Donaldson-Thomas theory in [23].

The standard reference for the localization formula in the literature is [14]. That paper is written mostly in the language of schemes and the authors note in the appendix how the proof is adapted to stacks. The aim in this thesis is to develop the technical language needed to state and prove the result entirely in the language of stacks, which is the right setting for applications in Gromov-Witten theory and other areas of modern enumerative geometry. The proof in this thesis is mostly extracted from the more general proof given in [6] with some simplifications. For a reader with some familiarity with the basics of Gromov-Witten theory, this thesis can hopefully serve as an introduction to some of the more technical aspects underlying modern enumerative geometry, which are often ignored or only treated in special cases in introductory material (e.g. in [12]). Let us now describe the contents of each section in more detail.

The first technical difficulty of Gromov-Witten theory is that the space of stable maps $\overline{M}_{g,n}(X,\beta)$ is generally a stack and not a scheme. Thus, in section 2 we will introduce the language of stacks using definitions of Mumford [7] and Artin [2]. In general, the moduli spaces $\overline{M}_{g,n}(X,\beta)$ will not be "nice" even as Deligne-Mumford stacks. They will often be singular and of impure dimension, but they will however be proper (compact), which is a crucial thing we want.

In Gromov-Witten theory, we want to compute intersections on the stack $\overline{M}_{g,n}(X,\beta)$ and thus one has to show that the essential parts of Fulton's intersection theory from [11] can be transferred to stacks. The theory of Chow groups on stacks was introduced by Vistoli in the case of Deligne-Mumford stacks [27] and later by Kresch for algebraic stacks [18]. In section 3 we will summarize the essential definitions and results of these papers. We will mostly omit the proofs.

Since the stacks $\overline{M}_{g,n}(X,\beta)$ are in general badly behaved, we define the notion of a virtual fundamental class which should in some sense be analogous to the usual fundamental class of a smooth stack. Defining Gromov-Witten invariants by intersecting with a virtual fundamental class is crucial for Gromov-Witten invariants to satisfy the axioms proposed in [16]. In section 4, we will define virtual fundamental classes in a general algebro-geometric setting following the work of Behrend and Fantechi [4].

In section 5, we turn to the main result of this thesis. We will state and prove the virtual localization formula in the generality of [14] (see Theorem 5.2 for the precise statement). The proof is very close to the one given in [6], but we will prove the result in a slightly less general context, avoiding the language of cosections.

In section 6, we will give an introduction to Gromov-Witten theory. In particular, we will give a rigorous definition of the stacks $\overline{M}_{g,n}(X, \beta)$ and prove some of their geometric properties. We will also construct virtual fundamental classes for these stacks using the definitions from section 4.

Section 7 is dedicated to showing how localization turns integrals over $\overline{M}_{g,n}(\mathbb{P}^1, d)$ to graph combinatorics and computations in various moduli spaces of curves. As the first application of this, we will first show how Hurwitz numbers are related to integrals in the moduli space of curves, following [24]. The second application is to so-called Hodge integrals in $\overline{M}_{g,1}$, following [9].

2 Introduction to Stacks

Modern moduli theory makes heavy use of the language of stacks which we now introduce. Let us first explain on a high level why stacks are needed.

2.1 Moduli spaces as stacks

Suppose we want to parametrize some class of algebro-geometric objects (e.g. smooth curves of genus g or vector bundles on a scheme) by some kind of algebro-geometric space. In the ideal situation, we are able to find a scheme M that does the job. This means that a morphism of schemes $T \to M$ should correspond to a family of the geometric objects over T. Thus we want

$$\{\text{families over } T\} \leftrightarrow \{\text{morphisms } T \to M\}.$$

In modern algebraic geometry, this can be expressed by saying that if we look at the category of families of the geometric objects over schemes then M represents this category.

So it is natural to start with the category and try to find a scheme representing it. However, due to presence of automorphisms and other technical problems, we cannot always construct such a representative as a scheme. Thus it is important to study these categories by themselves. This is precisely where stacks come in. With this motivation in mind, we will present the precise definitions.

2.2 Stacks

If $\mathcal{X}, \mathcal{S}$ are categories and $p\colon \mathcal{X} \to \mathcal{S}$ is a functor, we say $\mathcal{X}$ is a *category over* $\mathcal{S}$. For an object $U \in \mathcal{S}$, we denote by $\mathcal{X}(U)$ the objects in $\mathcal{X}$ over U, i.e. objects $T \in \mathcal{X}$ such that $p(T) = U$. $\mathcal{X}(U)$ is a category when taking the morphisms to be morphisms over id_U.

Definition 2.1. Let $\mathcal{S}$ be a category. A category $\mathcal{X}$ over $\mathcal{S}$ is said to be *fibered in groupoids* over $\mathcal{S}$ if

(i) Given a morphism $\varphi\colon U \to V$ in $\mathcal{S}$ and an object $Y \in \mathcal{X}(V)$, there is $X \in \mathcal{X}(U)$ and a morphism $f\colon X \to Y$ over φ.

(ii) These lifts satisfy the following universal property. Given morphsisms $h\colon X \to Z$, $g\colon Y \to Z$ in $\mathcal{X}$ over morphsims $\varphi\colon U \to W$ and $\phi\colon V \to W$ in $\mathcal{S}$. If $\psi\colon U \to V$ is a morphism s.t. $\phi\psi = \varphi$ in $\mathcal{S}$ then there is a unique lift $f\colon X \to Y$ of ψ s.t. $gf = h$.

Note that property (ii) above implies that the lifts of property (i) are unique up to unique isomorphism. Suppose from now on, that for a morphism $\varphi\colon U \to V$ in $\mathcal{S}$ and an object $Y \in \mathcal{X}(V)$ we fix a lift $f\colon X \to Y$ of φ and denote this by $X = \varphi^* Y$. For a composition $U \xrightarrow{\varphi} V \xrightarrow{\psi} W$, $(\psi\varphi)^*$ and $\varphi^*\psi^*$ are the same up to unique isomorphism. We assume in what follows that a category $\mathcal{S}$ has products and fiber products.

Definition 2.2. Let $\mathcal{S}$ be a category. A *Grothendieck topology* on $\mathcal{S}$ assigns to each object U in $\mathcal{S}$, a set of families of morphisms $\{U_i \to U\}_i$, called *coverings* of U, that satisfy the following properties:

(i) If $V \to U$ is an isomorphism in $\mathcal{S}$ then $\{V \to U\}$ is a covering of U.

(ii) If $\{U_i \to U\}$ is a covering of U, then for a morphism $V \to U$ the set $\{U_i \times_U V \to V\}$ is a covering of V (we assumed fiber products exist in $\mathcal{S}$).

(iii) If $\{U_i \to U\}_i$ and $\{U_{ij} \to U_i\}_j$ are coverings, then the compositions $\{U_{ij} \to U_i \to U\}_{i,j}$ form a covering.

A *site* is a category with a Grothendieck topology. More material on Grothendieck topologies can be found in [28].

Definition 2.3. Let Sch be the category of schemes. Given a scheme X, an étale cover is a family of étale morphisms $\{U_i \to X\}_i$ such that $\bigsqcup U_i \to X$ is surjective. This defines a Grothendieck topology on Sch and we call the resulting site the *big étale site* and denote it by $\mathrm{Sch}_{\text{ét}}$.

We now give the definition of a stack.

Definition 2.4. Let $\mathcal{X}$ be a category fibered in groupoids over a site $\mathcal{S}$. $\mathcal{X}$ is called a *stack* if

(a) (Morphism glue) Given $V \in Ob(\mathcal{S})$ and $X, Y \in \mathcal{X}(V)$, the functor

$$\begin{aligned} \mathcal{S}(V) &\to \text{Sets} \\ (U \xrightarrow{\varphi} V) &\mapsto \text{Hom}_{\mathcal{X}(U)}(\varphi^* X, \varphi^* Y) \end{aligned}$$

is a sheaf.

(b) (Objects glue) Let $\{U_i \to U\}$ be a covering of U in $\mathcal{S}$. Given $X_i \in \mathcal{X}(U_i)$ and isomorphisms $\varphi_i \colon X_{ij} \to X_{ji}$ over U_{ij} satisfying the cocycle conditions, there is an object $X \in \mathcal{X}(U)$ s.t. X_i is (isomorphic to) the pullback of X via $U_i \to U$.

Morphisms of stacks $\mathcal{X}, \mathcal{Y}$ over $\mathcal{S}$ is a functor $f \colon \mathcal{X} \to \mathcal{Y}$ that (strictly) commutes with the morphsism $p_{\mathcal{X}} \colon \mathcal{X} \to \mathcal{S}$, $p_{\mathcal{Y}} \colon \mathcal{Y} \to \mathcal{S}$. From now on, we consider $\mathcal{S} = \text{Sch}_{\text{ét}}$.

Example. Given a scheme X, we can consider the category over $\text{Sch}_{\text{ét}}$ whose objects over a scheme T are morphisms $T \to X$ and the morphisms are X-morphisms of schemes. One can check that this is a stack over $\text{Sch}_{\text{ét}}$. We will denote this stack by the same letter X and speak of a scheme also in the case we actually mean the stack associated to a scheme.

Example. Let $g \geq 2$ be an integer. Consider the category M_g whose objects over a scheme T are smooth proper morphisms $C \to T$, whose geometric fibers are connected genus g curves. Morphisms between families are Cartesian diagrams

$$\begin{array}{ccc} C' & \longrightarrow & C \\ \downarrow & & \downarrow \\ T' & \longrightarrow & T \end{array}$$

Then M_g is a stack.

Definining the fiber product of stacks is quite easy. Let $f \colon \mathcal{X} \to \mathcal{Z}, g \colon \mathcal{Y} \to \mathcal{Z}$ be morphisms of stacks. Then the fiber product $\mathcal{X} \times_{\mathcal{Z}} Y$ is the category consisting of objects (x, y, φ), where $x \in \mathcal{X}(T)$, $y \in \mathcal{Y}(T)$ and an isomorphism $\varphi \colon f(x) \xrightarrow{\sim} g(y)$. A morphism $(x, y, \varphi) \to (x', y', \varphi')$ of objects over S and T respectively is defined by a triple (h, α, β), consisting of a morphism $h \colon S \to T$ and morphisms $\alpha \colon x \to x'$ and $\beta \colon y \to y'$ over h s.t.

$$\begin{array}{ccc} f(x) & \xrightarrow{\varphi} & g(y) \\ \downarrow{\scriptstyle f(\alpha)} & & \downarrow{\scriptstyle g(\beta)} \\ f(x') & \xrightarrow{\varphi'} & g(y') \end{array}$$

commutes. We get a 2-commutative diagram

$$\begin{array}{ccc} \mathcal{X} \times_{\mathcal{Z}} Y & \xrightarrow{p_2} & \mathcal{Y} \\ \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle g} \\ \mathcal{X} & \xrightarrow{f} & \mathcal{Z} \end{array}$$

that satisfies the usual universal property (where the diagrams now only 2-commute).

Definition 2.5. A morphism $F\colon \mathcal{X} \to \mathcal{Y}$ of stacks is representable by schemes if for any scheme S and a morphism $S \to \mathcal{Y}$, the fiber product $\mathcal{X} \times_{\mathcal{Y}} S$ is a scheme.

Proposition 2.1. *Let $\mathcal{X}$ be a stack. The diagonal $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is representable if and only if for any schemes S, T and morphisms $S \to \mathcal{X}$ and $T \to \mathcal{X}$, the fibre product $S \times_{\mathcal{X}} T$ is a scheme or equivalently any morphism $S \to \mathcal{X}$ from a scheme is reprentable by schemes.*

Proof. For schemes S and T and morphisms $S \to \mathcal{X}$ and $T \to \mathcal{X}$, the square

$$\begin{array}{ccc} S \times_{\mathcal{X}} T & \longrightarrow & S \times T \\ \downarrow & & \downarrow \\ \mathcal{X} & \longrightarrow & \mathcal{X} \times \mathcal{X} \end{array}$$

is Cartesian so if the diagonal is representable then $S \times_{\mathcal{X}} T$ is a scheme.

Conversely, suppose that every $S \to \mathcal{X}$ is representable by schemes. Let $f\colon T \to \mathcal{X} \times \mathcal{X}$ be a morphism from a scheme. Then the projections $f_i\colon T \to \mathcal{X}$ define a morpshim $p = f_1 \times f_2\colon T \times T \to \mathcal{X} \times \mathcal{X}$ and $p \circ \Delta_T = f$. We can thus decompose the Cartesian square of $T \times_{\mathcal{X} \times \mathcal{X}} T$ into two Cartesian squares

$$\begin{array}{ccccc} T \times_{\mathcal{X} \times \mathcal{X}} T & \longrightarrow & T \times_{\mathcal{X}} T & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow & & \downarrow \Delta \\ T & \xrightarrow{\Delta_T} & T \times T & \xrightarrow{p} & \mathcal{X} \times \mathcal{X} \end{array}$$

By assumption $T \times_{\mathcal{X}} T$ is a scheme so everything in the left square is a scheme. In particular $T \times_{\mathcal{X} \times \mathcal{X}} T$ is a scheme which is what we wanted to show. ☐

Definition 2.6. Let **P** be property of morphisms of schemes that is stable under base change and étale local on the target. A morphism $F\colon \mathcal{X} \to \mathcal{Y}$ representable by schemes is said to have property **P** if for any scheme S and a morphism $S \to \mathcal{Y}$, the morphism $\mathcal{X} \times_{\mathcal{Y}} S \to S$ has property **P**.

2.3 Deligne-Mumford stacks

Next, we define the notion of a Deligne-Mumford stack. There are slightly different conventions depending on the author but we have chosen a definition given by Deligne and Mumford in [7]. This is also the definition used in [4], [27] and [18].

Definition 2.7. We say a stack $\mathcal{X}$ is *quasi-separated* if the diagonal morphism is representable, quasi-compact and separated.

Definition 2.8. A stack $\mathcal{X}$ is a *Deligne-Mumford stack* (DM stack for short) if

(a) the diagonal morphism $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is representable (by schemes) and

(b) there is a scheme U and a morphism $U \to \mathcal{X}$ that is surjective and étale.

The morphism $U \to \mathcal{X}$ above is called an *atlas* of $\mathcal{X}$ and is representable by (a) using Proposition 2.1.

Deligne and Mumford note that this is the right definition for quasi-separated DM stacks. The stacks appearing in applications are indeed usually quasi-separated so this definition suffices for our purposes. Next, we introduce several properties of Deligne-Mumford stacks that will be needed.

Definition 2.9. A stack $\mathcal{X}$ is *separated* if the diagonal morphism $\mathcal{X} \to \mathcal{X} \times_{\mathrm{Spec}(\mathbb{Z})} \mathcal{X}$ (which is representable by schemes) is proper.

Let us now see how properties of schemes and morphisms of schemes transfer to DM stacks. Let $\mathbf{P}$ be a property of morphisms of schemes that is stable under base change and étale local on the domain and codomain. More precisely, this means that if $f\colon U \to V$ is a morphism of schemes and $\{U_i \to U\}$ and $\{V_j \to V\}$ are étale covers then for diagrams

$$\begin{array}{ccc} U_i & \longrightarrow & U \\ {\scriptstyle f_i}\downarrow & & \downarrow{\scriptstyle f} \\ V_i & \longrightarrow & V \end{array}$$

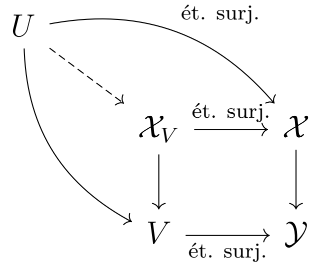
we have $\mathbf{P}(f_i)$ for all i if and only if $\mathbf{P}(f)$. Such properties include e.g flat, smooth, locally of finite type, unramified étale, etc...

Proposition 2.2. *Let $F\colon \mathcal{X} \to \mathcal{Y}$ be a representable morphism of DM stacks. Then $\mathbf{P}$ holds for F if and only if for atlases $U \to \mathcal{X}$ and $V \to \mathcal{Y}$ and a commutative diagram*

$$\begin{array}{ccc} U & \longrightarrow & \mathcal{X} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle F} \\ V & \longrightarrow & \mathcal{Y} \end{array}$$

$\mathbf{P}$ *holds for f.*

Proof. Suppose that we're given $U \to \mathcal{X}, V \to \mathcal{Y}$ as in the statement. From the universal property of fiber products, we have the diagram



In fact, the dashed arrow is also an étale surjection. Since the property $\mathbf{P}$ is étale local on the source, by descent, the morphism $\mathcal{X}_V \to V$ has property $\mathbf{P}$ if and only if

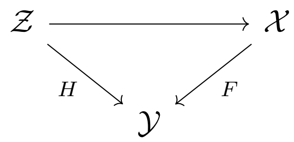
$U \to V$ does. It is easy to show that this suffices. □

Motivated by this, a (not necessarily representable) morphism $F\colon \mathcal{X} \to \mathcal{Y}$ of DM stacks is said to have the property $\mathbf{P}$ if for every commutative diagram of atlases as in the above proposition, the map f has the property $\mathbf{P}$. In fact, by a descent argument, it suffices to find only one such diagram.

In a similar fashion, we say that a DM stack $\mathcal{X}$ has some étale local property of schemes if there is an atlas $U \to \mathcal{X}$ where U has that property. We say also that $\mathcal{X}$ is *quasi-compact* if there is an atlas $U \to \mathcal{X}$ such that U is quasi-compact and a morphism $F\colon \mathcal{X} \to \mathcal{Y}$ is quasi-compact if for any quasi-compact scheme T, the morphsism $\mathcal{X} \times_{\mathcal{Y}} T$ is quasi-compact.

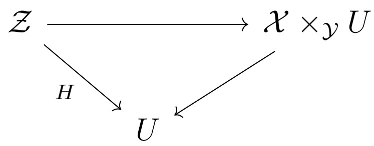
An important and nice property of DM stacks that we want our moduli stacks to have is properness. First, a morphism $F\colon \mathcal{X} \to \mathcal{Y}$ of DM stacks is of finite type if it is quasi-compact and locally of finite type.

Definition 2.10. A morphism $F\colon \mathcal{X} \to \mathcal{Y}$ of DM stacks is *proper* if it is separated, finite type and, locally over $\mathcal{Y}$, there is a DM stack $\mathcal{Z}$ and a commutative diagram



with G surjective, H representable and proper.

When we say some property $\mathbf{P}$ of morphisms holds for $F\colon \mathcal{X} \to \mathcal{Y}$ locally over $\mathcal{Y}$ we mean that there is an atlas $U \to \mathcal{Y}$ s.t. the property holds for $\mathcal{X} \times_{\mathcal{Y}} U \to U$. So if F is proper, there is an atlas $U \to \mathcal{Y}$ and a commutative diagram:



so that H is representable and proper and G is surjective.

2.4 Algebraic stacks

The notion of an algebraic stack is weaker than Deligne-Mumford in two respects. First, we require that the diagonal is representable by algebraic spaces. Recall that an algebraic space is a *sheaf* of sets X on $\mathrm{Sch}_{\text{ét}}$ s.t. there is a surjective étale representable morphism $U \to X$ from a scheme U. We say a morphism $\mathcal{X} \to \mathcal{Y}$ of stacks is *representable by algebraic spaces* if for any $\mathcal{Y}$-scheme T, the fiber product $\mathcal{X} \times_{\mathcal{Y}} T$ is an algebraic space (thought of as a stack). The second thing is that the presentation by a scheme is now only smooth and not necessarily étale.

Definition 2.11. A stack $\mathcal{X}$ over $\mathrm{Sch}_{\text{ét}}$ is an *algebraic stack* if

(i) The diagonal $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is representable by algebraic spaces.

(ii) There is a scheme U and a smooth surjective morphism $U \to \mathcal{X}$.

Note now that property (i) above implies $U \to \mathcal{X}$ from a scheme U is automatically representable by algebraic spaces. Thus if $T \to \mathcal{X}$ is a morphism from a scheme $U \times_{\mathcal{X}} T \to T$ is a morphism from an algebraic space. Saying this is smooth and surjective means that if we take an étale presentation $W \to U \times_{\mathcal{X}} T$, then the composition

$$W \to U \times_{\mathcal{X}} T \to T$$

is a smooth and surjective morphism of schemes.

3 Chow Groups on Stacks

This section aims to summarize the basic definitions and results regarding Chow groups on stacks. As a reference we use the papers of Vistoli [27] and Kresch [18]. The constructions of Vistoli are preformed on Deligne-Mumford stacks and are natural generalizations of the theory developed by Fulton in [11]. It is imporant to note that with the definitions of Vistoli, some important constructions from Fulton's theory e.g. intersection products and Chern classes work only with $\mathbb{Q}$-coefficients. Another important thing to note is that push-forward by a proper non-representable morphism is defined only with rational coefficients. This is important when we want to do enumerative geometry on a Deline-Mumford stack X proper over k. The degree map is defined as the push-forward of the non-representable proper structure morphism $p\colon X \to \operatorname{Spec} k$ and hence defined only with rational coefficients.

Kresch's definition is a bit more technical, but produces a theory of integral Chow groups on algebraic stacks. An important thing to note is that the push-forward in this theory is in general defined for so called projective morphisms which are in particular representable. Representability is required if we want integer coefficients, as noted in the previous paragraph.

If the reader is not concerned about technicalities, in many cases one can assume that the basic results that hold for intersection theory on schemes are valid for stacks too. However, one has to be careful whether some result holds for algebraic stacks or only Deligne-Mumford stacks and whether one has to work with $\mathbb{Q}$- coefficients instead of $\mathbb{Z}$.

3.1 Naïve Chow groups

In this section we summarize the basic construction given in Vistoli. The construction of the Chow groups and basic operations takes place on Deligne-Mumford stacks. The construction of Chow groups follows closely the definition for schemes in [11], i.e. we simply take integral closed substacks modulo rational equivalence. For algebraic stacks this definition is not enough, but these "naïve" Chow groups will serve as a starting point of the more general definition.

We assume in this section that all stacks are DM stacks and all stacks and schemes are of finite type over a fixed field k. We also assume that for a DM stack X, there is a scheme M and a proper morphism $X \to M$.

Definition 3.1. Let X be a stack. A k-cycle is an element of the free abelian group $Z_k(X)$ generated by integral closed substacks of X of dimension k. We set

$$Z_*(X) = \bigoplus_{k\geq 0} Z_k(X).$$

We define

$$W_k(X) = \bigoplus_{Z} k(Z)^*,$$

where the sum is taken over integral substacks Z of dimension $k+1$ and $k(Z)$ denotes the field of rational functions consisting of morphisms $U \to \mathbb{A}^1$ for open substack U of Z. The *group of rational equivalences* is then

$$W_*(X) = \bigoplus_{k\geq 0} W_k(X).$$

As in the case of schemes we should be able to define a divisor associated with a rational function and therefore a homomorphism

$$W_k(X) \to Z_k(X)$$

and the cokernel of this map is the Chow group.

A novel way of formulating this is presented in [13] and goes as follows. We consider a stack X and the associated étale site $X_{\text{ét}}$ (the category of étale morphisms $U \to X$). Since étale is in particular flat, we get a presheaf $\mathscr{Z}$ of groups on the étale site defined simply by

$$(U \xrightarrow{\text{ét}} X) \mapsto Z(U).$$

Similarly we get a presheaf $\mathscr{W}$ corresponding to the groups $W(U)$ for schemes.

In fact, one can pretty easily show that this is a sheaf. The map $\operatorname{div}\colon W(U) \to Z(U)$ for schemes gives a morphism of sheaves $\operatorname{div}\colon \mathscr{Z} \to \mathscr{W}$. In particular we have a map of the groups of global sections and we claim that these are equal to $Z(X)$ and $W(X)$ defined previously.

Let $U \to X$ be an atlas for X. Then the group of global sections of a sheaf F on the étale site $X_{\text{ét}}$ is given by

$$\Gamma(X, F) = Eq(F(U \to X) \rightrightarrows F(U \times_X U \to X))$$

where the two maps are induced by the two projections. For $\mathscr{Z}$, we have a natural map $Z(X) \to Z(U)$ and showing that $Z(X) \cong \Gamma(X, F)$ amounts to showing that the sequence

$$Z(X) \to Z(U) \rightrightarrows Z(U \times_X U)$$

is exact. This is the content of Lemmas 4.2 and 4.3 in [13].

Thus we get a map $\operatorname{div}\colon W(X) \to Z(X)$ and

Definition 3.2. we define the kth Chow group $A_k(X)$ to be the cokernel of

$$\mathrm{div}\colon W_k(X) \to Z_k(X)$$

and the Chow group of X to be

$$A_*(X) = \bigoplus_{k\geq 0} A_k(X)$$

and the Chow group with rational coefficients is

$$A_*(X)_{\mathbb{Q}} = A_*(X) \otimes \mathbb{Q}.$$

Let $Z \to X$ be a closed substack (not necessarily integral) and let $U \to X$ be a presentation. We note that $Z_U = Z \times_X U$ is a closed subscheme of U and defines a cycle $[Z_U] \in Z(U)$. We have of course $p_1^*[Z_U] = [Z \times_X (U \times_X U)] = p_2^*[Z_U]$ and since $\mathscr{Z}$ is a sheaf, $[Z_U]$ comes from a unique cycle $[Z] \in Z(X)$ called the *cycle associated to Z*.

Next, we define the pullbacks and pushforwards. To define pushforwards we need to know what the degree of a morphism of stacks means. In [27], the definitions are motivated by two properties for schemes. First, we recall how the degree of a morphism of schemes is defined. Let $f\colon X \to Y$ be a separated dominant morphism of finite type and Y integral. If f generically quasi-finite, there is a nonempty open subset $U \subseteq Y$ s.t. $f^{-1}(U) \to U$ is flat and finite. In this case we define $\deg(X/Y)$ be the degree of $f_*\mathcal{O}_X$ as a locally free sheaf on U. If f is not generically quasi-finite, we set $\deg(X/Y) = 0$.

Suppose now that $X \to Y, V \to Y$ are separated dominant morphism of integral schemes. We have

$$\deg(X/Y) = \deg(X \times_Y V/V).$$

Also, if $U \to X$ is a separated dominant morphism of integral schemes then

$$\deg(U/Y) = \deg(U/X)\deg(X/Y)$$

so that

$$\deg(X/Y) = \deg(U/Y)/\deg(U/X).$$

The first one can be used as a definition for representable morphism of stacks and the second in the non-representable case.

Definition 3.3. Let $f\colon X \to Y$ be a separated dominant morphism of finite type of integral stacks. Let $U \to X$ and $V \to Y$ be integral atlases. If f is representable we define

$$\deg(X/Y) = \deg(X \times_Y V/V)$$

and if not, we define

$$\deg(X/Y) = \deg(U/Y)/\deg(U/X).$$

Lemma 1.16 of [27] shows that these notions of degree are well-defined. With our assumption that X admits proper morphism to a scheme, the map $I_X \to X$ from the inertia stack is representable, finite and separated. Thus we may define

$$\delta(X) = \deg(I_X/X).$$

For the next result, recall that for a field extension K/L, we have $[K : L] = \deg(K/L)$ if the extension is finite and $[K : L] = 0$ if the extension is infinite.

Proposition 3.1. *Let $f\colon X \to Y$ be a separated dominant morphism of finite type of integral stacks. Then*

$$\deg(X/Y) = \frac{\delta(X)}{\delta(Y)}[K(X) : K(Y)].$$

Now can define f^* for f flat, f_* for f proper. And after this we transfer the basic construction in [11] to stacks. This gives us an intersection product and importantly it gives us the refined Gysin homomorphisms.

The definitions are now as follows:

Definition 3.4. Let $f\colon X \to Y$ be a morphism of DM stacks.

(a) (Push-forward) If f is proper, then for an integral closed substack V of X we define

$$f_*[V] = \deg(V/W)[W],$$

where W is the stack theoretic image of V in Y. Note that since the degree might be rational in general we get

$$f_*\colon Z(X)_{\mathbb{Q}} \to Z(Y)_{\mathbb{Q}}.$$

In case f is representable we have

$$f_*\colon Z(X) \to Z(Y).$$

(b) (Pull-back) If f is flat we define

$$f^*[W] = [W \times_Y X]$$

for an integral closed substack W of Y. This defines

$$f^*\colon Z(Y) \to Z(X)$$

Theorem 3.1. *The above homomorphisms pass to rational equivalence and we obtain pull-backs and push-forwards for DM stacks.*

3.2 Chow groups on algebraic Stacks

Next, we will introduce Kresch's construction of Chow groups on algebraic stacks [18]. The approach of Vistoli does not work directly with algebraic stacks, since in many cases there are not enough integral substacks to get a meaningful theory. We present this more general definition and note that in the new Chow groups will reduce to ones defined earlier when we restrict to DM stacks and work with rational coefficients.

We will then show that these more general Chow groups have flat pull-backs and *projective* push-forwards and that they satisfy the usual properties. We will also construct the top Chern class operator and based on that define the other Chern classes. Importantly, the defining property and the other basic properties (e.g. commutativity, projection and pull-back formulas) of Chern classes continues to hold in this more general context. Lastly, we need to define refined Gysin pull-backs for regular local immersions and then see that these in fact satisfy the functoriality, commutativity and compatibility with pull-backs and push-forwards.

3.2.1 The Chow group

Although we noted that Vistoli's naïve Chow groups do not in themselves give a nice theory for algebraic stacks, they can be defined also in the category of algbraic stacks and we denote these by

$$A_k^\circ(X) = \operatorname{coker}(W_k(X) \to Z_k(X))$$

following [18]. The Chow groups $A_*(X)$ will be constructed via two limits. First we take a direct limit of the naïve Chow groups of vector bundles of X. We obtain groups denoted $\widehat{A}_k(X)$. The second direct limit is a limit of certain quotients of $\widehat{A}_k(Y)$ taken over projective morphisms $Y \to X$ which finally gives us $A_k(X)$. Here are the details.

In what follows, a stack means an algebraic stack of finite type over a field. The definitions of Edidin and Graham in [8] can be rephrased to define *Edidin-Graham-Totaro Chow groups* as

$$\widehat{A}_k(X) = \varinjlim_{\mathfrak{B}_X} A^\circ_{k+\operatorname{rank}(E)}(E),$$

where $\mathfrak{B}_X$ is the set of isomorphism classes of vector bundles on X. By Remark 2.1.5 in [18], this is a direct system and thus the limit makes sense. This is a quite natural definition but this is a sufficient definition only for algebraic stacks that are quotient stacks $X = [Y/G]$. In this case we recover the groups which Edinin-Graham denote by $A_k^G(Y)$ as noted in Remarks 2.1.7 in Kresch.

For example, we will see that stacks of stable maps – central objects in Gromov-Witten theory – are quotient stacks and the above definition suffices. However, in some cases, we have an algebraic stack X that only has a *stratification* by quotient stacks. This means that there are locally closed substacks $U_i = [Y_i/G_i]$ of X that form a stratification. Roughly speaking, we then take the Chow group of X to be

$$\bigsqcup \widehat{A}(U_i)$$

modulo some reasonable identifications. Let us make this precise.

Definition 3.5. A morphism $f\colon Y \to X$ is *projective* if it factors (up to 2-isomorphism) as

$$Y \to \mathbf{P}(\mathcal{E}) \to X,$$

where the first arrow is a closed immersion, $\mathcal{E}$ is a coherent sheaf of $\mathcal{O}_X$-modules and the second arrow is the projection.

We now define the directed set of projective X-stacks.

Definition 3.6. A *projective X-stack* is a pair (Y, f), where $f\colon Y \to X$ is a projective morphism. A *morphism of projective X-stacks* $(Y, f) \to (Y', f')$ is a morphism $\varphi\colon Y \to Y'$ together with a 2-morphism α from f to $f' \circ \varphi$. We say (φ, α) is an *inclusion of components* if φ is an isomorphism onto a union of connected components of Y'. We denote by $\mathfrak{A}_X$, the set of isomorphism classes of projective X-stacks. We define $(Y, f) \preceq (Y', f')$ whenever there is an inclusion of components $(Y, f) \to (Y', f')$. With this partial ordering, $\mathfrak{A}_X$ forms a directed set.

Given a morphism $f\colon X \to Y$ of stacks, we often consider the groups

$$\widehat{A}_k^f(X) = \varinjlim_{\mathfrak{B}_Y} A^{\circ}_{k+\operatorname{rank}(E)}(f^*E),$$

called the *restricted Edinin-Graham-Totaro* Chow groups. Note that we have natrual morphims $\iota_f\colon \widehat{A}_k^f(X) \to \widehat{A}_k(X)$ and $f_*\colon \widehat{A}_k^f(X) \to \widehat{A}_k(Y)$. With this in mind, let (Y, f) be a projective X-stack. Given a pair of projective morphisms $p_1, p_2\colon T \to Y$ of stacks, we define the set

$$\widehat{B}_k^{p_1,p_2}(Y) = \{p_{1*}\alpha_1 - p_{2*}\alpha_2 \mid \beta_i \in \widehat{A}_k^{p_i}(T) \text{ and } \iota_{p_1}(\alpha_1) = \iota_{p_2}(\alpha_2)\}$$

Then the union

$$\widehat{B}_k(Y) = \bigcup_{f\circ p_1 \simeq f \circ p_2} \widehat{B}_k^{p_1,p_2}(Y),$$

where $\simeq$ means 2-isomorphism, is a subgroup and it turns out that $(\widehat{A}_k(Y)/\widehat{B}_k(Y))_{(Y,f)\in\mathfrak{A}_X}$ is a direct system and we make our final definition of the Chow groups of algebraic stacks.

Definition 3.7. For an algebraic stack X, the Chow groups $A_k(X)$ are defined by

$$A_k(X) = \varinjlim_{\mathfrak{A}_X} (\widehat{A}_k Y / \widehat{B}_k Y).$$

We let

$$A_*(X) = \bigoplus_k A_k(X)$$

and

$$A_*(X)_{\mathbb{Q}} = A_*(X) \otimes \mathbb{Q}$$

For concreteness, let us see how we represent elements of $A_*(X)$. By definition an element $\alpha \in A_k(X)$ is represented by a pair (f, α'), where $f\colon Y \to X$ is a projective morphism and $\alpha' \in \widehat{A}_k Y/\widehat{B}_k Y$. We may choose a representative $\alpha' \in \widehat{A}_k Y$. We usually denote an element of $A_k(X)$ by this pair (f, α'). Sometimes we go further and choose a representative $\alpha'' \in A^\circ_{k+\mathrm{rank}(E)}(E)$ of α' for some vector bundle $E \to Y$.

3.2.2 Basic operations

The first thing to do now is, of course, to define push-forwards and pull-backs. These are very natural and one can easily guess what the right definitions are.

Definition 3.8. Let $f\colon X \to Y$ be a projecive morphism. Then given (h, α) where $h\colon Z \to X$ and $\alpha \in \widehat{A}_k(Z)$ we simply set

$$f_*(h, \alpha) = (f \circ h, \alpha)$$

Next, suppose $f\colon X \to Y$ is flat. Then given (h, α) as above, think of $\alpha \in A^\circ(F)$ for a vector bundle F on Z. These form the Cartesian diagram

$$\begin{array}{ccc} f'^*F & \xrightarrow{\tilde{f}'} & F \\ \downarrow & & \downarrow \\ Z' & \xrightarrow{f'} & Z \\ \downarrow h' & & \downarrow h \\ X & \xrightarrow{f} & Y \end{array}$$

Then we simply define

$$f^*(h, \alpha) = (h', \tilde{f}'^* \alpha)$$

Note that $\tilde{f}'^*$ descends to $f'^*\colon \widehat{A}_*(Z) \to \widehat{A}_*(Z')$ and we may equivalently define

$$f^*(h, \alpha) = (h', f'^* \alpha).$$

These definitions are independent of the representatives chosen and therefore gives us the desired homomorphisms

$$f^*\colon A(Y) \to A(X)$$

and

$$f_*\colon A(X) \to A(Y).$$

The following result is useful for dealing with push-forwards. We will see a consequence of this shortly, when proving an imporant identity of top Chern classes. This is Remark 2.1.16 in [18].

Lemma 3.1. *Let $f\colon X \to Y$ and $p_1, p_2\colon T \to X$ be projective morphisms s.t. $f \circ p_1$ and $f \circ p_2$ are 2-isomorphic. Given $\beta_i \in \widehat{A}^{p_i}(X)$ for $i = 1, 2$ we have*

$$(f, p_{2*}\beta_2 - p_{1*}\beta_1) = (p_1 \circ f, \iota_{p_2}(\beta_2) - \iota_{p_1}(\beta_1))$$

in $A_*(Y)$. In particular,

$$(f \circ p_{1*}) = (f \circ p_1, \iota_{p_1}(\beta_1))$$

Proof. As hinted by Kresch we consider $q_1, q_2\colon T \sqcup T \to X \sqcup T$ where and $q_1 = p_1 \sqcup 1_T$ and $q_2 = 1_T \sqcup p_2$. Then we notice that $\iota_{q_1}(\beta_1, \beta_2) = \iota_{q_2}(\beta_1, \beta_2)$ and hence we have by definition of the subgroups $\widehat{B}_*$ that

$$\begin{aligned} q_{1*}(\beta_1, \beta_2) &= (p_{1*}\beta_1, \iota_{p_2}(\beta_2)) \\ &= q_{2*}(\beta_1, \beta_2) = (p_{2*}\beta_2, \iota_{p_1}(\beta_1)) \\ &\in \widehat{A}_*(X \sqcup T)/\widehat{B}_*(X \sqcup T). \end{aligned}$$

Rearranging gives

$$p_{2*}\beta_2 - p_{1*}\beta_1 = \iota_{p_2}(\beta_2) - \iota_{p_1}(\beta_1) \in \widehat{A}_*(X \sqcup T)/\widehat{B}_*(X \sqcup T).$$

Now $X \to X \sqcup T$ and $T \to X \sqcup T$ are inclusions of components and the above shows that $(f, p_{2*}\beta_2 - p_{1*}\beta_1)$ and $(f \circ p_1, \iota_{p_2}(\beta_2) - \iota_{p_1}(\beta_1))$ agree in $X \sqcup T$ so this shows that when we pass to the limit, we obtain

$$(f, p_{2*}\beta_2 - p_{1*}\beta_1) = (f \circ p_1, \iota_{p_2}(\beta_2) - \iota_{p_1}(\beta_1))$$

□

The next theorem tells us that push-forwards and pull-backs have the nice basic properties that we would expect. In particular they are functorial and satisfy the push-pull formula.

Theorem 3.2. *The Chow groups $A_*(X)$ defined above satisfy the following properties.*

(i) (Functoriality) If f, g are flat, then

$$(f \circ g)^* = g^* \circ f^*.$$

If f, g are projective then

$$(f \circ g)_* = f_* \circ g_*.$$

(ii) (Push-pull formula) If the diagram

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

is Cartesian, f is projective and g is flat, then we have

$$g^* f_* = f'_* g'^*.$$

Proof. Immediate from the definitions. □

3.2.3 Chern classes

Next, we define the top Chern class operator for vector bundle E on a stack X. This definition is also extremely simple.

Definition 3.9. Let $\pi\colon E \to X$ be a vector bundle on a stack X. An element of $A_k(X)$ is represented by (f, α), where $f\colon Y \to X$ is a projective morphism $\pi'\colon F \to Y$ is a vector bundle and $\alpha \in A^{\circ}_{k+\mathrm{rank}(F)}(F)$. Let $s\colon F \to F \oplus f^*E$ be the zero section. We define

$$c_{\mathrm{top}}(E) \cap (f, \alpha) = (f, s_*\alpha).$$

Here s_* is defined on the naïve Chow groups.

This is well-defined and descends to rational equivalence to give

$$c_{\mathrm{top}}(E) \cap -\colon A_k(X) \to A_{k-r}(E),$$

where $r = \mathrm{rank}(E)$.

Here's the basic property for the top Chern class that will be used many times in the proof of the virtual localization formula.

Proposition 3.2. *Let $E \to X$ be a vector bundle on a stack X and let s be the zero section of X. Then*

$$s_*\alpha = \pi^*(c_{\mathrm{top}}(E) \cap \alpha)$$

for $\alpha \in A(X)$.

Proof. Let us see what the right hand side gives us. We consider α as (f, α), where $f\colon Y \to X$ projective and $\alpha \in \widehat{A}_k(Y)$. We can further think of α as being an element of $A^{\circ}(F)$ for a vector bundle F on Y. We form the three level Cartesian square

$$\begin{array}{ccc} f^*E \oplus F & \longrightarrow & F \\ \downarrow & & \downarrow \\ f^*E & \xrightarrow{\pi'} & Y \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ E & \xrightarrow{\pi} & X \end{array}$$

We have by definition that

$$c_{\mathrm{top}}(E) \cap (f, \alpha) = (f, s''_*\alpha),$$

where s'' is the zero section of the bundle $f^*E \oplus F \to F$. Next, we look at the diagram

$$\begin{array}{ccc} f^*E \oplus f^*E \oplus F & \longrightarrow & f^*E \oplus F \\ \downarrow & & \downarrow \\ f^*E & \xrightarrow{\pi'} & Y \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ E & \xrightarrow{\pi} & X \end{array}$$

and note that by definition

$$\pi^*(f, \beta) = (f', p_{2,3}^*\beta)$$

where $p_{2,3}\colon f^*E \oplus f^*E \oplus F \to f^*E \oplus F$ is the projection onto the second and third factors. If $p_{1,3}$ is the projection onto the first and third factors then Lemma 3.1 shows that $(f', p_{2,3}^*\beta) = (f', p_{1,3}^*\beta)$ and since $p_{1,3}^*\beta$ and β define the same class in the limit over vector bundles, we see that

$$\pi^*(f, \beta) = (f', p_{2,3}^*\beta) = (f', \beta'),$$

where $\beta' \in \widehat{A}(f^*E)$ is the element determined by β. But note that β' is simply $s'_*\alpha$. Thus we've shown

$$\pi^*(c_{\text{top}}(E) \cap (f, \alpha)) = (f', s'_*\alpha).$$

Now we simply note that

$$s_*(f, \alpha) = (s \circ f, \alpha) = (f' \circ s', \alpha) = (f', s'_*\alpha)$$

where the last equality is again Lemma 3.1. This completes the proof. □

With this definition, we can define Segre classes exactly as they are defined in Fulton and then general Chern classes are defined also as in Fulton. For a vector bundle E on a stack X one defines the projective bundle $p\colon P(E) \to X$ just as for schemes. Then there is a tautological line bundle $\mathcal{O}_E(1)$ on $P(E)$ and we define

$$s_i(E) \cap \alpha = p_*(c_{\text{top}}(\mathcal{O}_E(1))^i \cap p^*\alpha).$$

Then one forms the total Segre class

$$s(E) = \sum s_i(E)t^i$$

and the Chern classes $c_i(E)$ come from the total Chern class

$$c(E) = s(E)^{-1} = \sum c_i(E)t^i$$

The following properties of Chern classes are the most important for our purposes. The statements are almost identical to those of Fulton.

Theorem 3.3. *Let $f\colon X \to Y$ be a morphism of stacks and let E be a vector bundle on Y.*

(i) (Pull-back formula) If f is flat then

$$f^*(c_i(E) \cap \alpha) = c_i(f^*E) \cap f^*\alpha$$

for $\alpha \in A(Y)$.

(ii) (Projection formula) If f is flat then

$$f_*(c_i(f^*E) \cap \alpha) = c_i(E) \cap f_*\alpha$$

for $\alpha \in A(X)$.

(iii) (Whitney sum formula) If

$$0 \to F \to E \to F' \to 0$$

is a short exact sequence of vector bundles on X, then

$$c(E) = c(F)c(F').$$

Remark. The formula $s_*\alpha = \pi^*(c_{\text{top}}(E) \cap \alpha)$ holds for $c_r(E)$ too and thus $c_r(E) = c_{\text{top}}(E)$ and we can use these interchangeably. In particular all the nice properties (projection, pull-back, commutativity etc...) hold for the top Chern class.

The next result is crucial for a good theory of Chow groups. It states that if we have a vector bundle or more generally a vector bundle stack (to be defined in next section) the pull-back of the projection morphism gives an isomorphism of Chow groups (up to an appropriate shift in degree).

Theorem 3.4. *Let E be a vector bundle (or more generally a vector bundle stack) on a Deligne-Mumford stack X. Then $\pi^*\colon A_k(X) \to A_{k+\text{rank}(E)}(E)$ is an isomorphim.*

A curious reader can find the proof in Kresch as Proposition 4.3.2. The proof uses Noetherian induction as the correponding proof in Fulton, but uses an extended excision sequence which requires some homological algebra which we will not introduce in this thesis.

Given a vector bundle E on a Deligne-Mumford stack X with a zero section $s\colon X \to E$, we denote by s^* the inverse of of the pull-back π^*. Applying s^* is often referred to as "interesecting with the zero section s". Note that Proposition 3.2 takes the form

$$s^*s_*\alpha = c_{\text{top}}(E) \cap \alpha.$$

3.2.4 Refined Gysin Homomorphisms

Recall that a morphism $f\colon X \to Y$ is a local immersion if f is representable and there is a smooth atlas $V \to Y$ and closed immersion of schemes $g\colon S \to T$ and a commutative diagram

$$\begin{array}{ccc} S & \xrightarrow{g} & T \\ \downarrow & & \downarrow \\ X \times_Y V & \xrightarrow{f'} & V \end{array}$$

with the vertical maps being étale surjections. We say that f is a regular local immersion if g is a regular immersion. To define the refined Gysin homomorphism, we shall need a specialization homomorphism coming from deformation to the normal cone as in Fulton. Let us recall the construction.

Given a local immersion $f\colon X \to Y$ we can form the deformation space $M^\circ = M^\circ_X Y$ with

$$\begin{array}{ccc} X \times \mathbb{P}^1 & \hookrightarrow & M^\circ \\ & \searrow \quad \swarrow & \\ & \mathbb{P}^1 & \end{array}$$

so that restricted to $\mathbb{A}^1 = \mathbb{P}^1 \setminus \{\infty\}$ we get the immersion $X \times \mathbb{A}^1 \to Y \times \mathbb{A}^1$ and over ∞ we get a closed immersion $X \hookrightarrow C_X Y = C$. We consider the diagram

$$\begin{array}{ccccccc} A_{k+1}(C) & \xrightarrow{i_*} & A_{k+1}(M^\circ) & \xrightarrow{j^*} & A_{k+1}(Y \times \mathbb{A}^1) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle i^*} & & \uparrow{\scriptstyle p^*} & & \\ & & A_k(C) & \xleftarrow[\sigma]{\quad\cdots\quad} & A_k(Y) & & \end{array}$$

where the top row is the excision sequence, i^* is given by intersection with the divisor C in M° and p^* is the pull-back of the projection $Y \times \mathbb{A}^1 \to Y$. Then $i^* i_* = 0$ and by exactness of the top row, we get σ as a composition

$$\sigma\colon A_k(Y) \xrightarrow{p^*} A_{k+1}(Y \times \mathbb{A}^1) \xrightarrow{(j^*)^{-1}} A_{k+1}(M^\circ)/i_*(A_{k+1}(C)) \xrightarrow{i^*} A_k(C)$$

which is called the *specialization homomorphism*. We can describe it more explicitly as follows.

Let $g\colon T \to Y$ be projective, $E \to T$ a vector bundle and consider $[V] \in A^\circ(E)$. Then unwinding definitions and copying the argument from Proposition 5.2 in [11] gives

$$\sigma(g, [V]) = (g', [C_V W]).$$

Now we finally come to the construction of refined Gysin pull-backs which is the most important construction in intersection theory.

Let $i\colon X \to Y$ be a regular local immersion so that $C = C_X Y$ is a vector bundle stack and let $g\colon Y' \to Y$ be any morphism. We form the Cartesian diagram

$$\begin{array}{ccc} X' & \xrightarrow{i'} & Y' \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ X & \xrightarrow{i} & Y \end{array}$$

Now $C' = C_{X'} Y' \hookrightarrow g^* C$ and if $s\colon X' \to g^* C$ is the zero section, we define $i^!$ as the composition

$$i^!\colon A_k(Y') \xrightarrow{\sigma} A_k(C') \to A_k(g^* C) \xrightarrow{s^*} A_k(X').$$

4 Introduction to Virtual Fundamental Classes

The reference for this section is the original paper of Behrend and Fantechi [4]. The idea here is to give a minimal summary of the basic definitions and results needed for the constructions. The localization formula in Section 5 concerns the objects defined in this section. Let us now give some motivation for the rather abstract constructions that we're about to present.

In Gromov-Witten theory, a central goal is to compute Gromov-Witten invariants which are intersections of certain classes in the moduli space of stable maps. However, the moduli space is often very ill-behaved, e.g. reducible, of impure dimension and highly singular and computing intersections on the whole moduli space does not produce invariants with nice properties. The "nice properties" here mean that the Gromov-Witten invariants should produce a so-called *cohomological field theory* formulated axiomatically by Kontsevich and Manin in [16]. In [4], Behrend and Fantechi constructed a so called virtual fundamental class which has the property that taking intersections against this class makes the axioms of cohomological field theory hold for GW invariants. In this section we describe the necessary constructions for arbitrary DM stacks and later look at the special case of the moduli of stable maps. We now give a very brief summary of the construction and then hop into more details. A more concrete introduction to virtual classes with lots of examples can be found in [3].

Given a DM stack X of finite type over a field k, we construct the so called intrinsic normal sheaf $\mathfrak{N}_X$ and a closed subcone $\mathfrak{C}_X$ of $\mathfrak{N}_X$, called the intrinsic normal cone. The cone $\mathfrak{C}_X$ will be of pure dimension 0 and is an object related to the singularities of X. An obstruction theory is a map of complexes $\phi\colon E^\bullet \to L_X^\bullet$, where $L_X^\bullet$ is the cotangent complex of X and $E^\bullet$ and ϕ satisfy certain conditions in cohomology. The obstruction theory is said to be perfect if $E^\bullet$ is isomorphic to a two-term complex of vector bundles and in this case we can construct a vector bundle stack $\mathfrak{E}$ with closed immersions

$$\mathfrak{C}_X \hookrightarrow \mathfrak{N}_X \hookrightarrow \mathfrak{E}.$$

We obtain the virtual class $[X]^{vir}$ by intersecting the class of $\mathfrak{C}_X$ in $A_0(\mathfrak{E})$ with the zero section of $\mathfrak{E}$.

We will now explain the construction in more detail. In this section we assume all stacks to be quasi-separated and locally of finite type over a field k. The constructions use derived categories and the reader can find a brief introduction to derived categories in the appendix.

4.1 Cone stacks

Let us begin by introducing the notion of a cone stack.

Definition 4.1. Let X be a DM stack.

(a) (Cones) Let $S = \bigoplus_{i\geq 0} S^i$ be a graded quasi-coherent sheaf of $\mathcal{O}_X$-algebras, s.t.

$S^0 = \mathcal{O}_X$ and S^1 generates S. Then we can form an X-stack

$$C = \mathrm{Spec}(S)$$

called a *cone* over X. A morphism of cones is an X-morphism induced by a graded morphism of graded $\mathcal{O}_X$-algebras. A closed immersion of cones defines a *closed subcone*. The morphsim $0\colon X \to C$ defined by projection $S \to \mathcal{O}_X$ is the *vertex* of C and there is an $\mathbb{A}^1$-action $\gamma\colon \mathbb{A}^1 \times C \to C$ given by a moprhism $S \to S[x]$ defined by $s \in S^i \mapsto sx^i$. (The axioms of an $\mathbb{A}^1$-action are the natural ones and are spelled out in [4])

(b) (Abelian cones) A cone of the form

$$C(\mathscr{F}) = \mathrm{Spec}(\mathrm{Sym}(\mathscr{F}))$$

for a coherent sheaf of $\mathcal{O}_X$-modules $\mathscr{F}$ is called an *abelian cone*.

Example. A vector bundle E on X is an abelian cone $E = C(\mathcal{E}^\vee)$, where $\mathcal{E}$ is the sheaf of sections of E.

Example. If $i\colon X \to Y$ is a closed immersion with ideal sheaf $\mathscr{I}$ then

$$C_{X/Y} = \mathrm{Spec}\left(\bigoplus_{n\geq 0} \mathscr{I}^n/\mathscr{I}^{n+1}\right)$$

is the *normal cone* of X in Y and

$$N_{X/Y} = C((\mathscr{I}/\mathscr{I}^2)^\vee)$$

is the *normal sheaf* of X in Y.

Exact sequences of cones are defined in a natural way.

Definition 4.2. Let E be a vector bundle and C, D cones over X. A sequence

$$0 \to E \to C \to D \to 0$$

is *exact* if locally over X the sequence splits, i.e., there is a morphism $C \to E$ inducing an isomorphism $C \to E \times D$.

Given a vector bundle E and a cone C on X with a morphism $d\colon E \to C$, we define what it means for E to act on C. This gives rise to the notion of C being an E-cone and imporantly we can form a quotient $[C/E]$ which will be an object called a *cone stack*. Here are the precise definitions.

Let $d\colon E \to C$ be a morphism where $E = \mathrm{Spec}(\mathrm{Sym}(\mathcal{E}^\vee))$ is a vector bundle and $C = \mathrm{Spec}(\bigoplus S^i)$ is a cone. Then d is defined by a morphism

$$\bigoplus S^i \to \mathrm{Sym}(\mathcal{E}^\vee)$$

which gives

$$S^1 \to \mathcal{E}^\vee$$

and hence

$$S^1 \to \mathcal{E}^\vee \oplus S^1$$

and finally this induces

$$\gamma\colon E \times A(C) = \operatorname{Spec}(\operatorname{Sym}(\mathcal{E}^\vee \oplus S^1)) \to A(C) = \operatorname{Spec}(\operatorname{Sym}(S^1))$$

Definition 4.3. If E and C are as above, we say C is an E-*cone*, if C is invariant under the action $\gamma\colon E \times A(C) \to A(C)$ defined above.

We can now define a quotient $[C/E]$ for an E-cone C as follows. The construction is completely analogous to how we define quotient stacks of group actions.

Constuction. *Given an X-scheme T, we define the objects over T in $[C/E]$ to be diagrams*

$$\begin{array}{ccc} P & \longrightarrow & C \\ \big\downarrow & & \\ T & & \end{array}$$

where P is a E-torsor over T and $P \to C$ is E-equivariant. Recall that an E-torsor over T is a T-stack P with an E-action s.t. $P \to T$ is E-invariant and locally trivial. $[C/E]$ is an X-stack and comes with a vertex $0\colon X \to [C/E]$ and an $\mathbb{A}^1$-action which are defined in the obvious way.

We will show that these quotients are examples of *cone stacks* which we now define.

Definition 4.4 (Cone, abelian cone and vector bundle stacks)**.** Let $(\mathfrak{C}, 0, \gamma)$ be an algebraic X-stack together with a section $0\colon X \to \mathfrak{C}$ and an $\mathbb{A}^1$-action $\gamma\colon \mathbb{A}^1 \times \mathfrak{C} \to \mathfrak{C}$ is a *cone stack* if étale locally there is a smooth surjective $\mathbb{A}^1$-equivariant morphism $C \to \mathfrak{C}$ from a cone C over X such that $C \times_{\mathfrak{C},0} X$ is a vector bundle. We call such $C \to \mathfrak{C}$ a *local presentation.* If the local presentations can be chosen to be abelian cones (resp. vector bundles) then $\mathfrak{C}$ is a an abelian cone (resp. vector bundle) stack.

Example. If $d\colon E \to C$ is a morphism that makes C an E-cone then the diagram

$$\begin{array}{ccc} E \times C & \xrightarrow{\sigma} & C \\ \big\downarrow{\scriptstyle p} & & \big\downarrow \\ C & \longrightarrow & [C/E] \end{array}$$

is Cartesian, where σ is the action and p is the projection. Note that we also have the Cartesian diagram

$$\begin{array}{ccc} E & \xrightarrow{id \times 0} & E \times C \\ \big\downarrow & & \big\downarrow{\scriptstyle p} \\ X & \xrightarrow{0} & C \end{array}$$

and combining with the previous diagram shows that

$$\begin{array}{ccc} E & \xrightarrow{d} & C \\ \downarrow & & \downarrow \\ X & \xrightarrow{0} & [C/E] \end{array}$$

is Cartesian. Thus $C \to [C/E]$ is a global presentation for $[C/E]$ so that $[C/E]$ is a cone stack. We will meet this Cartesian diagram again when proving the localization formula (see Lemma 5.2).

4.2 Intrinsic normal cone

Let X be a Deligne-Mumford stack. We let $X_{\text{ét}}$ be the small étale site and we let X_{fppf} be the big fppf site of X and denote the corresponding topoi (categories of sheaves) by the same letters. A sheaf on X_{fppf} can be restricted to a sheaf on $X_{\text{ét}}$, denote this functor by v_*. We also have the functor v^{-1}

$$(v^{-1}F)(U \to X) = \lim_{V \xrightarrow{\text{ét}} X} F(V \to X)$$

where the limit is taken over étale morphisms $V \to X$ s.t. $U \to X$ factors through $V \to X$. These functors define a morphism of topoi

$$v\colon X_{\text{fppf}} \to X_{\text{ét}}.$$

In fact, if $\mathcal{O}_{\text{fppf}}$ and $\mathcal{O}_{\text{ét}}$ are the sheaves of rings given by $\mathcal{O}_X$, then we have a morphism of sheaves

$$v^{-1}\mathcal{O}_{\text{ét}} \to \mathcal{O}_{\text{fppf}}$$

and so v is a morphism of ringed topoi. Thus we get an induced functor

$$v^*\colon \mathrm{Mod}(X_{\text{ét}}) \to \mathrm{Mod}(X_{\text{fppf}})$$
$$v^*(M) = \mathcal{O}_{\text{fppf}} \otimes_{v^{-1}\mathcal{O}_{\text{ét}}} v^{-1}M$$

This is a right exact functor and we have enough projectives so we may take the left derived functor in the derived category

$$Lv^*\colon D^-(\mathcal{O}_{\text{ét}}) \to D^-(\mathcal{O}_{\text{fppf}}).$$

We write $M^\bullet_{\text{fppf}} = Lv^*M^\bullet$ for $M^\bullet \in D^-(\mathcal{O}_{\text{ét}})$. We also have the right derived functor

$$R\mathscr{H}om(-, \mathcal{O}_{\text{fppf}})\colon D^-(\mathcal{O}_{\text{fppf}}) \to D^+(\mathcal{O}_{\text{fppf}})$$

and we denote the image of $M^\bullet \in D^-(\mathcal{O}_{\text{fppf}})$ by

$$M^{\bullet\vee} := R\mathscr{H}om(M^\bullet, \mathcal{O}_{\text{fppf}}).$$

In what follows, we will be interested in complexes $M^{\bullet\vee}_{\text{fppf}} = R\mathscr{H}om(Lv^*M^\bullet, \mathcal{O}_{\text{fppf}})$. Also, given a complex $E^\bullet$ of abelian sheaves on a topos X, we introduce the notation

$$h^1/h^0(E^\bullet) := [\ker(E^1 \to E^2)/\mathrm{coker}(E^{-1} \to E^0)].$$

Now given a Deligne-Mumford stack of finite type over a field k, it has a *cotangent complex* $L_X^\bullet \in D^-(\mathcal{O}_{\text{ét}})$ [25]. Using the above notation, we can make the following definition.

Definition 4.5. Let X be a DM stack of finite type over a field k. We define the *intrinsic normal sheaf* of X to be

$$\mathfrak{N}_X := h^1/h^0(((L_X^\bullet)_{\text{fppf}})^\vee)$$

where we recall that

$$((L_X^\bullet)_{\text{fppf}})^\vee = (Lv^*L_X^\bullet)^\vee = R\mathscr{H}om(Lv^*L_X^\bullet, \mathcal{O}_{\text{fppf}}).$$

We construct the intrinsic normal cone $\mathfrak{C}_X$ of a DM stack locally of finite type over k as follows. Consider the diagram

$$\begin{array}{ccc} U & \xrightarrow{f} & M \\ \big\downarrow{\scriptstyle i} & & \\ X & & \end{array}$$

where U affine k-scheme of finite type, i étale, M smooth affine k-scheme of finite type and f local immersion. We call (U, M) a *local embedding* of X.

From basic algebraic geometry we get the sequence

$$T_U \to f^*T_M \to N_{U/M} \to 0$$

and we know that there is an embedding $C_{U/M} \hookrightarrow N_{U/M}$. The interesting thing to note is that $C_{U/M}$ is invariant under the action of f^*T_M and hence is a f^*T_M-cone. We can thus form the quotient $[C_{U/M}/f^*T_M]$. One can the glue together these cone stacks coming from all local embeddings to form a cone stack $\mathfrak{C}_X$ which is the *intrinsic normal cone.*

Proposition 4.1. *Let X be a DM stack locally of finite type over a field k. Then there is a cone stack $\mathfrak{C}_X$ over X s.t. for any local embedding (U, M) of X, we have*

$$\mathfrak{C}_X\Big|_U = [C_{U/M}/f^*T_M].$$

We also have

$$\mathfrak{N}_X\Big|_U = [N_{U/M}/f^*T_M].$$

We obtain a closed embedding

$$\mathfrak{C}_X \hookrightarrow \mathfrak{N}_X.$$

So the above proposition gives a fairly concrete geometric local description of the intrinsic normal sheaf and intrinsic normal cone.

4.3 Obstruction theories and the virtual fundamental class

The final thing in the construction of the virtual fundamental class involves a choice. We need to construct a so called perfect obstruction theory which then naturally defines a vector bundle stack where $\mathfrak{N}_X$ and hence $\mathfrak{C}_X$ live as closed substacks.

Let us first give the abstract general definition and then see a result of Behrend and Fantechi that gives a deformation theoretic meaning to the construction.

Definition 4.6. Let X be a DM stack and let $E^\bullet \in D(\mathcal{O}_{X_{\text{ét}}})$ satisfy the following two conditions.

(1) $h^i(E^\bullet) = 0$ for $i > 0$.

(2) $h^i(E^\bullet)$ coherent for 0 and -1.

Then a morphism $\phi\colon E^\bullet \to L_X^\bullet$ is called an *obstruction theory for X* if $h^0(\phi)$ is an isomorphism and $h^{-1}(\phi)$ is surjective.

Let us now see why these objects deserve the name "obstruction theory". Let X be a DM stack and let $g\colon T \to X$ be a morphism from a scheme. In deformation theory, we're interested in studying the infinitesimal deformations of such maps. This is formally done by taking a square zero extension of T i.e. a scheme $\overline{T}$ and a closed embedding $T \to \overline{T}$ s.t. the ideal sheaf J of T has $J^2 = 0$ and asking whether g can be extended to $\overline{g}\colon \overline{T} \to X$. Note that g gives rise to

$$g^* L_X^\bullet \to L_T^\bullet$$

and

$$L_T^\bullet \to L_{T/\overline{T}}^\bullet \xrightarrow{\sim} \tau_{\geq -1} L_{T/\overline{T}}^\bullet = J[1].$$

Composition of these gives

$$\omega(g) \in \operatorname{Hom}(g^* L_X^\bullet, J[1]) = \operatorname{Ext}^1(g^* L_X^\bullet, J).$$

Basic deformation theory says that $\omega(g) = 0$ iff extensions $\overline{g}$ exist. $g^* L_X^\bullet \to L_T^\bullet \to L_{T/\overline{T}}^\bullet$ defines a morphism

$$ob(g)\colon C(J) = h^1/h^0(L_{T/\overline{T}}^{\bullet\vee}) \to h^1/h^0(g^* L_X^{\bullet\vee}) = g^*\mathfrak{N}_X.$$

We also get a morphism

$$g^*\phi^\vee\colon g^*\mathfrak{N}_X \to g^*\mathfrak{E}$$

and we denote $\phi^\vee(ob(g)) = g^*\phi^\vee \circ ob(g)$. Finally, we have the following result.

Theorem 4.1 (Behrend-Fantechi). *Let $\phi\colon E^\bullet \to L_X^\bullet$ be a morphism as in definition 4.9. Then the following are equivalent.*

(a) $\phi\colon E^\bullet \to L_X^\bullet$ is an obstruction theory.

(b) $\phi^\vee$ is a closed immersion.

(c) *The obstruction* $\phi^*(\omega(g)) = \omega(g) \circ \phi = 0$ *iff an extension* $\overline{g}$ *exists. In this case the extensions are a* $\mathrm{Ext}^0(g^*E^\bullet, J)$*-torsor.*

(d) *The sheaf of extensions* $\underline{\mathrm{Ext}}(g, \overline{T})$ *(a sheaf on* $T_{\acute{e}t}$*) is isomorphic to the sheaf* $\underline{\mathrm{Hom}}(\phi^\vee(ob(g)), 0)$ *of* $\mathbb{A}^1$ *equivariant morphisms, where* 0 *denotes the vertex* $0\colon C(J) \to g^*\mathfrak{E}$*.*

So obstruction theories contain information about the deformations and their obstructions. We finally come to the main definitions of this section.

Recall that an element $E^\bullet \in D(\mathcal{O}_{X_{\acute{e}t}})$ is of *perfect amplitude contained in* $[-1, 0]$ if it is locally isomorphic (in $D(\mathcal{O}_{X_{\acute{e}t}})$) to a two term complex $E^{-1} \to E^0$ of finite rank vector bundles.

Definition 4.7. We say an obstruction theory $E^\bullet \to L_X^\bullet$ is *perfect* if $E^\bullet$ is of perfect amplitude contained in $[-1, 0]$.

This is a way of guaranteeing that $\mathfrak{E}$ above is actually a vector bundle stack. Indeed, if $\phi\colon E^\bullet \to L_X^\bullet$ is a perfect obstruction theory, then locally we have

$$\mathfrak{E} = h^1/h^0((E^\bullet_{\mathrm{fppf}})^\vee) \cong h^1/h^0([E_0 \to E_1]) = [E_1/E_0]$$

and this is precisely by definition what it means to be a vector bundle stack. Thus by the intersection theory of Kresh, we can intersect with the zero section of $\mathfrak{E}$.

Definition 4.8. Given a perfect obstruction theory $\phi\colon E^\bullet \to L_X^\bullet$ of X, we define the *virtual fundamental class* of X to be

$$[X]^{vir} := 0^!_{\mathfrak{E}}[\mathfrak{C}_X].$$

Up to this point, we have basically just given a whole lot of definitions, so let us now turn to the main techical result of this thesis, namely the torus localization formula for the virtual classes.

5 Virtual Localization

In this section our goal is to prove a very important theorem concerning Deligne-Mumford stacks equipped with a torus action. Roughly speaking, the theorem says that we can compute the virtual fundamental class defined above by computing the virtual fundamental class of the fixed stack up to some normal bundle contributions.

As we will see later, in case of Gromov-Witten invariants of $\mathbb{P}^1$, this reduces integrals on the moduli of stable maps to integrals on the moduli stacks of stable curves and combinatorics of certain graphs.

The result was proven for torus actions on schemes by Graber and Pandharipande in [14] and they noted in the appendix why the statement is true also for DM stacks. They made the technical assumption that the scheme or a stack admits an embedding into a smooth scheme or stack. This assumption can be dropped and we will present the proof given in the more recent paper by Chang, Kiem and Li [6] with some modifications. We start by giving the definitions and a few crucial lemmas needed for the precise statement and proof.

5.1 The setup and statement

Let X be a proper DM stack over $\mathbb{C}$ with an action of a torus $T = \mathbb{C}^*$. In [26] a quotient stack $[X/T]$ is constructed as T-torsors as usual. We shall assume that $[X/T]$ is also a Delinge-Mumford stack and of locally finite type over $\mathbb{C}$. We shall define the *equivariant Chow group* of X of the torus action to be

$$A_*^T(X) \coloneqq A_*([X/T]).$$

Our starting point is the following result.

Theorem 5.1 ([18], Theorem 5.3.5)**.** *Let X be a DM stack with a torus action. Let X^T denote the fixed stack and let $t = c_1(\mathcal{O}(1)) \in A_*(BT)$. Then the inclusion $X^T \to X$ induces an isomorphism*

$$A_*(X^T) \otimes_{\mathbb{Q}} \mathbb{Q}[t, t^{-1}] \to A_*^T(X) \otimes_{\mathbb{Q}[t]} \mathbb{Q}[t, t^{-1}].$$

To use this result for a virtual fundamental class $[X]^{vir}$, we need an *equivariant perfect obstruction theory.* We note that there is an induced T-action on $L_X^\bullet$ s.t. $L_X^\bullet$ is equivariant and hence $L_X^\bullet$ can be viewed as an element of $D([X/T])$.

Definition 5.1. A T-equivariant perfect obstruction theory of X is an object $E^\bullet \in D([X/T])$ and a morphism $E^\bullet \to L_X^\bullet$ s.t. this is a perfect obstruction theory for X.

Note that a T-equivariant perfect obstruction theory defines a T-equivariant virtual fundamental class $[X]_T^{vir} \in A_*([X/T])$ which we shall denote by $[X]^{vir}$ by slight abuse of notation.

Let $E^\bullet \to L_X^\bullet$ be a T-equivariant perfect obstruction theory for X. In [6] it is proved that $E_i^{\bullet,f} \to L_{X_i}^\bullet$ is a T-equivariant perfect obstruction theory, where X_i is a connected component of the fixed locus X^T and $E_i^{\bullet,f}$ is the T-fixed part of the pull-back of $E^\bullet$ to X_i.

The virtual normal bundle N_i^{vir} of X_i is defined to be the dual of the complex $E_i^{\bullet,m}$. We assume that each N_i^{vir} has a locally free global resolution $[N_{i,0} \to N_{i,1}]$ i.e. N_i^{vir} is quasi-isomorphic to $[N_{i,0} \to N_{i,1}]$. In this case we write $e(N_i^{vir}) = e(N_{i,0})/e(N_{i,1})$, where e denotes the top Chern class.

The main theorem we are about to prove is the following.

Theorem 5.2 (Torus localization)**.** *Let X be a Delinge-Mumford stack equipped with a torus action (with above assumptions). With notation as above, we have*

$$[X]^{vir} = \sum \iota_{i*}\left(\frac{[X_i]^{vir}}{e(N_i^{vir})}\right) \in A_*^T(X) \otimes_{\mathbb{Q}[t]} \mathbb{Q}[t, t^{-1}].$$

We start with a few lemmas.

5.2 Preliminary results

Given a closed immersion $\iota\colon Y \to X$ and a vector bundle stack $\mathfrak{E}$ s.t. $C_{Y/X}$ embeds into $\mathfrak{E}$ as a closed substack, we define $\iota^!$ as the composition

$$\iota^!\colon A_*(X) \xrightarrow{\sigma} A_*(C_{Y/X}) \xrightarrow{i_*} A_*(\mathfrak{E}) \xrightarrow{s^*} A_*(Y).$$

For a regular local immersion one can simply take $\mathfrak{E} = N_{X/Y}$ and we recover the definition of Kresh.

Lemma 5.1. *Let X be a DM stack, let $\mathcal{N}, \mathcal{E}$ be vector bundle stacks on X. Then for $\alpha \in A_*(\mathcal{E})$ we have*

$$0^!_{\mathcal{E}\oplus\mathcal{N}}(\alpha) = 0^!_{\mathcal{N}} 0_{\mathcal{N}*} 0^!_{\mathcal{E}}(\alpha)$$

In particular, if $\mathcal{N}$ is a vector bundle N then

$$0^!_{\mathcal{E}\oplus N}(\alpha) = e(N) \cap 0^!_{\mathcal{E}}(\alpha)$$

Proof. Consider the following Cartesian diagram of zero sections

$$\begin{array}{ccc} X & \xrightarrow{0_{\mathcal{E}}} & \mathcal{E} \\ \big\downarrow{\scriptstyle 0_{\mathcal{N}}} & & \big\downarrow{\scriptstyle s} \\ \mathcal{N} & \xrightarrow{s'} & \mathcal{E}\oplus\mathcal{N} \end{array}$$

Using compatibility and functoriality of refined Gysin maps we obtain

$$\begin{aligned} 0^!_{\mathcal{N}} 0_{\mathcal{N}*}(0^!_{\mathcal{E}}(\alpha)) &= 0^!_{\mathcal{N}}(s'^! s_*\alpha) \\ &= (s' \circ 0_{\mathcal{N}})^!(s_*\alpha) \\ &= 0^!_{\mathcal{E}\oplus\mathcal{N}}(s_*\alpha) \end{aligned}$$

Identify $s_*\alpha$ with α and the first part is proved. The second formula follows immediately from the basic formula for top Chern classes

$$0^!_N 0_{N*}\alpha = e(N) \cap \alpha.$$

□

Next, we provide a basic formula for quotient cones.

Lemma 5.2. *Let $f\colon N_0 \to N_1$ be a morphism of vector bundles on a DM stack X making N_1 an N_0-cone so that $[N_1/N_0]$ is defined. Then*

$$e(N_1) \cap \alpha = e(N_0) \cap 0^!_{[N_1/N_0]} 0_{[N_1/N_0]_*}\alpha$$

Proof. The square

$$\begin{array}{ccc} N_0 & \xrightarrow{f} & N_1 \\ \Big\downarrow{\scriptstyle \pi_{N_0}} & & \Big\downarrow{\scriptstyle g} \\ X & \xrightarrow{0_Q} & Q \end{array}$$

is Cartesian, where $Q = [N_1/N_0]$. Apply $\pi^!_{N_0}$ to the basic formula

$$0^!_{N_0} 0_{N_0 *}\alpha = e(N_0) \cap \alpha$$

to obtain

$$0_{N_0 *}\alpha = e(\pi^*_{N_0} N_0) \cap \pi^!_{N_0}\alpha = e(f^*\pi^*_{N_1} N_0) \cap \pi^!_{N_0}\alpha$$

by the pull-back formula. Using the projection formula we have

$$0_{N_1 *}\alpha = f_* 0_{N_0 *}\alpha = e(\pi^*_{N_1} N_0) \cap f_*\pi^!_{N_0}\alpha.$$

Now $f_*\pi^!_{N_0} = g^! 0_{Q*}$ and hence $0^!_{N_1} f_*\pi^!_{N_0} = 0^!_{N_1} g^! 0_{Q*} = 0^!_Q 0_{Q*}$ which implies

$$\begin{aligned} e(N_1) \cap \alpha = 0^!_{N_1}(0_{N_1 *}\alpha) &= e(0^*_{N_1}\pi^*_{N_1} N_0) \cap 0^!_{N_1} f_*\pi^!_{N_0}\alpha \\ &= e(N_0) \cap 0^!_Q 0_{Q*}\alpha. \end{aligned}$$

and this completes the proof. □

The next lemma is the familiar self-intersection formula. Suppose that in the definition of $\iota^!$ above, we can take $\mathfrak{C}$ to be a vector bundle N on Y.

Lemma 5.3. *For $\alpha \in A_*(Y)$ we have*

$$\iota^! \iota_* \alpha = e(N) \cap \alpha$$

Proof. By definition of $\iota^!$, we have

$$\iota^! \iota_*(\alpha) = 0^!_N(i_*(\sigma(\iota_*\alpha)))$$

Suppose $i_* \circ \sigma \circ \iota_* = 0_{N*}$. Then

$$\begin{aligned} \iota^! \iota_*(\alpha) &= 0^!_N(0_{N*}(\alpha)) \\ &= e(N) \cap \alpha \end{aligned}$$

by the basic formula for Euler classes and this is the desired result. To prove that

$$i_* \circ \sigma \circ \iota_* = 0_{N*}$$

we just recall that σ is defined as

$$\sigma[V] = [C_{V \cap X} V]$$

and hence for $[V] = \iota_*[V]$ we've

$$\sigma[V] = [C_V V] = [V]$$

so that it becomes clear that

$$i_* \circ \sigma \circ \iota_*[V] = 0_{N*}[V].$$

□

5.3 The proof

We shall now prove Theorem 5.2. Recall that we assume $[X/T]$ is actually Deligne-Mumford so that the lemmas proved above can be applied in the T-equivariant Chow. Since everything (obstruction theory, cones, vector bundle stacks) is defined T-equivariantly, they descend to $[X/T]$. For notational convenience we denote the objects on X and their counterparts on $[X/T]$ with the same symbols.

Proof of Theorem 5.2. First, we apply the theorem of Kresh stated above. If we denote by X_i the connected components of X^T with inclusions $\iota_i \colon X_i \to X$ then we have

$$[X]^{vir} = \sum_i \iota_{i*}(\alpha_i)$$

in the localized equivariant Chow. Thus

$$\iota_i^![X]^{vir} = \iota_i^!(\iota_{i*}(\alpha_i)) = e(N_0) \cap \alpha_i \tag{1}$$

by Lemma 5.3 (which we can apply since we assume $[X/T]$ is actually Deligne-Mumford). But we can also compute $\iota_i^![X]^{vir}$ as follows. By the definition of $[X]^{vir}$ and Lemma 4.7 in [21], we have

$$\iota_i^![X]^{vir} = \iota_i^! 0_{\mathcal{E}}^! [\mathfrak{C}_X] = (0_{\mathcal{E}} \circ \iota_i)^!_{\mathcal{E}_i \oplus N_0}[\mathfrak{C}_X]$$

By definition $(0_{\mathcal{E}} \circ \iota_i)^!_{\mathcal{E}_i \oplus N_0}$ gives

$$(0_{\mathcal{E}} \circ \iota_i)^!_{\mathcal{E}_i \oplus N_0}[\mathfrak{C}_X] = 0^!_{\mathcal{E}_i \oplus N_0}[\mathfrak{C}_{X_i \times_{\mathcal{E}} \mathfrak{C}_X / \mathfrak{C}_X}] = 0^!_{\mathcal{E}_i \oplus N_0}[\mathfrak{C}_{X_i}]$$

where we used $X_i \times_{\mathcal{E}} \mathfrak{C}_X = X_i$ and $[\mathfrak{C}_{X_i/\mathfrak{C}_X}] = [\mathfrak{C}_{X_i}]$. Now using Lemma 5.1, Lemma 5.2 and $\mathcal{E}_i = \mathcal{E}_i^f \oplus Q$ we get

$$\begin{aligned} 0^!_{\mathcal{E}_i \oplus N_0}[\mathfrak{C}_{X_i}] &= e(N_0) \cap 0^!_{\mathcal{E}_i^f \oplus Q}[\mathfrak{C}_{X_i}] \\ &= e(N_0) \cap 0^!_Q 0_{Q*}(0^!_{\mathcal{E}_i^f}[\mathfrak{C}_{X_i}]) \\ &= e(N_1) \cap (0^!_{\mathcal{E}_i^f}[\mathfrak{C}_{X_i}]) \\ &= e(N_1) \cap [X_i]^{vir}. \end{aligned}$$

So we've shown

$$\iota_i^![X]^{vir} = e(N_1) \cap [X_i]^{vir}. \tag{2}$$

Putting (1) and (2) together, we see that

$$\alpha_i = \frac{e(N_1) \cap [X_i]^{vir}}{e(N_0)}$$

and finally that

$$[X]^{vir} = \sum_i \iota_{i*}(\alpha_i) = \sum_i \iota_{i*}(\frac{e(N_1) \cap [X_i]^{vir}}{e(N_0)}).$$

In neater form

$$[X]^{vir} = \sum_i \iota_{i*}(\frac{[X_i]^{vir}}{e(N_i^{vir})}).$$

□

Remark. The proof differs from the one given in [6] in the way we proved Eq. (2). The attempt here was to give a somewhat simpler proof avoiding the arguments using various distinguished triangles.

6 The Stack of Stable Maps and Gromov-Witten theory

We will now define stacks of stable maps which are the basic object of interest in Gromov-Witten theory and prove some geometric properties of stable maps. The rest of the thesis is dedicated to showing how Gromov-Witten theory and localization can be used to prove identities in the moduli space of curves.

6.1 Stacks of stable Maps

Let X be a smooth projective variety over $\mathbb{C}$. In Gromov-Witten theory we think of curves as maps $C \to X$ from a curve C. Our goal is to define a proper moduli stack parametrizing such curves and then count curves in this space satisfying some constraints.

One naturally starts by considering smooth projective domain curves. However, these do not give us a proper moduli stack. To compactify, we allow the domains to become nodal. This turns out to be enough and the domain curves will be so called *prestable curves.*

Definition 6.1. An n-pointed, genus g *prestable curve* $(C, p_1, \dots, p_n)$ is a connected, projective, reduced, at worst nodal curve of (arithmetic) genus g together with n marked points lying on the non-singular locus of C. A *family of n-pointed, genus g prestable curves* is a flat and projective morphism $C \to T$ with n sections $p_i \colon T \to C$ s.t. for each geometric point $t \in T$, $(C_t, \{p_i(t)\})$ is a n-pointed, genus g prestable curve.

Next, we look at maps from such domains to X. More specifically, we will study maps such that $f_*[C] = \beta$ for a fixed homology class $\beta \in H_2(X)$. Since we're in the algebraic category, it should be stated precisely what we mean by a "homology class". The following definition is made in [5].

Definition 6.2. Let $f \colon C \to X$ be a morphism from a family $C \to T$ of prestable curves to a scheme X. Then for each $t \in T$, the morphism $f_t = f\big|_{C_t}$ gives a homomorphism

$$\begin{aligned} \operatorname{Pic}(X) &\to \mathbb{Z} \\ L &\mapsto \deg(f_t^* L). \end{aligned}$$

This defines a locally constant map

$$T \to \operatorname{Hom}_{\mathbb{Z}}(\operatorname{Pic}(X), \mathbb{Z})$$

which call *the homology class of* C and denote by $f_*[C]$.

More specifically, we define

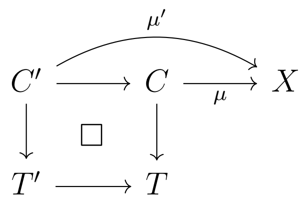
$$H_2(X)^+ = \{\alpha \in \operatorname{Hom}_{\mathbb{Z}}(\operatorname{Pic}(X), \mathbb{Z}) \mid \alpha(L) \geq 0 \text{ for ample } L\}$$

and one can see that $f_*[C] \in H_2(X)^+$.

We shall also need to impose a stablility condition on the morphisms to X to eliminate the possibility of infinitely many automorphisms. This gives rise to the notion of *stable maps* introduced for the first time in the literature by Kontsevich in [17].

Definition 6.3. Let $\beta \in H_2(X)^+$. A *stable map from n-pointed genus g* curve to X of class β is a triple $(C \to T, \{p_i\}, \mu\colon C \to X)$, where $(C, \{p_i\})$ is a family of n-pointed, genus g prestable curves and $\mu\colon C \to X$ is morphism s.t. $f_*[C] = \beta$ and any genus 0 collapsed component (i.e. component mapping to a points) of C has at least 3 special points and a genus 1 collapsed component has at least one special point. By a special point we mean a node or a marked point.

Morphisms between stable maps are given by commutative diagrams



compatible with all the structure (marked points, homology class, etc.). We denote the category of stable maps to X from genus g curves with n marked points having class β by $\overline{M}_{g,n}(X, \beta)$. This can be shown to be a stack and in fact we can say much more.

6.2 $\overline{M}_{g,n}(\mathbb{P}^r, d)$ as a quotient stack

Next, we will see that $\overline{M}_{g,n}(X, \beta)$ can be viewed as a quotient stack of a quasi-projective scheme. We follow arguments presented in [12] and [1]. We start with a lemma from [12].

Lemma 6.1. *Let $\pi\colon \mathcal{C} \to S$ be a flat family of quasi-stable curves over S. Let $\mathcal{L}$ and $\mathcal{M}$ be line bundles on $\mathcal{C}$ whose degrees on each irreducible component of a geometric fiber $\mathcal{C}_s$ coincide. Then there is a unique closed subscheme $T \to S$ satisfying*

(a) *There is a line bundle* $\mathcal{N}$ *on* T *s.t.* $(\mathcal{L} \otimes \mathcal{M}^{-1})\big|_T \cong \pi^*(\mathcal{N})$.

(b) T *is universal in the sense that if* $R \to S$ *is a morphism and* $\mathcal{N}$ *is a line bundle s.t.* $(\mathcal{L} \otimes \mathcal{M}^{-1})\big|_R \cong \pi^*(\mathcal{N})$, *then* $R \to S$ *factors as through* T.

We also need the following result. The proof relies on existence and quasi-projectivity of certain moduli spaces.

Lemma 6.2 ([1], Lemma 2.5.)**.** *Let* M, D *be positive integers. Let* $\overline{M}_{g,n}(\mathbb{P}^r, d)^D_M$ *denote the category of stable maps* $(C \to T, \{p_i\}, f : C \to \mathbb{P}^r)$ *together with a non-degenerate embedding* $C \hookrightarrow \mathbb{P}^M_T$ *of degree* D. $\overline{M}_{g,n}(\mathbb{P}^r, d)^D_M$ *is a stack that is representable by a quasi-projective scheme that has a natural* $PGL(M+1)$*-action (translating the embedding).*

Proof sketch. The domain curves are parametrized by the Hilber scheme H^P_M of closed subschemes of $\mathbb{P}^M$ with Hilbert polynomial $P(t) = Dt + 1 - g$. The data of marked points can be represented as elements of $(P^M)^n$. There is then a closed subscheme $H_1 \subseteq H^P_M \times (\mathbb{P}^M)^n$ defined by the condition that the points lie on the curve. Restricting further, there is a quasi-projective subscheme $H_2 \subseteq H_1$ consisting of prestable curves, i.e. the curve is at worst nodal and the marked points are distinct and lie on the non-singular locus. We pull the universal family $U^P_M \to H^P_M$ back to H_2 to get a family $U_2 \to H_2$. By a result of Grothendieck, there is a quasi-projective scheme S parametrizing morphisms $C \to \mathbb{P}^r$, where C is a fiber of $U_2 \to H_2$. Then we let $H_3 \subseteq H_2 \times S$ be the closed subscheme consisting of pairs where the first coordinate is the domain curve of the second coordinate. As the final step, we take the take the open subscheme $H \subseteq H_3$ where the sheaf

$$\omega_{U/H_3}(\textstyle\sum p_i) \otimes f^*\mathcal{O}(3)$$

is ample. This subscheme represents the stack $\overline{M}_{g,n}(\mathbb{P}^r, d)^D_M$. ☐

Finally we're ready to prove the following proposition.

Proposition 6.1. $\overline{M}_{g,n}(\mathbb{P}^r, d)$ *is the quotient stack of a quasi-projective scheme. In particular, it is an algebraic stack.*

Proof. Let $(C \to T, \{p_i\}, \mu\colon C \to \mathbb{P}^r)$ be a stable map of degree d from n-pointed curve of genus g to $\mathbb{P}^r$. We consider the line bundle

$$\mathcal{L} = \omega_{C/T}(p_1 + \ldots + p_n) \otimes \mu^*(\mathcal{O}_{\mathbb{P}^r}(3)).$$

There is an integer $f = f(g, n, r, d) > 0$ s.t. $\mathcal{L}^f$ very ample and $h^1(C_t, \mathcal{L}^f) = 0$. On each geometric fiber C_t we have

$$\deg(\mathcal{L}^f) = f \cdot (2g - 2 + n + 3d) = D$$

and

$$h^0(C_t, \mathcal{L}^f) = D - g + 1 = M + 1$$

In particular M, D depend only on (g, n, r, d). We consider the quasi-projective scheme H parametrizing stable curves embedded in $\mathbb{P}^M$ with degree D. This scheme has a natural action of $PGL(M+1)$. We take a closed subscheme V of H that corresponds to embeddings whose sheaf coincides with $\mathcal{L}^f$ which exists by Lemma 6.1. This subscheme is in fact invariant under the $PGL(M+1)$-action. We have the morphism

$$V \to \overline{M}_{g,n}(\mathbb{P}^r, d)$$

forgetting the embedding and clearly this map is invariant under the $PGL(M+1)$-action and hence descends to

$$[V/PGL(M+1)] \to \overline{M}_{g,n}(\mathbb{P}^r, d).$$

To construct the inverse, let $(\pi\colon C \to T, \{p_i\}, f\colon C \to \mathbb{P}^r)$ in $\overline{M}_{g,n}(\mathbb{P}^r, d)$ be given. For each $t \in T$ we let $B(H^0(C_t, \mathcal{L}^f))$ be the set of all bases of $H^0(C_t, \mathcal{L}^f) \cong \mathbb{C}^{M+1}$ and consider the bundle P over T whose fiber over $t \in T$ is $\mathbb{P}(B(H^0(C_t, \mathcal{L}^f)))$. P has an action of $PGL(H^0(C_t, \mathcal{L}^f)) \cong PGL(M+1)$ on the fibers. Each element of a fiber is a given by a basis for $H^0(C_t, \mathcal{L}^f)$ and thus defines an embedding $C_t \hookrightarrow \mathbb{P}^M$. This defines a map $P \to V$ which is easily seen to be $PGL(M+1)$-invariant. P is a principal $PGL(M+1)$-bundle and indeed

$$\begin{array}{ccc} P & \longrightarrow & V \\ \downarrow & & \\ T & & \end{array}$$

defines an element of $[V/PGL(M+1)]$. Thus we have defined a morphism

$$\overline{M}_{g,n}(\mathbb{P}^r, d) \to [V/PGL(M+1)].$$

These two maps are inverses of each other and thus we've proven

$$\overline{M}_{g,n}(\mathbb{P}^r, d) \cong [V/PGL(M+1)].$$

In particular, we conclude that $\overline{M}_{g,n}(\mathbb{P}^r, d)$ is an algebraic stack.

□

The stability condition of stable maps ensure that points in $\overline{M}_{g,n}(\mathbb{P}^r, d)$ have finite stabilizers and using this one can show that $\overline{M}_{g,n}(\mathbb{P}^r, d)$ is a DM stack. Alternatively, we can appeal to Proposition 4.1. in [5] which tells us that the diagonal of $\overline{M}_{g,n}(\mathbb{P}^r, d)$ is representable, unramified and finite. These properties allow us to say that $\overline{M}_{g,n}(\mathbb{P}^r, d)$ is a DM stack in the sense of Definition 2.8. Furthermore, one can actually prove that $\overline{M}_{g,n}(\mathbb{P}^r, d)$ is a proper DM stack. However showing properness requires quite a lot of work which we do not present in this thesis. The main steps are outlined in [1] and we will only state the final conclusion needed in this thesis.

Theorem 6.1. *$\overline{M}_{g,n}(\mathbb{P}^r, d)$ is a proper Deligne-Mumford stack over $\mathbb{C}$ that admits a projective coarse moduli scheme M over $\mathbb{C}$.*

This can be used to prove the same result for $\overline{M}_{g,n}(X, \beta)$ for any smooth projective variety X. This gives us enough structure on $\overline{M}_{g,n}(X, \beta)$ to be able to do intersection theory and construct the virtual fundamental classes on this stack. In particular, when $X = \mathbb{P}^r$ we have torus actions on $\overline{M}_{g,n}(\mathbb{P}^r, d)$ and we can employ the torus localization theorem as we will see later in the case $r = 1$.

6.3 Geometric properties of $\overline{M}_{g,n}(X, \beta)$

There are various geometrically interesting operations on the moduli stacks of stable maps. We will give a brief account of the basics and refer the reader to e.g. [5] for more details and complete proofs.

The first very useful geometric property of the moduli stack $\overline{M}_{g,n}(X, \beta)$ is that there is a universal family of curves over it. By this we mean that there is a stack $U_{g,n}(X, \beta)$ and a morphism $\mu\colon U_{g,n}(X, \beta) \to X$ that satisfies the following universal property. Given a stable map $(C \to T, \{p_i\}, f\colon C \to X)$ there are morphisms $T \to \overline{M}_{g,n}(X, \beta)$ and $C \to U_{g,n}(X, \beta)$ s.t. the diagram

$$\begin{array}{ccccc} & & \overset{f}{\frown} & & \\ C & \longrightarrow & U_{g,n}(X,\beta) & \longrightarrow & X \\ \downarrow & & \downarrow & & \\ T & \longrightarrow & \overline{M}_{g,n}(X,\beta) & & \end{array}$$

commutes and the square is Cartesian.

Proposition 6.2. *Let $U_{g,n}(X, \beta)$ be the stack consisting of quadruples (C, p, f, Δ), where (C, p, f) is a stable map from an n-pointed genus g curve over T to X and $\Delta\colon T \to C$ is a section. Let $\mu\colon U_{g,n}(X, \beta) \to X$ be the morphism $(C, p, f, \Delta) \mapsto f \circ \Delta$ and let $\pi\colon U_{g,n}(X, \beta) \to \overline{M}_{g,n}(X, \beta)$ be the morphism forgetting the section. This stack is the universal family over $\overline{M}_{g,n}(X, \beta)$ in the above sense.*

Proof. Let (C, p, f) be a stable map from an n-pointed genus g curve over T to X and $\Delta\colon T \to C$ a section. Given a C-scheme S, we have the composition $S \to C \to T$. We can pull-back $C \to T$ to a stable map over S given by $S \leftarrow C_S = S \times_T C \xrightarrow{f \circ p_2} X$ we have the obvious section $S \to C_S$ and hence we get an element in $U_{g,n}(X, \beta)$. This defines a morphism $C \to U_{g,n}(X, \beta)$ and hence a morphism

$$C \to T \times_{\overline{M}_{g,n}(X,\beta)} U_{g,n}(X, \beta)$$

Let

$$(S \to T, (C' \to S, p', f', \Delta), \alpha) \in T \times_{\overline{M}_{g,n}(X,\beta)} U_{g,n}(X, \beta).$$

By definition $\alpha\colon (C_S, q_i, f_S) \xrightarrow{\sim} (C', p', f')$ and thus we get a morphism

$$S \xrightarrow{\Delta} C' \xrightarrow{\alpha^{-1}} C_S \to C.$$

We have defined a morphism

$$T \times_{\overline{M}_{g,n}(X,\beta)} U_{g,n}(X,\beta) \to C.$$

One then checks that this is an inverse for the previous map and hence that the diagram

$$\begin{array}{ccccc} & & \overset{f}{\frown} & & \\ C & \longrightarrow & U_{g,n}(X,\beta) & \longrightarrow & X \\ \downarrow & & \downarrow & & \\ T & \longrightarrow & \overline{M}_{g,n}(X,\beta) & & \end{array}$$

is commutes. □

We can actually do better. It turns out that there is a more natural way to represent the universal curve. Namely, we will prove that $U_{g,n}(X,\beta) \cong \overline{M}_{g,n+1}(X,\beta)$. We need a two lemmas.

Lemma 6.3 (Contraction)**.** *Let $(C, x_1, \ldots, x_{n+1}, f)$ be a stable map. Then there is a stable map $(C' \to T, x'_1, \ldots, x'_n, f')$ and a morphism $p\colon C \to C'$ s.t. $p(x_i) = x'_i$ for $i \leq n$ and if $x = p(x_{n+1})$ then $p^{-1}(x)$ is either a rational component or a point and $p\big|_{p^{-1}(x)}$ is an isomorphism and $f = f' \circ p$.*

Proof sketch. Suppose $T = \operatorname{Spec}(\mathbb{C})$ Let E be the irreducible component containing x_{n+1}. If E remains stable after removing x_{n+1} we simply take $C' = C$, $x'_i = x_i$ for $i \leq n$ and $f' = f$. Suppose E becomes unstable when x_{n+1} is removed. Then E must be a rational collapsed component having exactly three special points.

If E has one node and two marks, one of which is x_{n+1}, then we take $C' = \overline{C - E}$, with marks $x_1, \ldots, x_n$ and the obvious map $p\colon C \to C'$. Since f maps E to a point, it descends to $f'\colon C' \to X$ s.t. $f = f'p$.

If E has two nodes y_1, y_2 then we take $C' = \overline{C - E}/(y_1 \sim y_2)$. I.e. we remove E and identify the two nodes. We let $p\colon C' \to C$ be the map collapsing E to a point.

The above process can be applied in families. □

Note that the contraction operation defines a morphims

$$\pi\colon \overline{M}_{g,n+1}(X,\beta) \to \overline{M}_{g,n}(X,\beta)$$

forgetting the last mark.

Lemma 6.4 (Stabilization)**.** *Let $(C, x_1, \dots, x_n, f)$ be a stable map and $\Delta\colon T \to C$ a section. Then there is a unique (up to isomorphism) stable map $(C' \to T, x'_1, \dots, x'_{n+1}, f')$ and a morphism $p\colon C' \to C$ s.t. C is a contraction of C' in the sense of the above lemma and $p(x'_{n+1}) = \Delta$.*

Proof sketch. Let $(C, x_1, \dots, x_n, f)$ be a stable map and $\Delta\colon T \to C$ a section. Then by adding sections $z_1, \dots, z_N$, we can make

$$(C, z_1, \dots, z_N, x_1, \dots, x_n)$$

a stable curve i.e. rational components have at least three special components and elliptic ones have at least one special point. Now by theorem 2.4. of [15], there is a stable curve

$$(C', z'_1, \dots, z'_N, x'_1, \dots, x'_n, x'_{n+1})$$

and a morphism

$$p\colon C' \to C$$

making C is a contraction of C' and so that $p(x_{n+1}) = \Delta$. Then we simply take $f' = fp$ and clearly

$$(C', z'_1, \dots, z'_N, x'_1, \dots, x'_{n+1}, f')$$

is a stable map. Just apply Lemma 6.3 several times to get rid of the z'_i to obtain the desired stable map

To prove uniqueness we must show that two $(n+1)$-pointed stable curves having the same contraction are isomorphic. This is straightforward by the construction of contractions. □

With these two lemmas we can prove the following.

Proposition 6.3. *We have $U_{g,n}(X, \beta) \cong \overline{M}_{g,n+1}(X, \beta)$ and the isomorphism is compatible with the forgetful morphisms $U_{g,n}(X, \beta) \to \overline{M}_{g,n}(X, \beta)$ and $\pi\colon \overline{M}_{g,n+1}(X, \beta) \to \overline{M}_{g,n}(X, \beta)$.*

Proof. Let $(C, x_1, \dots, x_{n+1}, f)$ be a stable map. Then Lemma 6.3 defines a stable map

$$(C', x'_1, \dots, x'_n, f') \in \overline{M}_{g,n}(X, \beta)$$

and a section $T \xrightarrow{x_{n+1}} C \xrightarrow{p} C'$. Thus we get a morphism

$$\overline{M}_{g,n+1}(X, \beta) \to U_{g,n}(X, \beta).$$

Now Lemma 6.4 shows that this morphism of stacks is (essentially) surjective. The uniqueness part of Lemma 6.4 shows that this is also fully fatihful. Thus the morphism is an equivalence. □

Another imporant collection of morphisms on $\overline{M}_{g,n}(X,\beta)$ are the evaluation morphisms,

$$\mathrm{ev}_i\colon \overline{M}_{g,n}(X,\beta) \to X.$$

The last operation on the stacks $\overline{M}_{g,n}(X,\beta)$ we will mention here is called *gluing* or *clutching*.

Proposition 6.4 (Clutching)**.** *Given $g_1, g_2, n_1, n_2 \geq 0$ and a partition $\{1,\dots,n\} = A \sqcup B$, we have a morphism*

$$\gamma_{g_1,g_2,A,B}\colon \overline{M}_{g_1,n_1+1}(X,\beta) \times \overline{M}_{g_2,n_2+1}(X,\beta) \to \overline{M}_{g,n+2}(X,\beta),$$

where $n = n_1 + n_2$ and $g = g_1 + g_2$. Applying contraction twice we get

$$\beta\colon \overline{M}_{g_1,n_1+1}(X,\beta) \times \overline{M}_{g_2,n_2+1}(X,\beta) \to \overline{M}_{g,n}(X,\beta).$$

β is called the clutching morphism.

To summarize, we have defined contraction that removes marks, a stabilization that adds marks and clutching that glues marks. Note that all of these are available already in the moduli space of curves. The only new things are the evaluation maps which are defined in terms of the maps to X.

6.4 The virtual fundamental class of $\overline{M}_{g,n}(X,\beta)$

We will give a general idea of the construction of the virtual class using tools of section 4. Let X be a smooth projective variety and $\beta \in H_2(X)^+$. Let $\pi\colon \mathcal{U} \to \overline{M}_{g,n}(X,\beta)$ be the universal family. By Proposition 6.3 above, we have $\mathcal{U} = \overline{M}_{g,n+1}(X,\beta)$ and we let $\mu\colon \mathcal{U} \to X$ be the evaluation at the last marked point. We let $\mathfrak{M}_{g,n}$ denote the category of prestable curves and we let $\varepsilon\colon \overline{M}_{g,n}(X,\beta) \to \mathfrak{M}_{g,n}$ be the morphism forgetting the map to X. We note that $\mathfrak{M}_{g,n}$ is a smooth algebraic stack. To simplify notation, we will denote $\mathfrak{M} = \mathfrak{M}_{g,n}$ and $\overline{M} = \overline{M}_{g,n}(X,\beta)$.

By properties in Theorem 8.1 in [25], the diagram

$$\begin{array}{ccc} \mathcal{U} & \xrightarrow{\mu} & X \\ \downarrow{\scriptstyle \pi} & & \downarrow \\ \overline{M} & \longrightarrow & \mathrm{Spec}(\mathbb{C}) \end{array}$$

induces a morphism

$$\mu^* L_X^\bullet \to L_{\mathcal{U}}^\bullet.$$

The diagram

$$\begin{array}{ccc} \mathcal{U} & \longrightarrow & \mathfrak{U} \\ \downarrow{\scriptstyle \pi} & & \downarrow \\ \overline{M} & \longrightarrow & \mathfrak{M} \end{array}$$

is Cartesian, where $\mathfrak{U}$ is the universal curve over $\mathfrak{M}$. π is flat so induces an isomorphism

$$\pi^* L^\bullet_{\overline{M}/\mathfrak{M}} \xrightarrow{\sim} L^\bullet_{\mathcal{U}/\mathfrak{U}}.$$

Putting these together gives

$$e\colon \mu^* L^\bullet_X \to L^\bullet_{\mathcal{U}} \to L^\bullet_{\mathcal{U}/\mathfrak{U}} \to \pi^* L^\bullet_{\overline{M}/\mathfrak{M}}.$$

Taking the dual and applying the higher direct image $R\pi_*$ functor gives

$$R\pi_*(e^\vee)\colon R\pi_*(\pi^* T^\bullet_{\overline{M}/\mathfrak{M}}) \to R\pi_*(\mu^* T^\bullet_X).$$

Now $R\pi_*(\pi^* T^\bullet_{\overline{M}/\mathfrak{M}})$ is isomorphic to $T^\bullet_{\overline{M}/\mathfrak{M}}$ and by taking duals once again, we obtain

$$\phi\colon (R\pi_*(\mu^* T^\bullet_X))^\vee \to L^\bullet_{\overline{M}/\mathfrak{M}}.$$

We denote

$$E^\bullet \coloneqq (R\pi_*(\mu^* T^\bullet_X))^\vee$$

and it can be shown that

Proposition 6.5. *The morphism $\phi\colon E^\bullet \to L^\bullet_{\overline{M}/\mathfrak{M}}$ is a perfect relative obstruction theory.*

The proof of the above propostion is a relative version of proposition 6.3. in [4].

Note that in section 4 we defined obstruction theories only in the absolute case, but the definitions are basically the same in the relative case, see [4] Section 7. Now as in the absolute case, we have closed immersions

$$\mathfrak{C}_{\overline{M}/\mathfrak{M}} \hookrightarrow \mathfrak{N}_{\overline{M}/\mathfrak{M}} \hookrightarrow \mathfrak{E} \coloneqq h^1/h^0(E^{\bullet\vee})$$

and the virtual fundamental class of $\overline{M}_{g,n}(X,\beta)$ is

$$[\overline{M}_{g,n}(X,\beta)]^{vir} = 0^!_{\mathfrak{E}}[\mathfrak{C}_{\overline{M}/\mathfrak{M}}].$$

6.5 Gromov-Witten Theory

Gromov-Witten theory can be broadly described as studying the stacks of stable maps $\overline{M}_{g,n}(X,\beta)$. One important aspect of the theory is the study of so-called Gromov-Witten invariants. To define these, recall that we have the evaluation maps

$$\mathrm{ev}_i\colon \overline{M}_{g,n}(X,\beta) \to X.$$

And putting these together we obtain

$$\mathrm{ev}\colon \overline{M}_{g,n}(X,\beta) \to X^n.$$

Then we make the following definition.

Definition 6.4. Given classes $\gamma_i \in A_*(X)$, we define the *Gromov-Witten invariants* to be

$$I^X_{g,n,\beta}(\gamma_1 \cdots \gamma_n) = \int_{[\overline{M}_{g,n}(X,\beta)]^{vir}} \mathrm{ev}^*(\gamma_1 \otimes \ldots \otimes \gamma_n).$$

Remark. The right hand side is by definition

$$\deg(\mathrm{ev}^*(\gamma_1 \otimes \ldots \otimes \gamma_n) \cap [\overline{M}_{g,n}(X,\beta)]^{vir}) \in \mathbb{Q}.$$

The result is a rational number in general by the remarks given in section 3. To reiterate, the reason was that we have $\deg = p_*$, where the structure morphism $p\colon \overline{M}_{g,n}(X,\beta) \to \mathrm{Spec}(\mathbb{C})$ is non-representable a and proper and hence the push-forward is defined only with rational coefficients.
Remark. The cap product in the above definition is defined as the refined intersection, i.e. if we have a morphism $f\colon \overline{M} \to X$ to a smooth projective variety, then we can define $f^*(\alpha) \cap \beta = \Gamma^!_f(\alpha \times \beta)$ where $\Gamma_f\colon \overline{M} \to \overline{M} \times X$ is the map $x \mapsto (x, f(x))$ which is a regular local immersion since X is smooth. That such a product exists when $\overline{M}$ is a DM stack can be extracted from results of section 3.

The invariant $I^X_{g,n,\beta}(\gamma_1 \cdots \gamma_n)$ is intuitively interpreted as counting the number of genus g, n-marked curves of class β on X that intersect the classes γ_i. This is not always geometrically meaningful since the counts may be rational numbers. However, even if the counts are rational, in some cases there are ways to extract the ”enumerative” information from the numbers.

Note that non-zero invariants occur only when

$$\sum \mathrm{codim}(\gamma_i) = (\dim X - 3)(1-g) - \int_\beta c_1(X) + n.$$

In the rest of the thesis we're mainly interested in other types of integrals on $\overline{M}_{g,n}(X,\beta)$. In particular, we will study the Gromov-Witten theory of the target $\mathbb{P}^1$.

As a first example we will study intergals of classes pulled back via the so called *branch morphism*

$$br\colon \overline{M}_{g,n}(\mathbb{P}^1, d) \to \mathrm{Sym}^r(\mathbb{P}^1)$$

constructed in [10]. In the second example we will study integrals of the classes

$$x = e(R^1\pi_*(\mu^*\mathcal{O}_{\mathbb{P}^1})), y = e(R^1\pi_*(\mu^*\mathcal{O}_{\mathbb{P}^1}(-1))).$$

On the moduli stack of genus g stable curves with n marks $\overline{M}_{g,n}$ we have the line bundles $\mathcal{L}_i = x_i^*(\omega_{\mathcal{U}_{g,n}/\overline{M}_{g,n}})$, where $\pi\colon \mathcal{U}_{g,n} \to \overline{M}_{g,n}$ is the universal curve and the *Hodge bundle* $\mathbb{E} = \pi_*(\omega_{\mathcal{U}_{g,n}/\overline{M}_{g,n}})$. We define the *tautological classes*

$$\psi_i := c_1(\mathcal{L}_i),$$
$$\lambda_i := c_i(\mathbb{E}).$$

Hodge integrals are then intgrals of these tautological classes over $\overline{M}_{g,n}$.

In both of the examples we will see how the torus localization formula applied to the mentioned integrals gives us information about hodge integrals in the moduli space of curves.

7 Applications

In this section, we will show how the localization formula reduces the Gromov-Witten theory of $\mathbb{P}^1$ to graph combinatorics and integrals involving the tautological classes in various moduli spaces of curves which we can compute e.g. via Kontsevich's theorem. First we will give the general formula and then apply this formula to the two examples mentioned above.

7.1 The torus action and fixed locus

On $\mathbb{P}^1$ we consider the $T = \mathbb{C}^*$-action given by $t \cdot [v_0, v_1] = [v_0, tv_1]$. We identify 0 and ∞ with $p_0 = [1,0]$, $p_1 = [0,1]$ respectively. This induces a T-action on $\overline{M}_{g,n}(\mathbb{P}^1, d)$ by acting on the image. In what follows, we denote $\overline{M} = \overline{M}_{g,n}(\mathbb{P}^1, d)$. An element in the fixed locus $\overline{M}^T$ over a scheme S is given by a tuple $(\mu, \{\alpha_g\}_{g \in T(S)})$, where $\mu \in \overline{M}(S)$ and $\alpha_g \colon \mu \xrightarrow{\sim} g \cdot \mu$ is an isomorphism and $\alpha_{gh} = \alpha_h \circ \alpha_g$. For precise definitions and constructions, see [26].

Let us see what the fixed stack looks like. Let μ denote a fixed point over $\mathbb{C}$. Note that an isomorphism $\alpha_g \colon \mu \xrightarrow{\sim} g \cdot \mu$ is just an automorphism of the domain of μ s.t. $\mu \circ \alpha_g = g \cdot \mu$. An automorphism of the domain preserves special points, ramification points and collapsed components and thus their images under μ are fixed points of $\mathbb{P}^1$, i.e. they are p_0 or p_1. The noncollapsed components are also nice. Let E be a noncollapsed component and consider the restriction $\mu \colon E \to \mathbb{P}^1$. By Hurwitz forumula

$$2g(E) - 2 = -2d(E) + \sum (m_x - 1).$$

Since ramification points map to 0 and ∞, we have

$$\sum_x (m_x - 1) = \sum_{\mu(x)=0} (m_x - 1) + \sum_{\mu(x)=\infty} (m_x - 1) = 2d(E) - \#\mu^{-1}(0) - \#\mu^{-1}(\infty).$$

$\mu\Big|_E$ is nonconstant so there must be zeros and poles and hence

$$2g(E) - 2 = -2d(E) + \sum (m_x - 1) = -\#\mu^{-1}(0) - \#\mu^{-1}(\infty) < 0.$$

Thus $g(E) = 0$ and $\#\mu^{-1}(0) + \#\mu^{-1}(\infty) = 2$ so there are two ramification points with full ramification and the possible marks or nodes lie at the ramification points.

Using the above information we can describe a T-fixed point in terms of a graph and some additional data $\Gamma = (V, E, N, \gamma, j, \delta)$, where

(i) V is the vertex set,

(ii) $g\colon V \to \mathbb{Z}_{\geq 0}$ is a genus map,

(iii) $j\colon V \to \{0,1\}$ is a bipartite structure,

(iv) E an edge set s.t. Γ connected,

(v) $d\colon E \to \mathbb{Z}_{>0}$ degree map,

(vi) $N = \{1,\ldots,n\}$ set of markings attached to vertices,

(vii) $g = \sum_v \gamma(v) + h^1(\Gamma)$,

(viii) and $d = \sum_{e\in E} \delta(e)$.

The set of such graphs is denoted $G_{g,n}(\mathbb{P}^1, d)$. The graph corresponding to a the T-fixed point μ is constructed as follows:

(i) $V = \pi^{-1}(\{p_0, p_1\})$

(ii) $\gamma\colon V \to \mathbb{Z}_{\geq 0}$ genus map, where the genus of an isolated point is by definition equal to zero.

(iii) $j\colon V \to \{0,1\}$ defined by $\mu(v) = p_{j(v)}$.

(iv) E is the set of non-collapsed irreducible components.

(v) $d\colon E \to \mathbb{Z}_{>0}$ degree map.

(vi) $N = \{1,\ldots,n\}$ the n marks. We denote by $N(v)$ the set of markings attached to vertex v.

This correspondence gives a bijection between $G_{g,n}(\mathbb{P}^1, d)$ and the $\mathbb{C}^*$-fixed components of $\overline{M}$.

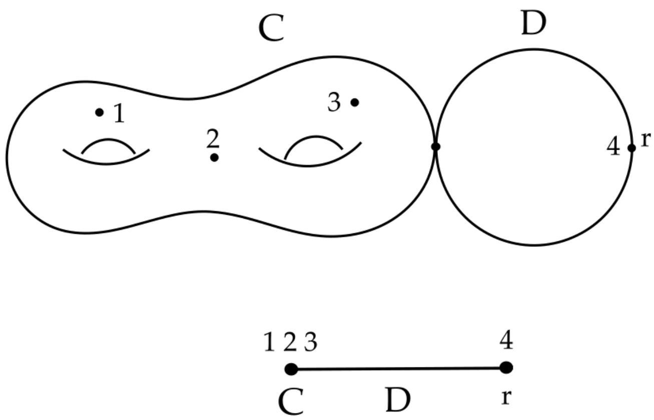


Figure 1: A $\mathbb{C}^*$-invariant curve and the corresponding graph. C is a collapsed component and D non-collapsed.

In the above figure, C is a collapsed component and D is a non-collapsed component. C has three marked points which are visible in the graph. In D the node and marked point are also the two ramification points of that component.

Given a vertex v and incident edge e, we call the pair (v, e) a *flag*. We denote by $F(v)$ the set of flags associated to a vertex v. Let $(C \to \mathrm{Spec}(\mathbb{C}), \{p_i\}, \mu \colon C \to \mathbb{P}^1)$ be a T-fixed point and Γ is the associated graph. Then the dividing the domain into components gives an element in

$$\overline{M}_\Gamma = \prod_{v \in V} \overline{M}_{g(v), N(v) \cup F(v)}.$$

If A is a finite set, we define $\overline{M}_{g,A}$ to be the stack of stable curves of genus g whose fixed points are labeled by A.

Conversely, let

$$(C_v)_{v \in V} \in \overline{M}_\Gamma$$

be $\mathbb{C}$-point. For each $e \in E$ we have the corresponding vertices $v_0, v_1 \in V$ s.t. $j(v_i) = i$. We take a $\mathbb{P}^1$ and glue 0 to $p_{(v_0,e)}$ (the marked point labeled by (v_0, e)) in C_{v_0} and glue ∞ to $p_{(v_1,e)}$ in C_{v_1}. Doing this for every edge, we obtain a prestable curve C whose corresponding graph is Γ and we can define a morphism $\mu \colon C \to \mathbb{P}^1$ to be constant $j(v)$ on a vertex v and to be $z \mapsto z^{d(e)}$ on a rational component corresponding to an edge. This process can be applied in families to obtain a morphism

$$\overline{M}_\Gamma \to \overline{M}.$$

The image is precisely a connected component F_Γ of the fixed locus $\overline{M}^T$ corresponding to a graph Γ.

7.2 Localization

Given a stable map μ with graph Γ there is a natural action by automorphisms of the Galois covers corresponding to edges i.e. an action by $\prod_{edges} \mathbb{Z}/d(e)\mathbb{Z}$. A second action is given by the symmeries of the graph $\mathrm{Aut}(\Gamma)$. $\mathrm{Aut}(\Gamma)$ actually acts on $\prod_{edges} \mathbb{Z}/d(e)$ and thus the semidirect product $A_\Gamma = \prod_{edges} \mathbb{Z}/d(e) \rtimes \mathrm{Aut}(\Gamma)$ on $\overline{M}_\Gamma$.

As a semidirect product, A_Γ admits

$$1 \to \prod_{e \in E} \mathbb{Z}/d(e) \to A_\Gamma \to \mathrm{Aut}(\Gamma) \to 1$$

We can form the quotient stack $Q_\Gamma = [\overline{M}_\Gamma / A_\Gamma]$. This descends to give a closed immersion of DM stacks

$$\tau_\Gamma / A_\Gamma \colon Q_\Gamma \to \overline{M}.$$

Note that $\prod \mathbb{Z}/d(e)$ acts trivially on $\overline{M}_\Gamma$. Here in fact Q_Γ is nonsingular so $[Q_\Gamma]^{vir} = [Q_\Gamma]$. We may now apply localization formula to obtain

$$\begin{aligned}[\overline{M}_{g,n}(\mathbb{P}^1, d)]^{vir} &= \sum_{\Gamma \in G_{g,n}(\mathbb{P}^1,d)} \frac{(\tau_\Gamma/A_\Gamma)_*[Q_\Gamma]}{e(N_\Gamma^{vir})} \\ &= \sum_{\Gamma \in G_{g,n}(\mathbb{P}^1,d)} \frac{1}{|A_\Gamma|} \frac{\tau_{\Gamma*}[\overline{M}_\Gamma]^{vir}}{e(N_\Gamma^{vir})}.\end{aligned}$$

Already one can see that we're reducing computing integrals over virtual class of stable maps to computing integrals over various stacks $\overline{M}_{g,n}$.

Next, we would like to compute $1/e(N_\Gamma^{vir})$ as explicitly as possible. We will present the computation here following [24] with the hope of clarifying some steps to the reader. We start with sequences 6.7 and 6.8 of [24] and Whitney formula. These give us that

$$\frac{1}{e(N^{vir})} = \frac{e(\mathrm{Ext}^0(\Omega_C(P), \mathcal{O}_C)^m)}{e(\mathrm{Ext}^1(\Omega_C(P), \mathcal{O}_C)^m)} \cdot \frac{e(H^1(C, \pi^*T\mathbb{P}^1)^m)}{e(H^0(C, \pi^*T\mathbb{P}^1)^m)},$$

where the superscript m denotes the moving part. Note that we're representing a vector bundle by its fibers. In the computations we shall have to distinguish between different type of vertices. We introduce the following notation:

$$\begin{aligned}V_S(\Gamma) &= \{v \in V \mid 2g(v) - 2 + |N(v)| + |F(v)| > 0\} \\ V_E(\Gamma) &= \{v \in V \mid g(v) = 0, |N(v)| = 0, |F(v)| = 2\} \\ V_U(\Gamma) &= \{v \in V \mid g(v) = 0, |N(v)| = 0, |F(v)| = 1\} \\ V_M(\Gamma) &= \{v \in V \mid g(v) = 0, |N(v)| = 1, |F(v)| = 1\}.\end{aligned}$$

V_S are the *stable* vertices, i.e. ones corresponding to collapsed components. V_E is the set of nodes between edge components, V_U are the unmarked isolated vertices and V_M are the marked isolated vertices.

Given a flag $F = (e, v)$ (edge and incident vertex) we denote by x_F the point of intersection of the edge and vertex. We have the exact normalization sequence

$$0 \to \mathcal{O}_C \to \bigoplus_{v \in V_S} \mathcal{O}_{C_v} \oplus \bigoplus_{e \in E} \mathcal{O}_{C_e} \to \bigoplus_{\substack{F=(v,e) \\ v \in V_S}} \mathcal{O}_{x_F} \oplus \bigoplus_{v \in V_E} \mathcal{O}_v \to 0$$

We twist the above sequence by $\pi^*T\mathbb{P}^1$ and take the long exact sequence in cohomology to get

$$\begin{aligned}0 &\to H^0(C, \pi^*T\mathbb{P}^1) \to \bigoplus_{v \in V_S} H^0(C_v, \pi^*T\mathbb{P}^1) \oplus \bigoplus_{e \in E} H^0(C_e, \pi^*T\mathbb{P}^1) \\ &\to \bigoplus_{\substack{F=(v,e) \\ v \in V_S}} T_{p_{j(v)}}\mathbb{P}^1 \oplus \bigoplus_{v \in V_E} T_{p_{j(v)}}\mathbb{P}^1 \to H^1(C, \pi^*T\mathbb{P}^1) \\ &\to \bigoplus_{v \in V_S} H^1(C_v, \pi^*T\mathbb{P}^1) \oplus \bigoplus_{e \in E} H^1(C_e, \pi^*T\mathbb{P}^1) \to 0.\end{aligned}$$

We want to compute the Euler classes of the direct sums. First consider $H^1(C_v, \pi^*T\mathbb{P}^1)$. This is just $H^1(C_v, \mathcal{O}_{C_v}) \otimes T_{p_{i(v)}}\mathbb{P}^1$. By Serre duality $H^1(C_v, \mathcal{O}_{C_v}) \cong H^0(C_v, \omega)^\vee$ and this is by definition the dual of the Hodge bundle $\mathbb{E}$ on $\overline{M}_{\gamma(v),val(v)}$. Thus taking the Euler class of the tensor product gives us

$$e(H^1(C_v, \pi^*T\mathbb{P}^1)^m) = \sum_{i=0}^{g(v)} (-1)^i c_1(T_{p_{j(v)}}\mathbb{P}^1)^{g(v)-i} c_i(E) = \sum_{i=0}^{g(v)} (-1)^i ((-1)^{j(v)} t)^{g(v)-i} \lambda_i$$

Next, consider an edge component C_e. We take the Euler sequence of $\mathbb{P}^1$, pull back to C_e via π and take cohomology and we get

$$0 \to \mathbb{C} \to H^0(\mathcal{O}(d(e))) \otimes \mathbb{C}^2 \to H^0(C_e, \pi^*T\mathbb{P}^1) \to 0$$

$\pi\colon C_e \to \mathbb{P}^1$ looks like $[x, y] \mapsto [x^{d(e)}, y^{d(e)}]$ and Thus $\mathbb{C}^*$ acts on C_e by $t \cdot [x, y] = [x, t^{1/d(e)} y]$ hence the induced $\mathbb{C}^*$-action on $H^0(\mathcal{O}(d(e)))$ is defined on monomials by $t \cdot x^{d(e)-i} y^i = t^{i/d(e)} x^{d(e)-i} y^i$. Thus the non-zero weights of the action on $H^0(\mathcal{O}(d(e)))$ are $\frac{i}{d(e)}$ for $i = 1, \ldots, d(e)$. The weights on the tensor product $H^0(\mathcal{O}(d(e))) \otimes \mathbb{C}$ are thus $\frac{i}{d(e)}$ and $\frac{i}{d(e)} - 1$. Since the sequence is equivariant, the nontrivial weights of the action on $H^0(C_e, \pi^*T\mathbb{P}^1)$ are thus determined. Recall that for the 1-dimensional representation $\mathbb{C}_a$ of T with character $t \mapsto t^a$ we have $c_1(\mathbb{C}_a) = c_1(\mathbb{C}_1^{\otimes a}) = ac_1(\mathbb{C}_1) = at$ so we get

$$\begin{aligned} e(H^0(C_e, \pi^*T\mathbb{P}^1)^m) &= \prod_{i=1}^{d(e)} \left(\frac{it}{d(e)}\right) \prod_{i=0}^{d(e)-1} \left((\frac{i}{d(e)} - 1)t \right) \\ &= t^{2d(e)} \frac{d(e)!}{d(e)^{2d(e)}} \prod_{i=0}^{d(e)-1} (i - d(e)) \\ &= (-1)^{d(e)} \frac{d(e)!^2}{d(e)^{2d(e)}} t^{2d(e)}. \end{aligned}$$

Notice that $H^0(C_v, \pi^*T\mathbb{P}^1) = T_{p_{i(v)}}\mathbb{P}^1$ and $H^1(C_e, \pi^*T\mathbb{P}^1)$ vanishes. Finally, the node contributions are $(-1)^{j(v)} t$. Thus using Whitney sum formula we have computed that

$$\begin{aligned} \frac{e(H^1(C, \pi^*T\mathbb{P}^1)^m)}{e(H^0(C, \pi^*T\mathbb{P}^1)^m)} &= \prod_{e \in E} \left((-1)^{d(e)} t^{-2d(e)} \frac{d(e)^{2d(e)}}{d(e)!^2} \right) \\ &\cdot \prod_{v \in V_S} \left(\sum_{i=0}^{g(v)} (-1)^i ((-1)^{j(v)} t)^{g(v)-i} \lambda_i \right) \\ &\cdot \prod_{v \text{ node}} (-1)^{j(v)} t \end{aligned}$$

Next, we need to handle the bundles $\mathrm{Ext}^0(\Omega_C(P), \mathcal{O}_C)^m, \mathrm{Ext}^1(\Omega_C(P), \mathcal{O}_C)^m$.

Let us first look at $\mathrm{Ext}^1(\Omega_C(P), \mathcal{O}_C)$. We have the Grothendieck local-to-global sequence

$$0 \to H^1(C, \mathscr{H}om(\Omega_C(P), \mathcal{O}_C)) \to \mathrm{Ext}^1(\Omega_C(P), \mathcal{O}_C) \to H^0(C, \mathscr{E}xt^1(\Omega_C(P), \mathcal{O}_C)) \to 0.$$

By the basic properties of Ext groups, we have

$$\mathscr{E}xt^1(\Omega_C(P), \mathcal{O}_C) \cong \mathscr{E}xt^1(\Omega, \mathcal{O}_C) \otimes \mathcal{O}_C(-P)$$

since $\mathcal{O}_C(-P)$ is invertible. If U be the nonsingular locus of C we have $\mathscr{E}xt^1(\Omega_C, \mathcal{O}_C)\Big|_U \cong \mathscr{E}xt^1(\Omega_C\Big|_U, \mathcal{O}_C\Big|_U) = 0$ since $\Omega_C\Big|_U$ is locally free. Thus $\mathscr{E}xt^1(\Omega_C, \mathcal{O}_C)$ is supported precisely at the nodes, so this reduces to a local calculation at the nodes.

At a node, C is étale locally $\mathrm{Spec}(\mathbb{C}[x,y]/(xy))$. Let $A = \mathbb{C}[x,y]/(xy)$ and let $X = \mathrm{Spec}(A)$. We have

$$\mathscr{E}xt^1_X(\widetilde{\Omega}_A, \widetilde{A}) \cong \widetilde{\mathrm{Ext}^1_A(\Omega_A, A)}$$

Consider the exact sequence of A-modules

$$0 \to (xdy + ydx) \to Adx + Ady \to \Omega_A \to 0$$

This is a free resolution of Ω_A and we get the exact sequence

$$\begin{aligned} 0 &\to \mathrm{Hom}_A(\Omega_A, A) \to \mathrm{Hom}_A(Adx + Ady, A) \\ &\to \mathrm{Hom}_A((xdy + ydx), A) \to \mathrm{Ext}^1_A(\Omega_A, A) \to 0 \end{aligned}$$

$\mathrm{Hom}_A(Adx + Ady, A) \cong A \oplus A$ since $f \in \mathrm{Hom}_A(Adx + Ady, A)$ is given by specifying $f(dx), f(dy)$. The image in $\mathrm{Hom}_A(xdx + ydx) \cong A$ is the morphism $f(xdy + ydx) = xf(dy) + yf(dx)$ and thus the cokernel is $A/(Ax + Ay) \cong A/(x) \otimes_A A/(y)$. Finally we have

$$\mathrm{Ext}^1_A(\Omega_A, A) \cong A/(x) \otimes_A A/(y)$$

Thus at the node $\mathfrak{p}$ we have

$$\mathscr{E}xt^1_X(\widetilde{\Omega}_A, \widetilde{A})_{\mathfrak{p}} \cong (A/(x) \otimes_A A/(y))_{\mathfrak{p}}$$

and this can be identified as

$$\mathscr{E}xt^1_X(\widetilde{\Omega}_A, \widetilde{A})_{\mathfrak{p}} = T_0Z_1 \otimes T_0Z_2$$

where T_0Z_i are the tangent spaces at the origin of the two components. Going back to the global case we see that

$$\mathscr{E}xt^1(\Omega_C, \mathcal{O}_C) = \bigoplus T_i \otimes T_i'$$

where the sum is taken over nodes and the tangent spaces are the tangent spaces of the two branches. Thus

$$\mathscr{E}xt^1(\Omega_C(P), \mathcal{O}_C) = (\bigoplus T_i \otimes T_i') \otimes \mathcal{O}_C(-P) \cong \bigoplus T_i \otimes T_i'$$

since the marked points do not lie on the nodes and

$$H^0(\mathscr{E}xt^1(\Omega_C(P), \mathcal{O}_C)) = \bigoplus T_i \otimes T_i'.$$

The space $H^1(\mathscr{H}om(\Omega_C(P), \mathcal{O}_C))$ is $\mathbb{C}^*$-fixed so

$$e(\mathrm{Ext}^1(\Omega_C(P), \mathcal{O}_C)^m) = e(\bigoplus T_i \otimes T_i') = \prod_{\substack{F=(v,e) \\ v \in V_S}} (\omega_F - \psi_F) \prod_{v \in V_E} (\omega_{(v,e)} + \omega_{(v,e')}),$$

where $\omega_F = \frac{(-1)^{j(v)} t}{d(e)}$ and in the second product e and e' are the two edges connected to vertex $v \in V_E$.

Let $\pi \colon \tilde{C} \to C$ be the normalization of C, then we have

$$\pi_* \mathscr{H}om(\Omega_{\tilde{C}}(P + \text{nodes}), \mathcal{O}_{\tilde{C}}) = \mathscr{H}om(\Omega_C(P), \mathcal{O}_C).$$

And thus

$$\begin{aligned} \mathrm{Hom}(\Omega_{\tilde{C}}(P + \text{nodes}), \mathcal{O}_{\tilde{C}}) &= H^0(\tilde{C}, \mathscr{H}om(\Omega_{\tilde{C}}(P + \text{nodes}), \mathcal{O}_{\tilde{C}})) \\ &\cong H^0(C, \mathscr{H}om(\Omega_C(P), \mathcal{O}_C)) \\ &= \mathrm{Hom}(\Omega_C(P), \mathcal{O}_C) \end{aligned}$$

Thus

$$\begin{aligned} \mathrm{Hom}(\Omega_C(P), \mathcal{O}_C) &= \bigoplus_e \mathrm{Hom}(\Omega_{C_e}(P_e), \mathcal{O}_{C_e}) \oplus \bigoplus_{v \in V_S} \mathrm{Hom}(\Omega_{C_v}(P_v), \mathcal{O}_{C_v}) \\ &= \bigoplus_e \mathrm{Hom}(\Omega_{C_e}(P_e), \mathcal{O}_{C_e}) \end{aligned}$$

since the collapsed components are stable.

The $\mathbb{C}^*$-moving part of the above comes from the tangent spaces at unmarked isolated vertices.

$$\mathrm{Hom}(\Omega_C(P), \mathcal{O}_C)^m = \bigoplus_{\substack{F=(v,e) \\ v \in V_U}} T_v C_e$$

So finally

$$e(\mathrm{Hom}(\Omega_C, \mathcal{O}_C)^m) = \prod_{\substack{F=(v,e) \\ v \in V_U}} \omega_F$$

Putting all of the above calculations together we obtain a formula for Euler class the virtual normal bundle

$$\begin{aligned}\frac{1}{e(N^{vir})} = &\prod_{\substack{F=(v,e)\\ v\in V_U}} \omega_F \prod_{\substack{F=(v,e)\\ v\in V_S}} (\omega_F - \psi_F)^{-1} \prod_{v\in V_E} (\omega_{(v,e)} + \omega_{(v,e')})^{-1} \\ &\cdot \prod_{e\in E} \left((-1)^{d(e)} t^{-2d(e)} \frac{d(e)^{2d(e)}}{d(e)!^2} \right) \\ &\cdot \prod_{v\in V_S} \left(\sum_{i=0}^{g(v)} (-1)^i ((-1)^{j(v)} t)^{g(v)-i} \lambda_i \right) \\ &\cdot \prod_{v \text{ node}} (-1)^{j(v)} t\end{aligned} \tag{3}$$

In general the above formula for the virtual normal bundle looks rather complicated, but we will see that in many applications it reduces to something very tractable.

7.3 Hurwitz numbers

The Hurwitz number $H_{g,d}$ is the number of degree d genus g covers of $\mathbb{P}^1$ étale over ∞ (in particular unramified) with

$$r = 2g - 2 + 2d$$

simple ramification points. There is a *branch morphism*

$$br\colon \overline{M}_{g,0}(\mathbb{P}^1, d) \to \operatorname{Sym}^r(\mathbb{P}^1)$$

and $H_{g,d}$ can be computed in $\overline{M}_g(\mathbb{P}^1, d)$ by

$$H_{g,d} = \int_{[\overline{M}_g(\mathbb{P}^1,d)]^{vir}} br^*(\xi_p),$$

where ξ_p denotes the class in $\operatorname{Sym}^r(\mathbb{P}^1)$ representing r points in $\mathbb{P}^1$ (see [10]). In fact, we can take $\xi_p = r[p_0]$.

To compute the RHS, we will apply the localization formula for $[\overline{M}_g(\mathbb{P}^1, d)]^{vir}$. We get

$$\int_{[\overline{M}_g(\mathbb{P}^1,d)]^{vir}} br^*(\xi_p) = \sum_{\Gamma\in G_g(\mathbb{P}^1,d)} \frac{1}{|A_\Gamma|} \int_{\overline{M}_\Gamma} \frac{\tau_\Gamma^*(br^*(\xi_p))}{e(N_\Gamma^{vir})}$$

Let us find the graphs Γ for which $\tau_\Gamma^*(br^*(\xi_p)) \neq 0$. Given a T-fixed $C \to \mathbb{P}^1$, the image under br is T-fixed so it is of the form $br(C \to \mathbb{P}^1) = (r-a)[p_0] + a[p_1]$ for some $0 \le a \le r$. But $\xi_p = r[p_0]$ so we have $br^*(\xi_p)\Big|_{\overline{M}_\Gamma} = 0$ when $a \neq 0$. Thus the surviving terms are ones for which $br(\overline{M}_\Gamma) = r[p_0]$.

Luckily there is only one connected component satisfying this requirement and it is easy to describe. $br(\overline{M}_\Gamma) = r[p_0]$ means that all ramification points, collapsed components and nodes map to p_0. This means that there is a single collapsed component C_v of genus g with d edges attached, each with degree 1. This graph has $\overline{M}_\Gamma = \overline{M}_{g,d}$. The formula for the virtual normal bundle reduces simply to

$$\begin{aligned}\frac{1}{e(N_\Gamma^{vir})} &= \frac{(-1)^d t^d}{\prod_{i=1}^d (t-\psi_i)} \cdot (-1)^d t^{-2d} \cdot \left(\sum_{i=0}^{g} (-1)^i t^{g-i} \lambda_i\right) \cdot t^d \\ &= \frac{\sum_{i=0}^g (-1)^i t^{g-i}\lambda_i}{\prod_{i=1}^d (t-\psi_i)}\end{aligned}$$

Then it is an easy fact that

$$\tau_\Gamma^* br^*(r[p_0]) = r! t^r.$$

Finally, the automorphisms of Γ are just permutations of the edges and hence

$$|A_\Gamma| = d!$$

Thus

$$\int_{[\overline{M}_g(\mathbb{P}^1,d)]^{vir}} br^*(\xi_p) = \frac{r!}{d!} \int_{\overline{M}_{g,d}} \frac{\sum_{i=0}^g (-1)^i \lambda_i}{\prod_{i=1}^d (1-\psi_i)}.$$

We have proved the following theorem.

Theorem 7.1. *Let $H_{g,d}$ denote the number of branched covers of $\mathbb{P}^1$ that are unramified over ∞ and have $r = 2g-2+2d$ simple ramification points. Then*

$$H_{g,d} = \frac{(2g-2+2d)!}{d!} \int_{\overline{M}_{g,d}} \frac{\sum_{i=1}^g (-1)^i \lambda_i}{\prod_{j=1}^d (1-\psi_j)}$$

With a more involved analysis one can generalize this formula to allow arbitrary ramification profile over ∞. The following formula is proved in [24].

Theorem 7.2. *Let $\mu = (m_1, \ldots, m_l)$ be a partition of a positive integer d. Let $H_{g,\mu}$ denote the number of branched covers of $\mathbb{P}^1$ that have profile μ over ∞ and $r = 2g-2+d+l$ simple ramification points away from ∞. Then*

$$H_{g,\mu} = \frac{(2g-2+d+l)!}{|\mathrm{Aut}(\mu)|} \prod_{i=1}^{l} \frac{m_i^{m_i}}{m_i!} \int_{\overline{M}_{g,l}} \frac{\sum_{i=0}^g (-1)^i \lambda_i}{\prod_{j=1}^l (1-m_j\psi_j)}$$

Since the Hurwitz numbers can be computed purely combinatorially, these results give us a lot of information about integrals in the moduli space of curves.

For example we can look at $g = 1$ and $\mu = (d)$. In this case

$$H_{1,\mu} = (d+1)! \frac{d^d}{d!} \int_{\overline{M}_{1,1}} \frac{1-\lambda_1}{1-d\psi_1}.$$

Now $\dim \overline{M}_{1,1} = 1$ so we have

$$\int_{\overline{M}_{1,1}} \frac{1-\lambda_1}{1-d\psi_1} = \int_{\overline{M}_{1,1}} -\lambda_1 + d\psi_1$$

by expanding the denominator as a geometric series and taking codimension 1 classes. So

$$H_{1,\mu} = (d+1)d^d \int_{\overline{M}_{1,1}} d\psi_1 - \lambda_1$$

First if $d = 1$, then LHS is the number of degree 1 covers simply ramified over two points which is impossible. So LHS vanishes and we obtain

$$\int_{\overline{M}_{1,1}} \psi_1 = \int_{\overline{M}_{1,1}} \lambda_1$$

Next, take $d = 2$. This time the Hurwitz number is the number of degree 2 covers of $\mathbb{P}^1$ simply ramified over ∞ and 3 other simple ramification points. There is one isomorphism class of such covers and since the cover has an automorphism group of order two $H_{1,(2)} = 1/2$ and hence

$$\frac{1}{2} = 3 \cdot 2^2 \int_{\overline{M}_{1,1}} 2\psi_1 - \lambda_1 = 12 \int_{\overline{M}_{1,1}} \psi_1$$

so we get

$$\int_{\overline{M}_{1,1}} \lambda_1 = \int_{\overline{M}_{1,1}} \psi_1 = \frac{1}{24}.$$

Thus for $\mu = (d)$

$$H_{1,\mu} = (d+1)d^d(\frac{d}{24} - \frac{1}{24}) = \frac{(d^2-1)d^d}{24}$$

is the number of genus 1, degree d covers fully ramified over ∞.

7.4 Hodge integrals

As a second application of localization formula for $\mathbb{P}^1$, we show how localization can be used to find relations for Hodge integrals

$$\int_{\overline{M}_{g,1}} \psi_1^{2g-2+i} \lambda_{g-i}$$

following the arguments of Faber and Pandharipande in [9].

We package the integrals in a generating series

$$f_\xi(t) = 1 + \sum_{g \geq 1} \sum_{i=1}^{g} t^{2g} \xi^i \int_{\overline{M}_{g,1}} \psi_1^{2g-2+i} \lambda_{g-i}.$$

and the important result proved in [9] is the following.

Proposition 7.1 (Faber-Pandharipande)**.** *For all* $\xi \in \mathbb{Z}$

$$f_\xi(t) = f_0(t)^{\xi+1}.$$

From this we can extract explicit formulas such as this.

Proposition 7.2. *We have*

$$1 + \sum_{g\geq 1} t^g \int_{\overline{M}_{g,1}} \psi_1^{3g-2} = \exp(t/24).$$

and thus

$$\int_{\overline{M}_{g,1}} \psi_1^{3g-2} = \frac{1}{24^g g!}$$

Proof. The integrals appear in $i = g$ terms in $f_\xi(t)$. Consider

$$f_\xi(\sqrt{t/\xi}) = 1 + \sum_{g\geq 1} \sum_{i=1}^{g} t^g \xi^{i-g} \int_{\overline{M}_{g,1}} \psi_1^{2g-2+i} \lambda_{g-i}$$

Taking $\xi \to \infty$, we're left with the relevant terms

$$\lim_{\xi\to\infty} f_\xi(\sqrt{t/\xi}) = 1 + \sum_{g\geq 1} t^g \int_{\overline{M}_{g,1}} \psi_1^{3g-2}$$

By Proposition 7.1, we have

$$\begin{aligned}\lim_{\xi\to\infty} f_\xi(\sqrt{t/\xi}) &= \lim_{\xi\to\infty} f_0(\sqrt{t/\xi})^{\xi+1} \\ &= \lim_{\xi\to\infty} \left(1 + \sum_{g\geq 1} \left(\frac{t}{\xi}\right)^{2g} \int_{\overline{M}_{g,1}} \psi_1^{2g-2} \lambda_g\right)^{\xi+1}\end{aligned}$$

For a power series

$$p(x) = \sum_{k\geq 0} a_k x^k$$

we have

$$\exp(a_1 x) = \lim_{n\to\infty} \left(p\left(\frac{x}{n}\right)\right)^n.$$

Using this and the fact that $\int_{\overline{M}_{1,1}} \lambda_1 = \frac{1}{24}$ we obtain

$$\lim_{\xi\to\infty} f_\xi(\sqrt{t/\xi}) = \exp(t/24).$$

This proves the proposition. □

Let us now finish by proving Proposition 4. We will consider $\overline{M} = \overline{M}_{g,0}(\mathbb{P}^1, 1)$. The fixed locus X is now particularly easy to describe. The maps have degree one so there is only one rational component in a fixed point. The edge connects two stable vertices of genus g_1 and g_2 with no marked points. Let us denote the connected component corresponding to genus splitting $g = g_1 + g_2$ by X_{g_1,g_2}. Localization and Eq. (3) gives us

$$\int_{[\overline{M}]^{vir}} \gamma = \sum_{g_1+g_2=g} \int_{X_{g_1,g_2}} (-1)^g \iota^*(\gamma) \frac{\Lambda_1(-1)}{1-\psi_1} \frac{\Lambda_2(-1)}{1-\psi_2},$$

where we use notation

$$\Lambda_j(k) = \sum_{i=0}^{g_j} k^i \lambda_{g_j - i} \in A_*(\overline{M}_{g_j,1})$$

following [9].

Let $\pi\colon U \to \overline{M}$ be the universal family and $\mu\colon U \to \mathbb{P}^1$ the universal map. We have the following natural classes

$$x = e(R^1\pi_*(\mu^*\mathcal{O}_{\mathbb{P}^1})), y = e(R^1\pi_*(\mu^*\mathcal{O}_{\mathbb{P}^1}(-1))).$$

The pull-backs of these in the localization requires choosing equivariant lifts of the torus action. These are induced by equivariant lifts of the action on $\mathbb{P}^1$ to $\mathcal{O}_{\mathbb{P}^1}$ and $\mathcal{O}_{\mathbb{P}^1}(-1)$. These are determined by the weights on the fibers over p_0 and p_1. For any $\alpha, \beta \in \mathbb{Z}$ we can choose weights $[\alpha, \alpha]$ for the action on $\mathcal{O}_{\mathbb{P}^1}$ and weigts $[\beta, \beta+1]$ on $\mathcal{O}_{\mathbb{P}^1}(-1)$.

One can then show that

$$\begin{aligned} \iota^* x &= \Lambda_1(-\alpha)\Lambda_2(\alpha) \\ \iota^* y &= \Lambda_1(-\beta)\Lambda_2(\beta+1) \end{aligned}$$

and therefore localization gives

$$\begin{aligned} \int_{[\overline{M}]^{vir}} x \cap y &= (-1)^g \sum_{g_1+g_2=g} \int_{X_{g_1,g_2}} \frac{\Lambda_1(-1)\Lambda_1(-\alpha)\Lambda_1(-\beta)}{1-\psi_1} \frac{\Lambda_2(-1)\Lambda_2(\alpha)\Lambda_2(\beta+1)}{1-\psi_2} \\ &=: (-1)^g I_g(\alpha,\beta). \end{aligned}$$

and

$$\begin{aligned} \int_{[\overline{M}]^{vir}} y \cap y &= (-1)^g \sum_{g_1+g_2=g} \int_{X_{g_1,g_2}} \frac{\Lambda_1(-1)\Lambda_1(-\alpha)\Lambda_1(-\beta)}{1-\psi_1} \frac{\Lambda_2(-1)\Lambda_2(\alpha+1)\Lambda_2(\beta+1)}{1-\psi_2} \\ &=: (-1)^g J_g(\alpha,\beta) \end{aligned}$$

In particular, I_g, J_g are independent of α and β. Along with this independence we will utilize Mumford's identities

$$\begin{aligned} \Lambda_j(1)\Lambda_j(-1) &= (-1)^{g_j} \\ \Lambda_j(0)\Lambda_j(0) &= \delta_{g_j 0}. \end{aligned}$$

With these tools, we're ready to prove Proposition 7.1.

Proof of Proposition 7.1. We can immediately notice that

$$f_0(it) = 1 + \sum_{g\geq 1} t^{2g}(-1)^g \int_{\overline{M}_{g,1}} \psi_1^{2g-2}\lambda_g = 1 + \sum_{g\geq 1} t^{2g} \int_{\overline{M}_{g,1}} \frac{(-1)^g\lambda_g}{1-\psi_1}$$

and by Mumford's identities

$$I_g(0,0) = \int_{\overline{M}_{g,1}} \frac{(-1)^g \lambda_g}{1-\psi_1}$$

so

$$f_0(it) = 1 + \sum_{g \geq 1} t^{2g} I_g(0,0)$$

Similarly one can check that

$$f_0^2(it) = 1 + \sum_{g \geq 1} t^{2g} J_g(0,-1).$$

Now we utilize the independence of I_g, J_g of the inputs to relate $f_0(it)$ and $f_0^2(it)$ to $f_\xi(it)$ and $f_{\xi+1}(it)$ respectively. By looking at the expressions for I_g and J_g obvious choices are $I_g(\xi,0)$ and $J_g(0,\xi)$. Indeed, by defining

$$g_\xi(t) = 1 + \sum_{g \geq 1} t^{2g} \int_{\overline{M}_{g,1}} \frac{\Lambda(-1)\Lambda(-\xi)\Lambda(0)}{1-\psi_1}$$

we have

$$f_0(it) = 1 + \sum_{g \geq 1} t^{2g} I_g(\xi,0) = g_\xi(t) f_\xi(it)$$
$$f_0^2(it) = 1 + \sum_{g \geq 1} t^{2g} J_g(0,\xi) = g_\xi(t) f_{\xi+1}(it)$$

Thus

$$g_\xi(t) f_{\xi+1}(it) = f_0(it) f_0(it) = f_0(it) g_\xi(t) f_\xi(it)$$

and the result follows. □

8 Summary

In this thesis we introduced the technical foundations of Gromov-Witten theory, namely the notion of stacks, their Chow groups and then the notion of virtual fundamental classes. The aim was to gather material scattered in several papers to give a shorter exposition of the main definitions. Stacks are the central geometric objects in modern moduli theory and hence increasingly important for algebraic geometers. We saw how the Chow theoretic constructions of Fulton could be transferred to stacks. For enumerative geometry on stacks, e.g. Gromov-Witten theory, the ordinary fundamental group isn't nice enough and we had to define a so called virtual fundamental class, against which we evaluate integrals.

Having covered the rigorous foundations of Gromov-Witten theory, we proceeded to prove an important formula regarding the virtual fundamental classes of Delinge-Mumford stacks, called the torus localization formula. The formula allowed us to write the virtual fundamental class of a DM stack with a torus action in terms of the virtual class of the fixed stack and normal bundle contributions. The normal bundle contribution was computed for $\overline{M}_{g,n}(\mathbb{P}^1, d)$ and then applied in special cases to obtain information about the integrals of the tautological classes in the moduli space of stable curves. The torus localization is still one of the only powerful tools for computations in Gromov-Witten theory. For more recent computations using localization, see e.g. [22] or [19].

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A Appendix: Derived categories

In section 4 we consider various derived categories. For the interested reader, We will quickly give the necessary definitions and ideas without going into many details. A standard reference for derived categories is the book by Gelfand and Manin[20].

Let $\mathcal{A}$ be an abelian category (zero objects, zero morphsims, cokernel and kernels exist). We start with a category that is easy to understand. The category of complexes $K(\mathcal{A})$ whose objects are complexes $E^\bullet$, i.e. sequences $(E^i)_{i\in\mathbb{Z}}$ of objects of $\mathcal{A}$ with morphisms $d^i\colon E^i \to E^{i+1}$

$$\cdots \to E^{i-1} \xrightarrow{d^{i-1}} E^i \xrightarrow{d^i} E^{i+1} \to \cdots$$

such that $d^{i+1}\circ d^i = 0$. Usually we simply denote $d = d^i$. Morphisms $\phi\colon E^\bullet \to F^\bullet$ are simply maps $\phi^i\colon E^i \to F^i$ for each i and these commute with the maps d^i. The ith cohomology of a complex $E^\bullet$ is

$$h^i(E^\bullet) = \frac{\ker(d^i)}{\mathrm{im}(d^{i-1})}.$$

Note that a morphism $\phi\colon E^\bullet \to F^\bullet$ defines a morphims $h^i(\phi)\colon h^i(E^\bullet) \to h^i(F^\bullet)$.

The following kind of morphism is important.

Definition A.1. A morphism $\phi\colon E^\bullet \to F^\bullet$ is a quasi-isomorphim is the induced maps $h^i(\phi)$ on cohomology are isomorphisms.

There is also a nice situation in which we can say that two maps induce the same map on cohomology.

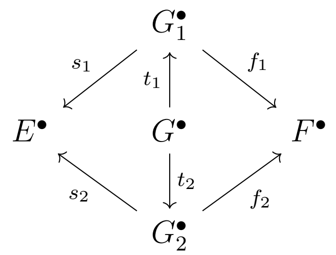
Definition A.2. We say two maps $f, g\colon E^\bullet \to F^\bullet$ are homotopic if $f - g = ds + sd$ for a collection of morphisms $s^i\colon E^i \to F^{i+1}$.

It is now natural to work up to homotopy, i.e., we consider the homotopy category $H(\mathcal{A})$ whose objects are those of $K(\mathcal{A})$ and the morphisms are homotopy classes of moprhims in $K(\mathcal{A})$. Next we would like to define a category where quasi-isomorphisms are invertible so that in this new category a morphism of complexes is an isomorphism if and only if the induced map on cohomology is an isomorphism. This new category is the *derived category* of $\mathcal{A}$. The definition of this category is easy, but many details have to be checked.

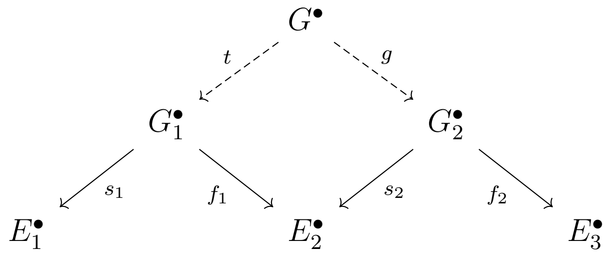
Definition A.3. Let $\mathcal{A}$ be an abelian category. Let Σ be the set of quasi-isomorphsims in $H(\mathcal{A})$. The derived category $D(\mathcal{A})$ of $\mathcal{A}$ is the category whose objects are the objects of $H(\mathcal{A})$ and a morphism $E^\bullet \to F^\bullet$ is a diagram

$$E^\bullet \xleftarrow{\ s\ } G^\bullet \xrightarrow{\ f\ } F^\bullet$$

where $s, f \in H(\mathcal{A})$ and $s \in \Sigma$. Two such morphisms $E^\bullet \to F^\bullet$, are considered equivalent whenever there are $t_1, t_2 \in \Sigma$ s.t. the following diagram commutes



Composition of morphisms $E_1^\bullet \to E_2^\bullet \to E_3^\bullet$ is defined as $(s_1 \circ t, f_2 \circ g)$ in the diagram



Proposition A.1. *Let notation be as in the above definition.*

(1) The equivalence of morphisms in $D(\mathcal{A})$ above is an equivalence relation.

(2) The composition rule above is well-defined, i.e. does not depend on the choice of arrows.

(3) The composition rule is associative.

We shall need two things for the proof. First one is that the shifting functor $A^\bullet \mapsto A[1]^\bullet$ makes $H(\mathcal{A})$, and hence $H(\mathcal{A})$, a *triangulated category.* This means that for any morphism $\phi\colon A^\bullet \to B^\bullet$ in $K(\mathcal{A})$ there is a *mapping cone* $M[\phi]^\bullet$ that fits into

$$A^\bullet \xrightarrow{\phi} B^\bullet \to M[\phi]^\bullet \to A[1]^\bullet.$$

usually abbreviated as

$$A^\bullet \xrightarrow{\phi} B^\bullet \to M[\phi]^\bullet \xrightarrow{+1}$$

A sequence of morphisms $A^\bullet \to B^\bullet \to C^\bullet \to A[1]^\bullet$, is called a *distinguished triangle* if we have $\tilde{\phi}\colon \tilde{A}^\bullet \to \tilde{B}^\bullet$ and isomorphisms in the commutative diagram

$$\begin{array}{ccccccc}
\tilde{A}^\bullet & \xrightarrow{\tilde{\phi}} & \tilde{B}^\bullet & \longrightarrow & M[\tilde{\phi}]^\bullet & \longrightarrow & \tilde{A}[1]^\bullet \\
\downarrow\wr & & \wr\downarrow & & \wr\downarrow & & \wr\downarrow \\
A^\bullet & \xrightarrow{\phi} & B^\bullet & \longrightarrow & C^\bullet & \longrightarrow & A[1]^\bullet
\end{array}$$

where the rightmost map is the shifted version of the leftmost map. A distinguished triangle induced a long exact sequence in cohomology.

The second thing is that

Proposition A.2. Σ *forms a so called multiplicative system in $H(\mathcal{A})$ which means that it satisfies the following three properties:*

(a) *Identity morphisms are in Σ and it is closed under composition.*

(b) *The following property and its dual hold. Given $s \in \Sigma$ and f in $H(\mathcal{A})$ there is $t \in \Sigma$ and g in $H(\mathcal{A})$ forming the diagram*

$$\begin{array}{ccc} W^\bullet & \overset{g}{\dashrightarrow} & F^\bullet \\ {\scriptstyle t}\downarrow & & \downarrow{\scriptstyle s} \\ E^\bullet & \xrightarrow{f} & G^\bullet \end{array}$$

(c) *Given morphisms f, g in $K(\mathcal{A})$, there exists $s \in \Sigma$ such that $sf = sg$ if and only if there exists $t \in \Sigma$ such that $ft = gt$.*

Remark. We need to work in the homotopy category $H(\mathcal{A})$ for (c) to hold. One could define $D(\mathcal{A})$ as the localization of $K(\mathcal{A})$ but the proof of Proposition A.1 is simplified greatly when using (c) above.

Proof. (a) is obvious. We'll prove (b) in $K(\mathcal{A})$ and prove only the claim stated above since the dual proof gives the dual claim.

Keep the notation from the statement. We form the mapping cone

$$F^\bullet \overset{s}{\to} G^\bullet \overset{u}{\to} M[s]^\bullet \to F[1]^\bullet.$$

We get morphism $uf \colon E^\bullet \to M[s]^\bullet$. Taking the cone for this morphism (and rotating) gives

$$\begin{array}{ccccccc} M[uf][-1]^\bullet & \xrightarrow{t} & E^\bullet & \xrightarrow{uf} & M[s]^\bullet & \longrightarrow & M[uf]^\bullet \\ \downarrow{\scriptstyle g} & & {\scriptstyle f}\downarrow & & {\scriptstyle id}\downarrow & & \downarrow \\ F^\bullet & \xrightarrow{s} & G^\bullet & \xrightarrow{u} & M[s]^\bullet & \longrightarrow & F[1]^\bullet \end{array}$$

We have to show t is a quasi-isomorphsim. One can show that

$$F^\bullet \overset{s}{\to} G^\bullet \overset{u}{\to} M[s]^\bullet \to F[1]^\bullet.$$

gives a long exact sequence

$$\cdots \to h^i(F^\bullet) \overset{\sim}{\to} h^i(G^\bullet) \to h^i(M[s]^\bullet) \to h^{i+1}(F^\bullet) \overset{\sim}{\to} h^{i+1}(G^\bullet) \to \cdots$$

which implies that $h^i(M[s]^\bullet) = 0$ for each i. Similarly taking cohomology of the triangle of uf gives

$$\cdots \to \underbrace{h^i(M[s]^\bullet)}_{=0} \to h^i(M[uf]^\bullet) \to h^{i+1}(E^\bullet) \to \underbrace{h^{i+1}(M[s]^\bullet)}_{=0} \to \cdots$$

which tells us that t is a quasi-isomorphism. This completes the proof of (b).

For (c), it suffices to show that if $ft = 0$ in $H(\mathcal{A})$ for $t \in \Sigma$ then there is $s \in \Sigma$ s.t. $sf = 0$ in $H(\mathcal{A})$ since the other implication is the dual statement.

So suppose $ft \sim 0$ in $K(\mathcal{A})$, where $t \in \Sigma$. Then clearly we can form the diagram

$$\begin{array}{ccccccc}
A^\bullet & \xrightarrow{t} & E^\bullet & \xrightarrow{u} & M[t]^\bullet & \longrightarrow & A[1]^\bullet \\
\downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow \\
0 & \xrightarrow{s} & F^\bullet & \xrightarrow{id} & F^\bullet & \longrightarrow & 0
\end{array}$$

which commutes in $H(\mathcal{A})$. g is defined by $(b, a) \mapsto f(a)$. So $f = gu$. Next, we take the mapping cone of g

$$M[t]^\bullet \xrightarrow{g} F^\bullet \xrightarrow{i} M[g]^\bullet \to M[t][1]^\bullet.$$

Then one can check that

$$\begin{aligned} s^i \colon M[t]^i &\to M[g]^{i-1} \\ a &\mapsto (0, a) \end{aligned}$$

satisfies

$$ig = sd + ds$$

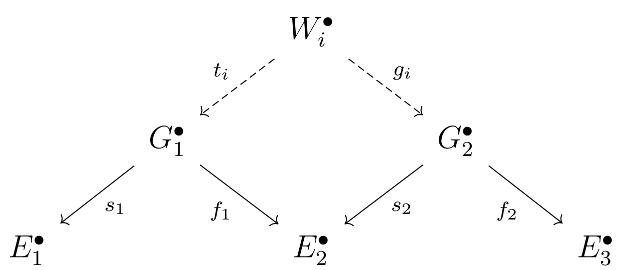
and thus $ig \sim 0$ (this is why we need to work in $H(\mathcal{A})$ and not $K(\mathcal{A})$). Arguments above in the proof of (b) show that $h^i(M[t]^\bullet) = 0$ for each i since t is a quasi-isomorphsim and also that i must be a quasi-isomorphism. Finally,

$$if = igu \sim 0u = 0$$

and this completes the proof. □

Now we're ready to prove Propostition 4.1.

Proof of Proposition A.1. We will refer to properties of Proposition A.2 just by their letters. Now part (1) follows easily from (b). For part (2) we take the diagrams



for $i = 1, 2$. We use property (b) to obtain

$$\begin{array}{ccc}
W^\bullet & \overset{h_2}{\dashrightarrow} & W_2^\bullet \\
{\scriptstyle h_1}\downarrow & & \downarrow{\scriptstyle t_2} \\
W_1^\bullet & \xrightarrow{t_1} & G_1^\bullet
\end{array}$$

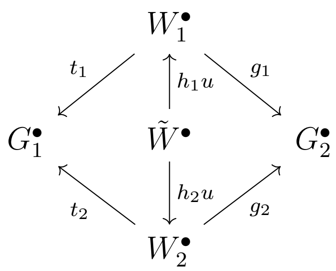
where $h_i \in \Sigma$. We note that

$$s_2 g_1 h_1 = f_1 t_1 h_1 = f_1 t_2 h_2 = s_2 g_2 h_2$$

and by property (c) there is $u \in \Sigma$ such that

$$g_1 h_1 u = g_2 h_2 u$$

and hence a commutative diagram



where $h_1 u, h_2 u \in \Sigma$ by (a). □

Remark. The same proof works for any triangulated category $\mathcal{C}$ with a multiplicative system S to show that the localization of $\mathcal{C}$ at S is a well-defined category.

Now we have the obvious functor $Q\colon K(\mathcal{A}) \to D(\mathcal{A})$ and we note that the image of a quasi-isomorphism is an isomorphism. Furthermore, we have the following universal property. If $\mathcal{C}$ is any category and $\mathcal{C}$ $F\colon K(\mathcal{A}) \to \mathcal{C}$ a functor taking quasi-isomorphisms to isomorphisms, then F factors uniquely through Q. Sometimes we define $D(\mathcal{A})$ to be a category satisfying this universal property. Then the above discussion shows that it exists and by the universal property it is unique up to unique isomorphism.