Cyclic sieving in rational q,t-Catalan combinatorics

Lauri Karanko

School of Science

Bachelor's thesis
Espoo 9.4.2024

Supervisor and advisor

Prof. Oscar Kivinen

Aalto University
School of Science

Aalto University
School of Science

Aalto University, P.O. BOX 11000, 00076 AALTO
www.aalto.fi
Abstract of the bachelor's thesis

Author Lauri Karanko

Title Cyclic sieving in rational q,t-Catalan combinatorics

Degree programme Bachelor's Programme in Science and Technology

Major Mathematics and Systems Sciences **Code of major** SCI3029

Teacher in charge and advisor Prof. Oscar Kivinen

Date 9.4.2024 **Number of pages** 15+3 **Language** English

Abstract

Catalan numbers, which count many combinatorial objects, can be generalized by deforming them to rational functions depending on one or more complex variables. The resulting q-analogs are interesting, since their specializations at roots of unity tell us about the combinatorial properties of cyclic group actions on sets of objects counted by the Catalan numbers. This exhibited effect is called the cyclic sieving phenomenon.

The two-variable q,t-Catalan numbers can be defined using statistics from combinatorial objects. One such set of combinatorial objects is the Dyck paths. They are lattice paths going from (0,0) to (n, n) without going below the diagonal $x = y$. If we make these paths to go to (n, m) instead, with $\gcd(n, m)$=1, we get rational Dyck paths that define the rational q,t-Catalan numbers.

This thesis looks at the cyclic sieving phenomenon from the rational q,t-Catalan numbers' viewpoint, with deliberation on possible generalizations for cyclic sieving. This is done by calculating various q,t-Catalan numbers and their evaluations mechanically. The results show novel values around the known q-Catalan numbers. The meaning for these is nontrivial. Some of the results seem to follow some rule, others are more sporadic and harder to draw conclusions from. In conclusion, we find many unexplained values that should be studied along with their combinatorial relations.

Keywords rational q,t-Catalan numbers, cyclic sieving, root of unity

Aalto University
School of Science

Aalto-yliopisto, PL 11000, 00076 AALTO
www.aalto.fi

Tekniikan kandidaatintyön tiivistelmä

Tekijä Lauri Karanko	
Työn nimi Cyclic sieving rationaalisten q,t-Catalan-lukujen kombinatoriikassa	
Koulutusohjelma Teknillistieteellinen kandidaattiohjelma	
Pääaine Matematiikka ja systeemitieteet	**Pääaineen koodi** SCI3029
Vastuuopettaja ja ohjaaja Prof. Oscar Kivinen	
Päivämäärä 9.4.2024	**Sivumäärä** 15+3 · **Kieli** Englanti

Tiivistelmä

Erinäisiä kombinatorisia objekteja laskevista Catalanin luvuista voidaan tehdä yleistyksiä lisäämäällä niihin kompleksisia muuttujia. Näin määritetyt q-Catalan-luvut ovat kiinnostavia, sillä niiden evaluoinnit ykkösenjuurilla pystyvät kuvaamaan syklisen ryhmän toimintaa joukossa, jonka objekteja vastaavat Catalan luvut laskevat. Kyseistä ilmiötä kutsutaan Cyclic sieving -ilmiöksi.

Kahden muuttujan q,t-Catalan-luvut määritellään kombinatoristen objektien avulla. Eräs tällainen kombinatoristen objektien joukko on nimeltään Dyck-polut. Dyck-polut kulkevat pisteestä (0,0) pisteeseen (n, n) koostuen n määrästä pohjoisaskelia (N) ja n määrästä itäaskelia (E) ylittämättä koskaan diagonaalia $x = y$. Jos nämä polut laitetaan kulkemaan pisteeseen (n, m), kun $\gcd(n, m)=1$, puhutaan rationaalisista Dyck-poluista ja vastaavasti rationaalisista q,t-Catalan-luvuista.

Tämän tutkielman tarkoituksena on tarkastella cyclic sieving -ilmiötä rationaalisten q,t-Catalan-lukujen yhteydessä, sekä pohtia yleistyksen mahdollisuutta ilmiölle. Tämä tehdään laskemalla erilaisia q,t-Catalan-lukuja ja niiden evaluaatioita koneellisesti. Tulokset näyttävät, että tunnettujen q-Catalan-lukujen ympäriltä löytyy arvoja, joiden merkitykset ovat epätriviaaleja. Jotkut saaduista arvoista vaikuttavat säännönmukaisilta, kun taas toiset ovat enemmän yksittäisiä ja niistä on vaikeampi vetää johtopäätöksiä. Tuloksista löytyy siis paljon nykyisen tiedon valossa selittämättömiä arvoja, joiden merkitystä ja yhteyksiä kombinatoriikkaan tulisi tutkia lisää.

Avainsanat rationaaliset q,t-Catalan-luvut, cyclic sieving, ykkösenjuuri

Contents

1 Introduction

This thesis aims to look at cyclic sieving in the context of the q,t-Catalan numbers. First, we will go through the theory behind the concepts and calculations in the thesis. Then we will explore the cyclic sieving phenomenon for q-Catalan numbers. Finally, we will evaluate q,t-Catalan numbers at various roots of unity and attempt to find some consistency in the results with the goal of finding overlaying cyclic sieving like structures.

As we shall see from the q-Catalan numbers and the connected cyclic sieving phenomenon, there are some intriguing applications for the rational q,t-Catalan numbers in the case where $m = n + 1$ and $t = 1/q$. In their paper, Bodnar and Rhoades [2] discovered an exhibited cyclic sieving effect on a set of homogeneous non-crossing partitions using the rational q-Catalan numbers. These findings raise questions about the existence of a higher generalization of the phenomenon on Catalan numbers and their analogs.

1.1 Catalan numbers

The Catalan number sequence, generally denoted by

$$C_n = \frac{1}{n+1}\binom{2n}{n} \qquad \text{for } n \geq 0, \tag{1}$$

has numerous applications in different fields of mathematics. The first few Catalan numbers are $(1, 1, 2, 5, 14, 42, 132, 429, \dots)$ and they count many combinatorial objects. For example, C_n counts the number of *Dyck paths* in an $n \times n$ grid.

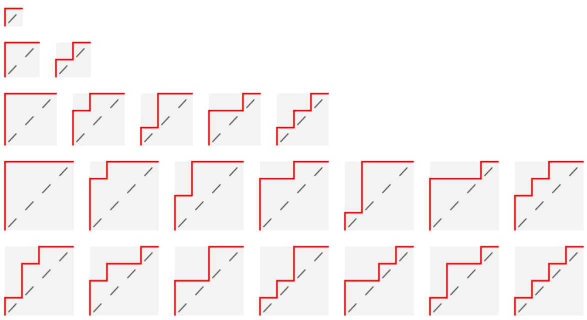

Figure 1: Dyck paths for $n = 1, 2, 3, 4$

Definition 1.1. The Dyck paths ($\mathcal{D}_{n,n}$) are lattice paths consisting of n north (N) and n east (E) steps going from $(0, 0)$ to (n, n) without ever going below the diagonal $x = y$.

Figure (1) shows all the possible Dyck paths for $n = 1, 2, 3, 4$, demonstrating how the number of such paths is indeed the corresponding Catalan number.

1.2 q-analog

A q-analog of an identity or expression is a generalization where a new variable $q \in \mathbb{C}$ is added. Usually this generalization is formulated so that it returns the original expression upon $\lim_{q \to 1}$. A classical q-analog for non-negative integers is

$$[n]_q = \frac{1 - q^n}{1 - q} = 1 + q + q^2 + ... + q^{n-1}.$$

One can also define q-analogs for the factorial

$$[n]_q! = [1]_q \cdot [2]_q \cdot ... \cdot [n]_q,$$

and the binomial coefficient

$$\begin{bmatrix} a \\ b \end{bmatrix}_q = \frac{[a]_q!}{[a-b]_q![b]_q!}.$$

Using these definitions, we can define a q-analog for the Catalan numbers (1), namely the *q-Catalan numbers*:

$$C_n(q) = \frac{1}{[n+1]_q} \begin{bmatrix} 2n \\ n \end{bmatrix}_q. \tag{2}$$

We will use the definition (2) above for the purposes of this thesis. Another frequently used definition is the Carlitz-Riordan analog [3].

1.3 q,t-Catalan numbers

To generalize the q-Catalan numbers, we introduce another variable $t \in \mathbb{C}$ to form the q,t-Catalan numbers. The q,t-Catalan numbers are involved in many fields of mathematics, tying together different q-analogs of the Catalan sequence. To define the q,t-Catalan numbers, we will use the Dyck paths we defined earlier and construct some statistics based on them. Conforming to Haglund's [7] derivations, we define the q,t-Catalan numbers

$$C_n(q,t) = \sum_{\pi \in \mathcal{D}_{n,n}} q^{area(\pi)} t^{dinv(\pi)} \tag{3}$$

where the sum is over all the possible Dyck paths in (n, n), using the two statistics *area* and *dinv* as exponents for the variables. Let's define the statistics next.

Starting with the simpler one, for a Dyck path $\pi \in \mathcal{D}$, the statistic *area* is the number of whole squares between π and the diagonal $x = y$. For the Dyck path shown in figure (2), $area = 8$.

With the statistic *dinv*, we do the following: For a given path $\pi \in \mathcal{D}$, for each pair of east and north steps (E, N), where the east step appears before the north step, we draw a line with slope k_1 from the east step's first coordinate to the north step's first coordinate and another line with slope k_2 from the east step's second coordinate to the north step's second coordinate. To compute $dinv(\pi)$, we count the

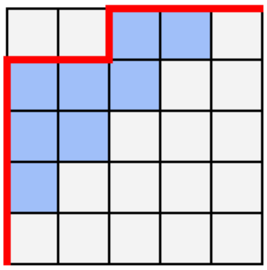

Figure 2: A Dyck path in (5,5), where the squares counted towards the *area* statistic are colored blue.

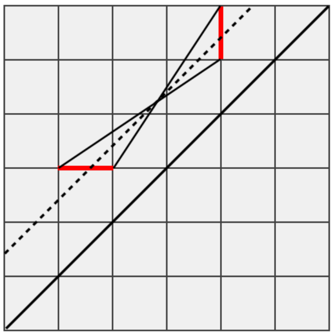

Figure 3: A pair of east and north steps, with the *dinv* requirement satisfied.

number of (E, N) pairs of π, where $k_1 \leq k \leq k_2$, k being the slope of the diagonal, (namely $k = 1$). In figure (3) we have highlighted one east and one north step from some Dyck path where the dotted line parallel to the diagonal goes through both of the highlighted steps.

For the Dyck path in figure (2), we have $dinv = 2$, meaning that the q,t-Catalan number $C_5(q, t)$ would have one term as $q^8 t^2$, corresponding the the path in question.

For certain values of t, we get familiar identities out of the q,t-Catalan numbers. As Haglund [7] points out, $C_n(q, 1/q)$ gives us the original q-Catalan numbers (2), whereas $C_n(q, 1)$ equals the Carlitz-Riordan analog [3].

2 Rational q,t-Catalan numbers

We will next generalize our definition of the q,t-Catalan numbers to the *rational q,t-Catalan numbers* where instead of $C_n(q, t)$ we have $C_{n,m}(q, t)$. The theory of rational q,t-Catalan numbers in this thesis is from Armstrong et al. [1].

We will start the generalization from the beginning. Let's define the rational Catalan numbers as

$$C_{n,m} = \frac{1}{n+m}\binom{n+m}{n}.$$

Naturally, this changes the Catalan number counting problems. In case of the Dyck paths, however, there is a clear counting interpretation.

Definition 2.1. Let $a, b \in \mathbb{N}$ be coprime. The *rational Dyck paths* ($\mathcal{D}_{a,b}$) are lattice paths consisting of b north (N) and a east (E) steps going from (0,0) to (a, b) without ever going below the diagonal $x = \frac{b}{a}y$.

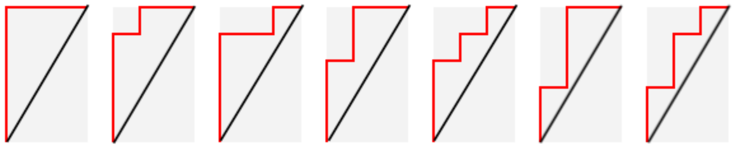

Figure 4: Rational Dyck paths in (3,5)

As an example, figure (4) shows the $C_{3,5} = 7$ rational Dyck paths with $(a, b) = (3, 5)$. Next, Armstrong et al. [1] proposes a q-analog for the rational Catalan numbers:

$$C_{n,m}(q) = \frac{1}{[n+m]_q} \begin{bmatrix} n+m \\ n \end{bmatrix}_q .$$

Ultimately, expanding to the q,t-Catalan numbers, we have the definition of the rational q,t-Catalan numbers:

Definition 2.2.

$$C_{n,m}(q,t) = \sum_{\pi \in \mathcal{D}_{n,m}} q^{area(\pi)} t^{dinv(\pi)} \tag{4}$$

defines the rational q,t-Catalan numbers, for all coprime $n, m \in \mathbb{N}$. Here, the difference to the classical version (3) is that the Dyck paths are of course rational, meaning that the diagonal used in the *area* and *dinv* statistics is the line from $(0, 0)$ to (n, m) with slope $k = m/n$.

To reduce the classical case to the rational case, instead of $m = n$ (where n and m would not be coprime), we use $m = n + 1$. In this case, the Dyck paths are in fact trivially in bijection to the (n, n) -case with the exception of there being an extra north step at the beginning of each path.

2.1 Representation theory

Since the theory of q,t-Catalan numbers is essentially based on a paper on representation theory by Garsia and Haiman [5], we will include a small outline on the connection of q,t-Catalan numbers to representation theory.

Let $DR_n = \mathbb{C}[x_1, ..., x_n, y_1, ..., y_n]/I$ be the ring of diagonal coinvariants, where I is the ideal generated by the nonconstant S_n-invariant polynomials for the action of the symmetric group S_n permuting the variables [7]. Roughly speaking, the weighted count of independent alternating polynomials in the space DR_n, counted with their x- and y-degrees, yields the q,t-Catalan number $C_n(q, t)$. This follows from [4]. For the rational case (n, m), DR_n should be replaced by a certain finite-dimensional representation of a Cherednik algebra. For more details, see [6].

3 Cyclic sieving

Definition 3.1. Let G be a finite cyclic group with a generator $g \in G$ acting on a finite set X. Let $f(q) \in \mathbb{Q}[q]$ be a polynomial and let $\zeta \in \mathbb{C}$ be a root of unity with the same order as g. Then the triple $(X, G, f(q))$ exhibits the *cyclic sieving phenomenon* if, for all $k \in \mathbb{N}$, $f(\zeta^k)$ equals the cardinality of g^k.

To get a better understanding of this definition, let's remind ourselves of the q-Catalan analog (2) and let's introduce another combinatorial object, namely the *non-crossing matchings*.

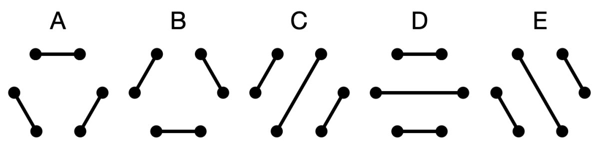

Figure 5: The five non-crossing matchings of $n = 3$

Definition 3.2. The Non-crossing matchings describe ways of connecting pairwise $2n$ points arranged in a circle such that none of the lines connecting two points intersect.

As we can see from figure (5), there are five non-crossing matchings for $n = 3$. Unsurprisingly, the number of non-crossing matchings follows the Catalan number sequence.

Now, consider rotation as a group action acting on the non-crossing matchings. The $2n$-sided polygon created by having the points as vertices has rotational symmetry of order $2n$. Let G be this rotational symmetry group acting on the non-crossing matchings with elements $g^k \in G$, where k is the order of the element. In the figure (5), g^1 sends A to B, B to A, C to D, D to E and E to C. Written in cycle notation: (AB)(CDE). With g^2, C is sent to E, E is sent to D, which in turn is sent to C. However, A and B are sent to themselves, being the two fixed points of the permutation. In cycle notation: (A)(B)(EDC). Upon three rotations (action by g^3), C, D and E become fixed points and A switches places with B again.

What is interesting, is that the q-Catalan numbers are capable of describing this orbit structure of the non-crossing matchings. Specifically, we get the number of fixed points of the action upon evaluating $C_n(\zeta^k)$, where $\zeta \in \mathbb{C}$ is a root of unity of order 6 and k denotes the number of rotations. For example, $C_3(\zeta^1) = 0$, $C_3(\zeta^2) = 2$, $C_3(\zeta^3) = 3$, $C_3(\zeta^4) = 2$, $C_3(\zeta^5) = 3$ and $C_3(\zeta^6) = 5$. This exhibited effect is the cyclic sieving phenomenon.

In fact, for many objects counted by the Catalan numbers, there exists such a cyclic action coming from the rotation of the non-crossing matchings that can be represented by the q-Catalan numbers and exhibits cyclic sieving.

4 Methods

To calculate various q,t-Catalan numbers, we need a way for depicting rational Dyck paths and for doing statistics on them. This is done with a Python class that can be examined fully in the appendix (A). The code is written in a SageMath environment to utilize the mathematical packages that are included with it. Armstrong et al. [1] have also written their own code to calculate rational q,t-Catalan numbers along with other functions that are defined in their paper. However, due to the somewhat different and broader context of their code, we will create our own Python class focused more explicitly on representing the Dyck paths and the q,t-Catalan numbers.

A Dyck class object takes the dimensions of a Dyck path as parameters **x** and **y**. Then it uses the function Creation to generate all the Dyck paths in that space. It does this recursively, starting from position (0,0) and generating two paths after each step, one continuing north and another going east, abandoning the paths that go over bounds. Using this strategy, it creates two lists containing all the Dyck paths. The list paths has each path as a list of **y** numbers, where the ith number is distance of the path's ith north step from the y-axis. The list NEpaths has each path as a string of N's and E's, for each respective north and east step the path has. As an example, the Dyck path in figure (2) would have the corresponding paths element as (0,0,0,0,2) and the NEpaths element as 'NNNNEENEEE'.

The class method area works as we defined earlier (3). Using the paths representation of the Dyck path, it simply sums the numbers together and subtracts that from the sum of the first path in the list, which happens to be the path with the minimum area (=0) due to the order in which the paths are created.

The class method dinv also works as we defined earlier (3). This time using the NEpaths representation of the Dyck path, we check each pair of E's and N's, where the N appears after the E, keeping track of their distances to each other with respect to both axes. When we find such a pair, we check the slopes they create and see if the slope **y**/**x** fits in between. If so, we count that pair towards the statistic and return the amount of these pairs at the end.

We also have the method catalan, which produces the q,t-Catalan polynomial, using both of the previous methods on all the Dyck paths in that grid. This is achieved simply by summing over the formula $q^{area(\pi)}t^{dinv(\pi)}$ for each Dyck path π.

Using these tools we can create rational q,t-Catalan numbers. Since it's easiest to start with something small, and since we are also looking for reasonable sized combinatorial representations, let's choose the parameters n and m in a way that the number of Dyck paths stays at moderate levels. The (n, m) pairs used in this thesis are (3,4), (4,5), (2,3), (2,5), (2,7), (2,9), (3,5) and (3,7).

As stated in the introduction, Bodnar and Rhoades [2] found a case of cyclic sieving with rational q-Catalan numbers. In their paper they used root of unity of order $a + b - 1$, to exhibit the phenomenon. We want to try out the same order for the roots of unity, as well as some others like $2n$, $2m$ or $n + m$. With the root ζ, we can try $C_{n,m}(\zeta^k, \zeta^l)$ and $C_{n,m}(\zeta^k, 1/\zeta^l) = C_{n,m}(\zeta^k, \zeta^{-l})$, for $k, l \in \mathbb{N}$. The latter corresponds to the rational q-Catalan number $C_{n,m}(q, 1/q)$ as $k = l$.

5 Results

All the results of the calculations are documented in the table in the appendix (B). At the top of the table are the 8 different grid sizes that the Dyck paths and the respective q,t-Catalan numbers are created for. Below each of them is either one or multiple integers corresponding to the order of the root of unity that is used. Here the root of unity is always the same for both parameters q and t. Different unequal combinations can also be tried for the orders of the root of unity, but this seems to result in complex solutions in the nontrivial cases, so we aren't interested in them here. The third row of the table tells whether the column is calculated using $C_{n,m}(\zeta^k, \zeta^l)$ or $C_{n,m}(\zeta^k, \zeta^{-l})$.

The first two columns indicate the exponents of the variables. The powers of the roots of unity are periodic, meaning that for the nth root of unity, we know that $\zeta^{n+1} = \zeta^n \cdot \zeta = \zeta$. This means that we only need to check exponents up to the order of the root of unity, since they will cycle around after that. In addition, we are using the theorem, proven recently by Mellit [8], that the rational q,t-Catalan numbers are symmetric with respect to q and t, meaning that $C_{n,m}(q,t) = C_{n,m}(t,q)$. This eases our computations, since we only have to compute the unordered pairs $\{k, l\}$.

One more thing to note is that we are listing only the integer values. The cyclic sieving phenomenon matches the solutions with cardinalities, which are integers, and integer solutions are also more interesting regarding the possible combinatorial descriptions. Thus, the non-integer values are left as empty cells.

$C_{3,4}(q,t) = q^3 + q^2 t + qt + qt^2 + t^3$, 6th root

	ζ^{-l}
k=l=1	0
k=l=2	2
k=l=3	-3
k=l=4	2
k=l=5	0
k=l=6	5

Table 1: The q-Catalan sequence appearing with $C_{3,4}(\zeta^k, \zeta^{-k})$.

First thing to notice from the results are the parts corresponding to the q-catalan numbers. In the table (1) are the collected values of $C_{3,4}(\zeta^k, \zeta^{-k})$ from the calculations. This corresponds to the combinatorics of the non-crossing matchings we explored earlier with the $2n$th root of unity, where $n = 3$ and cyclic sieving is present. The same can be seen from the equivalent parts of $C_{4,5}(\zeta^k, \zeta^{-k})$ with the 8th root of unity and $C_{2,3}(\zeta^k, \zeta^{-k})$ with the 4th root of unity.

If we have the same q,t-Catalan number, but change the root of unity to have a different order, we notice something interesting. For example, table (2) has the root of unity of order $2m$ instead of $2n$. Comparing to table (1), we see them agreeing on the last and the middle elements. This isn't surprising, since $z^{2n} = 1 = z^{2m} \Rightarrow z^n = z^m$. However, all of the other powers in table (2) get a value of 1, which doesn't really

$C_{3,4}(q,t) = q^3 + q^2t + qt + qt^2 + t^3$, 8th root

	ζ^{-l}
k=l=1	1
k=l=2	1
k=l=3	1
k=l=4	-3
k=l=5	1
k=l=6	1
k=l=7	1
k=l=8	5

Table 2: $C_{3,4}(\zeta^k, \zeta^{-k})$, with the 8th root of unity.

have a combinatorial explanation by our current knowledge. This could be related to an unknown action on some combinatorial object that works with $2m$ or $n + m + 1$.

$C_{3,5}(q,t) = q^4 + q^3t + q^2t + q^2t^2 + qt^2 + qt^3 + t^4$

	6th	7th	8th	10th
k=l=1	0	0		
k=l=2	-2	0	1	
k=l=3	3	0		
k=l=4	-2	0	3	
k=l=5	0	0		3
k=l=6	7	0	1	
k=l=7		7		
k=l=8			7	
k=l=9				
k=l=10				7

Table 3: $C_{3,5}(\zeta^k, \zeta^{-k})$, with varying orders for the root of unity.

With $C_{3,5}(\zeta^k, \zeta^{-k})$ and $C_{3,7}(\zeta^k, \zeta^{-k})$ we have the same situation as mentioned above, but in the rational case. We know there to be applications, where cyclic sieving is exhibited via rational q-Catalan numbers [2], but this doesn't explain the results produced using different roots of unity. Table (3) shows the results for $C_{3,5}(\zeta^k, \zeta^{-k})$, using four different orders for the root of unity. The 7th root of unity corresponds naturally to the $n + m - 1$ case, whereas the other three have no clear explanation for the arising values. Although, the computed results using the 6th root of unity have similarities to the ones in the (3,4) case. Using the 8th root of unity generates integer values only on the even powers, which correspond to the ones that the powers of the 4th root of unity would also have. Using the 10th root of unity doesn't bring any new solutions.

On the other hand, looking at the cases $C_{n,m}(\zeta^k, \zeta^l)$, equation $k = l$ rarely produces anything interesting. However, an observation can be made, that when $k + l$ equals the order of the root of unity used, $C_{n,m}(\zeta^k, \zeta^l)$ does often result in an integer solution. This is explained by the fact that the powers of the root of unity cycle naturally onto the negative integers also, namely $\zeta^a = 1 \Rightarrow \zeta^k = \zeta^{k-a} = \zeta^{-(a-k)}$. So in the case of a 6th root of unity, for example, $\zeta^4 = \zeta^{-(6-4)} = \zeta^{-2}$. In general, this means that $C_{n,m}(\zeta^k, \zeta^l) = C_{n,m}(\zeta^k, \zeta^{l-a})$, where a is the order of the root of unity.

$C_{2,7}(q,t) = q^3 + q^2t + qt^2 + t^3$, 8th root

k,l	1	2	3	4	5	6	7	8
1	0		0		0			
2		0		0				0
3	0		0				0	
4		0		-4		0		0
5	0				0		0	
6				0		0		0
7			0		0		0	
8		0		0		0		4

Table 4: Values of $C_{2,7}(\zeta^k, \zeta^{-l})$ with the 8th root of unity.

Next thing we want to point out are the repeating zeros, when $n = 2$. As an example, let's look at $C_{2,7}(\zeta^k, \zeta^{-l})$, values of which are listed in table (4). The repeating of the values can sometimes be an indication of there being a lower order root of unity that the values are cycling over. However, if that's the case, the missing zeros on the table's diagonal wouldn't make sense. Also, with a root of unity of order 4, there would still be additional zeros that would have no explanation, such as $C_{2,7}(\zeta^1, \zeta^{-3})$. The meaning of these values is therefore left unclear.

For the rest of the results, it's harder to find any consistent patterns or explanations. Some of the results may appear as integers without having any underlying combinatorial reason. For example, -1 is a second order root of unity, while still being an integer. This might be the reason for $C_{3,4}(\zeta^3, \zeta^{-6}) = -1$, using the 6th root of unity and for $C_{4,5}(\zeta^5, \zeta^{-10}) = 0$, using the 10th root of unity. However, for values such as $C_{4,5}(\zeta^1, \zeta^{-5}) = 2$ and $C_{3,5}(\zeta^5, \zeta^{-7}) = -1$ with ζ as the 8th root of unity, we have no explanation.

6 Summary

We construct a code (A) based on the definitions by Armstrong et al. [1] to study the rational Dyck paths and the rational q,t-Catalan numbers. We create a few of these numbers using different beforehand chosen values for the dimensions of the Dyck paths. Then we try out some roots of unity, evaluating their powers as the rational q,t-Catalan numbers' parameters. We list the integer results of the calculations (B) and analyze the content, trying to make sense from it. Some of the results can easily be explained. As expected, we can see the q-Catalan numbers as $k = l$. However, changing the root of unity changes the results and brings out new unexplained values. We notice more interesting results with the $n = 2$ cases. These cases produce repeating zeros, with quite a clear structure, but with no clear meaning. On top of these, we have outlying values, with no apparent logic or consistency.

Although most of the results on evaluating roots of unity as the q,t-Catalan parameters are not integers, the ones that are, are not fully explainable with our current knowledge. In the context of cyclic sieving, there might exist more combinatorial objects and actions on them that exhibit the phenomenon using different roots of unity. In many cases, varying orders of roots of unity seem to produce values for all $k = l$. Upon studying these cases more, we might find correlation to some actions on for example Dyck paths. Experimentations with new actions on Dyck paths might also lead us to cyclic sieving like behavior with different ordered roots of unity. The other unexplained results might be harder to link to cyclic sieving, because of their not so clear structure. However, since its unlikely that they appear as integers accidentally, more research is needed to put meaning behind them.

Finally, as we have emphasized, this thesis regards only the integer solutions produced by the computations. Nevertheless, most of the results are not integers and there is the possibility of defining some rationalized cyclic sieving phenomenon that includes the cases where $k \neq l$ and handles also complex numbers. Further speculations are left for the reader.

References

[1] D. Armstrong, N. Loehr, and G. Warrington. Rational parking functions and Catalan numbers, March 2014. URL http://arxiv.org/abs/1403.1845.

[2] M. Bodnar and B. Rhoades. Cyclic sieving and rational Catalan theory, October 2015. URL http://arxiv.org/abs/1510.08502.

[3] L. Carlitz and J. Riordan. Two element lattice permutation numbers and their q -generalization. 31:371–388, September 1964. URL https://doi.org/10.1215/S0012-7094-64-03136-9.

[4] E. Carlsson and A. Mellit. A proof of the shuffle conjecture, December 2018. URL http://arxiv.org/abs/1508.06239.

[5] A. Garsia and M. Haiman. A remarkable q, t-Catalan sequence and q-Lagrange inversion. *Journal of Algebraic Combinatorics*, 5(3):191–244, July 1996. URL https://doi.org/10.1007/BF00243787.

[6] I. Gordon and S. Griffeth. Catalan numbers for complex reflection groups, December 2009. URL http://arxiv.org/abs/0912.1578.

[7] J. Haglund. *The q, t-Catalan Numbers and the Space of Diagonal Harmonics*, volume 41. University of Pennsylvania, Philadelphia, 2008.

[8] A. Mellit. Toric braids and (m,n)-parking functions. *Duke Mathematical Journal*, 170(18), December 2021. URL http://arxiv.org/abs/1604.07456.

A Code

```
import copy

# Initialize the variables
R=QQ['q','t']
(q,t)=R.gens()

# Class for creating rational Dyck paths
# Takes parameters (x,y) as the width and height of the grid
class Dyck:
    def __init__(self, x: int, y: int):
        self.x = x
        self.y = y
        paths = self.creation()
        self.paths = paths[0] # Paths in numerical form with the distances of each N step from the y-axis (Eg. (0,0,1,1,3))
        self.NEpaths = paths[1] # Paths in text form with the N and E steps listed out (Eg. 'NNENNEENEE')

    # Function used to initialize and list the Dyck paths
    def creation(self):
        # Recursive function generates two paths each cycle (with appended N or E step) and discards the ones going out of bounds
        def iterate(pos: [int, int], path: list[int], NE: list[str]):
            if pos[0] <= self.x/self.y*pos[1] and pos[1] <= self.y:    # Are we inside the bounds
                if pos[0] == self.x and pos[1] == self.y:    # Are we at the destination (x,y)
                    allPaths.append(path)
                    NEPaths.append(NE)
                else:    # split into two paths
                    newPath = copy.copy(path)
                    newPath.append(pos[0])
                    northNE = copy.copy(NE)
                    northNE.append('N')
                    NE.append('E')
                    iterate((pos[0]+1, pos[1]), path, NE)
                    iterate((pos[0], pos[1]+1), newPath, northNE)

        allPaths: list[list[int]] = []
        NEPaths: list[list[str]] = []
        iterate((0,0), [], [])
        return allPaths, NEPaths

    # Method to calculate the area-statistic
    def area(self, path: list[int]):
        max = sum(self.paths[0])
        return max - sum(path)
```

```

    # Method to calculate the dinv-statistic
    def dinv(self, NEpath: list[str]):
        res: int = 0
        for e in range(len(NEpath)):
            if 'E' == NEpath[e]:
                a = 0
                b = 0
                for n in range(e+1, len(NEpath)):
                    if 'E' == NEpath[n]: a += 1
                    if 'N' == NEpath[n]:
                        b += 1
                        if (b-1)/(a+1) <= self.y/self.x and (not a or self.y/self.x <= b/a):
                            res += 1
        return res

    # Use the area and dinv methods to create the q,t-Catalan number
    def catalan(self):
        sum = 0
        for i in range(len(self.NEpaths)): # Sum over all of the paths
            sum += q^(self.area(self.paths[i]))*t^(self.dinv(self.NEpaths[i]))
        return sum
```

Listing 1: Python code used in the thesis.

The code explained and used in the thesis is shown in listing (1). The code is written in Python in a SageMath environment. It contains a class for creating rational Dyck paths and q,t-Catalan numbers. The class uses the creation method to initialize the Dyck paths. It also has the methods area and dinv for calculating statistics on them. The method catalan returns a rational q,t-Catalan number using the aforementioned statistics on the Dyck paths.

B Results

(n,m)		(3,4)				(4,5)				(2,3)		(2,5)		(2,7)		(2,9)		(3,5)								(3,7)			
root		6th		8th		8th		10th		4th		6th		8th		10th		6th		7th		8th		10th		9th		10th	
		ζ^l	ζ^{-l}	ζ^l	ζ^{-l}	ζ^l	ζ^{-l}	ζ^l	ζ^{-l}	ζ^l	ζ^{-l}	ζ^l	ζ^{-l}	ζ^l	ζ^{-l}	ζ^l	ζ^{-l}	ζ^l	ζ^{-l}	ζ^l	ζ^{-l}	ζ^l	ζ^{-l}	ζ^l	ζ^{-l}	ζ^l	ζ^{-l}	ζ^l	ζ^{-l}
k=1	l=1		0		1		0		1		0		0		0		0		0		0						0		
k=1	l=2								0			-2																	
k=1	l=3			-1		2				0		0	0	0	0	0	0						-1						
k=1	l=4												-2																
k=1	l=5	0			-1		2					0		0	0	0	0	0				-1							
k=1	l=6																			0									
k=1	l=7			1		0								0		0	0												
k=1	l=8							0																		0			
k=1	l=9							1								0													
k=1	l=10																												
k=2	l=2		2		1		-2		-1	-2	-2		0		0		0		-2		0		1				0		
k=2	l=3																												
k=2	l=4	2								0	0	0		0	0	0	0	-2											
k=2	l=5												-2							0									
k=2	l=6			1		-2						0	0	0		0	0					1							
k=2	l=7																									0			
k=2	l=8							-1						0	0	0													
k=2	l=9							0																					
k=2	l=10															0	0												
k=3	l=3	-3	-3		1		0		1		0	3	3		0		0	3	3		0						3		
k=3	l=4							0												0									
k=3	l=5			1		0						0	0	0		0	0												
k=3	l=6	-1	-1						0			1	1					1	1							3			
k=3	l=7				-1		2	1						0	0	0						-1							
k=3	l=8																												
k=3	l=9															0	0									0	0		
k=3	l=10																												
k=4	l=4		2	-3	-3	6	6		-1	2	2		0	-4	-4		0		-2		0	3	3				0		
k=4	l=5											-2														0			
k=4	l=6							-1				0	0	0	0	0													
k=4	l=7								0																				
k=4	l=8			-1	-1	0	0							0	0	0	0					1	1						
k=4	l=9																												
k=4	l=10															0	0												
k=5	l=5		0		1		0	6	6				0		0	5	5		0		0			3	3		0	4	4
k=5	l=6																												
k=5	l=7			-1		2								0	0	0	0						-1						
k=5	l=8																												
k=5	l=9															0	0												
k=5	l=10								0							1	1							1	1			2	2
k=6	l=6	5	5		1		-2		-1			3	3		0		0	7	7		0		1				3		
k=6	l=7							0																					
k=6	l=8													0	0	0	0												
k=6	l=9																									0	0		
k=6	l=10															0	0												
k=7	l=7				1		0		1						0		0			7	7						0		
k=7	l=8																												
k=7	l=9															0	0												
k=7	l=10																												
k=8	l=8			5	5	14	14		-1					4	4		0					7	7				0		
k=8	l=9								0																				
k=8	l=10															0	0												
k=9	l=9								1								0									12	12		
k=9	l=10																												
k=10	l=10							14	14							5	5							7	7			12	12

Table B1: Results of the calculations.

Table (B1) holds the results of the computations, which are explained in more detail in the results section. The first row tells the dimensions (n, m). The second row tells the order of the root of unity that is used for that column. The third row tells if the power of the second parameter is positive or negative. The two leftmost columns correspond to the exponents of the roots of unity, ranging from one up to the order of the root of unity. The table lists only the integer results.