Algebraic closure of local fields in characteristic $p > 0$

Bachelor Thesis

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Abstract

In this thesis, we study stucture theory of complete discrete valued fields to understand the absolute Galois group of $\mathbb{C}((x))$ as well as describe some of its structure for the local field $\mathbb{F}_p((x))$. We approach the theory through the concept of ramification, which is a fundamental concept in the theory of complete discrete valued fields, which characterizes the way in which the prime ideals behave in extensions. In the first chapter, we cover topics such as Hensel's Lemma, unramified, totally ramified, and tamely ramified extensions, ramification groups, and Artin-Schreier extensions. In the second chapter, we apply the theory to the field of formal Laurent series. Towards the end, we look at specific extensions of characteristic $p > 0$ fields and their Galois groups as well as describe the algebraic closure of $\overline{\mathbb{F}_p}((x))$ through the introduction of generalized power series.

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Contents

Chapter 1

Preliminaries

Let $K \subseteq L$ be a finite field extension. Then L is a $K-$vector space.

Definition 1.1 *The* ***trace*** $Tr_{L|K}(\alpha)$ *and* ***norm*** $N_{L|K}(\alpha)$ *of* $\alpha \in L$ *are the trace and determinant of the matrix of the* $K-$*linear transformation* $m_\alpha : L \longrightarrow L,\ m_\alpha(x) = \alpha x$ *for all* $x \in L$.

Let K be a field. Let $K((x))$ denote the field of formal Laurent series over K, that is, $K((x)) := \left\{ \sum_{i=m}^{\infty} \alpha_i x^i \mid m \in \mathbb{Z}, \alpha_i \in K \right\}$. Equivalently, $K((x))$ is the field of fractions of the ring of power series $K[[x]]$, or it is the localization of $K[[x]]$ with respect to the set of positive powers of x.

1.1 Galois extensions

1.1.1 Kummer Theory

Definition 1.2 *A field extension* L/K *is a* ***Kummer extension*** *if there exists an integer* $n > 1$ *such that:*

1. *K contains n distinct n-th roots of unity.*
2. *$Gal(L/K)$ is an abelian group with exponent n.*

If K contains n distinct n-th roots of unity, adjoining n-th root of any element α of K gives a splitting field of the polynomial $x^n - \alpha$, hence creating a Kummer extension with a cyclic Galois group of order dividing n.

Kummer Theory states that when K contains n distinct n-th roots of unity, any abelian extension of K of exponent dividing n is formed by adjoining roots of elements of K.

1.1.2 Artin-Schreier extensions

Let K be a field of characteristic $p > 0$ and $a \in K$. The polynomial of the form

$$z^p - z - a$$

is called an Artin-Schreier polynomial. If the polynomial is irreducible in $K[z]$, then its splitting field L over K is a cyclic extension of K of degree p. This is because if b is a root, for any $1 \leq i \leq p$, $(b+i)^p - (b+i) - a = b^i + i^p - b - i - a = b^i - b - a + (i^p - i) = 0$ by Fermat's Little theorem, and $b + i$ is also a root, hence the splitting field is $K(b)$.

Then L/K is Galois and the order of the Galois group divides p. Hence when p is prime $Gal(L/K) \cong \mathbb{Z}/p\mathbb{Z}$.

Lemma 1.3 (Artin-Schreier extensions) *Let L/K be a Galois extension of fields of characteristic $p > 0$ with Galois group $\mathbb{Z}/p\mathbb{Z}$, then L is a splitting field of the Artin-Schreier polynomial $z^p - z + a$ for some $a \in K$. This extension is called an Artin-Schreier extension.*

1.2 Direct and Inverse limits

1.2.1 Direct and inverse systems

Let us define direct and inverse systems. Let $(I, \leq)$ be a directed set (directed poset respectively). Let $(G_i)_{i \in I}$ be a family of groups and $f_{ij} : G_i \to G_j$ ($f_{ij} : G_j \to G_i$ resp.) a family of group homomorphisms for all $i \leq j$ with the following properties:

1. f_{ii} is the identity on G_i.
2. $f_{ik} = f_{jk} \circ f_{ij}$ ($f_{ik} = f_{ij} \circ f_{jk}$ resp.) for all $i \leq j \leq k$

The pair $\big((G_i)_{i \in I}, (f_{ij})_{i \leq j \in I}\big)$ is a **directed system** (an **inverse system** resp.) of groups G_i and transition morphisms f_{ij} over the directed set (poset resp.) I.

1.2.2 Direct Limit

The **direct limit** $\varinjlim G_i$ of the direct system $\big((G_i)_I, (f_{ij})_I\big)$ is the disjoint union $\bigsqcup_i G_i$ of the G_i's modulo an equivalence relation $x_i \sim x_j$ for $x_i \in G_i, x_j \in G_j \iff \exists k \in I$ with $i \leq k$ and $j \leq k$ such that $f_{ik}(x_i) = f_{jk}(x_j)$:

$$\varprojlim_{i \in I} G_i = \bigsqcup_i G_i/\sim.$$

There are a canonical functions $\varphi_j : G_j \longrightarrow \varinjlim G_i$ that send each element to its equivalence class.

Example 1.4 *The field of Puiseux series with coefficients in a field K : $\bigcup_{n>0} K((x^{1/n}))$ is the union of formal Laurent series in $x^{1/n}$. It is the direct limit of the direct system of $\left(K((x^{1/n})), f_{mn} : x^{1/m} \mapsto (x^{1/m})^{m/n}\right)$ whenever m divides n, where the map $f_{mn} : K((x^{1/m})) \to K((x^{1/n}))$ is a field homomorphism.*

1.2.3 Inverse Limit

The **inverse limit** $\varprojlim G_i$ of the inverse system $\left((G_i)_I, (f_{ij})_I\right)$ is a subgroup of the direct product of the G_i's:

$$\varprojlim_{i \in I} G_i = \left\{ (g_i)_I \in \prod_{i \in I} G_i \mid g_i = f_{ij}\left(g_j\right) \text{ for all } i \leq j \text{ in } I \right\}.$$

There is a natural projection $\pi_i : G \longrightarrow G_i$ such that $\pi_i = f_{ij} \circ \pi_j$ for all $i \leq j$. The inverse limit satisfies the universal property.

Examples:

Example 1.5 *The ring of p-adic integers*

$$\mathbb{Z}_p = \varprojlim_{i \in I} \mathbb{Z}/p^i\mathbb{Z}$$

where $f_{ij} : \mathbb{Z}/p^j\mathbb{Z} \overset{mod p^i}{\longrightarrow} \mathbb{Z}/p^i\mathbb{Z}$

Example 1.6 *The group of profinite integers $\hat{\mathbb{Z}}$ is the inverse limit of the inverse system consisting of the finite groups $\mathbb{Z}/n\mathbb{Z}$ where $n \in \mathbb{Z}^+$ and the maps f_{ij} : $\mathbb{Z}/j\mathbb{Z} \to \mathbb{Z}/i\mathbb{Z}$ for $i|j$.*

Proposition 1.7 *Let K be a field. Let I be a set of finite Galois extensions of K. For $L, M \in I$, we say that $L \leq M$ if $L \subseteq M$. This puts a directed partial order on I. For all $L \leq M$ we have the natural surjections*

$$\begin{aligned} \varphi_{LM} : Gal(M/K) &\to Gal(L/K) \\ \sigma_M &\longmapsto \sigma_M|_L. \end{aligned}$$

Then

$$Gal(K^{sep}/K) = \varprojlim_{L \in I} Gal(L/K)$$

where the inverse limit is taken over the inverse system $\left((Gal(L/K)_I), (\varphi_{LM})_I\right)$.

Proof By definition, we have

$$\varprojlim_{L \in I} Gal(L/K) = \left\{ (\sigma_L)_I \in \prod_{L \in I} Gal(L/K) \;\middle|\; \begin{array}{l} \text{For } L, M \in I \text{ such that} \\ L \subseteq M \text{ we have } \sigma_M|_L = \sigma_L \end{array} \right\}$$

Let $\sigma \in Gal(K^{sep}/K)$. Let $(\sigma|_L)_{L \in I} \in \prod_{L \in I} Gal(L/K)$ be an element of the direct product, obtained by restricting σ to each finite Galois sub-extension of $K \subseteq K^{sep}$, then if $L \subseteq M$ for $L, M \in I$, $\sigma|_M|_L = \sigma|_L$ hence $(\sigma|_L)_I$ is an element of $\varprojlim_{L \in I} Gal(L/K)$.

Now, for the opposite direction, let $(\sigma_L) \in \varprojlim_{L \in I} Gal(L/K)$. Let $x \in K^{sep}$, then the minimal polynomial of x over K is separable. Let L be the splitting field of this minimal polynomial. Then L is the smallest Galois extension of K such that $x \in L$. We define $\sigma \in Gal(K^{sep}/K)$ to be such that $\sigma(x) := \sigma_L(x)$. We have that for any other finite extension M of K such that $x \in M$, $\sigma_M(x) = \sigma_L(x)$ since $L \subseteq M$ and hence $\sigma_M(x) = \sigma(x)$. Hence the choice of $\sigma(x)$ does not depend on the choice of a finite extension of K containing x, which shows that σ is well-defined.

1.3 Infinite Galois Theory

In the thesis, we will be dealing with Galois groups of infinite algebraic extensions. In this case, the Galois correspondence fails since not all subgroups of an infinite Galois group $Gal(L/K)$ correspond to subextensions of L/K. We put a topology on $Gal(L/K)$ that allows us to state a modified version of the Fundamental theorem of Galois theory for infinite extensions.

Definition 1.8 *Let L/K be a Galois extension and $G := Gal(L/K)$. The Krull topology of G is such that its basis consists of all cosets of subgroups $H_E := Gal(L/E)$ where E ranges over finite normal extensions of K in L.*

Remark 1.9 *Under the Krull topology every open normal subgroup of G has a finite index.*

Theorem 1.10 (Fundamental theorem of Galois theory) *Let L/K be a Galois extension and $G := Gal(L/K)$ endowed with the Krull topology. There is an inclusion reversing bijection between subextensions $K \subseteq E \subseteq L$ and closed subgroups H of G, hence $E = L^H$.*

1.4 Discrete Valuation Ring

Definition 1.11 *A **discrete valuation ring (DVR)** A is a principal ideal domain that has a unique non-zero prime (hence maximal) ideal $\mathfrak{p}(A)$.*

Equivalently, A has one and only one (up to multiplication by an invertible element of A) irreducible element π called a **uniformizing element**. If $x \neq 0$ is an element of A, then $x = \pi^n u$, with $n \in \mathbb{N}$ and u a unit.

Since $\mathfrak{p}(A)$ is a maximal ideal, $A/\mathfrak{p}(A)$ is a field, called a **residue field** of A.

The invertible elements or **units** of A are the elements that do not belong to $\mathfrak{p}(A)$. They form a multiplicative group.

Proposition 1.12 *Let $\mathcal{O}_K$ be a commutative ring. Then $\mathcal{O}_K$ is a discrete valuation ring if and only if it is a Noetherian local ring and its maximal ideal is generated by a non-nilpotent element.*

1.5 Norms and Valuations

A **norm** (or absolute value) on a field K is a map

$$||\cdot|| : K \longrightarrow \mathbb{R}_{\geq 0}$$

with the following properties:

(i) $||x|| = 0 \Longleftrightarrow x = 0.$

(ii) $||x \cdot y|| = ||x|| \cdot ||y||$

(iii) $||x + y|| \leq ||x|| + ||y||$ (triangle inequality).

Definition 1.13 *A norm on a field K is called* ***non-archimedean*** *if it satisfies the stronger triangle inequality: $||x + y|| \leq max\{||x||, ||y||\}, \forall x, y \in K$.*

A **(rank one) valuation** on a field K is a map

$$v : K \longrightarrow \mathbb{R} \cup \infty$$

with the following properties:

(i) $v(x) = \infty \Longleftrightarrow x = 0.$

(ii) $v(xy) = v(x) + v(y)$, $\forall x, y \in K$

(iii) $v(x + y) \geq min\{v(x), v(y)\}$ $\forall x, y \in K$

Proposition 1.14 *The subset*

$$\mathcal{O}_K := \{x \in K \mid v(x) \geq 0\} = \{x \in K \mid \|x\| \leq 1\}$$

is a ring, called the ***valuation ring of*** *K, with a group of units*

$$\mathcal{O}_K^\times = \{x \in K \mid v(x) = 0\} = \{x \in K \mid \|x\| = 1\}$$

and with the unique maximal ideal

$$\mathfrak{p}(\mathcal{O}_K) := \{x \in K \mid v(x) > 0\} = \{x \in K \mid \|x\| < 1\}$$

From this point, we use the above notations and if there is no confusion, we shorten $\mathfrak{p}(\mathcal{O}_K)$ with $\mathfrak{p}$.

We have that $\Gamma := v(K^\times)$ is a subgroup of $(\mathbb{R}, +)$, called the **value group**. A valuation is **discrete** if Γ is a nontrivial discrete subgroup of $(\mathbb{R}, +)$ hence if it admits a smallest positive value s such that $\Gamma = s\mathbb{Z}$.

If $s = 1$ then the discrete valuation is said to be **normalized**. We may rescale a discrete valuation such that $s = 1$ and the $\mathcal{O}_K, \mathcal{O}_K^\times, \mathfrak{p}$ are unchanged. Then an element $\pi \in \mathcal{O}_K$ such that $v(\pi) = 1$ is a **prime element** (or a uniformizing element) and every element $x \in K^\times$ admits a unique representation $x = u\pi^m$ with $m \in \mathbb{Z}$ and $u \in \mathcal{O}_K^\times$.

If v is a discrete valuation then $\mathcal{O}_K$ is a **discrete valuation ring** and we denote by k the residue field of $\mathcal{O}_K$.

Example 1.15 *(**P-adic integers**). Let p be a prime. For any $x \in \mathbb{Z}\backslash\{0\}$ write $x = p^k x'$ such that $p \nmid x'$. Then the function*

$$ord_p : \mathbb{Q} \to R \cup \{\infty\}$$
$$x \mapsto \begin{cases} k & x \neq 0 \\ \infty & x = 0 \end{cases}$$

*is a valuation on $\mathbb{Q}$ with the non-archimedean norm defined as $|x|_p := p^{-ord_p(x)}$. The completion of $\mathbb{Q}$ with respect to this norm is denoted as $\mathbb{Q}_p$. The ring of **p-adic integers***

$$\mathbb{Z}_p := \{x \in \mathbb{Q}_p \mid |x|_p \leq 1\}$$

is a discrete valuation ring with the unique maximal ideal $m_p = \{x \in \mathbb{Q}_p \mid |x|_p \leq \frac{1}{p}\} = p\mathbb{Z}_p$ and residue field $\mathbb{Z}/p\mathbb{Z}$.

Example 1.16 *(**Formal Laurent series**). Let K be a field. Let $K((x))$ denote the field of formal Laurent series over K, that is, $K((x)) := \left\{\sum_{i=m}^{\infty} \alpha_i x^i \mid m \in \mathbb{Z}, \alpha_i \in K\right\}$. Let $\alpha = \sum_{i=m}^{\infty} \alpha_i x^i \in K((x))$. Then the function:*

$$v : K((\alpha)) \to R \cup \{\infty\}$$
$$\alpha \mapsto \begin{cases} m & \alpha \neq 0 \\ \infty & \alpha = 0 \end{cases}$$

is a valuation on $K((x))$. The ring of integers $K[[x]]$ is a discrete valuation ring with the unique maximal ideal (x) and residue field K.

1.5.1 Completions

Definition 1.17 *A normed field $(K, ||\cdot||)$ is complete if every Cauchy sequence is convergent to an element in K.*

Example 1.18 *$\mathbb{F}_p((x))$ is a completion of the ring $\mathbb{F}_p(x)$.*

1.5.2 Local fields

Definition 1.19 *A field K is (non-Archimedean)* ***local field*** *if it is complete with respect to the norm from the discrete valuation and its residue field k is finite.*

Examples and non-examples:

1. $\mathbb{F}_p((x))$ with the ring of integers $\mathbb{F}_p[[x]]$ which has the maximal ideal (x) is local since the residue field is $\mathbb{F}_p$.
2. $\mathbb{C}((x))$ is not a local field. Its residue field is $\mathbb{C}$ which is not finite.

1.6 Hensel's Lemma

In this section, let K be a non-archimedean complete field. We present Hensel's Lemma, which holds true for such fields. In general, the collection of non-archimedean valued fields that fulfill Hensel's Lemma is referred to as **Henselian fields**, and they encompass complete fields.

Lemma 1.20 (Hensel-Kurschak lemma) *Let $f \in K[x]$ be an irreducible polynomial whose leading and constant coefficients lie in $\mathcal{O}_K$. Then $f \in \mathcal{O}_K[x]$.*

Lemma 1.21 (Hensel's Lemma I) *Let K be a CDVF, $\mathcal{O}_K$ its DVR with the maximal ideal $\mathfrak{p}$ and k residue field. Let $f \in \mathcal{O}_K[x]$ be a monic polynomial such that $\bar{f} \in k[x]$ has a simple root $\bar{\alpha} \in k$. Then $\bar{\alpha}$ can be lifted to a root of f in $\mathcal{O}_K$*

Let $k = \mathcal{O}_K/\mathfrak{p}$.

Theorem 1.22 (Hensel's Lemma) *Let $f \in \mathcal{O}_K[x]$ be a primitive polynomial such that*

$$f \equiv \bar{g}\bar{h} mod\, \mathfrak{p}$$

for $\overline{g}, \overline{h} \in k[x]$ relatively prime polynomials, then f admits a factorization $f = gh$ for $g, h \in \mathcal{O}_K[x]$ such that $deg(g) = deg(\overline{g})$ and

$$g \equiv \overline{g} \text{ mod } \mathfrak{p} \text{ and } h \cong \overline{h} \text{ mod } \mathfrak{p}.$$

Theorem 1.23 *Let K be a henselian field with respect to the norm $||\cdot||_K$. Then $||\cdot||_K$ can be extended uniquely to a norm $||\cdot||_L$ of any given algebraic extension $K \subseteq L$. If the extension is of finite degree, $[L : K] = n$, then the extension is given by $||\alpha||_L = \sqrt[n]{||N_{L|K}(\alpha)||_K}$.*

Lemma 1.24 (Nakayama's Lemma) *Let R be a local ring with maximal ideal $\mathfrak{p}$ and let M be a finitely generated R-module. Let $x_1, x_2, ..x_n \in M$ be such that the images of x_i generate $M/\mathfrak{p}M$ as an $(R/\mathfrak{p})$-vector space. Then $x_1, ..., x_n$ generate M as an $R-$module.*

Let R be a discrete valuation ring with maximal ideal $\mathfrak{p}$ and residue field k. Let $f \in R[x]$ be a monic polynomial of degree n. Let $S_f = R[x]/(f)$ and $\bar{S_f} = S_f/\mathfrak{p}S_f = R[x]/(\mathfrak{p}, f)$. If $\bar{f}$ is the image of f in $k[x]$ then $\bar{S_f} = k[x]/(\bar{f})$ as it can be seen from the following diagram:

$$\begin{tikzcd} f \in R[x] \arrow[r, two heads, "/\mathfrak{p}"] \arrow[d, two heads] & k[x] \ni \bar{f} \arrow[d, two heads, "/(\bar{f})"] \\ S_f = R[x]/(f) \arrow[r, two heads, "/\mathfrak{p}S_f"] & \bar{S_f} = S_f/\mathfrak{p}S_f \end{tikzcd}$$

Let $\bar{f} = \prod_{i=1}^{g} \bar{f}_i^{e_i}$ and let f_i be a lift of $\bar{f}_i$ to $R[x]$.

Proposition 1.25 (Serre) *Let $\mathfrak{p}_i = (\mathfrak{p}, f_i)$. Then $\mathfrak{p}_i$ are the distinct maximal ideals of S_f and $S_f/\mathfrak{p}_i \cong k[x]/(\bar{f}_i)$.*

Proof We have that $\mathfrak{p}_i$ is the lift of $\bar{m} := (\bar{f}_i) \in \bar{S}_f$ in S_f. We have that $\bar{S}_f/(\bar{f}_i) = k[x]/(\bar{f}_i)$, which implies that $S_f/\mathfrak{p} = k[x]/(\bar{f}_i)$ and that $\mathfrak{p}_i$ is a maximal ideal. Now let us show that every maximal ideal $\mathfrak{n}$ of S_f is equal to one of $\mathfrak{p}_i$. It is enough to show that $\mathfrak{n}$ contains $\mathfrak{p}$ since then $\mathfrak{n}$ is the inverse image of a maximal ideal $\bar{f}_i$ of S_f. Assume for a contradiction that $\mathfrak{n}$ does not contain $\mathfrak{p}$. Then, $\mathfrak{n}+\mathfrak{p}S_f = S_f$ and by Nakayama's Lemma $\mathfrak{n} = S_f$, which is a contradiction. □

Remark 1.26 *If a field is* ***henselian*** *with respect to v, and $K \subset L$ is an algebraic extension, then the valuation ring $\mathcal{O}_L$ of the extended valuation w is the* ***integral closure*** *of the valuation ring $\mathcal{O}_K$ in L. Hence, this also holds for* ***complete*** *fields.*

1.7 Ramification

Let $(L, w)/(K, v)$ be an extension of valued fields, meaning that there is a field homomorphism $\iota : K \hookrightarrow L$ such that $w \circ \iota = v$. We have that ι induces an embedding of valuation rings $\mathcal{O}_K \hookrightarrow \mathcal{O}_L$ and of maximal ideals $\mathfrak{p}(\mathcal{O}_K) \hookrightarrow \mathfrak{p}(\mathcal{O}_L)$.

Therefore we have the following homomorphism:

$$\bar{\iota} : k = \mathcal{O}_K/\mathfrak{p}(\mathcal{O}_K) \hookrightarrow \mathcal{O}_L/\mathfrak{p}(\mathcal{O}_L) = l,$$

called a residual extension and the degree of the extension $[l : k]$ is the **residual degree** $f = f(L/K)$.

Furthermore, $v(K) \subset w(L)$ and the index $[w(L) : v(K)] = \frac{||L^\times||_L}{||K^\times||_K}$ of the subgroup is the **ramification index** $e = e(L/K)$.

In case v, and hence its extension $w = \frac{1}{n} v \circ N_{L|K}$, is discrete and if $\mathcal{O}_K, \mathfrak{p}, \pi$, respectively $\mathcal{O}_L, \mathfrak{B}, \Pi,$, are the valuation ring, the maximal ideal and a prime element of K, resp. L then we have

$$e = [w(\Pi)\mathbb{Z} : v(\pi)\mathbb{Z}].$$

Hence $v(\pi) = ew(\Pi)$ and therefore $\pi = u\Pi^e$ for some unit $u \in \mathcal{O}_K^\times$. Therefore we have $\mathfrak{p}\mathcal{O}_L = \pi\mathcal{O}_L = \Pi^e\mathcal{O}_L = \mathfrak{B}^e$.

Proposition 1.27 *$[L : K] \geq ef$ and if v is discrete and $K \subseteq L$ is separable then $[L : K] = ef$.*

Let K be a henselian field with respect to a non-Archimedean valuation v. Let p be the **characteristic exponent** of the residue field k, that is we take p to be 1 if k has zero characteristic and p if k has characteristic $p > 0$.

Definition 1.28 *A finite extension $K \subseteq L$ is*

(i) ***unramified*** *if the extension of residue fields $k \subseteq l$ is separable and $e(L/K) = 1$.*

(ii) ***totally ramified*** *if $e(L/K) = [L : K]$, equivalently $l = k$.*

(iii) ***tamely ramified*** *if $e(L/K)$ is prime to p.*

An arbitrary algebraic extension $K \subseteq L$ is unramified/totally ramified/tamely ramified if all of its finite subextensions are - respectively.

Remark that every unramified extension is tamely ramified and if the characteristic of k is 0 then every algebraic extension of K is tamely ramified. An extension is **wildly ramified** if it is not tamely ramified.

An extension is **totally tamely ramified (TTR)** if it is both totally ramified and tamely ramified.

1.7.1 Galois Extensions and Ramification groups

Let K be a DVF, with valuation ring $\mathcal{O}_K$, maximal ideal $\mathfrak{p}$ and residue field k. Let $K \subset L$ be a finite Galois extension with Galois group G. Let $\mathcal{O}_L$ be the **integral closure** of $\mathcal{O}_K$ in L and $\mathfrak{B} \subset \mathcal{O}_L$ a maximal ideal lying over $\mathfrak{p}$. Let $l_{\mathfrak{B}} := \mathcal{O}_L/\mathfrak{B}$. Note that since K is not complete, L is not necessarily a DVF, as the valuation isn't necessarily extended to L uniquely, hence there can be more than one primes lying over $\mathfrak{p}$.

1. The **decomposition group** of $\mathfrak{B}$ is the subgroup $D = \{\sigma \in G \mid \sigma(\mathfrak{B}) = \mathfrak{B}\}$
2. The **inertia group** of $\mathfrak{B}$ is the kernel I of the map $D \to Aut(l_{\mathfrak{B}}/k)$

Lemma 1.29 *$l_{\mathfrak{B}}/k$ is a normal. extension*

Proof Let $\bar{\alpha} \in l_{\mathfrak{B}}$ and α be a lift of $\bar{\alpha}$ to $\mathcal{O}_L$. We have that for any $\sigma \in G$, $\sigma(\alpha)$ is still a root of the minimal polynomial of α over $\mathcal{O}_K$, hence $\sigma(\alpha) \in \mathcal{O}_L$ as $\mathcal{O}_L$ is the integral closure of $\mathcal{O}_K$ in L.

The monic polynomial $f(x) := \prod_{\sigma \in G}(x - \sigma(\alpha))$ is invariant under the action of G, hence it has coefficients in $\mathcal{O}_K$. Since f splits into linear factors in $\mathcal{O}_L$, the reduction $\overline{f}$ of f in $l_{\mathfrak{B}}[x]$ also splits into linear factors.

Furthermore, $\alpha \in \mathcal{O}_L$ is a root of f, hence $\overline{\alpha} \in l_{\mathfrak{B}}$ is a root of $\overline{f} \in l_{\mathfrak{B}}[x]$ which is the same as reduction of f in $k[x]$ as $f \in \mathcal{O}_K \subseteq \mathcal{O}_L$. Hence the minimal polynomial $m_{\overline{\alpha}}$ of $\overline{\alpha}$ over k divides $\overline{f}$. Since $\overline{f}$ splits into linear factors in $l_{\mathfrak{B}}[x]$ we conclude that $m_{\overline{\alpha}}$ also splits into linear factors in $l_{\mathfrak{B}}[x]$ and hence $l_{\mathfrak{B}}/k$ is a normal extension. □

Furthermore $D \to Aut(l_{\mathfrak{B}}/k)$ is surjective.

Remark 1.30 *If k is a perfect field then $l_{\mathfrak{B}}/k$ is a Galois extension and $D/I \cong Gal(l_{\mathfrak{B}}/k)$.*

For the remaining section let K denote a **complete** discretely valued field with valuation v and let us assume k is a perfect field. Then L is also a CDVF and let us denote by w the unique extended valuation on L.

Theorem 1.31 *Let K be as above and $K \subset L$ an extension. If $k \subset l$ is separable then $\mathcal{O}_L = \mathcal{O}_K[x]$ for some $x \in \mathcal{O}_L$.*

Let x denote the element of $\mathcal{O}_L$ generating it as a $\mathcal{O}_K$-algebra.

Lemma 1.32 *Let $\sigma \in G = Gal(L/K)$, and $i \geq 1$ be an integer. Let G_i be the set of $\sigma \in G$ satisfying the following equivalent conditions.*

1. *σ acts trivially on the quotient ring $\mathcal{O}_L/\mathfrak{B}^{i+1}$.*
2. *$w(\sigma(a) - a) \geq i + 1$ for all $a \in \mathcal{O}_L$.*
3. *$w(\sigma(x) - x) \geq i + 1$.*

Each G_i forms a group and it is called the ***i-th ramification group*** *of G.*

Proof Let $a \in \mathcal{O}_L$. Part 1. is equivalent to $\sigma(a) = a + (\mathfrak{B}^{i+1})$ which is equivalent to part 2. since elements of $(\mathfrak{B}^{i+1})$ have valuation greater or equal to $i + 1$ as $\mathfrak{B}$ is a uniformizing element of $\mathcal{O}_L$ which is a discrete valuation ring. Let $x_i = x + (\mathfrak{B}^{i+1})$. Then x generates $\mathcal{O}_L/\mathfrak{B}^{i+1}$ as an $\mathcal{O}_K$-algebra, hence parts 2. and 3. are equivalent. Lastly, it can be seen that each G_i satisfies group axioms. □

Proposition 1.33 *The i-th ramification groups G_i form a decreasing sequence of normal subgroups of G. $G_{-1} = G$ and $G_i = \{1\}$ for i sufficiently large, where 1 denotes identity of G.*

Proof We see that G_i forms a decreasing sequence of subgroups from part 3. since if $\sigma \in G_{i+1}$, then $w(\sigma(x)-x) \geq i+2$, but then also $w(\sigma(x)-x) \geq i+1$ hence $\sigma \in G_i$.

We see that $G_i = ker(G \xrightarrow{\varphi} Aut(\mathcal{O}_L/\mathfrak{B}^{i+1}))$ where φ is a group homomorphism since $\sigma(\mathfrak{B}) = \mathfrak{B}$ for all $\sigma \in G$. Let $\sigma \in G$ and $\sigma_i \in G_i$, then $\varphi(\sigma\sigma_i\sigma^{-1}) = \varphi(\sigma)\varphi(\sigma_i)\varphi(\sigma^{-1}) = \varphi(\sigma)\cdot 1 \cdot \varphi(\sigma^{-1}) = \varphi(\sigma)\varphi(\sigma^{-1}) = 1$ hence $\sigma\sigma_i\sigma^{-1} \in G_i$ hence G_i is a normal subgroup of G.

We have that $G_{-1} = G$ since $\mathcal{O}_L/\mathfrak{B}^0 = \mathcal{O}_L/(1) = 0$ and for all $\sigma \in G$, $\sigma(0) = 0$.

We have that the decreasing chain of G_i stabilizes to $\cap_i G_i$. Let $\sigma \in \cap_i G_i$, then for all $\alpha \in \mathcal{O}_L$, $\sigma(\alpha) - \alpha \in \cap_{i\geq -1}\mathfrak{B}^{i+1} = 0$ as $\mathcal{O}_L$ is a local ring. Hence $\sigma(\alpha) = \alpha$ for all $\alpha \in \mathcal{O}_L$ and hence $\sigma = 1$. $\square$

Thus we have the following exact sequence:

$$1 \to G_1 \to G_0 \to G_0/G_1 \to 1.$$

The ramification groups define a filtration of G. Note that when L is a complete discrete valued field, the integral closure $\mathcal{O}_L$ of $\mathcal{O}_K$ is a DVR, hence there is only one $\mathfrak{B}$ lying over $\mathfrak{p}$. Hence decomposition group $D = G$ is the whole $Gal(K/L)$.

G_0 is the subgroup of G that fixes elements of $\mathcal{O}_L/\mathfrak{B} = k(\mathfrak{B})$ hence it is in the inertia group I of $\mathfrak{B}$.

We obtain that when the residual extension is separable the quotient $G/G_0 \cong Gal(l/k)$.

Hence we have that ramification groups of G determine those of a subgroup H.

Definition 1.34 *The wild inertia subgroup $P \subset I$ is $Gal(L/K_L^{tame})$ where K_L^{tame} is the maximal tamely ramified subextension. P is a normal subgroup of I and it is a unique $p-$Sylow subgroup.*

Theorem 1.35 *Let l have characteristic $p > 0$. Then the wild inertia group is the same as 1st ramification group $G_1 = P$.*

Proof Let us define a map the following way $\varphi : G_0 \to l^\times, \varphi(\sigma) = \sigma(\pi)/\pi \, mod \, \mathfrak{B}$ for any uniformizer $\pi \in L$. The map is a homomorphism and the kernel of φ is G_1. Note that $G_1 \subset ker(\varphi)$ since $\sigma(\pi) \equiv \pi \, mod \, \mathfrak{B}^2$ is equivalent to $\sigma(\pi)/\pi \cong 1 \, mod \, \mathfrak{B}$ which is same as $\sigma \in ker(\varphi)$. Also the map $G_0/G_1 \to l^\times$ induced by φ is injective. We get that G_0/G_1 canonically injects into $l^\times$ and hence it has order prime to p. $G_1 \subseteq G_0$ is a p-group and since G_0/G_1 has order prime to p we have that G_1 is a p-Sylow subgroup of G_0 and this is the unique one, hence $G_1 = P$. $\square$

If the characteristic of l is zero, then $G_1 = \{1\}$ and the group G_0 is cyclic.

Proposition 1.36 *Let $H \subseteq G$ be a subgroup. Let w_H be the restriction of the valuation w to $L^H \subseteq L$. For every $s \in H, w_H = (s(x) - x) = w(s(x) - x)$ and $H_i = G_i \cap H$.*

Proof This follows directly from part 1. of Lemma 1.32. □

Corollary 1.37 *Let K_L^{unr} denote the largest unramified extension of K in L and H the corresponding subgroup of G. Then $H = G_0$ and the ramification groups $G_i, i \geq 0$ of G are equal to those of H.*

Proof We have $H = Gal(L/K_L^{unr})$. Let k_l^{sep} be the largest separable extension of k in l. We have that $Gal(K_L^{unr}/K) = Gal(k_l^{sep}/k)$ and $Gal(l/k) = Gal(k_l^{sep}/k)$. Also $G/G_0 = Gal(l/k)$. Hence $Gal(K_L^{unr}/K) = G/G_0$. Since

$$1 \to Gal(L/K_L^{unr}) \to G = Gal(L/K) \to G/G_0 = Gal(K_L^{unr}/K) \to 1$$

we obtain that $H = G_0$.

The ramification groups of G and H coincide for $i \geq 0$ since $G_i \subset G_0 = H$ for $i \geq 0$. Proposition 1.36 shows that $G_i = H_i$. □

Remark 1.38 *Assume the residue field k is perfect. Then $k^{sep} = \bar{k}$.*

The extension K^{unr}/K is galois and $Gal(K^{unr}/K) = Gal(\bar{k}/k)$. The extension K^{sep}/K^{unr} is a totally ramified Galois extension. Hence we have a short exact sequence:

$$1 \to Gal(K^{sep}/K^{unr}) \to Gal(K^{sep}/K) \to Gal(\bar{k}/k) \to 1$$

Let K^{tame} denote the maximal tamely ramified extension of K. Then similarly, K^{tame}/K^{unr} is a Galois extension and we have an exact sequence:

$$1 \to Gal(K^{tame}/K^{unr}) \to Gal(K^{tame}/K) \to Gal(K^{unr}/K) \to 1$$

.

1.7.2 Unramified Extensions

Let $(K, ||\cdot||)$ be a complete, non-Archimedean field whose valuation ring R is a DVR.

The finite unramified extensions L of K with $K-$algebra homomorphisms form a catefory $\mathcal{C}_K^{unr}$. Finite separable extensions l of k with $k-$algebra homomorphisms form a category $\mathcal{C}_k^{sep}$.

Theorem 1.39 *The categories $\mathcal{C}_K^{unr}$ and $\mathcal{C}_k^{sep}$ are equivalent via the functor:*

$$\mathcal{F} : \mathcal{C}_K^{unr} \to \mathcal{C}_k^{sep}, \mathcal{F}(L) = l, \quad \mathcal{F}(\varphi : L_1 \to L_2) = \overline{\varphi} : l_1 \to l_2$$

such that $\overline{\varphi}(\overline{\alpha}) := \overline{\varphi(\alpha)}$ where α is a lift of $\bar{\alpha}$ as shown on the diagram:

$$\begin{tikzcd}
\alpha \in \mathcal{O}_{L_1} \arrow[r, two heads] \arrow[d, "\varphi"'] & l_1 \ni \bar{\alpha} \arrow[d, dashed, "\bar{\varphi}"] \\
\varphi(\alpha) \in \mathcal{O}_{L_2} \arrow[r, two heads] & l_2 \ni \overline{\varphi(\alpha)}
\end{tikzcd}$$

$\mathcal{F}$ gives a bijection between the isomorphism classes in $\mathcal{C}_K^{unr}$ and $\mathcal{C}_k^{sep}$ and it induces a bijection between $Hom_K(L_1, L_2) \cong Hom_k(l_1, l_2)$.

Corollary 1.40 *$K \subset L$ is an unramified extension if and only if $\mathcal{O}_L = \mathcal{O}_K[\alpha]$ for some $\alpha \in L$ whose minimal polynomial $f \in \mathcal{O}_K[x]$ has a separable image $\overline{f}$ in $k[x]$.*

Corollary 1.41 *Let K have a residue field k of characteristic exponent p, and let ζ_n be a primitive nth root of unity in an algebraic closure $\overline{K}$ of K, where $gcd(p, n) = 1$. The extension $K \subset K(\zeta_n)$ is unramified.*

Proof Let $L = K(\zeta_n)$. We see that $\mathcal{O}_L = \mathcal{O}_K[\zeta_n]$. The field $K(\zeta_n)$ is a splitting field of $f = x^n - 1 \in K[x]$. By 1.20 $f \in \mathcal{O}_K[x]$. The minimal polynomial f_n of ζ_n in $\mathcal{O}_K$ divides f. To show that $\bar{f}_n$ is separable it suffices to show that $\bar{f}$ is separable. The latter is true since gcd($\bar{f}, \bar{f}'$)$\neq 1$ if and only if $\bar{f}' = nx^{n-1} = 0$, equivalently when $p|n$, however we have that gcd(p, n)$= 1$. □

Corollary 1.42 *Let K have a finite residue field $\mathbb{F}_q$ and let $K \subset L$ be a degree n extension. L/K is unramified if and only if $L \cong K(\zeta_{q^n-1})$. In this case L/K is a Galois extension with $Gal(L/K) \cong \mathbb{Z}/n\mathbb{Z}$.*

Proof ($\Leftarrow$) From Corollary 1.41

($\Rightarrow$) Assume $K \subseteq L$ is an unramified extension. Then $[l : k] = [L : K] = n$ and hence $l \cong \mathbb{F}_{q^n}$. Then $l^\times$ is cyclic of order $q^n - 1$ generated by $\bar{\alpha}$. The minimal polynomial $\bar{g} \in \mathbb{F}_q[x]$ of $\bar{\alpha}$ divides $x^{q^n-1} - 1 \in \mathcal{O}_K$. By Hensel's Lemma 1.22, we can lift $\bar{g}$ to $g \in \mathcal{O}_K$ which will divide $x^{q^n-1} - 1$ and by Hensel's Lemma 1.21 we can lift $\bar{\alpha}$ to a root α of g. Since α is then also a root of $x^{q^n-1} - 1$ it must be a primitive $(q^n - 1)$th root of unity. By the Theorem 1.31 $\mathcal{O}_L = \mathcal{O}_K[\zeta_{q^n-1}]$. Since $\mathbb{F}_{q^n}$ is a splitting field of x^{q^n}, by Hensel's lemma L is a splitting field of $x^{q^n-1} - 1$ and $K \subset L$ is a Galois extension and $Gal(L/K) \cong \mathbb{Z}/n\mathbb{Z}$. Furthermore one can see that $Gal(L/K) \cong Gal(l/k)$. □

Now we add the assumption that the residue field k of K is **perfect**.

Proposition 1.43 (p. 51 Lang) *Let L be a finite extension of K. Let L_{unr} be the compositum of all unramified subfields over K. Then L_{unr} is unramified over K and L is totally ramified over L_{unr}.*

$$\begin{array}{c} L \\ \Big| \text{tot. ram.} \\ L_{unr} \\ \Big| \text{unr} \\ K \end{array}$$

Proposition 1.44 (Lang, p.49) *Let $L \subseteq L$ be a finite extension.*

(i) If $K \subset F \subset L$ are field extensions, then E/K is unramified if and only if E/F and F/K are unramified.

(ii) If E is unramified over K and K_1 is a finite extension of K, then EK_1 is unramified over K_1.

(iii) If E_1 and E_2 are finite unramified over K, then so is E_1E_2.

These properties hold for when "unramified" is replaced by "tamely ramified".

Definition 1.45 *The maximal unramified extension K^{unr} of K (in K^{sep}) is the subfield*

$$\bigcup_{\substack{K\subseteq E\subseteq L \\ K\subseteq E \text{ fin. unram.}}} E \subseteq K^{sep}$$

where union is taken over the finite unramified subextensions $K \subseteq K^{sep}$.

The residue field of K^{unr} is k^{sep}.

1.7.3 Totally Ramified extensions

Definition 1.46 *Let $\mathcal{O}_K$ be a discrete valuation ring with maximal ideal $\mathfrak{p}$. A monic polynomial*

$$f(x) = x^n + a_{n-1}x^{n-1} + ... + a_1x + a_0 \in \mathcal{O}_K[x]$$

*is **Eisenstein** if $a_i \in \mathfrak{p}$ for all $i \in \{0, 1, ..., n-1\}$ and $a_0 \notin \mathfrak{p}^2$.*

Proposition 1.47 *Let $f \in \mathcal{O}_K[x]$ be an Eisenstein polynomial. Then $\mathcal{O}_K[x]/(f)$ is a discrete valuation ring with maximal ideal generated by the image of x under the map $\mathcal{O}_K[x] \longrightarrow \mathcal{O}_K[x]/(f)$ and the residue field k. Hence, the extension $K \subseteq L$, where L is the fraction field of $\mathcal{O}_K[x]/(f)$ is totally ramified.*

Proof We have the image of f in $k[x]$ is $\bar{f} = t^n$. By Lemma 1.25, $\mathcal{O}_K[x]/(f)$ has only one maximal ideal $(\mathfrak{p}, t)$ hence it is a local ring. The fact that $a_0 \in \mathfrak{p}$ but $a_0 \notin \mathfrak{p}^2$ implies that a_0 is the uniformizing element of $\mathcal{O}_K$, or $\mathfrak{p} = (a_0)$.

Let $\bar{x}$ be the image of x in $\mathcal{O}_K[x]/(f)$. Then

$$a_0 = -\bar{x}^n - a_{n-1}\bar{x}^{n-1} - \ldots - a_1\bar{x},$$

hence $a_0 \in (\bar{x})$ and thus, $(\mathfrak{p}, \bar{x}) = (a_0, \bar{x}) = (\bar{x})$. Hence the unique maximal ideal in $\mathcal{O}_K[x]/(f)$ is a principal maximal ideal and since a_0 is not nilpotent, neither is $\bar{x}$ and by Proposition 1.12 $\mathcal{O}_K[x]/(f)$ is a discrete valuation ring.$\square$

Theorem 1.48 *(Theorem 2.11 in (Clark, n.d.)) Let $K \subseteq L$, $\mathcal{O}_K$ the discrete valuation ring and π a uniformizer of the integral closure $\mathcal{O}_L$ of $\mathcal{O}_K$ in L. L/K is totally ramified if and only if $\mathcal{O}_L = \mathcal{O}_K[\pi]$ and the minimal polynomial of π is Eisenstein.*

1.7.4 Tamely ramified extensions

Theorem 1.49 *Let L/K be totally tamely ramified, with $[L : K] = e$. Then, there exist uniformizers π and Π of K and L respectively such that $\Pi^e = \pi$.*

Theorem 1.50 *Let K be a henselian DVF with **algebraically closed** residue field k of characteristic p. For each $e \in \mathbb{Z}^+$ prime to p, there exists a unique **tamely ramified extension** $K \subset L_e$ of degree e, obtained by taking the eth root of any uniformizing element of K. Moreover, $K^{tame} = \bigcup_e L_e$ and $Gal(K^{tame}/K) \cong \prod_{l \neq p} \mathbb{Z}_l$.*

Proof We have that $k = \bar{k}$ therefore there are no proper algebraic, hence separable extensions of k. Since any unramified extension of K gives a separable extension of k we may deduce that there are no proper unramified extensions of K. Adjoining any primitive $n-$th root of unity to K when gcd(p, n)= 1 gives an unramified extension of K, and therefore all such roots of unity are contained in K. Since k is algebraically closed it is a perfect field and since there are no proper unramified extensions of k by the Proposition 1.43 every extension of K is totally ramified. Thus, every tamely ramified extension of K is totally tamely ramified, in other words, every degree e extension, where gcd(p, e)= 1, is totally tamely ramified. Let us show that there is a unique such degree e extension L_e of K.

Let e be coprime with p and $K \subset L$ an extension of degree e. Then by the Theorem 1.49, there exists a uniformizer π of K such that $L = K[\pi^{\frac{1}{e}}]$. Conversely, if e is coprime with p, for any uniformizer π' of K, $K \subset K[\pi'^{\frac{1}{e}}]$ gives a degree e tamely ramified extension. We have to show that $K[\pi^{\frac{1}{e}}] = K[\pi'^{\frac{1}{e}}]$. Since π, π' are uniformizing elements $\pi\pi'^{-1}$ is a unit in $\mathcal{O}_K$. Let $f(x) = x^e - \pi^{-1}\pi' \in \mathcal{O}_K[x]$. Since k is algebraically closed with characteristic p, $\bar{f}$ has a simple root in k. By 1.21 this root can be lifted to a root of f in $\mathcal{O}_K$, hence K contains eth root of $\pi^{-1}\pi'$. This shows that $K[\pi^{\frac{1}{e}}] = K[\pi'^{\frac{1}{e}}]$ and there is a unique degree e extension L_e of K when gcd(p, e)= 1.

By Kummer theory we have $Gal(L_e/K) \cong \mathbb{Z}/e\mathbb{Z}$. If $e|e'$ there is a natural surjection $Gal(L_{e'}/K) \to Gal(L_e/K)$ and the following diagram commutes:

$$\begin{tikzcd} Gal(L_{e'}/K) \arrow[r, "\sim"] \arrow[d, two heads] & \mathbb{Z}/e'\mathbb{Z} \arrow[d, two heads, "\mathrm{mod}\ e"] \\ Gal(L_e/K) \arrow[r, "\sim"] & \mathbb{Z}/e\mathbb{Z} \end{tikzcd}$$ □

Then, $Gal(K^{tame}/K) \cong \varprojlim \mathbb{Z}/e\mathbb{Z} = \prod_{l \neq p} \mathbb{Z}_l$.

In Theorem 1.50 if the residue field is not necessarily algebraically closed but of characteristic exponent p then K^{tame}/K can be split $K^{tame}/K^{unr}/K$. Then $K^{tame} = \bigcup_e L_e$ where L_e is a unique degree e tamely ramified extension of K^{unr}. Finally $Gal(K^{tame}/K^{unr}) \cong \prod_{l \neq p} \mathbb{Z}_l$.

1.7.5 Wildly Ramified extensions

Theorem 1.51 *The wild ramification group $Gal(K^{sep}/K^{tame})$ is a pro-p-group.*

Proof We have that $Gal(K^{sep}/K)$ is a profinite group endowed with Krull topology and since $G := Gal(K^{sep}/K^{tame})$ is a closed subgroup, then the latter is also a profinite group. Let N be an open normal subgroup of $Gal(K^{sep}/K^{tame})$, then it has a finite index. Let K^N be a fixed field of N, so $K^{tame} \subset K^N \subset K^{sep}$ and $N = Gal(K^{sep}/K^N)$. Then $G/N \cong Gal(K^N/K^{tame})$ and since the extension K^N/K^{tame} is wildly ramified it has a degree that is a power of p. Hence, $|Gal(K^N/K^{tame})|$ is a power of p which is equivalent to it being a $p-$group as it is a finite group. □

Chapter 2

The field of formal Laurent series

2.1 Absolute Galois group of $\mathbb{C}((x))$

In the case when K is an algebraically closed, characteristic 0 field, an algebraic closure of $K((x))$ is isomorphic to the Puiseux field

$$\bigcup_{n=0}^{\infty} K((x^{\frac{1}{n}})).$$

Theorem 2.1 *Let $K = \mathbb{C}((x))$. Then the absolute Galois group $\mathfrak{g}_K := Gal(K^{sep}/K) = Gal(\overline{K}/K) \cong \hat{\mathbb{Z}}$.*

Proof The residue $\mathbb{C}$ field of K is algebraically closed, hence $K^{unr} = K$. Since the residue characteristic is zero, there are no wildly ramified extensions, hence $K^{sep} = K^{tame}$. Since the characteristic of K is zero, it is a perfect field, hence $K^{sep} = \overline{K}$. Then, $Gal(\overline{K}/K) = Gal(K^{tame}/K) = \prod_l \mathbb{Z}_l = \hat{\mathbb{Z}}$. □

2.2 Filtration of the Absolute Galois group of $\mathbb{F}_p((x))$

In this section let $K = \mathbb{F}_p((x))$ for some prime p.

Theorem 2.2 *The maximal unramified extension of K is $K^{unr} = \overline{\mathbb{F}_p}((x))$ and the Galois group of this extension $Gal(K^{unr}/K) = \hat{\mathbb{Z}}$.*

Proof Let L_n be a degree n unramified extension of K. By Corollary 1.42 $L_n = \mathbb{F}_p((x))(\zeta_{p^n-1}) = \mathbb{F}_{p^n}((x))$ and $Gal(L_n/K) \cong Gal(l/k) \cong Gal(\mathbb{F}_{q^n}/\mathbb{F}_q) \cong \mathbb{Z}/n\mathbb{Z}$. Since the algebraic closure of $\mathbb{F}_p$ is the union $\bigcup_{n=1}^{\infty} \mathbb{F}_{p^n}$, then $K^{unr} = \bigcup_{n=1}^{\infty} L_n = \bigcup_{n=1}^{\infty} \mathbb{F}_{p^n}((x)) = \overline{\mathbb{F}_p}((x))$. Hence $Gal(K^{unr}/K) = \varprojlim Gal(L_n/K) = \varprojlim \mathbb{Z}/n\mathbb{Z} = \hat{\mathbb{Z}}$ □

Proposition 2.3 *The maximal tamely ramified extnesion of K is*

$$K^{tame} = \bigcup_{n=1}^{\infty} \overline{\mathbb{F}_p}((x^{\frac{1}{n}}))_{(n,p)=1}$$

and its Galois group over K is such that $Gal(K^{tame}/K)/\prod_{l\neq p}\mathbb{Z}_l = \hat{\mathbb{Z}}$.

Proof Since $K^{unr} = \overline{\mathbb{F}_p}((x))$, from the Theorem 1.50 it follows that the maximum tamely ramified extension of K^{unr} which is the same as K^{tame} is $K^{tame} = \bigcup_n^{\infty} \overline{\mathbb{F}_p}((x))(x^{\frac{1}{n}})_{(n,p)=1} = \bigcup_{n=1}^{\infty} \overline{\mathbb{F}_p}((x^{\frac{1}{n}}))_{(n,p)=1}$ and $Gal(K^{tame}/K^{unr}) = \prod_{l\neq p}\mathbb{Z}_l$.

By the exact sequence:

$$1 \to Gal(K^{tame}/K^{unr}) \to Gal(K^{tame}/K) \to Gal(K^{unr}/K) \to 1$$

we obtain that $Gal(K^{tame}/K)/\prod_{l\neq p}\mathbb{Z}_l = \hat{\mathbb{Z}}$.

From the above results, we have that the Puiseux field $\bigcup_{n=1}^{\infty} \overline{\mathbb{F}_p}((x^{\frac{1}{n}}))$ is the perfect closure of the maximal tamely ramified extension $\bigcup_{n=1}^{\infty} \overline{\mathbb{F}_p}((x^{\frac{1}{n}}))_{(n,p)=1}$ of K, as it is obtained by adjoining all p^r-th roots of the uniformizing element x where $r \in \mathbb{N}$.

2.3 Extensions of $\mathbb{F}_p((x))$

There are infinitely many **separable** extensions of degree p of $\mathbb{F}_p((x))$ namely the extensions generated by the Artin-Schreier polynomial $z^p - z - x^{-m}$ for any $m \in \mathbb{Z}^+$.

There are infinitely many **inseparable extensions** of $\mathbb{F}_p((x))$, namely by adjoining $x^{\frac{1}{p^m}}$ to the field for any $m \in \mathbb{Z}^+$. The minimal polynomial of $x^{\frac{1}{p^m}}$ over $\mathbb{F}_p((x))$ is $z^{p^m} - x$, and the degree of the extension is n since each root of the polynomial is a p times repeated root.

Let $L/\mathbb{F}_p((x))$ be a degree p extension. We have that $p = ef$ hence either $f = p$ or $e = p$. Hence we have that every degree p extension of $\mathbb{F}_p((x))$ is either totally ramified or unramified.

Furthermore, by the Corollary 1.42, $L/\mathbb{F}_p((x))$ is unramified if and only if $L = \mathbb{F}_p((x))(\zeta_{p^p-1})$. This gives us that there are infinitely many totally ramified degree p extensions of $\mathbb{F}_p((x))$.

Now, let m be prime to p. Then, there are exactly m totally ramified extensions of $\mathbb{F}_p((x))$ of degree m, namely $\mathbb{F}_p((\zeta_m^i x^{\frac{1}{m}}))$ for $1 \leq i \leq m$ and where ζ_m is the primitive mth root of unity in the algebraic closure of $\mathbb{F}_p$.

We also state a theorem from (Brown et al., 2015) that we will use in the next example to classify the extensions of $\mathbb{F}_p((x))$:

Theorem 2.4 *Let ζ be a primitive $\left(p^f - 1\right)$ st root of unity contained in K, and let $g = \gcd\left(p^f - 1, e\right)$. Set $m = e/g$. There are exactly e totally and tamely ramified extensions of K of degree e. Furthermore, these extensions can be split into g classes of m many K-isomorphic extensions, all extensions in the same class being generated over K by the roots of the polynomials*

$$f_r(x) = x^e - \zeta^r \pi_K$$

for $r = 0, ..., g-1$.

Example 2.5 *Let us look at a specific example of a degree $n = 6$ extension L of $\mathbb{F}_5((x))$. We have that $L/\mathbb{F}_5((x))$ has to be one of the following:*

1. *a degree 6 unramified extension*
2. *a degree 2 totally ramified extension of a degree 3 unramified extension of $\mathbb{F}_5((x))$.*
3. *a degree 3 totally ramified extension of a degree 2 unramified extension of $\mathbb{F}_5((x))$.*
4. *a degree 6 totally tamely ramified extension.*

For each case there is a unique unramified subextension of degree dividing 6 or of zero degree as in the 4th case. The extensions are formed by adjoining $(5^m - 1)$st root of unity to $\mathbb{F}_5((x))$ where m is 6, 3 and 2 in the first three cases above. By the Theorem 2.4 for $g = gcd(e, 5^m - 1)$ there are g non-isomorphic totally tamely ramified extensions of degree e. For the first case there is only one unique extension. For 2, 3, 4 cases there are $gcd(2, 5^3 - 1) = 2$, $gcd(3, 5^2 - 1) = 3$ and $gcd(6, 5^1 - 1) = 2$ non-isomorphic extensions respectively. We use the Theorem 2.4 to find the generating polynomials of the totally and tamely ramified extensions of the unramified subextension K of $L/\mathbb{F}_5((x))$ and summarize the results in the table:

Case	e	f	*Gal* $(K/\mathbb{F}_5((x)))$	*Polynomial for* L/K
1	1	6	$\mathbb{Z}/6\mathbb{Z}$	
2	3	2	$\mathbb{Z}/2\mathbb{Z}$	$z^2 - x$
2	3	2	$\mathbb{Z}/2\mathbb{Z}$	$z^2 - \zeta_{124}x$
3	2	3	$\mathbb{Z}/3\mathbb{Z}$	$z^3 - x$
3	2	3	$\mathbb{Z}/3\mathbb{Z}$	$z^3 - \zeta_{24}x$
3	2	3	$\mathbb{Z}/3\mathbb{Z}$	$z^3 - \zeta_{24}^2x$
4	6	1		$z^6 - z$
4	6	1		$z^6 - \zeta_4x$

Example 2.6 *Let us find the the Artin-Schreier extension of $\mathbb{F}_5((x))$ generated by the polynomial $F(z) = z^5 - z - x^{-1} \in \mathbb{F}_5((x))[z]$.*

Let $f = \sum_{i=1}^{\infty} x^{\frac{-1}{5^i}}$.

$$(\sum_i x^{\frac{-1}{5^i}})^5 - \sum_i x^{\frac{-1}{5^i}} - x^{-1} = x^{-1} + x^{\frac{-1}{5}} + x^{\frac{-1}{25}} + ... - \sum_i x^{\frac{-1}{5^i}} - x^{-1} = 0.$$

Hence f *is a root of* $F(z)$*. We see that* $f + i$ *for* $1 \leq i \leq 4$ *are also the roots of* $F(z)$*. The Galois group of the extension is then* $\mathbb{Z}/5\mathbb{Z}$*.*

In the above example, we see that the solution $f = \sum_{i=1}^{\infty} x^{\frac{-1}{5^i}}$ of the Artin-Schreier polynomial does not belong to the Puiseux field $\bigcup_{n=1}^{\infty} \overline{\mathbb{F}_5}((x^{\frac{1}{n}}))$ since the denominators of the exponents are not bounded. This suggests that the algebraic closure of $\mathbb{F}_p((x))$ is larger than the Puiseux field. In order to describe the algebraic closure of $\mathbb{F}_p((x))$ we need to introduce generalized power series.

2.4 Generalized power series

When K is a field with characteristic $p > 0$, Chevalley remarked that the Artin-Schreier polynomial $z^p - z - x^{-1} \in K((x))[z]$ has no root in the Puiseux field. However, an algebraic closure of $K((x))$ can be described explicitly through the introduction of *generalized power series* which consist of expressions of the form $\sum_{i \in \mathbb{Q}} x_i t^i$ where $x_i \in K$ and the support of the series $\{i \in \mathbb{Q} \mid x_i \neq 0\}$ is a **well-ordered** subset of $\mathbb{Q}$ (meaning that each non-empty subset has a minimum element).

We put a ring structure on the set of generalized power series:

$$\sum_i x_i t^i + \sum_j y_j t^j = \sum_k (x_k + y_k) t^k$$
$$\sum_i x_i t^i \cdot \sum_j y_j t^j = \sum_k (\sum_{i+j=k} x_i y_j) t^k$$

where the multiplication makes sense since supports of the series are well-ordered subsets of $\mathbb{Q}$ hence there are finitely many i, j such that $i + j = k$ and $x_i \neq 0, y_j \neq 0$. Then, we see that the Artin-Schreier polynomial factors in the following way:

$$z^p - z - x^{-1} = \prod_{i=0}^{p-1} \left(z - i - \sum_{j=1}^{\infty} x^{\frac{-1}{p^j}} \right).$$

Note that $\sum_{j=1}^{\infty} x^{\frac{-1}{p^j}}$ is not an element of the Puiseux field as the denominators of the exponents are not bounded. If $K = \mathbb{F}_p((x))$ then the Artin-Schreier polynomial is irreducible, and the extension generated by the polynomial is cyclic of degree p:

$$Gal\left(K[z]/(z^p - z - x^{-1})/K\right) \cong \mathbb{Z}/p\mathbb{Z}.$$

We have the following theorem which we study further in the next section:

Theorem 2.7 (Huang, Rayner, Stefanescu) *The field of generalized power series is algebraically closed. In particular, an algebraic closure of the field $\mathbb{F}_p((x))$ is contained in the field of generalized power series.*

2.4.1 Galois groups of extensions

Let us look at an example from Vaidya's paper (Vaidya, 1997) that studies another extension of $\mathbb{F}_p((x))$ in the generalized power series field:

Theorem 2.8 *Let $K = \overline{\mathbb{F}_p}$. Let $m \in \mathbb{Z}^+$ be coprime with p and $n \in \mathbb{Z}+$. Let $f = \sum_{i=1}^{\infty} x^{\frac{-1}{mp^{in}}}$. If $m|p^n - 1$ then we have*

1. *The field $K((x))(f)$ a finite Galois extension of $K((x))$ of degree mp^n. Let G denote its Galois group.*
2. *There exist subgroups H and L of G such that $H \cong \bigoplus_{i=1}^{n}(\mathbb{Z}/p\mathbb{Z})$ and $L \cong \mathbb{Z}/m\mathbb{Z}$. Moreover, if $m > 1$, then $G = H \rtimes K$.*

Proof Let us first show that the polynomial $F(x) = z^{p^n} - z - x^{\frac{-1}{m}}$ is a minimal polynomial of f over the field $K((x^{\frac{1}{m}}))$. Firstly:

$$F(f) = \left(\sum_{i=1}^{\infty} x^{\frac{-1}{mp^{in}}}\right)^{p^n} - \sum_{i=1}^{\infty} x^{\frac{-1}{mp^{in}}} - x^{\frac{-1}{m}} = x^{\frac{-1}{m}} + x^{\frac{-1}{mp^n}} + ... - \sum_{i=1}^{\infty} x^{\frac{-1}{mp^{in}}} - x^{\frac{-1}{m}} = 0.$$

Now we use Huang's following result replacing x with $x^{\frac{1}{m}}$: Let $f = \sum_{i=1}^{\infty} x^{\frac{-1}{p^i}}$ with $a_i \in K$. If f satisfies a polynomial of the form $z^{p^n} + b_{n-1}z^{p^{n-1}} + ... + b_1 z^p + b_0 z + b(x)$, where $b_i \in K$, $b(x) \in K((x))$ and n is minimal. Then this polynomial is a minimal polynomial of the element f over the field $K((x))$.

We have left to show that n is minimal as $F(x)$ is a polynomial of this form. Assume for a contradiction that f satisfies $G(z) = z^{p^r} + b_{n-1}z^{p^{r-1}} + ... + b_1 z^p + b_0 z + b(x)$, for some $b_i \in K$, $b(x) \in K((x^{\frac{1}{m}}))$ then:

$$\sum_{j=1}^{\infty} x^{\frac{-1}{mp^{nj-r}}} + ... + b_i \sum_{j=1}^{\infty} x^{\frac{-1}{mp^{nj-i}}} + b_0 \sum_{j=1}^{\infty} x^{\frac{-1}{mp^{nj}}} = -b$$

hence we get a contradiction. We conclude that the polynomial $F(z)$ is the minimal polynomial of the element f over the field $K((x^{\frac{1}{m}}))$ and the degree of the extension is $[K((x))(f) : K((x^{\frac{1}{m}}))] = p^n$. All the roots of $F(z)$ are given by $f + \omega^i, 1 \leq i \leq p^n - 1$ where ω is $(p^n - 1)$th primitive root of unity, since $(w^i)^{p^n} - w^i = 0$. Since all the roots of $F(z)$ are distinct, $K((x))(f)$ is separable and algebraic over $K((x^{\frac{1}{m}}))$ and furthermore it is a splitting field of f over $K((x^{\frac{1}{m}}))$. Hence it is a Galois extension and let us denote by H its Galois group. We see from the roots of f that H is isomorphic to the direct sum of n copies of $\mathbb{Z}/p\mathbb{Z}$.

Now, $x^{\frac{1}{m}}$ is algebraic over $K((x))$ and since p and m are coprime, $K((x^{\frac{1}{m}}))$ is a finite separable extension of $K((x))$ of degree m.

Hence $[K((x))(f) : K((x))] = mp^n$. Let $F'(z) = (z^{p^n} - z)^m - x^{-1}$. Then $H(f) = 0$ and since the degree of the polynomial is mp^n it is the minimal polynomial of f over $K((x))$. Let u be mth primitive root of unity, then we see that the set is the solutions of $F'(z)$: $\{u^i \mid 1 \leq i \leq m\} \cup \{u^i f + \omega^j \mid 1 \leq i \leq m, 1 \leq j \leq p^n - 1\}$. Hence the extension $K((x))(f)/K((x))$ is Galois.

Let $\sigma \in H$ such that $\sigma(f) = f + \omega$ and let $\tau \in G$ such that $\tau(f) = uf$. Let $K \subseteq G$ be the subgroup generated by τ. We see that K is cyclic of m elements and $K \cong \mathbb{Z}/m\mathbb{Z}$.

When $m > 1$ the intersection $H \cap K$ is a trivial subgroup and every element of G is a product of some σ and τ. Since $K((x^{\frac{1}{m}}))$ is a normal extension of $K((x))$ we have that H is a normal subgroup of G. Hence G is isomorphic to a semidirect product of H and K: $G = H \rtimes K$.

In the above theorem, H is a p-group and we obtained the following exact sequence from the semidirect product:

$$1 \to H \to G \to K \to 1.$$

The degree of the extension of $K((x))(f)$ over $K((x))$ is mp^n. We see that the maximal tamely ramified subextension over $K((x))$ has to be of degree m and since tamely ramified extensions of $K((x))$ of degree m are obtained by adjoining mth root of a uniformizer of $K((x))$ we see that $K((x^{\frac{1}{m}}))$ is exactly the maximal tamely ramified subextension and hence H is the wild inertia group.

2.5 Algebraic closure of $\mathbb{F}_p((x))$

Let $K = \overline{\mathbb{F}_p}$. In this section, we give an explicit description of elements of the generalized power series that are algebraic over $K((x))$. Let us first prove a useful lemma for showing this result.

Lemma 2.9 *Every finite normal extension of $K((x))$ is contained in a tower of Artin-Schreier extensions over $K((x^{1/n}))$ for some $n \in \mathbb{N}$.*

Proof Let L be a finite normal extension of $K((x))$, then there exists an intermediate extension F such that it is purely inseparable over $K((x))$ and over which L is separable. Let S be the separable closure of $K((x))$ in L. Then $[S : K((x))] = [L : F]$ and hence $[L : S] = [F : K((x))] =: q$. Since $K((x))$ is an imperfect field and the degree of extension $F/K((x))$, is q then $F = K((x^{\frac{1}{q}}))$. Also, since $L/K((x))$ is normal, $L/K((x^{\frac{1}{q}}))$ is also normal and therefore it is Galois.

Let M denote the maximal subextension of L tamely ramified over $K((x^{\frac{1}{q}}))$ and $m := [M : K((x^{\frac{1}{q}}))]$. Since K is algebraically closed it contains m distinct $m-$th roots of unity (m is coprime with p) and by Kummer theory $M = K((x^{1/q}))(a^{1/m})$ for some $a \in K((x^{1/q}))$. Hence $M = ((x^{1/qm}))$.

Since the wild inertia group $Gal(L/M)$ is a p-group, L is a p-power extension of M (degree is a power of p). Hence, we have left to prove that L is contained in a tower of Artin-Schreier extensions over M. Since every nontrivial p-group has a nontrivial center, we can find a normal series

$$Gal(L/M) = G_0 \rhd G_1 \rhd ... \rhd G_n = 1$$

such that $[G_{i-1} : G_i] = p$ for $i = 1, .., n$. The corresponding fixed subfields of G_i form a tower of degree p Galois extensions from M to L. Every such extension in a characteristic p field is an Artin-Schreier extension hence showing the result.

Hence we have that every finite normal extension of $\mathbb{F}_p((x))^{unr} = \overline{\mathbb{F}_p}((x))$ is contained in a tower of Artin-Schreier extensions over $\mathbb{F}_p((x^{1/n}))^{unr}$ for some $n \in \mathbb{N}$.

Now we give some preliminaries in order to be able to define elements algebraic over $K((x))$.

Definition 2.10 *A sequence $(c_n)_n$ satisfies a* ***linearized recurrence relation (LRR)*** *if there exists $d_0, ...d_k$ such that $\sum_{i=0}^{k} d_i c_{n+i}^{p^i} = 0$ for all $n \geq 0$. We have that if $(c_n)_n$ and $(c'_n)_n$ satisfy LRRs, then their sum and product also satisfy LRRs.*

Definition 2.11 *Let $a \in \mathbb{N}$ and $b, c \geq 0$ and define the set*

$$S_{a,b,c} := \left\{ \frac{1}{a} \left(n - \frac{b_1}{p} - \frac{b_2}{p^2} - ... \right) : n \geq -b, \quad b_i \in \{0, ..., p-1\}, \quad \sum b_i \leq c \right\}$$

Note that $\sum_i \frac{b_i}{p^i} < 1$.

Let $T_c = S_{1,0,c} \cap (-1,0) = \left\{ \left(n - \frac{b_1}{p} - \frac{b_2}{p^2} ... \right) : n \geq 0,\ b_i \in \{0, ..., p-1\},\ \sum b_i \leq c \right\} \cap (-1, 0)$.

Then if $w = \sum_i w_i x^i$ is supported on $S_{1,b,c}$ it can be written as $\sum_{m \geq b} \sum_{i \in T_c} w_{m+i} x^{m+1}$.

Definition 2.12 (Twist-recurrent function of order k) *A function $f : T_c \to K$ is twist-recurrent of order $k \in \mathbf{Z}^+$ if $\exists d_0, ...d_k \in K$ such that LRR holds for any sequence $(c_n)_n$ of the form*

$$c_n = f\left(-\frac{b_1}{p} - ... - \frac{b_{j-1}}{p^{j-1}} - \frac{1}{p^n} \left(\frac{b_j}{p^j} + ... \right) \right)$$

for some $j \in \mathbb{N}$ and $b_1, b_2 ... \in \{0, ..., p-1\}$ with $\sum b_i \leq c$.

Hence if $\exists \lambda_1, ..., \lambda_k \in K$ such that

$$c_n = f\left(-\frac{b_1}{p} - ... - \frac{b_{j-1}}{p^{j-1}} - \frac{1}{p^n} \left(\frac{b_j}{p^j} + ... \right) \right) = \sum_{i=1}^{k} z_i \lambda_i^{\frac{1}{p^n}}.$$

Definition 2.13 (Twist-recurrent series) *A series $w = \sum w_i t^i$ is **twist-recurrent** if:*

1. *$\exists a, b, c \in \mathbb{N}$ such that w is supported on $S_{a,b,c}$.*
2. *For some a, b, c for which w is supported on $S_{a,b,c}$ and for each $m \geq -b$ the function*

   $$\begin{aligned} f_m : T_c &\to K \\ z &\longmapsto w_{(m+z)/a}. \end{aligned}$$

   *is **twist-recurrent** of order k for some k.*
3. *The functions f_m span a finite-dimensional vector space over K.*

If f_m as defined above (note that it depends on w) is twist-recurrent of order k, then from the definitions it follows that LRR holds for any sequence $(c_n)_n$ of the form

$$c_n = w_{\frac{1}{a}\left(m - \frac{b_1}{p} - ... - \frac{b_{j-1}}{p^{j-1}} - \frac{1}{p^n} \left(\frac{b_j}{p^j} + ... \right) \right)}$$

Theorem 2.14 *The twist-recurrent series form an algebraic closure of $K((x))$.*

Proof We omit the proofs of the assertions that every twist-recurrent series is algebraic over $K((x))$ and that twist-recurrent series are closed under addition and scalar multiplication which are presented in Theorem 8 of (Kedlaya, 2001).

Let us show that if y is twist-recurrent and $w^p - w = y$, then w is twist-recurrent. Let y be supported on $S_{a,b,c}$ for some a, b, c. Let us write $y = y' + y''$

where y' is supported on $(-\infty, 0) \cap S_{a,b,c}$ and y'' on $S_{a,b,c} \cap (0, +\infty)$. Since the function $w \mapsto w^p - w$ is additive, we can find w' and w'' such that $y' = w'^p - w'$ and $y'' = w''^p - w''$.

We have that $w' = \sum_i \sum_{l=1}^{\infty} (y_i t^i)^{\frac{1}{p^l}} = \sum_i t^i \sum_l y_{ip^l}^{1/p^l}$ is supported on $S_{a,b,b+c}$. To show that w' is a twist-recurrent series, we have to show that if $-b \leq m \leq 0, b_i \in \{0, ..., p-1\}$ and $\sum b_i \leq c$, then for any $j \geq 0$, the sequence

$$c_n = w'_{m - \frac{b_1}{p} - ... - \frac{b_{j-1}}{p^{j-1}} - \frac{1}{p^n}\left(\frac{b_j}{p^j} + ...\right)}$$

satisfies some fixed LRR.

In case $m < 0$ or $j > 0$, then $c_n = \sum_l y_{ip^l}^{\frac{1}{p^l}}$, where $i = m - \frac{b_1}{p} - ... - \frac{b_{j-1}}{p^{j-1}} - \frac{1}{p^n}\left(\frac{b_j}{p^j} + ...\right)$. Since y_{ip^l} is nonzero for a bounded number of l, due to the condition $m \geq -b$ on the support of the series y, we have that c_n is a sum of a bounded number of sequences, each obtained from a sequence satisfying LRRs, by taking p^l-th roots for some l. Hence w' is twist-recurrent as a sum of twist-recurrent series as the possible values of l are uniformly bounded over all possible sequences.

In case $m = j = 0$, then

$$c_{n+1}^p - c_n = y_{-\frac{b_1}{p} - ... - \frac{b_{j-1}}{p^{j-1}} - \frac{1}{p^n}\left(\frac{b_j}{p^j} + ...\right)}$$

which satisfies LRRs implying that $(c_n)_n$ also satisfies LRRs from direct computation. Hence w' is twist-recurrent series.

We have that $w'' = -\sum_i \sum_{n=1}^{\infty} (y_i t^i)^{p^n} = -\sum_i t^i \sum_n y_{i/p^n}^{p^n}$ is also supported on $S_{a,b,c}$. For $i < p^k$, $y_{i/p^n} = 0$ for $n > k + c$. This implies that w'' is twist-recurrent, hence $w = w' + w''$ is twist-recurrent.

We obtain that the field of twist-recurrent series is closed under taking Artin-Schreier extensions, hence there are no proper finite normal extensions of the field of twist-recurrent series by Lemma 2.9, which implies that this field is algebraically closed. $\square$

Conclusion

In conclusion, when K is a Henselian discretely valued field of residue characteristic p, we split up the Galois extension K^{sep}/K into $K^{sep}/K^{tame}/K^{unr}/K$ and obtain the following filtration by normal subgroups

$$1 \subset Gal(K^{sep}/K^{tame}) \subset Gal(K^{sep}/K^{unr}) \subset Gal(K^{sep}/K).$$

When the residual field of K is finite then $Gal(K^{unr}/K) \cong \hat{\mathbb{Z}}$. For the middle filtration, $Gal(K^{tame}/K^{unr}) \cong \prod_{l \neq p} \mathbb{Z}_l$. And the wild ramification group $Gal(K^{sep}/K^{tame})$ is trivial if the residue field of K has characteristic 0 and a pro-p-group if characteristic $p > 0$.

With the above results we have obtained that when $K = \mathbb{C}((x))$, $Gal(\overline{K}/K) = \hat{\mathbb{Z}}$. For the case when $K = \mathbb{F}_p((x))$, the Galois group of the algebraic closure over the field is not characterized this easily and we have seen that the introduction of generalized power series is needed to describe an algebraic closure of $\mathbb{F}_p((x))$. An algebraic closure of $K((x))$ is described by Kedlaya when K is algebraically closed and of characteristic $p > 0$ by putting constraints on the support and coefficients of the generalized power series.

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