A LIE-THEORETIC GENERALIZATION OF SOME HILBERT SCHEMES

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Abstract. We define different versions of a class of varieties $X_{\mathfrak{g}}$ attached to a complex reductive Lie algebra $\mathfrak{g}$. The definitions are representation-theoretic, and there are trigonometric and elliptic versions attached to complex reductive groups as well. When the root system is of type A_{n-1}, these varieties are all versions of the Hilbert scheme of n points on a rational surface, and the general case can be regarded as a natural generalization to other root data. We also define the corresponding "isospectral" varieties $Y_{\mathfrak{g}}$.

We prove a Gordon-Stafford localization theorem for $X_{\mathfrak{g}}$ and the corresponding equal–parameter rational Cherednik algebras, relate these varieties to the affine Springer fiber–sheaf correspondence of [16], and discuss some examples.

We conjecture that the torus–fixed points of our varieties are in bijection with two–sided cells in the finite Weyl group prove our conjecture in types ABC. We relate these results to known results about Calogero–Moser spaces in these cases.

Contents

1. Introduction

In broad strokes, recent conjectures on knot theory [18, 36] and affine Springer theory [16, 27] suggest that there should be the following commutative triangle:

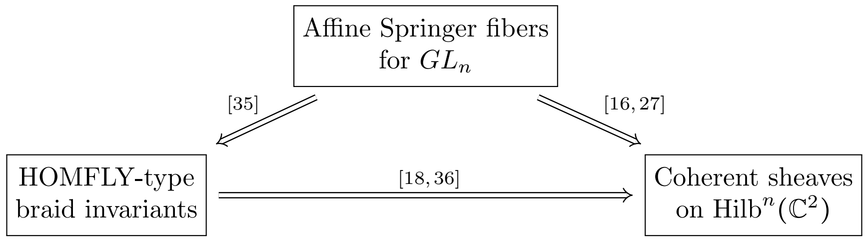

FIGURE 1.

We will not elaborate on the arrows in this diagram, which have been explained in detail in a variety of sources. See for example [17] for a recent survey.

The main theme of this article is to generalize the lower right vertex of this triangle, which we now motivate. The lower left vertex of this triangle has a generalization to other reductive groups G, which only depends on the root system. Recall that for a conjugacy class of braids on n strands, the HOMFLY polynomial is a two–variable polynomial which is an invariant of the braid closure. This polynomial invariant has been categorified to the HOMFLY or triply–graded homology of links in S^3. This homological invariant can be defined, for example, by using Hochschild homology of Rouquier complexes of Soergel bimodules for GL_n. We note that after fixing n, both of these theories have annular generalizations, which only give invariants of conjugacy classes of braids in $Br_n = \pi_1(\mathbb{C}^n \backslash \{\text{diagonals}\}/S_n)$.

Since the configuration space of $\mathbb{C}$ can be thought of as the regular part of a Cartan subalgebra of $\mathfrak{gl}_n(\mathbb{C})$, one can generalize to other G and consider the braid group $Br_W = \pi_1(\mathfrak{t}^{reg}/W)$, where $\mathfrak{t} \subset \mathfrak{g}$ is a Cartan and W is the Weyl group. Similar HOMFLY–like (annular) invariants for conjugacy classes in Br_W can be constructed using Markov traces in the polynomial case or the Hochschild homology of Rouquier complexes in the categorified setting. In many ways, these braid groups and the corresponding invariants are much less understood.

The top vertex of the triangle also generalizes to other reductive G, by considering affine Springer fibers for these groups. Now there is dependence on more data than simply the root system, which gives rise to various complications in establishing the still conjectural left and right arrows.

After these considerations, one may look at the lower right vertex and ask: is there a replacement for the variety $\mathrm{Hilb}^n(\mathbb{C}^2)$ for other G, coherent sheaves on which encode information about homology of affine Springer fibers or the above braid invariants?

The goal of this paper is to give definitions of such replacement varieties $X_{\mathfrak{g}}$ for arbitrary reductive Lie algebras, such that $X_{\mathfrak{gl}_n} \cong \mathrm{Hilb}^n(\mathbb{C}^2)$. These varieties come in three different versions, which we denote $X_{\mathfrak{g},sgn}, X_{\mathfrak{g},diag}$ and $X_{\mathfrak{g},symb}$. Another goal is to convince the reader that $X_{\mathfrak{g},symb}$ is the most relevant for applications. There are obvious trigonometric and elliptic generalizations of these notions attached to reductive groups. Not all results are established in these cases, but in view of further applications, it seems useful to allow for this flexibility in the definitions. Both of these generalizations also arise naturally in the context of Coulomb branches of some SUSY gauge theories, constructed using the mathematical version in [10]. For ease of exposition, we mostly stay in the rational case.

For people familiar with some of the above topics, it is no surprise that the geometry of $X_{\mathfrak{g}}$ is strongly related to the representation theory of rational Cherednik algebras. In particular, we conjecture that $X_{\mathfrak{g},symb}$ is a hyper-Kähler rotation of the Calogero–Moser space attached to the center of the RCA for $\mathfrak{g}$ at $t = 0$, and provably so in types ABC. As we prove in Theorem 4.3, in general $X_{\mathfrak{g},symb}$ is the commutative degeneration of a $\mathbb{Z}$–algebra attached to the RCA with equal parameters and $t = 1$.

We remark that the bottom vertices of the triangle in Figure 1 also generalize to other Coxeter groups and their reflection representations. Many of the notions in Section 4 even generalize to complex reflection groups, but we do not consider these cases.

The plan of the paper is as follows. We first recall some of Haiman's results on the Hilbert scheme of points on $\mathbb{C}^2$ in Section 2, and then define three different versions of $X_{\mathfrak{g}}$ in Section 3. In Section 4, we prove a localization theorem for these varieties. In Section 5, we pose some conjectures on the geometry of $X_{\mathfrak{g}}$ and prove most of these conjectures in types ABC in Appendix A.

Acknowledgments. The computer experiments in Appendix B were computed with the aptly named OSCAR computer algebra system [34], using generous help from Johannes Schmitt as well as Ulrich Thiel. We also thank Alex Weekes for correspondence, and Gwyn Bellamy and Cédric Bonnafé for comments on a draft of this paper.

2. Hilbert schemes

In this section, we recall some results on the Hilbert scheme of points on the plane, $\mathrm{Hilb}^n(\mathbb{C}^2)$. Recall that by definition, $\mathrm{Hilb}^n(\mathbb{C}^2)$ parametrizes subschemes of length n in the plane $\mathbb{C}^2$. On closed points we may write

$$\mathrm{Hilb}^n(\mathbb{C}^2) = \{I \subseteq \mathbb{C}[x,y] \,|\, \dim_{\mathbb{C}} \mathbb{C}[x,y]/I = n\}.$$

There is also a well-known quiver variety/ADHM description, realizing $\mathrm{Hilb}^n(\mathbb{C}^2)$ as the quotient by GL_n of the space of triples

$$\{(x,y,v) \in \mathfrak{gl}_n \times \mathfrak{gl}_n \times \mathbb{C}^n \,|\, [x,y] = 0, \mathbb{C}[x,y]v = \mathbb{C}^n\}.$$

Instead of these two descriptions, we seek to generalize the following Proj–construction by Haiman in [24].

Theorem 2.1.

$$\mathrm{Hilb}^n(\mathbb{C}^2) = \mathrm{Proj} \bigoplus_{d \geq 0} (\Delta A)^d,$$

where $\Delta A \subseteq \mathbb{C}[x_1,\dots,x_n,y_1,\dots,y_n]^{S_n}$ is the ideal consisting of products of the form Δf, where $\Delta = \prod_{i<j}(x_i - x_j)$ and $f \in \mathbb{C}[x_1,\dots x_n,y_1,\dots,y_n]^{sgn}$ is an alternating polynomial for the diagonal action.

Proof. (Sketch.) On $\mathrm{Spec}\,\mathbb{C}[x_1,\dots,x_n,y_1,\dots,y_n]^{S_n} = \mathrm{Sym}^n\mathbb{C}^2$ there is an open locus $V = (\mathrm{Sym}^n\mathbb{C}^2)^{reg}$ of collections of n points where all the points are distinct. Restricted to V, the Hilbert-Chow map $\mathrm{Hilb}^n(\mathbb{C}^2) \to \mathrm{Sym}^n\mathbb{C}^2$ is an isomorphism. For every partition $\mu \vdash n$, there are open charts $U_\mu \subseteq \mathrm{Hilb}^n(\mathbb{C}^2)$ formed of those ideals for which $\mathbb{C}[x,y]/I$ is spanned by the monomials corresponding to the diagram of $\mu \in \mathbb{N} \times \mathbb{N}$. On $U_\mu \cap Y$, one can construct regular functions Δ_D/Δ_μ, where $\Delta_D := \det(x_i^{p_j} y_i^{q_j})_{i,j=1}^n$ are alternating polynomials attached to n-element subsets $D = \{(p_1, q_1), \dots, (p_n, q_n)\} \subset \mathbb{N} \times \mathbb{N}$. One proves that these regular functions glue as we vary μ and also extend to regular functions on all of $\mathrm{Hilb}^n(\mathbb{C}^2)$. The universal property of blowing up then implies that there is a surjective map $\alpha : \mathrm{Hilb}^n(\mathbb{C}^2) \to \mathrm{Proj} \bigoplus_{d \geq 0} (\Delta A)^d$. Further, one checks that the pullbacks of the above Δ_D / Δ_μ give all the regular functions on U_μ, showing that the map on structure sheaves is also surjective and α has to be an isomorphism. □

Let us also recall an easy corollary of this result.

Corollary 2.2. *The isospectral Hilbert scheme, i.e. the reduced fiber product*

$$\begin{array}{ccc} Y_n & \longrightarrow & \mathbb{C}^{2n} \\ \downarrow & & \downarrow \\ \mathrm{Hilb}^n(\mathbb{C}^2) & \longrightarrow & \mathrm{Sym}^n \mathbb{C}^2 \end{array}$$

is isomorphic to

$$\mathrm{Proj} \bigoplus_{d \geq 0} (\Delta J)^d,$$

where $J = \mathbb{C}[x_1, \dots, y_n] \cdot A$ is the ideal in $\mathbb{C}[x_1, \dots, y_n]$ generated by the diagonally alternating polynomials.

The reader unfamiliar with the above constructions might wonder why we use ΔA. Of course, one might take the ideal fA, where f is any alternating polynomial, and get an isomorphic Proj. After a further Veronese twist, one could also consider the ideal A^2.

Remark 2.3. We will use this theorem as an inroad to defining $X_{\mathfrak{g}, sgn}$ in the next section. As Haiman remarks in [24], since an alternating polynomial vanishes on the diagonals, it is natural to expect that the radical of the ideal

$$A^2 \subseteq \mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_n]^{S_n}$$

coincides with the ideal of the locus in $\mathrm{Sym}^n \mathbb{C}^2$ where two or more points coincide. If two points coincide in $\mathbb{C}^2$, their $x-$ and $y-$coordinates have to be equal. Therefore, the ideal of this locus can be written as $\mathbf{e}I$, where $\mathbf{e} = \frac{1}{n!} \sum_{\sigma \in S_n} \sigma$ and I is the ideal $I = \bigcap_{i<j} \langle x_i - x_j, y_i - y_j \rangle \subseteq \mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_n]$. Lifting these considerations to Y_n, one can similarly ask whether the ideal J from Corollary 2.2 equals I.

Knowing the analogous equality of ideals in dimension 1 is almost trivial, so it might seem unsurprising that indeed we have

$$I = J.$$

However, the only known proof of this fact is a corollary of Haiman's polygraph theorem [25] and in fact equivalent to the Cohen-Macaulay/Gorenstein properties of the isospectral Hilbert scheme Y_n.

3. The analogs for other Lie algebras

Let G be a complex reductive group. We use standard notation such as $T, \mathfrak{t}$ for maximal tori and Cartans and $B, \mathfrak{b}$ for Borels. Similarly as we replaced $\mathbb{C}^n \backslash \{\text{diagonals}\}$ by $\mathfrak{t}^{reg}$ in the introduction, we may replace the ring

$$\mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_n]$$

and its diagonal S_n-action by the ring $\mathbb{C}[\mathfrak{t}\oplus\mathfrak{t}^*]$ of regular functions on $\mathfrak{t}\oplus\mathfrak{t}^*$ and its diagonal W-action, where W is the Weyl group of G.

Based on the previous section, a naive guess for defining $X_\mathfrak{g}$ for an arbitrary reductive $\mathfrak{g}$ is the following.

Definition 3.1. Let $A=\mathbb{C}[\mathfrak{t}\oplus\mathfrak{t}^*]^{sgn}\subseteq\mathbb{C}[\mathfrak{t}\oplus\mathfrak{t}^*]$ be the space of diagonally alternating polynomials for the W-action. Let $\Delta=\prod_{\alpha\in\Phi^+}\alpha^\vee$ and define

$$X_{\mathfrak{g},sgn}:=\operatorname{Proj}\bigoplus_{d\geq 0}(\Delta A)^d$$

In other words, $X_{\mathfrak{g},sgn}$ is the blow-up of $\mathbb{C}[\mathfrak{t}\oplus\mathfrak{t}^*]^W$ in the ideal ΔA, where Δ and A are as above. Note that we have dropped the subscript $\mathfrak{g}$ in the commutative algebraic objects,hoping this causes no further confusion.

Similarly to the definition for $\mathfrak{gl}_n$, one can consider the ideal J in $\mathbb{C}[\mathfrak{t}\oplus\mathfrak{t}^*]$ generated by A. This lets us define the *isospectral variety* of $X_{\mathfrak{g},sgn}$:

Definition 3.2. The isospectral variety $Y_{\mathfrak{g},sgn}$ of $X_{\mathfrak{g},sgn}$ is

$$\operatorname{Proj}\bigoplus_{d\geq 0}(\Delta J)^d$$

Since the W-action is bigraded, the construction of $X_{\mathfrak{g},sgn}$ and $Y_{\mathfrak{g},sgn}$ respects the natural $(\mathbb{C}^*)^2$-action on $\mathfrak{t}\oplus\mathfrak{t}^*$. Unpublished conjectures of Bonnafé[1] suggest that these varieties have desirable properties, for example that

Conjecture 3.3. *There is a bijection*

$$\{\textit{Two-sided cells in } W\}\leftrightarrow X_{\mathfrak{g},sgn}^{\mathbb{C}^\times}$$

In particular, there are only finitely many fixed points. That the fixed points only depend on W is clear, as the center of $\mathfrak{g}$ does not intervene in the construction. For $\mathfrak{g}=\mathfrak{gl}_n$ (or for example, $\mathfrak{sl}_n$) we have $W=S_n$, and the bijection in Conjecture 3.3 is then just the usual bijection between fixed points on the (balanced) Hilbert scheme and partitions. Bonnafé has also checked Conjecture 3.3 for the type B_2 case as well as some non–Weyl group versions.

Following the hint from Remark 2.3, an alternative possibility for defining $X_\mathfrak{g}$ is as follows. Let $I\subseteq\mathbb{C}[\mathfrak{t}\oplus\mathfrak{t}^*]$ be the ideal $I=\bigcap_{\alpha\in\Phi^+}\langle\alpha^\vee,\alpha\rangle$, and consider $\mathbf{e}\Delta I\subseteq\mathbb{C}[\mathfrak{t}\oplus\mathfrak{t}^*]^W$, where $\mathbf{e}:=|W|^{-1}\sum_{w\in W}w$. Blowing up $\mathfrak{t}\oplus\mathfrak{t}^*/W$ in the union of the codimension two root hyperplanes given by $\{\alpha^\vee=0\}\times\{\alpha=0\}$ ("diagonals") corresponding to this ideal gives

Definition 3.4.

$$X_{\mathfrak{g},diag}=\operatorname{Proj}\bigoplus_{d\geq 0}\mathbf{e}(\Delta I)^d$$

Similarly, one can define the isospectral variety by

Definition 3.5.

$$Y_{\mathfrak{g},diag}:=\operatorname{Proj}\bigoplus_{d\geq 0}(\Delta I)^d$$

Again, since a diagonally alternating polynomial vanishes on the codimension 2 root hyperplanes, there are inclusions $J\subseteq I$ and $\Delta A\subseteq\mathbf{e}\Delta I$, and in particular maps $Y_{\mathfrak{g},diag}\to Y_{\mathfrak{g},sgn}$ and $X_{\mathfrak{g},diag}\to X_{\mathfrak{g},sgn}$.

[1]Relayed to the author through Gorsky and Bezrukavnikov.

Interestingly, we prove in Corollary 4.5 that the latter map is an isomorphism, but check in Appendix B that the former map between isospectral varieties is not an isomorphism in type B_3. In particular, $Y_{\mathfrak{g},diag}$ is not the reduced fiber product of $X_{\mathfrak{g},diag}$ and $\mathfrak{t}\oplus\mathfrak{t}^*$ over $\mathfrak{t}\oplus\mathfrak{t}^*/W$.

The goal of the current paper is to convince the reader that for many purposes, $X_{\mathfrak{g},diag} = X_{\mathfrak{g},sgn}$ is still the wrong version. Instead of the powers of the ideal I, one should use the *symbolic powers*

$$I^{(d)} := \bigcap_{\alpha\in\Phi^+} \langle \alpha^\vee, \alpha\rangle^d$$

and define

Definition 3.6.

$$X_{\mathfrak{g},symb} := \operatorname{Proj} \bigoplus_{d\geq 0} \mathbf{e}(\Delta^d I^{(d)})$$

More geometrically, since I is a radical ideal, one can think of $I^{(d)}$ as the ideal of functions on $\mathfrak{t}\oplus\mathfrak{t}^*$ that vanish with multiplicity d along the locus defined by I. Now $X_{\mathfrak{g},symb}$ is the *symbolic* blow-up of $\mathfrak{t}\oplus\mathfrak{t}^*/W$ along the diagonals. In the trigonometric setting, which we review in Section 4.1, this was noticed in [16].

Remark 3.7. When $\mathfrak{g}$ is of type A, as mentioned in Remark 2.3, it follows from Haiman's results that $I^{(d)} = I^d = J^d$.

For general $\mathfrak{g}$, one of the immediate strengths of Definition 3.6 is the following theorem:

Theorem 3.8. *$X_{\mathfrak{g},symb}$ is normal. The same holds for the isospectral variety $Y_{\mathfrak{g},symb}$.*

Proof. The proof is a direct adaptation of [28, Theorem 4.7.] to the rational case (see also [16, Corollary 3.12.]). More precisely, $\alpha, \alpha^\vee$ is a regular sequence in $\mathbb{C}[\mathfrak{t}\oplus\mathfrak{t}^*]$, so $I_G^{(d)}$ is integrally closed for all d and so is the corresponding symbolic Rees algebra. This proves $Y_{\mathfrak{g},symb}$ is normal. Taking the quotient by the W-action preserves normality, so $X_{\mathfrak{g},symb}$ is also normal. □

Again, we note that from the inclusions $J^d \subseteq I^d \subseteq I^{(d)}$ it follows that there are maps

$$X_{\mathfrak{g},symb} \to X_{\mathfrak{g},diag} \to X_{\mathfrak{g},sgn}$$

as well as

$$Y_{\mathfrak{g},symb} \to Y_{\mathfrak{g},diag} \to Y_{\mathfrak{g},sgn}$$

Unless these are isomorphisms, it is not a priori easy to decide if $X_{\mathfrak{g},diag}$ or $X_{\mathfrak{g},sgn}$ are normal.

4. Cherednik algebras

In this section, we provide probably the most compelling evidence that $X_{\mathfrak{g},symb}$ is the "right" variety to consider. The main result is Theorem 4.3 which is an analog of [22, Proposition 1.7.]. In particular, we have an analog for all Lie types of the main Theorem of Gordon–Stafford [22, Theorem 1.4.].

In this section, we fix once and for all the Lie algebra $\mathfrak{g}$. Let $c : S \to \mathbb{C}$ be a conjugation–invariant function on the set of reflections S of W. We denote its values by subscripts, such as $c_s := c(s)$. For most of this paper c is constant. We let

H_c be the rational Cherednik algebra associated to the pair $(\mathfrak{t}, W)$. More precisely, following [6], we have

Definition 4.1. Fix c as above. The *rational Cherednik algebra* H_c of $\mathfrak{g}$ is the $\mathbb{C}$-algebra generated by $\mathfrak{t}, \mathfrak{t}^*$ and W with the relations $wxw^{-1} = w(x), wyw^{-1} = w(y)$ for all $y \in \mathfrak{t}, x \in \mathfrak{t}^*, w \in W$, $[y,x] = \langle y,x\rangle - \sum_{s\in S} c_s\langle y,\alpha_s\rangle\langle\alpha_s^\vee, x\rangle s$ and requiring the algebras generated by $\mathfrak{t}$ and $\mathfrak{t}^*$ to be commutative subalgebras.

We note that H_c is the specialization of $H_{\hbar,c}$ at $\hbar = 1$, where $H_{\hbar,c}$ is the $\mathbb{C}[\hbar]$–algebra with the same generators and relations as above, except that we modify the last relation to read

$$[y,x] = \hbar\langle y,x\rangle - \sum_{s\in S} c_s\langle y,\alpha_s\rangle\langle\alpha_s^\vee, x\rangle s$$

Some important facts about H_c are as follows:

Proposition 4.2. (1) *The algebra $H_{c,\hbar}$ has a triangular decomposition: the multiplication map*

$$\mathbb{C}[\mathfrak{t}]\otimes\mathbb{C}[W]\otimes\mathbb{C}[\mathfrak{t}^*]\otimes\mathbb{C}[\hbar]\to H_{c,\hbar}$$

is an isomorphism of vector spaces. In particular, for all c we have

$$\mathbb{C}[\mathfrak{t}]\otimes\mathbb{C}[W]\otimes\mathbb{C}[\mathfrak{t}^*]\xrightarrow{\cong} H_c$$

(2) *Let* $\deg(x) = \deg(y) = 1$ *for all* $y\in\mathfrak{t}, x\in\mathfrak{t}^*$ *and* $\deg(w) = 0$ *for* $w\in W$. *This gives a filtration on H_c whose associated graded equals the smash-product algebra $\mathbb{C}[\mathfrak{t}\oplus\mathfrak{t}^*]\rtimes W$.*

(3) *Let $ord(x) = ord(w) = 0, ord(y) = 1$. This defines another filtration on H_c, called the order filtration. The associated graded is still as above. We denote associated graded objects for this filtration by ogr.*

(4) *We have injections $\iota_c : H_c \hookrightarrow \mathcal{D}(\mathfrak{t}^{reg})\rtimes W$ for all c; this is the well-known Dunkl embedding. Moreover, these injections become a isomorphisms after inverting $\Delta\in H_c$. If we equip H_c with the order filtration and $\mathcal{D}(\mathfrak{t}^{reg})\rtimes W$ with the filtration induced by the order of differential operators, the injection ι_c is also filtration-preserving.*

(5) *Let* 1 *be the map $S\to\mathbb{C}$ taking all reflections to 1 and write* $\mathbf{e} = |W|^{-1}\sum_{w\in W} w$, $\mathbf{e}_- = |W|^{-1}\sum_{w\in W}(-1)^{\ell(w)}w$ *for the symmetrizing and antisymmetrizing idempotents. Then if $c-1$ is spherical in the sense that $\mathbf{e}H_{c-1}\mathbf{e}$ is Morita equivalent to H_{c-1}, we have algebra isomorphisms $\mathbf{e}H_{c-1}\mathbf{e}\cong\mathbf{e}_-H_c\mathbf{e}_-$.*

In the papers [22, 23], Gordon and Stafford show that when $\mathfrak{g} = \mathfrak{gl}_n$ or $\mathfrak{sl}_n$, the representation theory of H_c is closely connected to the geometry of $\mathrm{Hilb}^n(\mathbb{C}^2)$. More precisely, they show using the formalism of $\mathbb{Z}$-algebras that the spherical subalgebra $U_c = \mathbf{e}H_c\mathbf{e}\subseteq H_c$ is in a precise sense a non–commutative deformation of $\mathrm{Hilb}^n(\mathbb{C}^2)$. Many generalizations of this result have since appeared; the most well–understood cases seem to be the wreath product groups $G(\ell,1,n)$, see for example [19, 32] and [21] for further discussion. We will now define similar $\mathbb{Z}$–algebras for any H_c as above.

Following [22], we will write $U_c = \mathbf{e}H_c\mathbf{e}$ for the spherical subalgebra. By point (5) of Proposition 4.2 above, we can define $U_{c+1}-U_c$-bimodules $Q_c^{c+1} := \mathbf{e}H_{c+1}\mathbf{e}_-\delta$. These inject into the difference–reflection operators $\mathcal{D}(\mathfrak{t}^{reg})\rtimes W$ for all c via the maps ι_c. Using the multiplicative structure of the latter, one can define $U_{c+i}-U_{c+j}$-bimodules

$$B_{c+i\leftarrow c+j} = Q^{c+i}_{c+i-1}Q^{c+i-1}_{c+i-2}\cdots Q^{c+j+1}_{c+j}.$$

Sometimes for convenience we fix c and denote these by B_{ij}. When the parameter c is spherical, we have $B_{ij} \cong Q^{c+i}_{c+i-1}\otimes_{U_{c+i-1}}\cdots\otimes_{U_{c+j+1}} Q^{c+j+1}_{c+j}$, as noted in [22, (6.3.2)]. The main theorem of this section is the following.

Theorem 4.3. *Let c be constant and spherical. There exists a filtered $\mathbb{Z}$-algebra B_c such that*

(1) *There is an equivalence of categories from H_c-mod to $QCohB_c$*
(2) *The associated graded grB_c is isomorphic to the $\mathbb{Z}$-algebra associated to the graded ring*
$$\bigoplus_{d\geq 0}\mathbf{e}(\Delta^d I^{(d)}),$$
in other words the homogeneous coordinate ring of $X_{\mathfrak{g},symb}$.
(3) *In particular, there is a functor $H_c - filtmod \to QCoh(X_{\mathfrak{g},symb})$ from H_c-modules with a good filtration to quasicoherent sheaves on $X_{\mathfrak{g},symb}$.*

Proof. For the first part, since the parameters $c+i$ for all $i\geq 0$ are spherical, by [9, Section 5.3] the bimodules Q^{c+i+1}_{c+i} induce Morita equivalences, and by [22, Lemma 5.5.] the claim follows. For the second part, let $I^{(k)}$ be as before. Denote $N_k := B_{k0}\otimes_{\mathbf{e}H_c\mathbf{e}}\mathbf{e}H_c$, similar to [22, 6.6.]. We will construct a map $\mathbf{e}J^{(k)}\Delta^k \to ogrN_k$ for all $k\geq 0$, and prove that it is an isomorphism. The strategy is standard. We first show this for the $\mathfrak{sl}_2$-case, in which case it is obvious that $I^{(k)} = I^k = J^k$.

Denote $h_s := \Delta\alpha_s^{-1}$ for $s\in S$. We want to localize H_c in h_s in order to show that $H_c[(\Delta\alpha_s^{-1})^{-1}] \cong \mathcal{D}(\mathfrak{t}_s^{reg})\rtimes W_s\otimes_{\mathbb{C}} H^{\mathfrak{sl}_2}_{c_s}$. Here W_s is the pointwise stabilizer of the hyperplane perpendicular to α_s and $\mathfrak{t}_s$ is the non–fixed part of the s-action on $\mathfrak{t}$. See also [5, Theorem 2.20.] for the spherical case.

Applying ogr, we have isomorphisms $J^{(k)}[(\Delta\alpha_s^{-1})^{-2k}] \to ogrN_k[(\Delta\alpha_s^{-1})^{-2k}]$ given by the obvious maps on the $\mathfrak{sl}_2$-factor. Together the maps glue to a map $\Phi: J^{(k)}\to ogrN_k$, which over $\mathfrak{t}\backslash\left(\bigcup_{s_1,s_2}\mathfrak{t}^{s_1}\cap\mathfrak{t}^{s_2}\right)$ becomes an isomorphism.

But the module $ogrN(k)$ is free over $\mathbb{C}[\mathfrak{t}^*]$ (on the right), using an argument similar to [22, 6.11.]. Since $J^{(k)}$ is $\mathbb{C}[\mathfrak{t}^*]$-torsion-free, Φ is an isomorphism by the algebraic Hartog's theorem.

The third part is standard.

□

Remark 4.4. When $\mathfrak{g}$ is of type A, a combination of Theorem 4.3 and the fact that $I^{(k)} = J^k$ for all $k\geq 0$ reproves [22, Proposition 6.5.].

As a corollary of the proof, we also get

Corollary 4.5. $\mathbf{e}\Delta I = \mathbf{e}\Delta J = \Delta A$ *for all* $\mathfrak{g}$.

Analogously to [22, 6.5. and Corollary 6.22.] we have the following theorem:

Theorem 4.6. $ogr\bigoplus B_{k0}\mathbf{e}H_c \cong \bigoplus\Delta^k J^{(k)}$. *Therefore the sheaf associated to the spherical representation $\mathbf{e}H_c$ is exactly $\rho_*\mathcal{O}_{Y_{\mathfrak{g},symb}}$, the "Procesi bundle" for the symbolic blowup.*

Remark 4.7. We do not know if $\rho_*\mathcal{O}_{Y_{\mathfrak{g},symb}}$ satisfies the properties to be a Procesi bundle in the sense of Losev [29].

When $c = 1/h$ where h is the Coxeter number, H_c is known to possess a one-dimensional representation by [6]. Therefore by a similar argument as [23] we see that there is a surjection from $\mathbf{e}J^{(i)}/\mathbf{e}\mathfrak{m}J^{(i)} \twoheadrightarrow gr\mathbf{e}L_{c+i}$, where the associated graded is for the tensor product filtration arising from $\mathbf{e}L_{c+ih} \cong B_{i0} \otimes_{U_c} \mathbf{e}L_c$. When $i = 1$, $gr\mathbf{e}L_{c+1}$ is known as Gordon's canonical quotient. We infer that there is a surjection from the ring of functions on the "punctual" part of $X_{\mathfrak{g},symb}$ to this quotient.

4.1. **Coulomb branches.** We recall a natural incarnation of the trigonometric version of $X_{\mathfrak{g}}$ following [16]. Let again $I \subseteq \mathbb{C}[T^*T^\vee]$ be the ideal

$$I = \bigcap_\alpha \langle 1 - e^\alpha, \alpha^\vee \rangle$$

We denote these versions by X_G and in particular make the following definition, with obvious versions for "$sgn, diag$" in place of "$symb$".

Definition 4.8.

$$X_{G,symb} = \operatorname{Proj} \bigoplus_{d \geq 0} (\mathbf{e}I^{(d)})$$

We then consider the partially resolved Coulomb branch for the (G, Ad)-theory constructed in [16, Section 3]. More precisely, the ordinary/naive Coulomb branch algebra from [10] for a pair $(G, N), N \in Rep(G)$ is constructed as the spectrum of a commutative ring, which in turn is constructed using a convolution product in the $G[[t]]$–equivariant Borel–Moore homology of the space of triples

$$\mathcal{R} = \{[g, s] \in G((t)) \times^{G[[t]]} N[[t]] | gs \in N[[t]]\}$$

In *loc. cit.* this ind–scheme is replaced by

$$\mathcal{R}^d = \{[g, s] \in G((t)) \times^{G[[t]]} N[[t]] | gs \in t^{-d} N[[t]]\}$$

and a convolution product is constructed on the space

$$\bigoplus_{d \geq 0} H_*^{G[[t]]}(\mathcal{R}^d)$$

When $N = Ad$, we denote the Proj of this algebra as $X_{G,Coulomb}$.

One of the main theorems of the paper [16] is

Theorem 4.9. *We have an isomorphism* $X_{G,Coulomb} \cong X_{G,symb}$.

Remark 4.10. There is an obvious generalization of this result to K-theoretic Coulomb branches, identifying the Proj of

$$\bigoplus_{d \geq 0} K^{G[[t]]}(\mathcal{R}^d)$$

with the symbolic blow-up in $\bigcap_{\alpha \in \Phi^+} \langle 1 - e^\alpha, 1 - e^{\alpha^\vee} \rangle \subseteq \mathbb{C}[T \times T^\vee]$. We leave its study to the reader.

The upshot of the construction of X_G as a Coulomb branch is that it comes with a natural quantization. Indeed, as studied there, $T^*T^\vee/W$ quantizes to a spherical trigonometric Cherednik algebra and the partial resolution to a certain $\mathbb{Z}$–algebra built out of this. Moreover, the interpretation as a Hamiltonian reduction as explained e.g. in [37, Remark 5] gives $X_{G,symb}$ a natural Poisson structure. By [4, Theorem 1.1. and Lemma 2.1.], we have the following.

Proposition 4.11. *$X_{G,symb}$ has symplectic singularities.*

5. Geometric properties of $X_{\mathfrak{g}}$

To understand the eventual applications to coherent sheaves on the varieties $X_{\mathfrak{g}}$, one needs a better handle on their geometry. For $\mathrm{Hilb}^n(\mathbb{C}^2)$, many results use the modular intepretation, which is not available for general $\mathfrak{g}$.

First, we have an analog of Proposition 4.11 for $X_{\mathfrak{g},symb}$.

Proposition 5.1. *The variety $X_{\mathfrak{g},symb}$ has symplectic singularities.*

Proof. Recall from [16, Section 3] and [12] that locally in $T^*T^\vee/W$, the singularities are modeled by $\mathfrak{t}\oplus\mathfrak{t}^*/W_\mathfrak{t}$ for various Cartan subalgebras of centralizers of semisimple elements $a \in G$ with maximal semisimple rank. More precisely, as shown there, there are isomorphisms of formal neighborhoods $(T^*T^\vee/W)^{\wedge(0,a)} \cong (\mathfrak{t}\oplus\mathfrak{t}^*/W_\mathfrak{t})^{(0,0)}$ and these respect the quantizations to trigonometric, resp. rational, Cherednik algebras. This is an incarnation of Borel–de Siebenthal theory, and the $\mathfrak{t}$ which appear can be read off by removing a single vertex from the affine Dynkin diagram of G. So formally locally the singularities of $X_{\mathfrak{g},symb}$ are the same as those of $X_{G,symb}$. By Artin approximation, since we work with varieties (of finite type) over $\mathbb{C}$, we can extend the formal local isomorphism to étale local ones. As the singularities of $X_{G,symb}$ are symplectic, a fortiori so are those of $X_{\mathfrak{g},symb}$. □

Finally, we note that $X_{\mathfrak{g},symb}$ is only smooth in type A. Indeed, by above it has symplectic singularities, and smoothness would give that $X_{\mathfrak{g},symb} \to \mathfrak{t}\oplus\mathfrak{t}^*/W$ is a symplectic resolution in this case. By Kaledin [26] this can only happen in types ABC, but for types BC the variety $X_{\mathfrak{g},symb}$ constructed above is singular by construction, as can be seen from the comparison to quiver varieties in Proposition A.1.

In the simply laced cases, we still expect $X_{\mathfrak{g},symb}$ to be a $\mathbb{Q}$-factorial terminalization of $\mathfrak{t}\oplus\mathfrak{t}^*/W$. In general, we expect that the singularities are terminal, or equivalently in this case that

$$\mathrm{codim}_{X_{\mathfrak{g},symb}} X^{sing}_{\mathfrak{g},symb} \geq 4$$

We will now give evidence for these expectations. There is a fourth candidate for $X_{\mathfrak{g}}$, which is constructed in [7, 29]. Though its construction involves various choices and is not clearly unique, we will denote it by $X_{\mathfrak{g},Losev}$. We recall the construction briefly and refer to those two papers for further details. First, one picks a $\mathbb{Q}$-factorial terminalization

$$\pi : \widetilde{X} \to \mathfrak{t}\oplus\mathfrak{t}^*/W$$

as in [29]. It is not unique, but for example in types B_n, C_n a choice of $\widetilde{X}$ can be constructed using a quiver variety construction (or $\mathbb{Z}/2\mathbb{Z}$–Hilbert schemes). Moreover, Bellamy has classified the $\mathbb{Q}$–factorial terminalizations in [3].

For ADE root systems, one takes $X_{\mathfrak{g},Losev} = \widetilde{X}$. For non–simply laced types, $X_{\mathfrak{g},Losev}$ is the contraction of a certain divisor on $\widetilde{X}$. Denote the exceptional divisor of π by the letter D. Then as explained in [7, Proposition 2.1.]. $D = \sum_{s\in \mathrm{Refl}/\sim} D_s$, where D_s are irreducible components of D corresponding to conjugacy classes of simple reflections in W. By $\mathbb{Q}$-factoriality, ℓD is Cartier for some $\ell > 0$. One chooses $\widetilde{X}$ so that ℓD lies in the closure of the ample cone. By the contraction theorem, there is a unique intermediate (normal, terminal, symplectic) partial resolution $\widetilde{X} \to X \to \mathfrak{t}\oplus\mathfrak{t}^*/W$ for which $\mathcal{O}(\ell D)$ on $\widetilde{X}$ is lifted from an ample line bundle on X. We refer to the proof of [7, Proposition 2.1.] for details. We take $X_{\mathfrak{g},Losev} = X$.

The above construction has the obvious pitfall of being inexplicit. However, we conjecture that

Conjecture 5.2. *For a well-chosen $\widetilde{X}$, we have an isomorphism*

$$X_{\mathfrak{g},symb} \cong X_{\mathfrak{g},Losev}$$

Again in types ABC, this is true by [7, Remark 2.2.] and Proposition A.1.

5.1. **Calogero–Moser spaces.** As a more refined version of Conjecture 3.3, we expect

Conjecture 5.3.

$$X_{\mathfrak{g},symb}^{(\mathbb{C}^*)^2} \leftrightarrow \{\textit{Two-sided cells in } W\}$$

Here two–sided cells are meant in the original sense of Lusztig, for equal parameters. For type A, this follows from the usual bijection between fixed points in the Hilbert scheme and partitions. We prove the conjecture for types BC in the appendix.

The above conjecture fits well with a conjecture of Bonnafé and Rouquier [8], who have conjectured the existence of a natural bijection between fixed poins in Calogero–Moser spaces and two-sided cells. More precisely, we define

Definition 5.4. The Calogero–Moser space of $\mathfrak{g}$ is

$$\mathcal{CM}_c = \operatorname{Spec} Z(H_{c,0}^{rat})$$

where $H_{c,\hbar}$ is the rational Cherednik algebra of $\mathfrak{g}$ from Section 4 and $Z(-)$ denotes taking the center.

The conjecture we want to compare to is as follows. When c is constant and generic enough, there is a bijection

$$\mathcal{CM}_c^{(\mathbb{C}^*)^2} \leftrightarrow \{\text{Two-sided cells in } W\}$$

In fact, it is possible to define certain *Calogero–Moser cells* in W using the fixed points in $\mathcal{CM}_c$. The conjecture is that for Coxeter groups these coincide with the two–sided cells defined by Lusztig.

For types other than ABC, Bellamy has shown [2] that $\mathcal{CM}_c$ is singular for all values of c. Based on the type ABC-cases where $X_{\mathfrak{g}}$ admits a quiver variety description, we expect that

Conjecture 5.5. *The varieties $\mathcal{CM}_c, X_{\mathfrak{g},symb}$ admit hyper-Kähler structures such that there is a $U(1)$-equivariant homeomorphism $\mathcal{CM}_c \to X_{\mathfrak{g},symb}$, given by rotating the complex structure (see for example [19, 3.7.]). In particular, there is a bijection $(X_{\mathfrak{g},symb})^{(\mathbb{C}^*)^2} \leftrightarrow \mathcal{CM}_c^{(\mathbb{C}^*)^2}$.*

5.2. **Further conjectures.** In this section, we make some further speculation on $X_{\mathfrak{g},symb}$. There is another reason to believe Conjecture 5.3. It might seem contrived but was the original motivation for that conjecture before the author learned of Conjecture 3.3. As explained in [27], starting from an elliptic regular semisimple element in the loop Lie algebra of $\mathfrak{gl}_n$ i.e. $\gamma \in \mathfrak{gl}_n((t))$, one can construct a sheaf $\mathcal{F}_\gamma$ on $\mathrm{Hilb}^n(\mathbb{C}^2)$ supported on the punctual Hilbert scheme. It is quasi–coherent and $(\mathbb{C}^*)^2$-equivariant. Conjecturally, $\mathcal{F}_\gamma$ is coherent and the localization theorem in K–theory lets one write the K–class $[\mathcal{F}_\gamma] = \sum_\lambda a_\lambda(q,t)\widetilde{H}_\lambda$, where $\widetilde{H}_\lambda$ are the fixed point classes and $a_\lambda(q,t)$ are some rational functions in $\mathbb{C}(q,t)$. If one believes the triangle of connections from the introduction, as explained in [27] the functions $a_\lambda(q,t)$ are closely related to Shalika germs for the element γ.

The main construction in [16] (which is a precursor to the last section of [27]) extends to other groups, as discussed in the introduction. For elliptic regular semisimple $\gamma \in \mathfrak{g}((t))$, the pushforward of the sheaf $\mathcal{F}_\gamma$ to the naive Coulomb branch $T^*T^\vee/W$ is supported at finitely many points $(0,b) \in T^*T^\vee/W$, corresponding to the endoscopic decomposition of cohomology. As explained in [16, Theorem 2.16.], the stable part of cohomology corresponds to $(0,1)$.

Near this singularity, $T^*T^\vee/W$ is formally locally isomorphic $\mathfrak{t}\oplus\mathfrak{t}^*/W$, so pulling back $X_{G,Coulomb}$ and the sheaf $\mathcal{F}_\gamma$ along the inclusion of the formal neighborhood, one gets a (quasi-)coherent sheaf $\mathcal{F}_\gamma^{rat}$ on $X_{\mathfrak{g},symb}$ above the formal neighborhood $(\mathfrak{t}\oplus\mathfrak{t}^*/W)^{\wedge 0}$. Using the perverse filtration for the BM homology of affine Springer fibers, it should be possible to upgrade this construction to be equivariant for the $(\mathbb{C}^*)^2$–action on $\mathfrak{t}\oplus\mathfrak{t}^*/W$. Assuming this, we expect that the K–theory localization of $\mathcal{F}_\gamma^{rat}$ to the fixed points on $X_{\mathfrak{g},rat}$, which can be written as

$$[\mathcal{F}_\gamma] = \sum_x a_\lambda(q,t)[\delta_x],$$

encodes information about the Shalika germs of γ. Even without knowing which x appear, conjectures of Assem [1] suggest that the stable Shalika expansion only contains special nilpotent orbits in $\mathfrak{g}((t))$. These are also in bijection with two–sided cells in W.

The last conjecture we make is about the braid invariants. Recall that in [15, Theorem 1.5.] it was shown that the y-ified or monodromically deformed version of the Hochschild homology of the Rouquier complex of the k:th power of the full twist braid in Br_n is isomorphic as a doubly graded $\mathbb{C}[x_1,\dots,x_n,y_1,\dots,y_n]$-module to $J^{(k)}$. The proof of that theorem heavily uses the earlier computation of the ordinary Hochschild homology of this Rouquier complex by Elias and Hogancamp in [11], in particular the fact that this homological invariant is *parity*, or in other words that it is only supported in even homological degrees.

For the braid groups of other types, it should still be true that the Hochschild homology of the Rouquier complex of the full twist is parity. The only evidence we have for this is the purity of the corresponding affine Springer fibers. Though we do not do it here, it is also possible to define "link–splitting maps" as in Section 4 of [15] for other W and prove analogs of their flatness properties. Using these properties and the conjectural parity of the k:th powers of the full twists, one can show that in general this monodromically deformed invariant is isomorphic as a doubly graded $\mathbb{C}[\mathfrak{t}\oplus\mathfrak{t}^*]$-module to $J^{(k)}$. We leave this question open.

Appendix A. Proof of Conjectures 5.3, 5.5 in types B_n, C_n

Proposition A.1. *Let $\mathfrak{g} = \mathfrak{so}_{2n+1}$ or $\mathfrak{sp}_{2n}$. Then $X_{\mathfrak{g},symb} \cong M_\theta(Q)$, where Q is the cyclic quiver with two vertices and dimension vector (n,n) together with a dimension 1 framing at the extending vertex, and θ is a generic stability condition on the wall containing $(0,1)$. This identification respects the torus actions on both sides.*

Proof. This follows from Theorem 4.3, [19, Theorem 4.1.] and the uniqueness of quantizations of line bundles in [9, Proposition 5.2]. In the notations of Gordon, we have $c_1 = 2h$ and $c_\sigma = H_0 - H_1 = -2H_1$. Since $c_1 = c_\sigma = c$ i.e. we are in the equal parameter case, so the stability condition θ in loc. cit. becomes $(0,-2c)$, because $-h-H_1=0$ and $H_1=-2c$. Since the quiver varieties are isomorphic under negating the stability condition, we see we are on the positive vertical wall and therefore have the desired isomorhism. □

Corollary A.2. *Conjectures 5.2 and 5.5 is true in types B_n, C_n.*

Proof. The first one is [7, Remark 2.2.], and the second one follows from [19, 3.5.–3.7.]. □

For the second result we need to set up some notation. Recall the core–and–quotient construction establishing a bijection between ℓ-multipartitions of n and partitions of $n\ell+r$ with a fixed ℓ–core partition s of size r. This gives an injective map

$$\tau_s : P(\ell,n) \to P(n\ell+r)$$

In our case $r=0$ and $\ell=2$, in which case we write τ_s as a bijection

$$\tau : P(2,n) \to \{\lambda \in P(2n) | \lambda \text{ has trivial 2-core}\}$$

Given a partition λ of any size and $\ell \geq 1$, recall that for $0 \leq i \leq \ell-1$ a box in the Young diagram of λ is called i–addable (resp. i–removable) if its content is congruent to i mod ℓ and it is addable (resp. removable). Given a subset $J \subseteq \{0,\dots,\ell-1\}$, we call the J–heart of λ the partition obtained by repeatedly removing all i–removable boxes for all $i \in J$. Having the same J–heart sets up an equivalence relation on partitions of varying or fixed size, and partitions these sets into equivalence classes called J–classes.

Proposition A.3. *The torus–fixed points $M_\theta(Q)^{(\mathbb{C}^*)^2}$ defined as above are in bijection with J-classes in $\tau(P(2,n))$, where $J=\{0\}$.*

Proof. This is [19, Proposition 8.3.]. □

Proposition A.4. *There is a bijection between $\tau(P(2,n))$ and irreducible representations of W, such that the partition of $\tau(P(2,n))$ to 0-classes induces the partition of Irr(W) to families. In particular, Conjecture 5.3 is true in this case.*

Proof. This follows from [20, Theorem 3.3.] and the results of [13].

□

Example A.5. We illustrate the bijection of the previous proposition. Recall that in [30], the families can be described using symbols with the same entries. A natural bijection between bipartitions and symbols of rank n and defect 1 is easy to set up, so that a given bipartition (λ,μ) is associated to a symbol as follows: If

$$\lambda_1 \leq \ldots \leq \lambda_{\ell(\lambda)},\ \mu_1 \leq \ldots \leq \mu_{\ell(\mu)}$$

are the partitions written in increasing order (possibly empty), if necessary add k zeros to the front of λ or μ so that the resulting pair of nondecreasing sequences of integers (λ',μ') satisfies $\ell(\lambda')=\ell(\lambda)+k=\ell(\mu)+1$ or $\ell(\mu')=\ell(\mu)+k=\ell(\lambda)-1$ depending on whether $\ell(\lambda)\leq\ell(\mu)$ or $\ell(\lambda)>\ell(\mu)$. Now consider the symbol with rows

$$\begin{bmatrix} \lambda'_1+0 & & \lambda'_2+1 & & \cdots & & \lambda'_{\ell(\lambda')}+\ell(\lambda')-1 \\ & \mu'_1+0 & & \cdots & & \mu'_{\ell(\mu')}+\ell(\mu')-1 & \end{bmatrix}$$

Now let $n = 3$. Then $P(2,3)$ consists of 10 bipartitions. We list these, the partitions of 6 with trivial 2–core together with their contents mod 2, as well as the corresponding symbols.

Partition	0-Heart	Bipartition	Symbol
1 0 1 0 1 0	1 0 1 0 1 0	$(\varnothing, (1,1,1))$	$\begin{bmatrix}0&&1&&2&&3\\&1&&2&&3&\end{bmatrix}$
0 1 0 1 0 1	1 0 1 0 1	$((1,1,1), \varnothing)$	$\begin{bmatrix}1&&2&&3\\&0&&1&\end{bmatrix}$
1 0 1 0 0 1	1 0 1 0 1	$((1), (1,1))$	$\begin{bmatrix}0&&1&&3\\&1&&2&\end{bmatrix}$
1 0 1 0 1 0	1 0 1 0 1	$(\varnothing, (2,1))$	$\begin{bmatrix}0&&1&&2\\&1&&3&\end{bmatrix}$
0 1 0 1 0 1	1 0 1 0 1	$((2,1), \varnothing)$	$\begin{bmatrix}1&&3\\&0&\end{bmatrix}$
1 0 0 1 0 1	1 0 1 0 1	$((2), (1))$	$\begin{bmatrix}0&&3\\&1&\end{bmatrix}$
1 0 1 0 1 0	1 0 1 0 1	$(\varnothing, (3))$	$\begin{bmatrix}0&&1\\&3&\end{bmatrix}$
1 0 1 0 1 0	1 0 1 0 1 0	$((1), (2))$	$\begin{bmatrix}0&&2\\&2&\end{bmatrix}$
0 1 1 0 0 1	0 1 1 0 0 1	$((1,1), (1))$	$\begin{bmatrix}1&&2\\&1&\end{bmatrix}$
1 0 1 0 1 0	1 0 1 0 1 0	$((3), \varnothing)$	$\begin{bmatrix}3\\-\end{bmatrix}$

Appendix B. Computer calculations

We have included the code used for the G_2 and B_3–calculations below, hoping that they are illustrative of the general procedure.

Example B.1. The root system of type G_2 can be realized in the plane $x+y+z=0$ inside $\mathbb{R}^3$, with positive simple roots $\beta = 2x-y-z$ and $\alpha = x-y$. We can thus realize W as the reflection group generated by the two simple reflections about the corresponding root hyperplanes. In the code below, M_1 is the matrix for s_α and M_2 is the matrix for $s_{\beta+3\alpha}$.

```
julia> M1=matrix(QQ,[0 1 0;1 0 0; 0 0 1])
[0   1   0]
[1   0   0]
[0   0   1]
```

```
julia> M2=matrix(QQ,[-1//3 2//3 2//3; 2//3 2//3 -1//3; 2//3 -1//3
    2//3])
[-1//3    2//3    2//3]
[ 2//3    2//3   -1//3]
[ 2//3   -1//3    2//3]

julia> G=matrix_group([M1,M2]);length(G)
12

julia> M1D=block_diagonal_matrix([M1, transpose(inv(M1))]);M2D=
    block_diagonal_matrix([M2,transpose(inv(M2))]);GD=matrix_group([
    M1D,M2D]);

julia> F=abelian_closure(QQ)[1];IR=invariant_ring(GD);R=
    polynomial_ring(IR);x=gens(R);

julia> chi=Oscar.class_function(GD,[F(det(representative(c))) for c
    in conjugacy_classes(G)])
class_function(character table of GD, [1, -1, -1, 1, 1, 1])

julia> ri=relative_invariants(IR,chi);

julia> J=ideal(R,ri);minimal_generating_set(J)
8-element Vector{MPolyDecRingElem{QQFieldElem, QQMPolyRingElem}}:
 x[2]*x[4] - x[3]*x[4] - x[1]*x[5] + x[3]*x[5] + x[1]*x[6] - x[2]*x
     [6]
 2*x[4]^5*x[5] - 5*x[4]^4*x[5]^2 + 5*x[4]^2*x[5]^4 - 2*x[4]*x[5]^5 -
     2*x[4]^5*x[6] + 20*x[4]^3*x[5]^2*x[6] - 20*x[4]^2*x[5]^3*x[6] +
     2*x[5]^5*x[6] + 5*x[4]^4*x[6]^2 - 20*x[4]^3*x[5]*x[6]^2 + 20*x
     [4]*x[5]^3*x[6]^2 - 5*x[5]^4*x[6]^2 + 20*x[4]^2*x[5]*x[6]^3 -
     20*x[4]*x[5]^2*x[6]^3 - 5*x[4]^2*x[6]^4 + 5*x[5]^2*x[6]^4 + 2*x
     [4]*x[6]^5 - 2*x[5]*x[6]^5
 2*x[1]*x[4]^4*x[5] - 2*x[3]*x[4]^4*x[5] - 5*x[1]*x[4]^3*x[5]^2 + 5*x
     [3]*x[4]^3*x[5]^2 + 5*x[1]*x[4]*x[5]^4 - 5*x[3]*x[4]*x[5]^4 - 2*
     x[1]*x[5]^5 + 2*x[3]*x[5]^5 - 2*x[1]*x[4]^4*x[6] + 2*x[3]*x
     [4]^4*x[6] + 2*x[1]*x[4]^3*x[5]*x[6] - 2*x[3]*x[4]^3*x[5]*x[6] +
      15*x[1]*x[4]^2*x[5]^2*x[6] - 15*x[3]*x[4]^2*x[5]^2*x[6] - 20*x
     [1]*x[4]*x[5]^3*x[6] + 20*x[3]*x[4]*x[5]^3*x[6] + 5*x[1]*x[5]^4*
     x[6] - 5*x[3]*x[5]^4*x[6] + 3*x[1]*x[4]^3*x[6]^2 - 3*x[3]*x
     [4]^3*x[6]^2 - 18*x[1]*x[4]^2*x[5]*x[6]^2 + 18*x[3]*x[4]^2*x[5]*
     x[6]^2 + 15*x[1]*x[4]*x[5]^2*x[6]^2 - 15*x[3]*x[4]*x[5]^2*x[6]^2
      + 3*x[1]*x[4]^2*x[6]^3 - 3*x[3]*x[4]^2*x[6]^3 + 2*x[1]*x[4]*x
     [5]*x[6]^3 - 2*x[3]*x[4]*x[5]*x[6]^3 - 5*x[1]*x[5]^2*x[6]^3 + 5*
     x[3]*x[5]^2*x[6]^3 - 2*x[1]*x[4]*x[6]^4 + 2*x[3]*x[4]*x[6]^4 +
     2*x[1]*x[5]*x[6]^4 - 2*x[3]*x[5]*x[6]^4
```

```
2*x[1]^2*x[4]^3*x[5] - 4*x[1]*x[3]*x[4]^3*x[5] + 2*x[3]^2*x[4]^3*x
    [5] - 5*x[1]^2*x[4]^2*x[5]^2 + 10*x[1]*x[3]*x[4]^2*x[5]^2 - 5*x
    [3]^2*x[4]^2*x[5]^2 + 5*x[1]^2*x[5]^4 - 2*x[1]*x[2]*x[5]^4 - 8*x
    [1]*x[3]*x[5]^4 + 2*x[2]*x[3]*x[5]^4 + 3*x[3]^2*x[5]^4 - 2*x
    [1]^2*x[4]^3*x[6] + 4*x[1]*x[3]*x[4]^3*x[6] - 2*x[3]^2*x[4]^3*x
    [6] + 4*x[1]^2*x[4]^2*x[5]*x[6] - 8*x[1]*x[3]*x[4]^2*x[5]*x[6] +
     4*x[3]^2*x[4]^2*x[5]*x[6] + 10*x[1]^2*x[4]*x[5]^2*x[6] - 20*x
    [1]*x[3]*x[4]*x[5]^2*x[6] + 10*x[3]^2*x[4]*x[5]^2*x[6] - 20*x
    [1]^2*x[5]^3*x[6] + 8*x[1]*x[2]*x[5]^3*x[6] + 32*x[1]*x[3]*x
    [5]^3*x[6] - 8*x[2]*x[3]*x[5]^3*x[6] - 12*x[3]^2*x[5]^3*x[6] + x
    [1]^2*x[4]^2*x[6]^2 - 2*x[1]*x[3]*x[4]^2*x[6]^2 + x[3]^2*x[4]^2*
    x[6]^2 - 14*x[1]^2*x[4]*x[5]*x[6]^2 + 28*x[1]*x[3]*x[4]*x[5]*x
    [6]^2 - 14*x[3]^2*x[4]*x[5]*x[6]^2 + 25*x[1]^2*x[5]^2*x[6]^2 -
    12*x[1]*x[2]*x[5]^2*x[6]^2 - 38*x[1]*x[3]*x[5]^2*x[6]^2 + 12*x
    [2]*x[3]*x[5]^2*x[6]^2 + 13*x[3]^2*x[5]^2*x[6]^2 + 4*x[1]^2*x
    [4]*x[6]^3 - 8*x[1]*x[3]*x[4]*x[6]^3 + 4*x[3]^2*x[4]*x[6]^3 -
    12*x[1]^2*x[5]*x[6]^3 + 8*x[1]*x[2]*x[5]*x[6]^3 + 16*x[1]*x[3]*x
    [5]*x[6]^3 - 8*x[2]*x[3]*x[5]*x[6]^3 - 4*x[3]^2*x[5]*x[6]^3 + 2*
    x[1]^2*x[6]^4 - 2*x[1]*x[2]*x[6]^4 - 2*x[1]*x[3]*x[6]^4 + 2*x
    [2]*x[3]*x[6]^4
2*x[1]^3*x[4]^2*x[5] - 6*x[1]^2*x[3]*x[4]^2*x[5] + 6*x[1]*x[3]^2*x
    [4]^2*x[5] - 2*x[3]^3*x[4]^2*x[5] - 5*x[1]^3*x[4]*x[5]^2 + 15*x
    [1]^2*x[3]*x[4]*x[5]^2 - 15*x[1]*x[3]^2*x[4]*x[5]^2 + 5*x[3]^3*x
    [4]*x[5]^2 + 5*x[1]^2*x[2]*x[5]^3 - 2*x[1]*x[2]^2*x[5]^3 - 5*x
    [1]^2*x[3]*x[5]^3 - 6*x[1]*x[2]*x[3]*x[5]^3 + 2*x[2]^2*x[3]*x
    [5]^3 + 8*x[1]*x[3]^2*x[5]^3 + x[2]*x[3]^2*x[5]^3 - 3*x[3]^3*x
    [5]^3 - 2*x[1]^3*x[4]^2*x[6] + 6*x[1]^2*x[3]*x[4]^2*x[6] - 6*x
    [1]*x[3]^2*x[4]^2*x[6] + 2*x[3]^3*x[4]^2*x[6] + 6*x[1]^3*x[4]*x
    [5]*x[6] - 18*x[1]^2*x[3]*x[4]*x[5]*x[6] + 18*x[1]*x[3]^2*x[4]*x
    [5]*x[6] - 6*x[3]^3*x[4]*x[5]*x[6] + 5*x[1]^3*x[5]^2*x[6] - 15*x
    [1]^2*x[2]*x[5]^2*x[6] + 6*x[1]*x[2]^2*x[5]^2*x[6] + 18*x[1]*x
    [2]*x[3]*x[5]^2*x[6] - 6*x[2]^2*x[3]*x[5]^2*x[6] - 9*x[1]*x
    [3]^2*x[5]^2*x[6] - 3*x[2]*x[3]^2*x[5]^2*x[6] + 4*x[3]^3*x[5]^2*
    x[6] - x[1]^3*x[4]*x[6]^2 + 3*x[1]^2*x[3]*x[4]*x[6]^2 - 3*x[1]*x
    [3]^2*x[4]*x[6]^2 + x[3]^3*x[4]*x[6]^2 - 8*x[1]^3*x[5]*x[6]^2 +
    15*x[1]^2*x[2]*x[5]*x[6]^2 - 6*x[1]*x[2]^2*x[5]*x[6]^2 + 9*x
    [1]^2*x[3]*x[5]*x[6]^2 - 18*x[1]*x[2]*x[3]*x[5]*x[6]^2 + 6*x
    [2]^2*x[3]*x[5]*x[6]^2 + 3*x[2]*x[3]^2*x[5]*x[6]^2 - x[3]^3*x
    [5]*x[6]^2 + 3*x[1]^3*x[6]^3 - 5*x[1]^2*x[2]*x[6]^3 + 2*x[1]*x
    [2]^2*x[6]^3 - 4*x[1]^2*x[3]*x[6]^3 + 6*x[1]*x[2]*x[3]*x[6]^3 -
    2*x[2]^2*x[3]*x[6]^3 + x[1]*x[3]^2*x[6]^3 - x[2]*x[3]^2*x[6]^3
```

```
 2*x[1]^4*x[4]*x[5] - 8*x[1]^3*x[3]*x[4]*x[5] + 12*x[1]^2*x[3]^2*x
     [4]*x[5] - 8*x[1]*x[3]^3*x[4]*x[5] + 2*x[3]^4*x[4]*x[5] - 5*x
     [1]^4*x[5]^2 + 5*x[1]^2*x[2]^2*x[5]^2 - 2*x[1]*x[2]^3*x[5]^2 +
     20*x[1]^3*x[3]*x[5]^2 - 10*x[1]^2*x[2]*x[3]*x[5]^2 - 4*x[1]*x
     [2]^2*x[3]*x[5]^2 + 2*x[2]^3*x[3]*x[5]^2 - 25*x[1]^2*x[3]^2*x
     [5]^2 + 14*x[1]*x[2]*x[3]^2*x[5]^2 - x[2]^2*x[3]^2*x[5]^2 + 12*x
     [1]*x[3]^3*x[5]^2 - 4*x[2]*x[3]^3*x[5]^2 - 2*x[3]^4*x[5]^2 - 2*x
     [1]^4*x[4]*x[6] + 8*x[1]^3*x[3]*x[4]*x[6] - 12*x[1]^2*x[3]^2*x
     [4]*x[6] + 8*x[1]*x[3]^3*x[4]*x[6] - 2*x[3]^4*x[4]*x[6] + 8*x
     [1]^4*x[5]*x[6] - 10*x[1]^2*x[2]^2*x[5]*x[6] + 4*x[1]*x[2]^3*x
     [5]*x[6] - 32*x[1]^3*x[3]*x[5]*x[6] + 20*x[1]^2*x[2]*x[3]*x[5]*x
     [6] + 8*x[1]*x[2]^2*x[3]*x[5]*x[6] - 4*x[2]^3*x[3]*x[5]*x[6] +
     38*x[1]^2*x[3]^2*x[5]*x[6] - 28*x[1]*x[2]*x[3]^2*x[5]*x[6] + 2*x
     [2]^2*x[3]^2*x[5]*x[6] - 16*x[1]*x[3]^3*x[5]*x[6] + 8*x[2]*x
     [3]^3*x[5]*x[6] + 2*x[3]^4*x[5]*x[6] - 3*x[1]^4*x[6]^2 + 5*x
     [1]^2*x[2]^2*x[6]^2 - 2*x[1]*x[2]^3*x[6]^2 + 12*x[1]^3*x[3]*x
     [6]^2 - 10*x[1]^2*x[2]*x[3]*x[6]^2 - 4*x[1]*x[2]^2*x[3]*x[6]^2 +
      2*x[2]^3*x[3]*x[6]^2 - 13*x[1]^2*x[3]^2*x[6]^2 + 14*x[1]*x[2]*x
     [3]^2*x[6]^2 - x[2]^2*x[3]^2*x[6]^2 + 4*x[1]*x[3]^3*x[6]^2 - 4*x
     [2]*x[3]^3*x[6]^2
 2*x[1]^5*x[5] - 5*x[1]^4*x[2]*x[5] + 5*x[1]^2*x[2]^3*x[5] - 2*x[1]*x
     [2]^4*x[5] - 5*x[1]^4*x[3]*x[5] + 20*x[1]^3*x[2]*x[3]*x[5] - 15*
     x[1]^2*x[2]^2*x[3]*x[5] - 2*x[1]*x[2]^3*x[3]*x[5] + 2*x[2]^4*x
     [3]*x[5] - 15*x[1]^2*x[2]*x[3]^2*x[5] + 18*x[1]*x[2]^2*x[3]^2*x
     [5] - 3*x[2]^3*x[3]^2*x[5] + 5*x[1]^2*x[3]^3*x[5] - 2*x[1]*x[2]*
     x[3]^3*x[5] - 3*x[2]^2*x[3]^3*x[5] - 2*x[1]*x[3]^4*x[5] + 2*x
     [2]*x[3]^4*x[5] - 2*x[1]^5*x[6] + 5*x[1]^4*x[2]*x[6] - 5*x[1]^2*
     x[2]^3*x[6] + 2*x[1]*x[2]^4*x[6] + 5*x[1]^4*x[3]*x[6] - 20*x
     [1]^3*x[2]*x[3]*x[6] + 15*x[1]^2*x[2]^2*x[3]*x[6] + 2*x[1]*x
     [2]^3*x[3]*x[6] - 2*x[2]^4*x[3]*x[6] + 15*x[1]^2*x[2]*x[3]^2*x
     [6] - 18*x[1]*x[2]^2*x[3]^2*x[6] + 3*x[2]^3*x[3]^2*x[6] - 5*x
     [1]^2*x[3]^3*x[6] + 2*x[1]*x[2]*x[3]^3*x[6] + 3*x[2]^2*x[3]^3*x
     [6] + 2*x[1]*x[3]^4*x[6] - 2*x[2]*x[3]^4*x[6]
 2*x[1]^5*x[2] - 5*x[1]^4*x[2]^2 + 5*x[1]^2*x[2]^4 - 2*x[1]*x[2]^5 -
     2*x[1]^5*x[3] + 20*x[1]^3*x[2]^2*x[3] - 20*x[1]^2*x[2]^3*x[3] +
     2*x[2]^5*x[3] + 5*x[1]^4*x[3]^2 - 20*x[1]^3*x[2]*x[3]^2 + 20*x
     [1]*x[2]^3*x[3]^2 - 5*x[2]^4*x[3]^2 + 20*x[1]^2*x[2]*x[3]^3 -
     20*x[1]*x[2]^2*x[3]^3 - 5*x[1]^2*x[3]^4 + 5*x[2]^2*x[3]^4 + 2*x
     [1]*x[3]^5 - 2*x[2]*x[3]^5


julia> I1=ideal(R,[x[1]-x[2],x[4]-x[5]]);I2=ideal(R,[x[1]-x[3],x[4]-x
    [6]]);I3=ideal(R,[x[2]-x[3],x[5]-x[6]]);I4=ideal(R,[2*x[1]-x[2]-x
    [3],2*x[4]-x[5]-x[6]]);I5=ideal(R,[2*x[2]-x[1]-x[3],2*x[5]-x[4]-x
    [6]]);I6=ideal(R,[2*x[3]-x[1]-x[2],2*x[6]-x[4]-x[5]]);I=intersect
    (I1,I2,I3,I4,I5,I6);

julia> J==I
true
```

Example B.2. In type B_n, the root system can be realized in $\mathbb{R}^n$ by $\pm x_i, \pm(x_i \pm x_j), i, j = 1, \ldots, n$. Similarly, the type C_n root system can be realized by $\pm 2x_i, \pm(x_i \pm x_j)$ The hyperoctahedral group W acts in the obvious way. We can take $\mathfrak{t} \oplus \mathfrak{t}^*$ to be $\mathbb{C}^{2n}$ with coordinates $x_1, \ldots, x_n, y_1, \ldots, y_n$ and write

$$I = \cap_i \langle x_i, 2y_i \rangle \bigcap \cap_{i<j} \langle x_i \pm x_j, y_i \pm y_j \rangle.$$

Below is the code for the B_3-case, computing I and J. One can check that I has one extra generator in its minimal generating set, which is not W-invariant. In particular, one has $J \subsetneq I$ but $\mathbf{e}I = \mathbf{e}J$.

```
julia> M1=matrix(QQ,[0 1 0; 1 0 0; 0 0 1]);M2=matrix(QQ,[1 0 0; 0 0
    1; 0 1 0]);M3=matrix(QQ,[-1 0 0; 0 1 0; 0 0 1]);W=matrix_group([
    M1,M2,M3]);length(W)
48

julia> M1D=block_diagonal_matrix([M1,transpose(inv(M1))]);M2D=
    block_diagonal_matrix([M2,transpose(inv(M2))]);M3D=
    block_diagonal_matrix([M3,transpose(inv(M3))]);WD=matrix_group([
    M1D,M2D,M3D]);

julia> IR=invariant_ring(WD);R=polynomial_ring(IR);x=gens(R); F=
    abelian_closure(QQ)[1];

julia> chi=Oscar.class_function(WD,[F(det(representative(c))) for c
    in conjugacy_classes(W)]);

julia> ri=relative_invariants(IR,chi);J=ideal(R,ri);
    minimal_generating_set(J)

20-element Vector{MPolyElem_dec{fmpq, fmpq_mpoly}}:
 x[3]*x[4]^3*x[5] - x[3]*x[4]*x[5]^3 - x[2]*x[4]^3*x[6] + x[1]*x
     [5]^3*x[6] + x[2]*x[4]*x[6]^3 - x[1]*x[5]*x[6]^3
 x[1]*x[3]*x[4]^2*x[5] - x[2]*x[3]*x[4]*x[5]^2 - x[1]*x[2]*x[4]^2*x
     [6] + x[1]*x[2]*x[5]^2*x[6] + x[2]*x[3]*x[4]*x[6]^2 - x[1]*x[3]*
     x[5]*x[6]^2
 x[1]^2*x[3]*x[4]*x[5] - x[2]^2*x[3]*x[4]*x[5] - x[1]^2*x[2]*x[4]*x
     [6] + x[2]*x[3]^2*x[4]*x[6] + x[1]*x[2]^2*x[5]*x[6] - x[1]*x
     [3]^2*x[5]*x[6]
 x[2]^3*x[3]*x[4] - x[2]*x[3]^3*x[4] - x[1]^3*x[3]*x[5] + x[1]*x
     [3]^3*x[5] + x[1]^3*x[2]*x[6] - x[1]*x[2]^3*x[6]
 x[2]*x[4]^3*x[5]^2*x[6] - x[1]*x[4]^2*x[5]^3*x[6] - x[3]*x[4]^3*x
     [5]*x[6]^2 + x[3]*x[4]*x[5]^3*x[6]^2 + x[1]*x[4]^2*x[5]*x[6]^3 -
      x[2]*x[4]*x[5]^2*x[6]^3
 x[2]*x[3]*x[4]^3*x[5]^2 - x[1]*x[3]*x[4]^2*x[5]^3 - x[2]*x[3]*x
     [4]^3*x[6]^2 + x[1]*x[3]*x[5]^3*x[6]^2 + x[1]*x[2]*x[4]^2*x[6]^3
      - x[1]*x[2]*x[5]^2*x[6]^3
 x[2]^2*x[3]*x[4]^3*x[5] - x[1]^2*x[3]*x[4]*x[5]^3 - x[2]*x[3]^2*x
     [4]^3*x[6] + x[1]*x[3]^2*x[5]^3*x[6] + x[1]^2*x[2]*x[4]*x[6]^3 -
      x[1]*x[2]^2*x[5]*x[6]^3
 x[1]^3*x[3]*x[4]^2*x[5] - x[2]^3*x[3]*x[4]*x[5]^2 - x[1]^3*x[2]*x
     [4]^2*x[6] + x[1]*x[2]^3*x[5]^2*x[6] + x[2]*x[3]^3*x[4]*x[6]^2 -
      x[1]*x[3]^3*x[5]*x[6]^2
 x[1]*x[2]^3*x[3]*x[4]^2 - x[1]*x[2]*x[3]^3*x[4]^2 - x[1]^3*x[2]*x
     [3]*x[5]^2 + x[1]*x[2]*x[3]^3*x[5]^2 + x[1]^3*x[2]*x[3]*x[6]^2 -
      x[1]*x[2]^3*x[3]*x[6]^2
 x[1]^2*x[2]^3*x[3]*x[4] - x[1]^2*x[2]*x[3]^3*x[4] - x[1]^3*x[2]^2*x
     [3]*x[5] + x[1]*x[2]^2*x[3]^3*x[5] + x[1]^3*x[2]*x[3]^2*x[6] - x
     [1]*x[2]^3*x[3]^2*x[6]
 x[4]^5*x[5]^3*x[6] - x[4]^3*x[5]^5*x[6] - x[4]^5*x[5]*x[6]^3 + x[4]*
     x[5]^5*x[6]^3 + x[4]^3*x[5]*x[6]^5 - x[4]*x[5]^3*x[6]^5
 x[2]*x[4]^5*x[5]^2*x[6] - x[1]*x[4]^2*x[5]^5*x[6] - x[3]*x[4]^5*x
     [5]*x[6]^2 + x[3]*x[4]*x[5]^5*x[6]^2 + x[1]*x[4]^2*x[5]*x[6]^5 -
      x[2]*x[4]*x[5]^2*x[6]^5
```

```
 x[1]*x[3]*x[4]^4*x[5]^3 - x[2]*x[3]*x[4]^3*x[5]^4 - x[1]*x[2]*x
     [4]^4*x[6]^3 + x[1]*x[2]*x[5]^4*x[6]^3 + x[2]*x[3]*x[4]^3*x[6]^4
      - x[1]*x[3]*x[5]^3*x[6]^4
 x[1]^2*x[3]*x[4]^3*x[5]^3 - x[2]^2*x[3]*x[4]^3*x[5]^3 - x[1]^2*x[2]*
     x[4]^3*x[6]^3 + x[2]*x[3]^2*x[4]^3*x[6]^3 + x[1]*x[2]^2*x[5]^3*x
     [6]^3 - x[1]*x[3]^2*x[5]^3*x[6]^3
 x[2]^3*x[3]*x[4]^3*x[5]^2 - x[1]^3*x[3]*x[4]^2*x[5]^3 - x[2]*x[3]^3*
     x[4]^3*x[6]^2 + x[1]*x[3]^3*x[5]^3*x[6]^2 + x[1]^3*x[2]*x[4]^2*x
     [6]^3 - x[1]*x[2]^3*x[5]^2*x[6]^3
 x[1]^3*x[2]*x[3]*x[4]^2*x[5]^2 - x[1]*x[2]^3*x[3]*x[4]^2*x[5]^2 - x
     [1]^3*x[2]*x[3]*x[4]^2*x[6]^2 + x[1]*x[2]*x[3]^3*x[4]^2*x[6]^2 +
      x[1]*x[2]^3*x[3]*x[5]^2*x[6]^2 - x[1]*x[2]*x[3]^3*x[5]^2*x[6]^2
 x[1]^3*x[2]^2*x[3]*x[4]^2*x[5] - x[1]^2*x[2]^3*x[3]*x[4]*x[5]^2 - x
     [1]^3*x[2]*x[3]^2*x[4]^2*x[6] + x[1]*x[2]^3*x[3]^2*x[5]^2*x[6] +
      x[1]^2*x[2]*x[3]^3*x[4]*x[6]^2 - x[1]*x[2]^2*x[3]^3*x[5]*x[6]^2
 x[1]^3*x[2]^3*x[3]*x[4]^2 - x[1]^3*x[2]*x[3]^3*x[4]^2 - x[1]^3*x
     [2]^3*x[3]*x[5]^2 + x[1]*x[2]^3*x[3]^3*x[5]^2 + x[1]^3*x[2]*x
     [3]^3*x[6]^2 - x[1]*x[2]^3*x[3]^3*x[6]^2
 x[1]^4*x[2]^3*x[3]*x[4] - x[1]^4*x[2]*x[3]^3*x[4] - x[1]^3*x[2]^4*x
     [3]*x[5] + x[1]*x[2]^4*x[3]^3*x[5] + x[1]^3*x[2]*x[3]^4*x[6] - x
     [1]*x[2]^3*x[3]^4*x[6]
 x[1]^5*x[2]^3*x[3] - x[1]^3*x[2]^5*x[3] - x[1]^5*x[2]*x[3]^3 + x[1]*
     x[2]^5*x[3]^3 + x[1]^3*x[2]*x[3]^5 - x[1]*x[2]^3*x[3]^5

julia> I1=ideal(R,[x[1]-x[2],x[4]-x[5]]); I2=ideal(R,[x[1]-x[3],x[4]-
    x[6]]); I3=ideal(R,[x[2]-x[3],x[5]-x[6]]);I4=ideal(R,[x[1]+x[2],x
    [4]+x[5]]); I5=ideal(R,[x[1]+x[3],x[4]+x[6]]); I6=ideal(R,[x[2]+x
    [3],x[5]+x[6]]);I7=ideal(R,[x[1],x[4]]);I8=ideal(R,[x[2],x[5]]);
    I9=ideal(R,[x[3],x[6]]);

julia> I=intersect(I1,I2,I3,I4,I5,I6,I7,I8,I9);

julia> I==J
false
```

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