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The Classical BV-BRST Formalism via Derived Geometry

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Tiivistelmä — Referat — Abstract

The BV-BRST (Batalin-Vilkovisky / Becchi-Rouet-Stora-Tyutin) formalism was originally introduced as an algebraic tool for perturbatively quantizing gauge theories in the path integral formulation of quantum field theory. The power of the BV-BRST formalism comes from resolving the obstructions to the existence of a gauge-invariant space of observables, equivalently the existence of a phase space, using the tools of homological algebra for arbitrary gauge theories. It has since been expanded and recast in the language of derived geometry and L_∞ algebras, and reinterpreted as describing a formal deformation problem on the moduli space of gauge-invariant on-shell fields. This thesis provides a self-contained introduction to the derived geometric view of classical BV-BRST formalism.

Beginning with classical mechanics, the Hamiltonian phase space is shown to be equivalent to the Lagrangian critical locus of the action functional algebraically, by the equivalence of observables obtained as the transversal intersection of C^∞-rings and those obtained as the algebra of functions on the phase space. These formulations are then generalized to field theory using the variational bicomplex on the jet bundle, leading to the covariant phase space with the Peierls bracket, and the critical locus as a derived manifold described by the Koszul complex, which are then shown to be equivalent using the classical BV-BFV correspondence.

Gauge symmetries obstruct both the formulations above; they cause the equations of motion to no longer be independent, demonstrated by the Noether identities, while compactly supported gauge transformations obstruct the existence of Cauchy surfaces. The former is resolved by the Koszul-Tate resolution and the latter by the homological quotient by gauge orbits given by the complex of longitudinal differential forms, identified as the Chevalley-Eilenberg algebra of an L_∞-algebra action, which a theorem of Lurie and Pridham exhibits as a formal moduli problem.

The full BV-BRST complex combines both formulations as a derived critical locus inside the homotopy quotient, the shifted cotangent bundle $T^*[-1](\Gamma E/\mathfrak{g})$ with a canonical (-1)-shifted symplectic structure, with differential constructed by combining the differentials of both the complex of longitudinal differential forms and the Koszul-Tate complex using homological perturbation with nilpotency equivalent to the classical master equation. The gauge-invariant on-shell observables are recovered as its degree-zero cohomology, and the BV-BFV correspondence extends to gauge theories to produce a derived phase space on the Cauchy surface. Finally, the construction is demonstrated explicitly for free electrodynamics.

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1. Introduction

The BV-BRST (Batalin-Vilkovisky / Becchi-Rouet-Stora-Tyutin) formalism was originally introduced as an algebraic tool for perturbatively quantizing gauge theories in the path integral formulation of quantum field theory. The power of the BV-BRST formalism comes from resolving the obstructions to the existence of a gauge-invariant space of observables, equivalently the existence of a phase space, using the tools of homological algebra for arbitrary gauge theories. It has since been expanded and recast in the language of derived geometry and L_∞ algebras, and reinterpreted as describing a formal deformation problem on the moduli space of gauge-invariant on-shell fields. This formalism underpins many recent approaches to mathematical formulations of quantum field theory, such as the factorization algebra formulation as in [CG17, CG21], the perturbative algebraic formulation as in [FR12, Rej15], and the higher geometric formulations of [GS25, Sch16].

This thesis provides a self-contained introduction to the derived geometric view of classical BV-BRST formalism.

In Chapter 2, the classical mechanical picture of phase space and observables is reviewed anticipating later generalization to field theory. The Hamiltonian and Lagrangian formalisms for the point particle are presented, and the Hamiltonian phase space is shown to be equivalent to the Lagrangian critical locus of the action functional. The algebra of observables in the Lagrangian formalism is obtained using the formalism of C^∞-rings, expressing the algebra of observables in the Lagrangian view as a transversal intersection derived from the critical locus, which is shown to be equivalent to the observables obtained as the algebra of functions on the phase space.

In Chapter 3, both formulations are generalized to field theory. The infinite-dimensional nature of field theory is dealt with by introducing the variational bicomplex on the jet bundle, which restricts the field theory to local action functionals and observables. The Euler-Lagrange equations and a covariant phase space on Cauchy surfaces, with Poisson structure given by the Peierls bracket, are obtained. The critical locus is again expressed algebraically as an intersection; since transversality fails for general field theories, the intersection is replaced by a derived intersection in the category of derived manifolds, defined by non-positively graded differential graded C^∞-

algebras. The resulting derived manifold (the Koszul complex) view is equivalent to the covariant phase space view by the classical BV-BFV correspondence, which recovers the covariant phase space equipped with its Hamiltonian/BFV symplectic form as a boundary theory to the bulk derived space with its BV symplectic form.

In Chapter 4, gauge symmetries are introduced and shown to obstruct both formulations above. Gauge symmetries cause the equations of motion to no longer be independent, demonstrated by the Noether identities, which causes the Koszul complex to no longer be a resolution, while compactly supported gauge transformations obstruct the existence of Cauchy surfaces. The Koszul-Tate resolution resolves the former obstruction by adjoining extra generators, called antighost generators, to account for the dependencies, and the latter obstruction requires quotienting by the gauge orbits, performed homologically by the complex of longitudinal differential forms, identified as the Chevalley-Eilenberg algebra of an L_∞-algebra action. Formal deformation theory, by the theorem of Lurie and Pridham, exhibits this homological quotient as the algebra of functions of a formal deformation problem.

In Chapter 5, the two complexes are combined into the full BV-BRST complex as the derived intersection inside the homotopy quotient by gauge orbits, whose underlying graded manifold is the shifted cotangent bundle $T^*[-1](\Gamma E/\mathfrak{g})$ with canonical (-1)-shifted symplectic structure given by the BV symplectic form. The total differential is constructed by combining the differentials of both the complex of longitudinal differential forms and the Koszul-Tate complex using homological perturbation, with nilpotency equivalent to the classical master equation. The algebra of gauge-invariant on-shell observables is obtained as its degree-zero cohomology. The BV-BFV correspondence then extends to gauge theories, giving a derived phase space structure on the boundary Cauchy surface whose degree-zero cohomology recovers the expected initial data. The construction is carried out explicitly for free electrodynamics, where Gauss's law is imposed by the boundary differential.

2. Classical Mechanics

Classical mechanics describes the kinematics and dynamics of non-relativistic macroscopic systems. The most crucial aspects of classical mechanics required for its generalization to field theory can be seen in the study of the dynamics of a point particle, which is what this chapter will be solely focused on. The central object of study will be the phase space, whose power comes not just from its enumeration of all possible motions of the system in question, but also from its ability to describe the dynamics of the system in a geometric manner. This makes it particularly amenable to generalization by considering the geometry of our new systems. In fact, it will be seen in later chapters that this theory is a special case of the classical theory of fields when the underlying space consists of a single point.

2.1 Equations of Motion

Point particles are idealized models of bodies which neglect their spatial extent. The whole of the body, with all its mass m, is thought to be concentrated at a single point x in space Σ at any point in time. The collection of these points defines the trajectory $x(t)$ of a point particle, thought of as a map from the time axis $\mathbb{R}$ to space Σ. Sometimes it is more convenient to talk about the product space called spacetime, $M = \mathbb{R} \times \Sigma$, where the graph of the trajectory resides, assumed to be simply connected throughout.

The equations of motion for a point particle are given by Newton's second law, which states that the trajectory $x(t)$ of a point particle representing a body of mass m satisfies the following second-order differential equation

$$m\frac{d^2x}{dt^2} = F(x, \frac{dx}{dt}, t)$$

where F represents the effect of outside forces on the point particle.

Notice that forces only determine the acceleration of a point particle and not its velocity nor position directly, thus the data required to completely specify the trajectory is given by the position $x(t_0)$ and the velocity $x'(t_0)$ at some initial time t_0. The space of all trajectories in the presence of forces is then given by the space of positions and velocities at the initial time, (x_0, x_0', t_0). For the purposes of generalization, we instead work with initial momenta $p = mx_0'$ and reformulate Newton's second law as

$$\frac{dp}{dt} = F(x, \frac{p}{m}, t)$$

and obtain the space (x_0, p_0, t_0).

Since the dynamics of a point particle are completely specified by these initial conditions, any state of the system at any time can be described by its position and momentum at that time.

2.1.1 Symmetries and Time Translation

Isolated systems are systems invariant under time translation, meaning that any two trajectories $x_1(t)$ and $x_2(t)$ in an isolated system which coincide at any two times in their positions and momenta

$$x_1(t_1) = x_2(t_2), \quad x_1'(t_1) = x_2'(t_2),$$

are identical up to a shift in time $t_2 - t_1$,

$$x_1(t) = x_2(t + (t_2 - t_1)).$$

Time translation invariance can also be described using a one-parameter group $U_{\Delta t}$ of translations satisfying the group law

$$U_{\Delta t_1} U_{\Delta t_2} = U_{\Delta t_1 + \Delta t_2}$$

with a representation as

$$U_{\Delta t} x(t) = x(t + \Delta t)$$

where time-translation symmetry is represented by

$$(x_1(t_1) = x_2(t_2), \ x_1'(t_1) = x_2'(t_2)) \implies x_1(t) = U_{t_2 - t_1} x_2(t).$$

An immediate corollary is that there is a redundancy in the description of the states of our system. Instead of describing the state of the system as a triplet (x, p, t), it is sufficient to describe it as a doublet (x, p) at any point in time. The notion that symmetries represent redundancies in the description of physical systems that should be quotiented out will play a crucial role in the following chapters.

Recall that the graph of a trajectory $x(t)$ is a subset of spacetime $M = \mathbb{R} \times \Sigma$.

Definition 1 (Section 2.2 of [Olv93]) *A symmetry of a physical system is a transformation $p : M \to M$ of spacetime M that maps graphs of trajectories to graphs of trajectories. For all trajectories x_1 whose graphs are denoted Γ_{x_1},*

$$p(\Gamma_{x_1}) = \Gamma_{x_2}$$

for some trajectory x_2.

For a one-parameter group of symmetries U_ξ, this becomes

$$U_\xi \circ x = x_\xi$$

where x_ξ are a $\mathbb{R}$-parameterized collection of solutions to the equations of motion.

Since the one-parameter Lie group of symmetries U_ξ is one-dimensional, its elements can be written as the flow of the left-invariant vector field on the group U_ξ given by a representative element H of its one-dimensional Lie algebra, hence $U_\xi = \exp(\xi H)$. The representation of the element H on solutions is obtained by differentiating the action of U_ξ on solutions at $\xi = 0$. [Olv93, Proposition 1.48] For time translation symmetry,

$$Hx(t) = \frac{d}{d\xi}(x(t+\xi))|_{\xi=0} = \frac{d}{dt}x(t).$$

As expected, the representation for the vector field that generates the flow of time translation symmetry is the time flow along solutions. Note that this is not the same as the time-directed coordinate vector field on spacetime M. Since the states of the system are described by a doublet (x, p), the time flow along solutions is described by a specific vector field along position and momentum. The value of this vector field is immediate from the following reformulation of Newton's second law

$$\frac{dx}{dt} = \frac{p}{m},$$
$$\frac{dp}{dt} = F(x, \frac{p}{m}).$$

2.2 The Hamiltonian Formalism

Definition 2 *An observable quantity of a physical system is a function that maps solutions to the equations of motion to the real numbers* $\mathbb{R}$

Since the space (x, p) classifies all solutions to the equations of motion, in this case the trajectories, the algebra of functions on (x, p) is exactly the space of all observables. Furthermore, the vector field of time evolution can be represented in a purely algebraic way on this space.

Recall that on a manifold equipped with a non-degenerate 2-form, vector fields and 1-forms can be interchanged. This allows specifying the vector field in terms of a function on the space if the 1-form associated with a vector field can be written as a gradient, unique up to an additive constant.

From this point on, refer to x as q to match with the usual notation and refer to the vector field corresponding to the Lie algebra element H as X_H.

Definition 3 *A phase space is the space of all solutions to the equations of motion, equipped with a non-degenerate 2-form ω that is also closed*

$$d\omega = 0$$

called the symplectic form.

On the space (q, p), we define the symplectic form as

$$\omega = dq \wedge dp$$

and later note that this choice is unique up to a change of coordinates.

Taking the interior product of the symplectic form along the flow vector field X_H yields a covector field $\iota_{X_H}\omega$. Since phase space is simply connected, the vanishing of the exterior derivative of $\iota_{X_H}\omega$ implies that there is a function $H(p, q)$ such that $dH = \iota_{X_H}\omega$. Taking the exterior derivative of $\iota_{X_H}\omega$ results in

$$d(\iota_{X_H}\omega) = -\iota_{X_H}d\omega + L_{X_H}\omega = L_{X_H}\omega$$

where Cartan's magical formula for the Lie derivative $L_X\omega = \iota_X d\omega + d\iota_X\omega$ was used.

It can be shown that for time-invariant systems in one-dimension, the vanishing of this Lie derivative is equivalent to the statement that the force is momentum-independent. In coordinates,

$$L_{X_H}\omega = -L_{X_H}dp \wedge dq - dp \wedge L_{X_H}dq = -dF \wedge dq - \frac{1}{m}dp \wedge dp = -(\partial_p F)dp \wedge dq.$$

A quick calculation will show that when working with conservative forces F, that is $F = -\frac{dU(x)}{dx}$ for some function $U(x)$, the function $H(q, p)$ is in fact the total energy function $E(q, p) = \frac{p^2}{2m} + U(x)$ up to an additive constant.

Note that the above derivation also implies that if the Lie algebra of vector fields on phase space is restricted to vector fields that preserve the symplectic form under the Lie derivative, then every such element can be associated to a function on phase space.

Definition 4 *A Hamiltonian function f associated to the vector field X_f that preserves the symplectic form is the unique solution (up to a constant) to the equation*

$$df = \iota_{X_f}\omega.$$

The vector field X_f is then called the Hamiltonian vector field of f.

Even further, the algebra of Hamiltonian vector fields under the standard Lie bracket is equivalent to the Lie algebra of observables (up to a constant) under the so-called Poisson bracket defined as the inverse of ω as a tensor, $\{A, B\} = \omega^{-1}(dA, dB)$.

This allows another reformulation of Newton's second law

$$\{q, H\} = \frac{dq}{dt},$$
$$\{p, H\} = \frac{dp}{dt}.$$

Although the above discussion focused on a one-dimensional system, in fact the space above is the cotangent space $T^*\mathbb{R}$, the formalism is immediately generalizable.

Definition 5 *A physical system is a manifold equipped with a symplectic form ω and a privileged Hamiltonian function H.*

Darboux's theorem states that symplectic manifolds are locally similar to the case of $T^*\mathbb{R}^n$, in that there exists a neighbourhood around every point with a set of coordinates (q_i, p_i) such that

$$\omega = \sum_i dq_i \wedge dp_i.$$

2.3 The Lagrangian Formalism

Another useful reformulation of Newton's second law comes from calculus of variations. Instead of considering the local flow of trajectories in phase space along the Hamiltonian vector field corresponding to $H(q, p)$, consider the global characteristics of the trajectory in phase space $(x(t), p(t))$ as extremizing a certain functional S, called the action functional, with specific boundary conditions.

Definition 6 *The action functional S is the integral of a function of position, velocity, and time $L(x, x', t)$, called the Lagrangian with specified boundary conditions*

$$S[x] = \int_{t_0}^{t_1} L(x(t), x'(t), t)dt.$$

The full variational calculus required to extremize this functional rigorously and define the following variational derivatives will be introduced in the following chapter. Following the usual exposition as in [Tay05], consider any other continuous function $\eta(t)$ such that $\eta(t_0) = \eta(t_1) = 0$ which will serve as a variation from $x(t)$. The derivative $\delta S[x]$ can be defined as

$$\delta_\eta S = \frac{\partial}{\partial \epsilon}(S[x + \epsilon\eta])|_{\epsilon=0}$$

which can be evaluated in the following way,

$$\begin{aligned}\frac{\partial}{\partial \epsilon}(S[x+\epsilon\eta])|_{\epsilon=0} &= \int_{t_0}^{t_1} \frac{\partial}{\partial \epsilon} L(x+\epsilon\eta, x'+\epsilon\eta', t)|_{\epsilon=0} dt \\ &= \int_{t_0}^{t_1} (\eta \frac{\partial L}{\partial x} + \eta' \frac{\partial L}{\partial x'}) dt \\ &= \int_{t_0}^{t_1} \eta (\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial x'}) dt.\end{aligned}$$

Since this must hold for any arbitrary continuous function η, the extremum of this derivative reduces down to a set of ordinary differential equations

$$\frac{d}{dt}\frac{\partial L}{\partial x'} - \frac{\partial L}{\partial x} = 0$$

called the Euler-Lagrange equations.

Assuming that the forces in the system under consideration are conservative, in the form $F = -\frac{dU(x)}{dx}$, the Euler-Lagrange equations reduce to Newton's second law with Lagrangian

$$L(x, x') = \frac{m(x')^2}{2} - U(x)$$

since

$$\frac{d}{dt}\frac{\partial L}{\partial x'} = m\frac{dx'}{dt} = m\frac{d^2x}{dt^2},$$

and

$$\frac{\partial L}{\partial x} = -\frac{\partial U}{\partial x} = F.$$

The space of all physically realizable trajectories is now given as the space of all extremal functions of the derivative of the action functional δS, called its critical locus.

This critical locus can be represented in geometric terms as the intersection of the graph of δS with the zero section, in the cotangent space of trajectories

$$C_{\delta S=0} = \Gamma_{\delta S} \cap 0.$$

Since this space classifies all physical trajectories, it is expected to be equipped with a symplectic form to give it the structure of a phase space. In fact, the symplectic form falls out of the derivation of the Euler-Lagrange equations

$$\int_{t_0}^{t_1} (\eta \frac{\partial L}{\partial x} + \eta' \frac{\partial L}{\partial x'}) dt = \eta \frac{\partial L}{\partial x'}\Big|_{t_0}^{t_1} + \int_{t_0}^{t_1} \eta (\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial x'}) dt. \tag{2.1}$$

The term $\eta \frac{\partial L}{\partial x'}$ can be understood as a 1-form on the space of trajectories, where $\epsilon \mapsto \epsilon\eta$ is interpreted as a curve with tangent vector η, $\theta(\eta) = \eta\frac{\partial L}{\partial x'}$ and $\eta\frac{\partial L}{\partial x'}\Big|_{t_0}^{t_1} = \int_{t_0}^{t_1} d\theta dt$. Then, as a 1-form, the exterior derivative can be taken in the usual way

$$\delta\theta(\eta, \xi) = \left(\delta_\eta(\xi\frac{\partial L}{\partial x'}) - \delta_\xi(\eta\frac{\partial L}{\partial x'})\right) = \left(\xi\delta_\eta(\frac{\partial L}{\partial x'}) - \eta\delta_\xi(\frac{\partial L}{\partial x'})\right)$$

For the Lagrangian of classical mechanics, $\frac{\partial L}{\partial x'} = mx' = p$, then suggestively write $\frac{\partial L}{\partial x'}$ as p in general to get

$$\Omega = \delta\theta = \delta x \wedge \delta p.$$

This expression is independent of time when restricted to the critical locus which is derived from (2.1) by taking the variation δ and applying the Euler-Lagrange equations.

Specializing to the case of classical mechanics, restricting the critical locus is equivalent to restricting to a space (x_0, p_0) of initial conditions at some t_0. Since the 2-form Ω is time-independent, there is no privileged choice of t_0. In order to make contact with the Hamiltonian formalism, consider the solution map X_{t_0} from the space of initial conditions (x_0, p_0) at t_0 to the space of trajectories. This map acts as $X_{t_0}(x_0, p_0)(t) = x(t)$. This solution map will intertwine the exterior derivative on the space of trajectories δ with the exterior derivative d on phase space since pullbacks and exterior derivatives commute, and thus $X_{t_0}^*\Omega = \omega$, resulting in the same phase space obtained in the Hamiltonian formalism.

2.4 The Critical Locus

Since there is an isomorphism in the Hamiltonian formalism between phase space and the space of observables under the Poisson bracket, a similar isomorphism should be expected in the Lagrangian formalism. In order to talk about isomorphisms between spaces and their algebra of smooth functions, it is necessary to first introduce the formalism that allows us to both identify smooth manifolds with their algebra of smooth functions and describe the behavior of the algebras of smooth functions under intersection. We will mainly be following the presentation given in [Joy19] and [MR91].

2.4.1 C^∞-Rings

Algebraic objects are typically defined as sets equipped with operations satisfying certain relations. Groups, for example, are defined as a set G with a binary operation called multiplication, a unary operation called inversion, and a distinguished element e called the identity, satisfying certain relations such as associativity of multiplication.

Definitions in terms of operations and relations are suitable for many algebraic object of interest; however, consider the ring of smooth functions on some manifold M, denoted $C^\infty M$. On this ring, every smooth function $\phi : \mathbb{R}^n \to \mathbb{R}$ defines an n-ary operation $\Phi : (C^\infty M)^n \to C^\infty M$ given by $\Phi(f_1, f_2, f_3, \ldots, f_n)(x) = \phi(f_1(x), f_2(x), \ldots, f_n(x))$ of which there is an uncountable amount, and every relation between such ϕ, $F(\phi_1, \phi_2, \ldots, \phi_r) = 0$ defines a relation between the operators $\Phi_1, \Phi_2, \ldots, \Phi_r$ of which there is also an uncountable amount.

For example, let $\phi_1 : \mathbb{R}^2 \to \mathbb{R}$ be addition in $\mathbb{R}$, $\phi_2 : \mathbb{R}^2 \to \mathbb{R}$ be projection onto the first element, and $\phi_3 : \mathbb{R}^2 \to \mathbb{R}$ be projection onto the second element. Then, a relation $F(\phi_1, \phi_2, \phi_3) = \phi_1 - \phi_2 - \phi_3 = 0$ is obtained between their corresponding operations, $\Phi_1(f,g)(x) - \Phi_2(f,g)(x) - \Phi_3(f,g)(x) = f(x) + g(x) - f(x) - g(x) = 0$. Then, a definition that lifts operations and relations from an already existing set of objects and maps between them would be more suitable.

Definition 7 *A Lawvere theory is a category $\mathcal{C}$ with finite products, where every object $c \in Ob(\mathcal{C})$ is a finite product of a single generating object c_0.*

For C^∞ rings this is the category of cartesian spaces $\mathbb{R}^n$ with smooth functions between them.

Definition 8 *An algebra over a Lawvere theory is a functor $A : \mathcal{C} \to Set$ that preserves products.*

For the algebra of smooth functions on a manifold X, the functor A maps objects $\mathbb{R}^n$ to $C^\infty(X)^n$ and morphisms $\mathbb{R}^n \to \mathbb{R}^m$ to operations $C^\infty(X)^n \to C^\infty(X)^m$. In fact, there is a contravariant full subcategory inclusion.

Theorem 1 (Theorem 2.8 of [MR91]) *There is a contravariant full subcategory inclusion from the category of smooth manifolds into the category of C^∞ algebras mapping into finitely presented algebras, of the form of the quotient of the algebra of functions on $\mathbb{R}^k$ by a finitely generated ideal, $C^\infty(\mathbb{R}^k)/(f_1, \ldots, f_n)$, and sends transversal pullbacks to pushouts.*

2.4.2 Transversal Intersections

The construction relevant to the critical locus is the intersection of smooth manifolds and its corresponding algebra of smooth functions. It can be easily seen that the intersection of smooth manifolds $M_1 \cap M_2$ is generally not a smooth manifold; consider the intersection of the plane $z = 0$ with the saddle $z = x^2 - y^2$.

The condition for such an intersection to be well-defined is for the inclusion maps ι_{M_1}, ι_{M_2} of the factors to be transverse.

Definition 9 *Two maps $f : M_1 \to N$ and $g : M_2 \to N$ are called transverse if for all $y = f(x_1) = g(x_2)$, the span of their push-forwards is equal to the whole tangent space of N,*

$$\mathrm{im}(f_{x_1 *}) + \mathrm{im}(g_{x_2 *}) = T_y N.$$

Informally, this means that the images of f and g must not be tangent at any point in which they coincide.

This formalism can be used to describe the critical locus $C_{\delta S=0}$ purely algebraically. In classical mechanics, this space is finite dimensional and will be seen to be equivalent to phase space; however, the intermediate infinite-dimensional space of trajectories needs functional-analytic care which will be bypassed in the next chapter by working with jet bundles where the construction is finite-dimensional at every step. For the sake of exposition, suppose the space of trajectories is a finite-dimensional smooth manifold M.

The critical locus $C_{\delta S=0}$ is then the intersection of smooth manifolds. Note that transversality of the critical locus is not guaranteed in general, which will motivate the definition of the derived critical locus. Then, assuming transversality, $C^\infty(\Gamma_{\delta S} \cap 0) = C^\infty(\Gamma_{\delta S}) \otimes_{\infty, C^\infty(T^*M)} C^\infty(0)$. Computing this expression, the algebra of smooth functions on the critical locus is then

$$\begin{aligned} C^\infty(\Gamma_{\delta S}) \otimes_{\infty, C^\infty(T^*M)} C^\infty(0) = \frac{C^\infty(T^*M)}{(p_i - \delta_i S)} \otimes_{\infty, C^\infty(T^*M)} \frac{C^\infty(T^*M)}{(p_i)} \\ = \frac{C^\infty(T^*M)}{(p_i, \delta_i S)} = \frac{C^\infty(M)}{(\delta_i S)} \end{aligned}$$

where $\delta_i S$ are the components of δS, which states that the algebra of smooth functions on the critical locus are the smooth functions on trajectories modulo the equations of motion, as expected.

Then, the pullback via solution map X_{t_0} yields the same algebra of observables on phase space as in the Hamiltonian formalism, $C^\infty(q, p)$.

3. Classical Field Theory

Field theory is the theory of physical systems extended over the whole of space. Instead of dealing with a point particle concentrated at a single point in space with value the coordinate, fields can be thought of as uncountably many point particles spread over every point in space defining the value of the field at that point, varying in time. In this view, the range of values a field can hypothetically attain at any point in space is identical. Then, it is possible to describe the space of fields in terms of the range at a single point which is typically called a fiber F assembled into a geometric object called a fiber bundle E locally of the form of a product of a spacetime segment and a fiber, $U \times F$. Examples include the electromagnetic field, which assigns a 1-form at every point in space. It will be seen that many of the constructions of classical mechanics, specifically that of phase space and the algebras of observables, are almost directly generalizable to most field theories. The most consequential exception will be gauge field theories, whose study will take up the remaining chapters.

3.1 Fields

As mentioned in the previous chapter, the trajectory of a point particle is described as a smooth function $x : \mathbb{R} \to \Sigma$ from the time axis $\mathbb{R}$ to space Σ. The graph of this trajectory resides in the product space $\mathbb{R} \times \Sigma$, which means the trajectory can equivalently be defined as a function $x : \mathbb{R} \to \mathbb{R} \times \Sigma$ such that $\mathrm{pr}_1(x(t)) = t$, where pr_1 is the projection along the time axis. This description casts the trajectory of a point particle as a field over a spacetime that consists of a single point in space $M = \{*\} \times \mathbb{R} = \mathbb{R}$ taking values in Σ.

Definition 10 *A (smooth) bundle with fiber F over a spacetime M is a smooth manifold E with a projection map $p : E \to M$ such that, for every point $x \in M$ there exists a neighborhood $U \subset M$ and a diffeomorphism $\varphi : p^{-1}(U) \to U \times F$ called a local trivialization compatible with the projection onto U, $pr_1 \circ \varphi = p|_U$.*

Definition 11 *A field is a section of E, defined as a map $\sigma : M \to E$ compatible with projection, $pr(\sigma(x)) = x$. The set of all sections of a bundle E is denoted as ΓE.*

In the case of a point particle, $M = \mathbb{R}$, $F = \Sigma$, and $E = M \times F = \mathbb{R} \times \Sigma$. Although field theory can be developed in the Hamiltonian formulation, it is customary to use Lagrangian mechanics. In this case, the dynamics will be governed by an action functional on fields $S[\varphi]$. Just as the action functional on trajectories was written as an integral over spacetime $M = \{*\} \times \mathbb{R} = \mathbb{R}$ of a Lagrangian function $L(x, x', t)$, the action functional on fields will be the integral of Lagrangian function $L(\varphi, \varphi', \ldots, x)$ over arbitrary spacetime M,

$$S[\varphi] = \int_M L(\varphi, \varphi', \ldots, x) dV.$$

3.2 The Variational Bicomplex

In order to calculate the variation of this action functional, the formalism of the variational bicomplex will be introduced. We will mainly follow the exposition in [And89], [Kha14], and [GS25].

The fundamental insight is that instead of working in the infinite-dimensional space of all fields, it is possible to instead work in the so-called k-th order jet space of the fields of a field bundle, $J^k E$, locally.

Definition 12 *The jet space of order k, $J^k E$, is a bundle $p : J^k E \to M$ with fiber the germ of fields which match up to kth partial derivatives.*

For a bundle E that can be written as a product $F \times M$ with fiber coordinates ϕ^i and spacetime coordinates x_μ, it is thought of as the space $(x_\mu, \phi^i, \phi^i_{,\mu_1}, \phi^i_{,\mu_1 \mu_2}, \ldots)$ up to k lower indices, where $\phi^i_{,\mu_1 \cdots \mu_m}$ represents the coordinates of the mth partial derivatives of the field. This is a finite dimensional space which allows side-stepping the functional-analytic issues of dealing with infinite-dimensional spaces and disentangles the analytical relations between the field and its derivatives, allowing us to represent the space of fields and field derivatives that satisfy the equations of motion as a subset.

Definition 13 *Any field φ clearly determines its jet coordinates by taking partial derivatives, this map of sections $j_k : \Gamma E \to \Gamma J^k E$ is called the jet prolongation of order k.*

In coordinates, $j_k(\varphi^i) = (\varphi^i, \partial_\mu \varphi^i, \partial_\mu \partial_\nu \varphi, \ldots)$. This passage to the jet bundle is implicitly done in classical mechanics, where expressions involving the derivative of the Lagrangian with respect to the coordinate derivative $\frac{\partial L}{\partial \dot{q}}$ appear.

Then, if the Lagrangian L depends on fields and their derivatives up to order m, the Lagrangian can be represented as a function on the jet space $J^m E$. Then, the action functional can be written as the integral of the Lagrangian under the pull-back by the jet prolongation $j_m(\varphi) : M \to J^m E$,

$$S(\varphi) = \int_M j_m(\varphi)^* L dV = \int_M L(\varphi^i, \partial_\mu \varphi^i, \ldots) dV;$$

in fact, the quantity $\mathcal{L} = LdV$ can be thought of as a differential form on the jet space $J^m E$. However, the resulting Euler-Lagrange equations involve derivatives of order $2m$, so we set $k = 2m$ throughout and work on $J^k E$.

Since the jet bundle is a fiber bundle, and hence a (smooth) manifold, the space of differential forms on the jet bundle $\Omega(J^k(E))$ is well-defined, along with a grading provided by the de Rham differential d. The de Rham differential d acting on a function f on the jet bundle is, in local coordinates,

$$df = \frac{\partial f}{\partial x^i} dx^i + \frac{\partial f}{\partial \varphi^a} d\varphi^a + \frac{\partial f}{\partial \varphi^a_{,\mu}} d\varphi^a_{,\mu} + \frac{\partial f}{\partial \varphi^a_{,\mu\nu}} d\varphi^a_{,\mu\nu} + \ldots$$

The jet bundle being locally of the form $F \times U$ for some open $U \subset M$ suggests classifying differential forms by rank along the fibers F and along the space M.

Let $p : J^k E \to M$ be the projection map from the jet bundle to spacetime, then a vector field X is called vertical if it vanishes under the action push-forward, $p_*(X) = 0$.

Since the push-forward of the projection map along the spacetime factor is the identity by compatibility of the local trivialization with projection along this factor, this implies that X is locally of the form $X = X_i \partial_{\phi^i} + X_i^\mu \partial_{\phi^i_{,\mu}} + \ldots$.

Then, a differential form ω is naturally called horizontal if it vanishes along the interior product of every vertical vector field X, $\iota_X \omega = 0$, and vertical if the pull-back along the jet prolongation vanishes, $j_k(\varphi)^* \omega = 0$, for every field $\varphi \in \Gamma E$. A basis for vertical forms is given by the contact forms $\theta^a_{,\mu_1 \ldots \mu_n} = d\phi^a_{,\mu_1 \ldots \mu_n} - \phi^a_{,\mu_1 \ldots \mu_n \nu} dx^\nu$. The condition $j_k(\varphi)^* \theta = j_k(\varphi)^*(d\phi - \phi_{,\mu} dx^\mu) = d(j_k(\varphi)^* \phi) - \frac{\partial \varphi}{\partial x^\mu} dx^\mu = d\varphi - \frac{\partial \varphi}{\partial x^\mu} dx^\mu = 0$ can be seen as characterizing vertical differential forms as measuring the deviation from the analytical relations between φ and its derivatives, $d\varphi = \frac{\partial \varphi}{\partial x^\mu} dx^\mu$.

Define a bigrading of $\Omega(J^k E)$ by taking $\Omega^{r,s}$ to be the space of differential $(r+s)$-forms that contain the wedge product of s contact forms and vanish when acted on by $s+1$ interior products of vertical vector fields; the spaces $\Omega^{r,s}$ can be clearly seen to span the whole of $\Omega(J^k E)$ when represented in coordinates. The de Rham differential then acts on this bigrading as $d : \Omega^{r,s} \to \Omega^{r+1,s} \oplus \Omega^{r,s+1}$, and since the direct sum equals the direct product in the finite case, the de Rham d decomposes into a vertical differential $d_V : \Omega^{r,s} \to \Omega^{r,s+1}$, typically called the variational derivative δ, and a horizontal differential $d_H : \Omega^{r,s} \to \Omega^{r+1,s}$. It is sometimes convenient for calculation in coordinates to define the action of δ as the difference of the de Rham differential d and the horizontal derivative d_H. Under this bigrading, the contact forms are seen as the variational derivative of the fields $\theta^i = \delta \phi^i$.

In order to convert between differential forms on the finite-dimensional jet bundle $\Omega(J^kE)$ and differential forms on the horizontal spacetime and infinite-dimensional vertical space of fields $\Omega(M\times\Gamma E)$ it is useful to define the so-called prolonged evaluation function

$$\mathrm{ev}_k : M \times \Gamma E \to J^k E,$$

which evaluates fields up to their k-th derivatives $\mathrm{ev}_k(x,\varphi) = (\varphi^i(x), \partial_\mu\varphi^i(x), \ldots)$. For example, the pull-back of the contact form θ^i is

$$\mathrm{ev}_k^*\theta^i = \mathrm{ev}_k^* d\varphi^i - \frac{\partial\varphi}{\partial x^\mu}dx^\mu = \frac{\partial\varphi}{\partial x^\mu}dx^\mu + \delta\varphi^i - \frac{\partial\varphi}{\partial x^\mu}dx^\mu = \delta\varphi^i$$

where δ is the exterior derivative on ΓE.

Then, the action S is the integral of the horizontal top form $\mathcal{L} \in \Omega^{n,0}$ and the variation of the action $\delta S[\phi]$ can be evaluated

$$\delta S = \delta \int_M \mathrm{ev}_k^*\mathcal{L} = \int_M \mathrm{ev}_k^*\delta\mathcal{L}.$$

Since $\delta = d - d_H$ and $d_H\mathcal{L} = 0$,

$$\delta\mathcal{L} = (\frac{\partial L}{\partial\phi^i}\delta\phi^i + \frac{\partial L}{\partial\phi^i_{,\mu}}\delta\phi^i_{,\mu} + \ldots) \wedge dV.$$

Then notice that under the prolonged evaluation map, $\mathrm{ev}_k^*\delta\phi^i_{,\mu\nu\ldots} = \partial_{,\mu\nu\ldots}\delta\phi^i$, which suggests that $\delta\mathcal{L}$ can be written in the form $f(\delta\phi^i \wedge dV) \in \Omega^{n,1}$ by integration by parts as in the case of classical mechanics, $(\frac{\partial L}{\partial x} - \frac{\partial}{\partial t}\frac{\partial L}{\partial x'})\delta x$. In fact, this computation can be done purely homologically.

Theorem 2 (Corollary 3.2 of [And89]) *For any Lagrangian $\mathcal{L} \in \Omega^{n,0}$, there exist forms $\delta_{EL}\mathcal{L} \in \Omega^{n,1}$ and non-unique $\Theta \in \Omega^{n-1,1}$ called the Euler-Lagrange variation of $\mathcal{L}$ and the pre-symplectic potential respectively, such that the following 'integration-by-parts' formula holds*

$$\delta\mathcal{L} = \delta_{EL}\mathcal{L} - d_H\Theta.$$

In local coordinates,

$$\begin{aligned}\delta\mathcal{L} = \left(\frac{\partial L}{\partial\phi^i} - \partial_\mu\frac{\partial L}{\partial\phi^i_{,\mu}} + \partial_\mu\partial_\nu\frac{\partial L}{\partial\phi^i_{,\mu\nu}} - \cdots\right)\delta\phi^i \wedge \mathrm{dV} \\ + \left(\partial_\mu\frac{\partial L}{\partial\phi^i_{,\mu}} - \partial_\mu\partial_\nu\frac{\partial L}{\partial\phi^i_{,\mu\nu}} + \cdots\right)\delta\phi^i \wedge \mathrm{dV} \\ + \left(\frac{\partial L}{\partial\phi^i_{,\mu}}\,\delta\phi^i_{,\mu} + \frac{\partial L}{\partial\phi^i_{,\nu\mu}}\,\delta\phi^i_{,\nu\mu} + \cdots\right) \wedge \mathrm{dV}\end{aligned}$$

where $\partial_\mu f = \iota_{\partial_\mu} d_H f$. Notice that this formula is identical to (2.1) when considering Lagrangians $\mathcal{L}_1$ on the first-order jet bundle J^1E,

$$\begin{aligned}\delta\mathcal{L}_1 &= (\frac{\partial L}{\partial\phi^i} - \partial_\mu\frac{\partial L}{\partial\phi^i_{,\mu}})\delta\phi^i\wedge dV + (\partial_\mu\frac{\partial L}{\partial\phi^i_{,\mu}})\delta\phi^i\wedge dV + (\frac{\partial L}{\partial\phi^i_{,\mu}}\delta\phi^i_{,\mu})dV \\ &= (\frac{\partial L}{\partial\phi^i} - \partial_\mu\frac{\partial L}{\partial\phi^i_{,\mu}})\delta\phi^i\wedge dV + d_H(\frac{\partial L}{\partial\phi^i_{,\mu}}\delta\phi^i\wedge\iota_{\partial_\mu}dV).\end{aligned}$$

The solutions to the variational problem $\delta S[\phi] = 0$ are then the solutions to $\delta_{EL}\mathcal{L} = 0$ with boundary conditions imposed by $d_H\Theta$.

It is easy to see that the variational 1-form whose variational derivative was used to define the symplectic form is the $(n-1,1)$ form Θ, and in fact a similar derivation of a symplectic form occurs in the field theory case as well.

3.3 The Covariant Phase Space

The tangent bundle of fields $T\Gamma E$, along with the tangent bundle of fields that satisfy the equations of motion $T\,\Gamma E|_{\delta_{EL}\mathcal{L}=0}$ which the symplectic form acts on, can be defined using the vertical tangent bundle VE.

Definition 14 *The vertical tangent bundle $VE \to E$ of a bundle $p : E \to M$ is the subset of the tangent bundle $TE \to E$ formed by the kernel of the pushforward of the projection p,*

$$VE = \ker(p_* : TE \to TM) \hookrightarrow TE.$$

Notice the connection to the definition of vertical vector fields $p_*X = 0$. In fact, this is the bundle of vertical vector fields over E. The vertical cotangent space V^*E can be naturally defined as the dual to the vertical tangent space VE.

Definition 15 *The tangent space of fields $T_\varphi\Gamma E$ around a certain field $\varphi : M \to E$ is defined as the pullback*

$$T_\varphi\Gamma E = \Gamma(\varphi^* VE)$$

where $\varphi^*VE = V_{\varphi(x)}E$ for each $x \in M$, which pulls-back the bundle $VE \to E$ to spacetime $VE \to M$.

The above definition is interpreted as assigning a variation from the field $\varphi(x)$ at each point $x \in M$. The sections $\Gamma(\varphi^*VE)$ are then interpreted as the space of all possible variation from φ, which is naturally identified as $T_\varphi\Gamma E$.

Note then that δS can be written as a map $\Gamma E \times T\Gamma E \to \mathbb{R}$, just like a cotangent vector field $df : M \times TM \to \mathbb{R}$ on a manifold M.

Definition 16 *The tangent bundle to the space of fields that satisfy the Euler-Lagrange equations, $T_\varphi\, \Gamma E|_{\delta_{EL}\mathcal{L}=0}$, is defined as a subset of the tangent space $T_\varphi \Gamma E$, where the pushforward of the Euler-Lagrange equations acting on a tangent vector $Z \in T_\varphi \Gamma E$, $(ev_k^* \delta_{EL}\mathcal{L})_* Z = 0$ vanish.*

The equations $(ev_k^* \delta_{EL}\mathcal{L})_* Z = 0$ are called the linearized Euler-Lagrange equations, or Jacobi equations.

This is just as in the ordinary differential geometry of surfaces where the vanishing of the dot product of vectors $v \in T_pM$ with the gradient $\nabla f \cdot v = f_*(v) = 0$ defines the tangent space.

Definition 17 (Definition 7.12 of [GS25]) *The pull-back of the Euler-Lagrange variation of the Lagrangian along the prolonged evaluation map $ev_k^* \delta_{EL}\mathcal{L}$, typically called the EL operator, is a map from a field $\varphi \in \Gamma E$ to a variational 1-form V^*E times a volume form on M,*

$$\begin{aligned} EL(\varphi) &= (ev_k^* \delta_{EL}\mathcal{L})(\varphi) \\ &= (\frac{\partial L}{\partial \phi^i} - \partial_\mu \frac{\partial L}{\partial \phi^i_{,\mu}} + \ldots)(\varphi^i, \partial_\mu \varphi^i, \ldots)\delta\phi^i \wedge dV : \Gamma E \to \Gamma(\varphi^* V^* E \otimes \bigwedge^n T^* M). \end{aligned}$$

The pushforward of this map is of the form, for some curve ψ_t in ΓE with tangent vector $Z \in T_\varphi \Gamma E$,

$$\begin{aligned} EL_*(Z) &= \partial_t EL(\psi_t)|_{t=0} \\ &= \sum_{|I|\geq 0} \frac{\partial EL_i}{\partial \phi^j_{,I}}(j_k\varphi)\; \partial_t(\partial_I \psi^j_t)\Big|_{t=0} = (\sum_{|I|\geq 0} \frac{\partial EL_i}{\partial \phi^j_{,I}}(j_k\varphi)\partial_I Z^j)\delta\phi^i \wedge dV \end{aligned}$$

where EL_i are the components of the Euler-Lagrange operator.

These equations are linear, and can be seen as a higher dimensional generalization of the relations $f_*(v) = \nabla f \cdot v = 0$.

As in the Hamiltonian formulation of classical mechanics, in order to proceed with the derivation of the symplectic form it is necessary to first define a space of solutions.

Definition 18 *A Cauchy surface is an $(n-1)$-dimensional submanifold of spacetime $\Sigma \subset M$ such that the set of fields that satisfy the Euler-Lagrange equations $\Gamma E|_{\delta_{EL}\mathcal{L}=0}$ is isomorphic to the restriction to Σ of the fields of the jet prolongation satisfying the Euler-Lagrange equations*

$$\Gamma E|_{\delta_{EL}\mathcal{L}=0} \cong \left(\Gamma J^k E\Big|_{\delta_{EL}\mathcal{L}=0}\right)\Big|_\Sigma .$$

Informally, this means that a solution is uniquely determined by specifying the initial field values up to their k-th derivatives on some surface Σ. In the case of classical mechanics, $M = \mathbb{R}$ and Σ reduces to a choice of point (in time) at which to specify the initial position and momentum.

All field theories under consideration will have, or can be made to have via gauge fixing below, hyperbolic equations of motion and thus Cauchy surfaces. We will further assume that the spacetime M is a manifold without boundary, and neglect questions of convergence which can be remedied by restriction to spacelike compact fields. More on this in [Kha14]. The following section implicitly relies on the higher geometry of smooth sets to rigorously treat the space of fields, see [GS25] for a thorough account.

The variational derivative of the pre-symplectic potential $\Theta \in \Omega^{n-1,1}$ results in a variational 2-form $\Omega = \delta\Theta \in \Omega^{n-1,2}$ called the pre-symplectic current. The pullback along the evaluation map restricted to Σ, $\mathrm{ev}_{J^kE}|_\Sigma : \Sigma \times \Gamma J^kE\Big|_\Sigma \to J^kE$ yields the form $\mathrm{ev}_{J^kE}|_\Sigma^*\Omega \in \Omega^{n-1,2}(\Sigma \times \Gamma J^kE\Big|_\Sigma)$, which can then be integrated along Σ to obtain the symplectic form

$$\Omega^2(\Gamma J^kE|_\Sigma) \ni \omega = \int_\Sigma \mathrm{ev}_{J^kE}|_\Sigma^*\Omega$$

on the Cauchy surface Σ. This construction is usually called the transgression of Ω to the infinitesimal disk of Σ of order k, $\omega = \tau_\Sigma^k(\Omega)$. This terminology will become clear in the following chapters.

Definition 19 *The transgression of a differential form $\alpha \in \Omega^{p,q}$ to a p-dimensional surface $P \subset M$ is the differential form $\tau_P(\alpha) \in \Omega^q(\Gamma E|_P)$ defined using the prolonged evaluation map $\mathrm{ev}_k|_P : P \times \Gamma E|_P \to J^kE$*

$$\tau_P(\alpha) = \int_P \mathrm{ev}_k|_P^*\alpha$$

Definition 20 *The transgression of a differential form $\alpha \in \Omega^{p,q}$ to the infinitesimal disk of a p-dimensional surface $P \subset M$ of order k is the differential form $\tau_P(\alpha) \in \Omega^q(\Gamma J^kE|_P)$ defined using the evaluation map on J^kE restricted to P, $\mathrm{ev}_{J^kE}|_P : P \times \Gamma J^kE|_P \to J^kE$*

$$\tau_P^k(\alpha) = \int_P \mathrm{ev}_{J^kE}|_P^*\alpha$$

Restricting to field histories that satisfy the equations of motion,

$$\omega \in \Omega^2(\,\Gamma J^kE\Big|_{\delta_{EL}\mathcal{L}=0}\Big|_\Sigma) = \Omega^2(\,\Gamma E|_{\delta_{EL}\mathcal{L}=0})$$

turns the Cauchy surface Σ into a phase space for the field theory.

It is easily verified that this form is closed since transgression turns exterior derivatives on the space of fields into variational derivatives up to a sign,

$$d\omega = (-1)^n\tau_\Sigma^k(\delta\Omega) = (-1)^n\tau_\Sigma^k(\delta\delta\Theta) = 0.$$

It is also simple to verify that this construction does not depend on choice of Cauchy surface when restricting to solutions of the equations of motion [Kha14, Lemma 3.11].

3.3.1 Free Scalar Field

To show the above construction more concretely, consider the case of free scalar field theory on Minkowski space $M = (\mathbb{R}^{3,1}, \eta)$ with field bundle $E = M \times \mathbb{R} \to M$ and Lagrangian

$$\mathcal{L} = \frac{1}{2}(\eta^{\mu\nu}\phi_{,\mu}\phi_{,\nu} - m^2\phi^2)dV.$$

Note that

$$d_H\phi = d\phi - \delta\phi = d\phi - (d\phi - \phi_{,\mu}dx^\mu) = \phi_{,\mu}dx^\mu.$$

Then, the Euler-Lagrange variation of $\mathcal{L}$ is

$$\delta_{EL}\mathcal{L} = (\frac{\partial L}{\partial\phi} - \partial_\mu\frac{\partial L}{\partial\phi_{,\mu}})\delta\phi\wedge dV = -(\eta^{\mu\nu}\phi_{,\mu\nu} + m^2\phi)\delta\phi\wedge dV,$$

which reproduces the familiar Klein-Gordon equations.

The vertical tangent bundle of $E = M \times \mathbb{R} \to M$ consists of fields $f(x,\phi)\partial_\phi \in VE \subset TE$, which implies that a field $Z \in T_\varphi\Gamma E = \Gamma(\varphi^*VE)$ is a section of $\varphi^*VE = \varphi^*(\mathbb{R}\times\mathbb{R}\times M) = \mathbb{R}\times M = E$. This means that a variation of a scalar field φ consists of a function $f(x)$ from a point on spacetime M to a real number representing a scalar variation from φ, as expected.

The space of tangent vectors to the fields that satisfy the equations of motion $T\,\Gamma E|_{\delta_{EL}\mathcal{L}=0}$ are defined as the fields $Z \in T_\varphi\Gamma E = \Gamma(\varphi^*VE) = \Gamma E$ using the Jacobi equation

$$\begin{aligned}
\mathrm{EL}_*(Z) =& (\sum_{|I|\geq 0}\frac{\partial \mathrm{EL}_i}{\partial\phi^j_{,I}}(j_k\varphi)\partial_I Z^j)\delta\phi^i\wedge dV\\
=& ((\frac{\partial \mathrm{EL}_0}{\partial\phi}(j_k\varphi)Z) + (\frac{\partial \mathrm{EL}_0}{\partial\phi_{,i}}(j_k\varphi)\partial_i Z) + (\frac{\partial \mathrm{EL}_0}{\partial\phi_{,ij}}(j_k\varphi)\partial_i\partial_j Z))\delta\phi\wedge dV\\
=& (-m^2 Z + 0 - \eta^{ij}\partial_i\partial_j Z)\delta\phi\wedge dV\\
=& -(\eta^{\mu\nu}Z_{,\mu\nu} + m^2 Z)\delta\phi\wedge dV
\end{aligned}$$

which recover the same Klein-Gordon equations that the fields satisfy, as expected since the Klein-Gordon equation is linear.

The horizontal derivative of the associated pre-symplectic potential is of the form

$$d_H\Theta = (\partial_\mu\frac{\partial L}{\partial\phi^i_{,\mu}})\delta\phi\wedge dV + (\frac{\partial L}{\partial\phi^i_{,\mu}})\delta\phi_{,\mu}\wedge dV = (\eta^{\mu\nu}\phi_{,\mu\nu})\delta\phi\wedge dV + \eta^{\mu\nu}\phi_{,\mu}\delta\phi_{,\nu}\wedge dV$$

for which one arbitrary pre-symplectic potential is

$$\Theta = \eta^{\mu\nu}\phi_{,\mu}\delta\phi\wedge(\iota_\nu dV).$$

The pre-symplectic current is then

$$\Omega = \delta\Theta = \delta(\eta^{\mu\nu}\phi_{,\mu}\delta\phi \wedge (\iota_\nu dV)) = \eta^{\mu\nu}\delta\phi_{,\mu} \wedge \delta\phi \wedge (\iota_\nu dV).$$

One choice of Cauchy surface is the hyperplane at $t = 0$, $\Sigma = \{(x, y, z, 0) \mid x, y, z \in \mathbb{R}\}$. The transgression of the pre-symplectic current to a symplectic form on Σ evaluated along two tangent fields $\varphi, \psi \in T_\xi\, \Gamma E|_{\delta_{EL}L=0} = \Gamma E|_{\delta_{EL}L=0}$ is

$$\omega(\varphi, \psi) = \tau_\Sigma(\Omega)(\varphi, \psi) = \int_\Sigma (\mathrm{ev}_k|^*_\Sigma \Omega)(\varphi, \psi) = \int_\Sigma (\varphi_{,\mu}\psi - \varphi\psi_{,\mu})\eta^{\mu\nu}\iota_{\partial_\nu} dV$$

where $\varphi_{,\mu}\eta^{\mu\nu}\iota_{\partial_\nu}$ is the normal derivative, which in this case is the time-derivative

$$\omega(\varphi, \psi) = \int_\Sigma (\partial_t\varphi\psi - \varphi\partial_t\psi) dA = \int_\Sigma (p_\varphi\psi - \varphi p_\psi) dA.$$

3.4 Observables and the Peierls Bracket

Next, we discuss observables following [Kha14] and [FR12].

The observables of a field theory are functionals of the fields that satisfy the equations of motion, denoted as $(F : \Gamma E|_{\delta_{EL}\mathcal{L}=0} \to \mathbb{R}) \in \mathrm{Obs}(E, \mathcal{L})$.

Note that as in the case of classical mechanics, the algebra of observables is the algebra of functions on the phase space by definition, and the existence of this algebra of functions is intimately related to the existence of phase space.

We will refer to functionals of all sections of the field bundle as 'off-shell' observables $\mathrm{Obs}(E)$, where 'shell' is referring to the critical locus. It is possible to construct the set of observables explicitly, and restrict it enough to obtain a well-defined Poisson bracket structure.

Definition 21 *The point evaluation functionals $\Phi_x : \Gamma E \to \mathbb{R}$ defined as $\Phi_x(\varphi) = \varphi(x)$. Integrating these functionals against compactly supported distributions $f(x) \in D(M, \mathbb{R})$ are again off-shell observables called the linear observables $LinObs(E)$,*

$$A(\varphi) = (\int_M \Phi(x) f(x) dV)(\varphi) = \int_M \varphi(x) f(x) dV.$$

Note that the point evaluation observable is itself a linear observable

$$\Phi(y) = \int_M \delta(x - y)\Phi(x) dV.$$

Definition 22 *Polynomial off-shell observables $PolyObs(E)$ are observables of the form*

$$\sum_n \int_{M^n} \Phi(x_1)\Phi(x_2)\ldots\Phi(x_n) f_n(x_1, x_2, \ldots, x_n) d^nV$$

for integral kernels $f_n(x_1, x_2, \ldots, x_n) \in D(M^n, \mathbb{R})$.

Since the product of distributions is not well-defined in general, it is necessary to place restrictions on the polynomial observables to define a Poisson bracket. Although this will not be discussed in detail here, the correct condition is that the directions where the Fourier transform of each of the integral kernels f_n of a polynomial observable do not satisfy a particular decay condition (specified by the Paley-Wiener-Schwartz theorem) must not all be in the same future or past cone. Such observables are called the microcausal polynomial observables $\mathrm{PolyObs}_{\mu c}(E)$. More details in [FR12].

As in classical mechanics, the Poisson bracket is defined as the inverse of the symplectic form; however, the symplectic form in field theory acts on the infinite dimensional tangent space to the on-shell fields, which makes this operation difficult. Instead, we will follow Peierls's definition of a Poisson bracket on this space and use the relation to the inverse of the symplectic form to verify that it is the correct notion.

The basic idea of the Peierls-Poisson bracket $\{A, B\}(\Phi)$ is to calculate the change in A driven by a change of the action S by a small ('infinitesimal') perturbation of B. Explicitly, fix a solution $\Phi \in \Gamma E|_{\delta_{EL}\mathcal{L}=0}$ and consider a perturbation of the action $(S + \epsilon B)[\Phi]$. This shift in the action S will induce a perturbation of Φ, which we denote as $\Phi_\epsilon = \exp(\epsilon\delta_B)\Phi = \Phi + \epsilon\delta_B\Phi + \dots$, which can be calculated to first order via perturbation of the Euler-Lagrange equations

$$\begin{aligned}
0 =&\delta(S+\epsilon B)[\Phi+\epsilon\delta_B\Phi] + O(\epsilon^2)\\
=&\delta S[\Phi+\epsilon\delta_B\Phi] + \epsilon\delta B[\Phi+\epsilon\delta_B\Phi] + O(\epsilon^2)\\
=&\mathrm{EL}(\Phi) + \epsilon\mathrm{EL}_{*\Phi}(\delta_B\Phi) + \epsilon\delta B[\Phi] + O(\epsilon^2)\\
=&\epsilon(\mathrm{EL}_{*\Phi}(\delta_B\Phi) + \delta B[\Phi]) + O(\epsilon^2).
\end{aligned}$$

Collecting terms in ϵ, the equation for the perturbation of Φ to first order is obtained

$$\mathrm{EL}_{*\Phi}(\delta_B\Phi) = -\delta B[\Phi] \tag{3.1}$$

as the Jacobi equation with right-hand side $-\delta B[\phi]$.

Using the assumption of Green hyperbolic equations of motion in a globally hyperbolic spacetime, the operator $\mathrm{EL}_{*\Phi}$ as a differential operator admits Green's functions. Specifically, advanced and retarded Green's functions $G_{a,\Phi}$ and $G_{r,\Phi}$ which are supported in the past and future causal cones, respectively. The two solutions of 3.1 that are supported on the future and past causal cones are $\delta_B^a\Phi = -G_{a,\Phi}(\delta B[\Phi])$ and $\delta_B^r\Phi = -G_{r,\Phi}(\delta B[\Phi])$. Then, the Peierls-Poisson bracket is defined as

$$\{A, B\}(\Phi) = \delta A[\Phi](\delta_B^r\Phi - \delta_B^a\Phi).$$

Since the Green's functions can be expressed as integrals against corresponding kernels, called propagators, this is also written as

$$\{A, B\}(\Phi) = \int_M\int_M \left.\frac{\delta A}{\delta\phi^i}\right|_{\Phi(x)} (\Delta^{ij}_{r,\Phi}(x,y) - \Delta^{ij}_{a,\Phi}(x,y)) \left.\frac{\delta B}{\delta\phi^j}\right|_{\Phi(y)} dV_x dV_y$$

where the functional derivatives are defined similarly to the Euler-Lagrange derivatives $\delta_{EL}\mathcal{L}$, $\int_M \frac{\delta A}{\delta \phi^i}\Big|_{\Phi(x)} B^i(x)dV_x + \int_{\partial M} d\theta dV = \delta A[\Phi](B)$, where boundary terms are ignored and $\Delta^{ij}_{c,\Phi} = \Delta^{ij}_{r,\Phi}(x,y) - \Delta^{ij}_{a,\Phi}(x,y)$ is the causal propagator. However, note that this Poisson bracket is degenerate; any operator that shifts the action in a way that reproduces the linearized equations of motion, $\delta B[\Phi](\psi) = \int_M \frac{\delta B}{\delta \phi^i}\Big|_{\Phi(x)} \psi^i(x)dV_x = \mathrm{EL}_{*,\Phi}(\psi)$, is annihilated by the causal Green's function and thus $\{A,B\}(\Phi) = 0$ for all A. [Proposition 2.1 [Kha14]] Then, the set of observables on which the Peierls-Poisson bracket is non-degenerate and well-defined is the on-shell microcausal polynomial observables $\mathrm{PolyObs}_{\mu c}(E,S)$, obtained by the quotient of the microcausal polynomial observables by the image of the equations of motion. From the form of the Peierls-Poisson bracket given above, it is a simple calculation to show that the causal Green's function G_c inverts the symplectic form $\omega(G_c(\delta A[\Phi]),\psi) = \delta A[\Phi](\psi)$ and thus $\{A,B\} = \omega(G_c(\delta A), G_c(\delta B))$.

3.5 The Derived Critical Locus

The above construction is the Hamiltonian view of the Poisson algebra of observables, defining a phase space then inverting the symplectic form to obtain a Poisson bracket. Then, we should also expect a Lagrangian view of the Poisson algebra of observables as a tensor product of Lawvere algebras. We will follow [HT92], [AY23], and [CG21].

For the sake of demonstration, let the space of fields ΓE be a finite dimensional smooth manifold. This is equivalent to letting the space-time manifold be a point $M = \{*\}$, since E and ΓE would then simply be the fiber F. Then, the action functional is a smooth function $S : \Gamma E \to \mathbb{C}$ whose variation $\delta S : \Gamma E \times T\Gamma E \to \mathbb{C}$ is a section of the cotangent space $\Omega^{0,1}(\Gamma E) = T^*\Gamma E$. The graph of the cotangent vector field δS is a subset of the cotangent space $T^*\Gamma E$, whose intersection with the graph of the zero section is the critical locus $C_{\delta S=0}$. The smooth algebra of functions on the critical locus can then be expressed, since the graphs are both finite-dimensional smooth manifolds, as the tensor product of smooth functions on the graph of δS and the zero section,

$$C^\infty(C_{\delta S=0}) = C^\infty(\Gamma_{\delta S}) \otimes_{\infty, C^\infty(T^*\Gamma E)} C^\infty(\Gamma_0) = \frac{C^\infty(\Gamma E)}{(\mathrm{EL}_i)}$$

with the usual assumption of transversality of the intersection. These are identified with the on-shell observables $\mathrm{Obs}(E,S)$ with Poisson bracket the Peierls bracket above.

The above construction can be adapted for an infinite-dimensional space of fields, working locally using the jet bundle. Since the Lagrangian L depends on the fields and their derivatives up to order m, the graph of δS is a subset of the finite-dimensional bundle $\Omega^{0,1}$, locally modeling $V^*\Gamma E$. Then, the smooth functions on the critical locus in the jet space to order $k = 2m$, $C^k_{\delta S=0}$, are

$$C^\infty(C^k_{\delta S=0}) := C^\infty(\Gamma_{\delta S}) \otimes_{\infty, C^\infty(\Omega^{0,1})} C^\infty(\Gamma_0) = \frac{C^\infty(J^k E)}{(\mathrm{EL}_{i,\mathrm{jet}})}$$

where we use the notation $\mathrm{EL}_{i,\mathrm{jet}}$ for the components of $\delta_{EL}\mathcal{L}$.

The above algebra of functions is called the algebra of local functionals of order k. In order to form the full algebra of local functions on the critical locus, this algebra needs to be generalized to arbitrary order jets which requires that the ideal $(\mathrm{EL}_{i,\mathrm{jet}})$ be expanded to include derivatives of the equations of motion. Instead of talking about jets to arbitrary order m, denote by $J^\infty E$ the unique infinite-dimensional jet bundle of arbitrary order on E which agrees with the projections to lower-order jet bundles $J^\infty E \to J^m E$, explicitly formed as a projective limit in the opposite category of C^∞ algebras.

Then, define the prolonged shell $C^\infty_{\delta S=0}$ as

$$C^\infty(C^\infty_{\delta S=0}) := \frac{C^\infty(J^\infty E)}{(\mathrm{EL}_{i,\mathrm{jet}}, d_\mu \mathrm{EL}_{i,\mathrm{jet}}, d_\mu d_\eta \mathrm{EL}_{i,\mathrm{jet}}, \ldots)},$$

explicitly as a colimit of algebras of local functionals

$$C^\infty(C^\infty_{\delta S=0}) = \varinjlim_m \frac{C^\infty(J^m E)}{I_m}, \qquad I_m := \Big(\mathrm{EL}_{i,\mathrm{jet}},\, d_\mu \mathrm{EL}_{i,\mathrm{jet}},\, \ldots\Big) \cap C^\infty(J^m E).$$

This algebra can be transgressed to a subset of local observables on the global space of fields. We write

$$C^\infty(C_{\delta S=0}) \supseteq C^\infty(C_{\delta S=0})_{\mathrm{loc}} = \tau_M(C^\infty(C^\infty_{\delta S=0}) dV)$$

for these observables. Note that for manifolds without boundary, the kernel of the transgression of local functionals are the exact forms $\tau_M((d_H F) dV) = 0$, thus

$$C^\infty(C_{\delta S=0})_{\mathrm{loc}} \;=\; \frac{C^\infty(C^\infty_{\delta S=0})\, dV}{d_H\Big(\Omega^{n-1,0}(C^\infty_{\delta S=0})\Big)} \;=\; H^n_{d_H}\Big(C^\infty(C^\infty_{\delta S=0})\Big), \qquad n = \dim M.$$

The problem with the above construction comes from the transversality condition, which excludes virtually all fields with infinitesimal symmetries explained below. Without the transversality condition, it is not possible to guarantee that the resulting space is a smooth manifold. Then, it is necessary to move to a larger category of spaces. The requirement that this category of spaces be the unique minimal (freest) closure of the category of smooth manifolds under intersections forces us to move to derived manifolds, as shown in [CS19], defined by certain differential graded C^∞ algebras.

3.5.1 Derived Manifolds

In the finite-dimensional case, we will follow the formulation of derived geometry in terms of differential graded algebras developed by Carchedi and Roytenberg [Car23] and Taroyan [Tar23] as opposed to the usual presentation in terms of homotopy algebras.

Definition 23 *A graded algebra $A^\bullet$ over a commutative ring R is a sequence of R-modules*

$$A^\bullet = \bigoplus_{n\in\mathbb{Z}} A^n$$

where $1 \in A^0$ and whose product respects grading $A^n \cdot A^m \subseteq A^{n+m}$. It is called commutative if for any $a \in A^n$ and $b \in A^m$,

$$ab = (-1)^{nm} ba.$$

Definition 24 *A commutative differential graded algebra (CDGA) $A^\bullet$ is a graded algebra equipped with a function $d : A^\bullet \to A^\bullet$ of degree $+1$, where $a \in A^n \implies da \in A^{n+1}$, that squares to zero, $d^2 = 0$, and satisfies a graded Leibniz rule; for any $a \in A^n$ and $b \in A^m$,*

$$d(ab) = (da)b + (-1)^n a db.$$

For example, the algebra of differential forms $\Omega(M)$ on a smooth manifold M is a commutative differential graded algebra over the real numbers $\mathbb{R}$ with product the wedge product $\wedge$, and differential the usual exterior differential d. Just as in the case of differential forms, it is possible to similarly define a cohomology for an arbitrary commutative differential graded algebra $A^\bullet$.

Definition 25 *The n-th cohomology group $H^n(A)$ of $A^\bullet$ is the quotient of closed elements (cocycles) by exact elements (coboundaries)*

$$H^n(A) = \frac{ker(d : A^n \to A^{n+1})}{im(d : A^{n-1} \to A^n)}.$$

Then, it is possible to define isomorphisms between CDGAs as weak equivalences.

Definition 26 *A morphism of CDGAs is a function $f : A^\bullet \to B^\bullet$ that maps elements degree-wise, $f(A^n) \subseteq B^n$, intertwines the differentials of $A^\bullet$ and $B^\bullet$, $f \circ d_A = d_B \circ f$, and preserves multiplication $f(a \cdot_A b) = f(a) \cdot_B f(b)$.*

This immediately induces maps of cohomology groups $f_n^* : H^n(A) \to H^n(B)$.

Definition 27 *A (quasi-)isomorphism of CDGAs is a morphism of CDGAs that induces isomorphisms of cohomology groups for all n.*

This defines the category cdgAlg of commutative differential graded algebras. More precisely, this category is formed by simplicial localization but this will not be explained further in what follows.

It is possible to specialize the above commutative differential graded algebras to commutative differential graded C^∞-algebras by requiring that the A^0 be a C^∞-algebra, and that the differential d be compatible with any operation.

Definition 28 *A commutative differential graded C^∞ algebra is a CDGA such that A^0 be a C^∞-algebra, and for any $\varphi : (A^0)^n \to A^0$,*

$$d\varphi(a_1, a_2, \ldots, a_n) = \sum_i \frac{d\varphi}{dx_i}(a_1, a_2, \ldots, a_n) da_i.$$

It is then possible to similarly define the category of commutative differential graded C^∞ algebras, cdgC^∞Alg. The theory of non-negatively graded ($A^\bullet = \bigoplus_{n\geq 0} A^n$) CDG$C^\infty$As and non-positively graded ($A^\bullet = \bigoplus_{n\leq 0} A^n$) CDG$C^\infty$As are quite different, the former describes deformations and the latter derived manifolds. Both will play an important role below. For the rest of this chapter, we will concentrate on non-positively graded CDGC$^\infty$As, and denote the category by cdgC$^\infty$Alg$^{\leq 0}$.

Every smooth manifold M can be seen as a CDGC$^\infty$A where the algebra of functions $C^\infty(M)$ is the component in zeroth order, the components in non-zero order are all trivial, and the differential is the zero function.

Definition 29 *A differential graded smooth manifold is a smooth manifold M with a graded algebra of functions $\mathcal{O}(M)$ equipped with a vector field Q of degree $+1$, meaning $Q\mathcal{O}(M)^k \subseteq \mathcal{O}(M)^{k+1}$, that squares to zero $Q^2 = [Q, Q] = 0$. This means the algebra of functions is a CDGC$^\infty$A. We will call differential graded manifolds with non-positively graded algebras of functions derived manifolds, in which smooth manifolds form a full subcategory.*

We also fix the notation for shifting a general differential complex as follows.

Definition 30 *A shift of a differential complex X with components of degree n, X_n, by an integer i is the complex $X[i]$ with components $X[i]_n = X_{n+i}$. That is, the complex is shifted back by i spaces.*

Note that the algebra of functions of a differential-graded manifold is the algebras of duals on this space, which involves negating degrees. That is, the algebra of functions $C^\infty(X[1])$ on the shifted space $X[1]$ is generated by coordinate functions of degree 1.

Not every CDGC$^\infty$A defines a derived manifold; however, there is contravariant full subcategory inclusion of derived manifolds into cdgC$^\infty$Alg$^{\leq 0}$ just as in the case of smooth manifolds and C^∞ algebras. We will denote the category of derived manifolds as dMan.

Note that the category $\mathrm{cdgC}^\infty\mathrm{Alg}^{\leq 0}$ with the usual tensor product does not automatically resolve the issue of transversality, since for purely degree zero algebras it is equivalent to the C^∞ tensor product. Instead, it is necessary to impose further finiteness conditions and size restrictions on this category of differential graded algebras in order to obtain the correct notion of a derived tensor product, denoted by $\otimes^{\mathbb{L}}$.

Definition 31 *A non-positively graded commutative differential graded algebra is bounded if $A^k = 0$ for all k less than some $n \in \mathbb{Z}$.*

Definition 32 *A non-positively graded CDGC$^\infty$A is semi-free if as a graded algebra it is of the form $A^\bullet \cong C^\infty(M)[\xi_1, \xi_2, \ldots]$ for some finite-dimensional smooth manifold M and generators ξ_i.*

Definition 33 *A non-positively graded CDGC$^\infty$A is finitely-generated if it is semi-free and there are only a finite number of generators ξ_i.*

Let $\mathrm{sfcdgC}^\infty\mathrm{Alg}_f^{\leq 0}$ denote the category of bounded finitely-generated semi-free commutative differential graded C^∞ algebras in non-positive degrees. When M is restricted to cartesian spaces $\mathbb{R}^n$, these algebras are of the form $C^\infty(\mathbb{R}^n)[\xi_1, \xi_2, \ldots, \xi_k]$ of smooth functions on $\mathbb{R}^n$ adjoined with graded generators. The spaces these algebras represent are called derived cartesian spaces dCart which form a full subcategory of derived manifolds dMan. These spaces fill the same role for derived manifolds that cartesian spaces $\mathbb{R}^n$ fill for smooth manifolds, where derived manifolds can be represented by the gluing of local charts of derived cartesian spaces. [Tar23, Theorem 6.30] Note that the full subcategory inclusion from dCart to $\mathrm{sfcdgC}^\infty\mathrm{Alg}_f^{\leq 0}$ sends intersections of derived cartesian spaces to the tensor product of their corresponding algebras.

In order to talk about the intersection of (derived) manifolds in this formulation, it is necessary to find algebras in $\mathrm{sfcdgC}^\infty\mathrm{Alg}_f^{\leq 0}$ that are (quasi-)isomorphic to the algebra of functions of each local chart, which are generally algebras in $\mathrm{cdgC}^\infty\mathrm{Alg}^{\leq 0}$. For a general CDGA A, a (quasi-)isomorphic semi-free replacement is called the semi-free resolution $\tilde{A}$ of A, and is always possible to define, and is additionally finitely generated and bounded for finite-dimensional derived cartesian spaces. The derived cartesian spaces obtained from the tensor product of these algebras are then glued together to form the resulting intersection.

Consider the following example of the intersection in $\mathbb{R}^3$ of a plane $P : z = 0$ and a saddle surface $Q : z = x^2 - y^2$. The ordinary intersection $I = P \cap Q : x^2 = y^2$ is not a smooth manifold, since any punctured neighborhood of the origin forms four connected components.

As derived manifolds, these surfaces can be defined using a single derived coordinate chart. The algebra of functions of P is

$$C^\infty(P) = C^\infty(\mathbb{R}^3)/(z)$$

which has a semi-free resolution

$$\tilde{\mathcal{O}}(P) = C^\infty(\mathbb{R}^3)[\xi]$$

with ξ in degree -1 and differential $d\xi = z$. Note that $\xi^2 = (-1)^{(-1)\cdot(-1)}\xi\xi = -\xi^2$ which ensures that this semi-free resolution is bounded,

$$0 \longrightarrow C^\infty(\mathbb{R}^3)\,\xi \xrightarrow{d} C^\infty(\mathbb{R}^3) \longrightarrow 0.$$

Similarly, the semi-free resolution of the algebra of functions of Q is given by

$$\tilde{\mathcal{O}}(Q) = C^\infty(\mathbb{R}^3)[\eta]$$

with η in degree -1 and differential $d\eta = z - x^2 + y^2$.

The algebra of functions of the derived intersection $\tilde{\mathcal{O}}(I)$ is straightforwardly given by tensor product of these two algebras

$$\tilde{\mathcal{O}}(I) = C^\infty(\mathbb{R}^3)[\xi] \otimes^{\mathbb{L}}_{C^\infty(\mathbb{R}^3)} C^\infty(\mathbb{R}^3)[\eta] = C^\infty(\mathbb{R}^3)[\xi, \eta]$$

with ξ and η in degree -1 equipped with the differential $d\xi = z$ and $d\eta = z - x^2 + y^2$,

$$0 \longrightarrow C^\infty(\mathbb{R}^3)\,\xi\eta \xrightarrow{d} C^\infty(\mathbb{R}^3)\,\xi \oplus C^\infty(\mathbb{R}^3)\,\eta \xrightarrow{d} C^\infty(\mathbb{R}^3) \longrightarrow 0.$$

The cohomology groups of this differential-graded algebra are

$$H^0(\tilde{\mathcal{O}}(I)) = \frac{C^\infty(\mathbb{R}^3)}{(z, z - x^2 + y^2)}$$

which is the ordinary intersection, and vanishing cohomology groups H^{-1} and H^{-2}, which makes $\tilde{\mathcal{O}}(I)$ a semi-free resolution of $C^\infty(I)$ as expected.

The (semi-)free resolution of the quotient of an algebra R by the ideal generated by central elements $f_1, \ldots, f_n$ in the manner described above is called the Koszul complex. Since the algebras under consideration are commutative, every element of R is a central element.

Definition 34 ([Wei94]) *Given elements $f_1, \ldots, f_n \in R$, the Koszul complex is the tensor product of differential graded algebras $K(f_1, \ldots, f_n) = K(f_1) \otimes^{\mathbb{L}}_R \ldots \otimes^{\mathbb{L}}_R K(f_n)$ where $K(f_k)$ is the differential graded algebra $R[\xi_k]$ with ξ_k in degree -1 and differential $d\xi_k = f_k$.*

In order to guarantee that this semi-free Koszul complex is a resolution of the quotient, the elements f_k must form a regular sequence.

Definition 35 *The elements f_k form a regular sequence if each element f_k is not a zero-divisor, where $f_k a = 0$ for some non-trivial a, in any quotient $R/(f_1, \ldots, f_k)$ of R unless $R/(f_1, \ldots, f_k) = 0$.*

The necessity of this requirement can be made clear using the example of the derived intersection of the plane P with itself. The resulting derived algebra is similar to the above, however with differential $d\xi = z$ and $d\eta = z$. The zeroth cohomology group $H^0(\tilde{\mathcal{O}}(I))$ is again the ordinary intersection $C^\infty(\mathbb{R}^3)/(z)$. For the -1st cohomology group; a closed element x is one that satisfies $dx = d(a\xi + b\eta) = az + bz = (a+b)z = 0$, which implies $x = a(\xi - \eta)$, an exact element y is of the form $d(a\xi\eta) = az\eta - a\xi z = az(\eta - \xi)$. The quotient is then $H^{-1}(\tilde{\mathcal{O}}(I)) = \frac{C^\infty(\mathbb{R}^3)(\xi-\eta)}{(z)(\xi-\eta)} = \frac{C^\infty(\mathbb{R}^3)}{(z)} = H^0(\tilde{\mathcal{O}}(I))$ which shows that the Koszul complex in this case is not a (semi-)free resolution.

3.5.2 The Koszul Complex of the Critical Locus

We apply this formalism to the case of the critical locus of the action S in the finite-dimensional case. Given a good open cover $\{U_i\}$ of the space of fields ΓE such that the cotangent bundle $p : T^*\Gamma E \to \Gamma E$ has local trivializations $\varphi|_{U_i} : p^{-1}(U_i) \to U_i \times F$, with local coordinate (co)frame $(e_1^{*i}, \ldots, e_n^{*i}) = (\delta\phi_1^i, \ldots, \delta\phi_n^i)$, the variational derivative of the action δS as a section $s : \Gamma E \to T^*\Gamma E$, is locally of the form

$$s|_{U_i} = \sum_{k=1}^{n} \mathrm{EL}_k^i \delta\phi_k^i.$$

Then, the Koszul complex for each $\{U_i\}$ is of the form

$$K(\mathrm{EL}_1^i, \ldots, \mathrm{EL}_n^i) = C^\infty(U_i)[\xi_1^i, \ldots, \xi_n^i]$$

with differential $d_i\xi_k^i = \mathrm{EL}_k^i$,

$$0 \longrightarrow C^\infty(U_i)\, \xi_1^i \cdots \xi_n^i \xrightarrow{d_i} \cdots \xrightarrow{d_i} \bigoplus_{j<k} C^\infty(U_i)\, \xi_j^i \xi_k^i$$
$$\xrightarrow{d_i} \bigoplus_{k=1}^{n} C^\infty(U_i)\, \xi_k^i \xrightarrow{d_i} C^\infty(U_i) \longrightarrow 0.$$

Note that in the finite-dimensional case, $C^\infty(U_i) = C^\infty(\phi_1^i, \ldots, \phi_n^i)$.

These local complexes assemble into a global derived manifold explicitly by elevating the -1-graded generators to global objects on ΓE. The local differential d_i acting on the ξ^i_k to produce smooth functions on the U_i can be globally modeled by letting the ξ^i_k represent the local frame $(e^i_1 \dots, e^i_n) = (\frac{\delta}{\delta\phi^i_1}, \dots, \frac{\delta}{\delta\phi^i_n})$ for the dual of $T^*\Gamma E = T\Gamma E$, the tangent bundle on the space of fields, and letting the differential d_i act by contraction of s, $d_i\xi^i_a = \iota_s \frac{\delta}{\delta\phi^i_a} = \frac{\delta}{\delta\phi^i_a}(\sum_{k=1}^n \mathrm{EL}^i_k \delta\phi^i_k) = \mathrm{EL}^i_a$. The resulting algebra of this derived manifold is then of the form $\mathrm{Sym}_{C^\infty(\Gamma E)}(\Gamma(T\Gamma E)[1]) = \Gamma(\Lambda^\bullet T\Gamma E)$, the sections of the exterior algebra of tangent space called polyvector fields, with differential the contraction ι_s with the cotangent vector field s,

$$\begin{aligned} 0 \longrightarrow \Gamma\Big(\Lambda^n T\Gamma E\Big) \xrightarrow{\iota_s} \Gamma\Big(\Lambda^{n-1} T\Gamma E\Big) \xrightarrow{\iota_s} \cdots \\ \xrightarrow{\iota_s} \Gamma\Big(\Lambda^2 T\Gamma E\Big) \xrightarrow{\iota_s} \Gamma\Big(T\Gamma E\Big) \xrightarrow{\iota_s} C^\infty(\Gamma E) \longrightarrow 0. \end{aligned}$$

In the case of field theory, the -1-graded generators ξ^i_k for the global Koszul complex are typically called anti-field generators and denoted ${\phi^\ddagger_k}^i$, matching the coordinate functions on U_i denoted as ϕ^i_k in the finite-dimensional case.

The infinite-dimensional case can be handled locally using the jet bundle $J^kE \to M$, where the jet coordinates $(\phi^i_k, \phi^i_{k,\mu}, \dots, \phi^i_{k,\mu_1\cdots\mu_k})$ replace the finite-dimensional coordinate functions ϕ^i_k, and the algebra $C^\infty(U_i)$ becomes the finitely-generated algebra

$$C^\infty(J^kE|_{U_i}) = C^\infty\Big(\phi^i_k,\, \phi^i_{k,\mu},\, \dots,\, \phi^i_{k,\mu_1\cdots\mu_k}\Big)_{k=1,\dots,n}.$$

The components of the Euler–Lagrange operator EL^i_k are smooth functions on $J^kE|_{U_i}$ and the local Koszul complex becomes

$$K(\mathrm{E}^i_1, \dots, \mathrm{E}^i_n) = C^\infty(J^kE|_{U_i})[{\phi^\ddagger_1}^i, \dots, {\phi^\ddagger_n}^i].$$

Globally, these assemble into the polyvector-field complex on the jet bundle,

$$\Gamma\Big(\Lambda^\bullet VJ^kE\Big) \;=\; \mathrm{Sym}_{C^\infty(J^kE)}\Big(\Gamma(VJ^kE)[1]\Big),$$

with differential the contraction ι_s expressed locally, $s = \sum_k \mathrm{EL}_{k,\mathrm{jet}}\delta\phi_k$. Further define jet coordinates of the anti-field generators also in degree -1,

$${\phi^\ddagger}^i_{k,\mu_1\dots\mu_n} = \partial_{\mu_1\dots\mu_n}\frac{\delta}{\delta\phi^i_k}$$

such that

$$\iota_s{\phi^\ddagger}^i_{k,\mu_1\dots\mu_n} = \frac{\partial}{\partial x_{\mu_1}}\cdots\frac{\partial}{\partial x_{\mu_n}}\mathrm{E}^i_k.$$

Then, passing to jet bundles of higher order is possible by the addition of a sufficient number of coordinate derivatives of the anti-field generators. Note that adjoining an infinite number of coordinate derivatives of the anti-field generators would make the above Koszul complex no longer represent the algebra of functions on a derived manifold but a more general derived space defined as the projective limit of derived manifolds as in the non-derived case explained above.

3.5.3 The BV Antibracket

We will follow the exposition in [CMR14], [HT92], and [Sch] for the rest of the chapter.

This Koszul complex above is the algebra of functions of the shifted cotangent space $V^*[-1]J^kE$, modeling $T^*[-1]\Gamma E$. The cotangent space structure allows the definition of a natural pre-symplectic current, locally of the form $\Omega_{\mathrm{BV}} = \sum_a (\delta\phi_a^\ddagger \wedge \delta\phi_a)dV$.

The corresponding bracket obtained by taking inverses of the pre-symplectic current Ω_{BV} is called the local BV anti-bracket $(-,-)$.

Definition 36 *The local BV anti-bracket $(-,-)$ is the bracket corresponding to the BV pre-symplectic current Ω_{BV} acting on local generators as $(\phi_i dV, \phi_j^\ddagger dV) = \delta_{ij}dV$ and extended to arbitrary local functionals $A[\phi_i, \phi_i^\ddagger]dV$, $B[\phi_i, \phi_i^\ddagger]dV$ as*

$$(AdV, BdV) = \sum_i \left(\frac{\delta^R A}{\delta\phi_i}\frac{\delta^L B}{\delta\phi_i^\ddagger} - \frac{\delta^L A}{\delta\phi_i^\ddagger}\frac{\delta^R B}{\delta\phi_i}\right)dV$$

where $\frac{\delta^R A}{\delta\phi_i}$ and $\frac{\delta^L A}{\delta\phi_i}$ are defined as

$$\begin{aligned}\delta A[\Phi](B) = \int_M \left.\frac{\delta^R A}{\delta\phi^i}\right|_{\Phi(x)} B^i(x)\, dV_x &= \int_M (-1)^{|\phi^i|(|A|+|\phi^i|)} B^i(x) \left.\frac{\delta^R A}{\delta\phi^i}\right|_{\Phi(x)} dV_x \\ &= \int_M B^i(x) \left.\frac{\delta^L A}{\delta\phi^i}\right|_{\Phi(x)} dV_x.\end{aligned}$$

The differential $\iota_{\delta S}$ in the above Koszul complex can be expressed in terms of the BV anti-bracket by bracketing against the Lagrangian $\mathcal{L} = LdV$,

$$(\mathcal{L}, \phi_a^\ddagger dV) = \sum_i \left(\frac{\delta^R L}{\delta\phi_i}\frac{\delta^L \phi_a^\ddagger}{\delta\phi_i^\ddagger} - \frac{\delta^L L}{\delta\phi_i^\ddagger}\frac{\delta^R \phi_a^\ddagger}{\delta\phi_i}\right)dV = \frac{\delta^R L}{\delta\phi_a}dV = \iota_{\delta S}\phi_a^\ddagger dV,$$

and we write $\iota_s = (\mathcal{L}, -)$.

Passing to the global space of fields, the pre-symplectic current transgresses over spacetime to a symplectic form

$$\omega_{\mathrm{BV}} = \tau_M(\Omega_{\mathrm{BV}}) = \int_M \sum_a \mathrm{ev}_k^*(\delta\phi_a^\ddagger \wedge \delta\phi_a)dV.$$

The variation of the action δS can be expressed using the BV symplectic form ω_{BV} by defining a vector field of degree 1 as

$$Q = \int_M \sum_i \mathrm{EL}_{i,\mathrm{jet}} \frac{\delta}{\delta\phi_i^\ddagger}$$

which results in

$$\iota_Q\omega_{BV} = \int_M \sum_i \mathrm{EL}_{i,\mathrm{jet}}\delta\phi_i = \delta S$$

ignoring boundary terms. This makes Q the Hamiltonian vector field corresponding to the function S.

The local BV anti-bracket can be transgressed to a BV anti-bracket $\{-,-\}$ on local observables of the global space of fields by stating

$$\{\tau_M(AdV), \tau_M(BdV)\} = \tau_M((AdV, BdV)).$$

Then, the relation between the Lagrangian $\mathcal{L}$ and the Koszul differential ι_s transgresses to a relation between the action $S = \tau_M(\mathcal{L})$ and a global differential, $\{S, \tau_M(\phi_a^\ddagger dV)\} = \tau_M(\iota_s \phi_a^\ddagger dV) = \tau_M(\mathrm{EL}_{a,\mathrm{jet}} dV) = \delta S(\frac{\delta}{\delta \phi_a})$ as expected. To make the global differential in the above formula explicit, notice that for a general local functional $A[\phi_i, \phi_i^\ddagger]dV$ with $F = \tau_M(A)$,

$$\begin{aligned}\{S, \tau_M(AdV)\} = \tau_M((\mathcal{L}, AdV)) = \int_M \sum_i \mathrm{EL}_i \frac{\delta^L F}{\delta \phi_i^\ddagger} dV \\ = \delta F(\int_M \sum_i \mathrm{EL}_i \frac{\delta}{\delta \phi_i^\ddagger} dV) = \delta F(Q) = L_Q F.\end{aligned}$$

Therefore,

$$\{S, \tau_M(-)\} = \tau_M \circ \iota_s = L_Q \circ \tau_M.$$

The relation $(\iota_{\delta S})^2 = 0$ of the differential then corresponds to the relation $(\mathcal{L}, \mathcal{L}) = 0$ of the Lagrangian and the relation $\{S, S\} = 0$ of the action called the classical master equation. In its current form, it is trivially satisfied since the action and the Lagrangian are always independent of the antifield coordinates.

3.6 The Classical BV-BFV Correspondence

The relationship between the BV formalism and the Hamiltonian formalism above goes beyond simply equating their (local) algebras of functions under the Peierls bracket, the Hamiltonian formalism can be obtained as a boundary theory to the bulk BV formalism above.

The above discussion neglected the effect of boundary terms by requiring that spacetime have no boundary $\partial M = \emptyset$. Then, consider a spacetime M with boundary ∂M. Recall that the variation of the Lagrangian $\delta\mathcal{L}$ is of the form

$$\delta\mathcal{L} = \delta_{EL}\mathcal{L} - d_H \Theta.$$

The second variation of the Lagrangian imposes a relation between the pre-symplectic current and the equations of motion

$$0 = \delta^2 \mathcal{L} = \delta(\delta_{EL}\mathcal{L}) + d_H \Omega \implies d_H \Omega = -\delta(\delta_{EL}\mathcal{L})$$

which implies that the pre-symplectic current is closed on-shell. Notice that

$$\delta(\delta_{EL}\mathcal{L}) = \sum_i \delta \mathrm{EL}_{i,\mathrm{jet}}\delta\phi_i dV = \iota_s \sum_i \delta\phi_i^{\ddagger} \wedge \delta\phi_i dV = -\iota_s \Omega_{\mathrm{BV}},$$

which leads to the relation

$$d_H\Omega = \iota_s \Omega_{BV}$$

called the local BV-BFV relation. Transgressing to the spacetime M, the horizontal exterior derivative of the pre-symplectic current becomes

$$\begin{aligned}\tau_M(d_H\Omega) = \int_M \mathrm{ev}^*_{M,k} d_H\Omega = \int_M d_M(\mathrm{ev}^*_k\Omega) = \int_{\partial M} (-)|^*_{\partial M}\mathrm{ev}^*_k\Omega \\ = (-)|^*_{\partial M}\int_{\partial M} \mathrm{ev}^*_k\Omega = (-)|^*_{\partial M}\omega.\end{aligned}$$

The Koszul differential acting on the BV pre-symplectic current becomes

$$\tau_M(\iota_s\Omega_{\mathrm{BV}}) = L_Q\omega_{\mathrm{BV}}.$$

Therefore,

$$(-)|^*_{\partial M}\omega = L_Q\omega_{\mathrm{BV}}$$

the Hamiltonian symplectic form $\omega|_{\partial M}$ on the boundary ∂M of spacetime M is given by the BV differential of the BV symplectic form ω_{BV}. For a region of M bounded by two Cauchy surfaces Σ_{in} and Σ_{out}, $L_Q\omega_{\mathrm{BV}} = \omega|_{\Sigma_{in}} - \omega|_{\Sigma_{out}}$ which makes $L_Q\omega_{\mathrm{BV}}$ act as a potential for the symplectic form, and witness the covariance of the symplectic form on-shell (that is, in cohomology).

Continuing the example of classical scalar field theory, the section s is

$$s = -(\eta^{\mu\nu}\phi_{,\mu\nu} + m^2\phi)\delta\phi,$$

then the Koszul differential acts on an arbitrary local functional $A[\phi, \phi^{\ddagger}]dV$ as

$$\iota_s A[\phi, \phi^{\ddagger}]dV = -(\eta^{\mu\nu}\phi_{\mu\nu} + m^2\phi)(\frac{\delta^L A}{\delta\phi^{\ddagger}})dV$$

and thus, before passing to the prolonged shell,

$$H^0\Big(\iota_s\Big) = \frac{C^\infty(J^2E)}{(E)} = \frac{C^\infty(x^\mu, \phi, \phi_{,\mu}, \phi_{,\mu\nu})}{(\,(\Box + m^2)\phi)}$$

as expected. The BV-BFV relation is then

$$\begin{aligned}
L_Q\omega_{\mathrm{BV}} = \tau_M(\iota_s\Omega_{\mathrm{BV}}) &= -\tau_M\Big(\sum_i \delta(\mathrm{EL}_{i,\mathrm{jet}}) \wedge \delta\phi\, dV\Big) = -\int_M \sum_i \delta(\mathrm{EL}_i) \wedge \delta\phi\, dV \\
&= \int_M \Big(\eta^{\mu\nu}\,\delta(\partial_\mu\partial_\nu\phi) + m^2\,\delta\phi\Big) \wedge \delta\phi\, dV \\
&= \int_M \eta^{\mu\nu}\,\delta(\partial_\mu\partial_\nu\phi) \wedge \delta\phi\, dV + m^2 \int_M \delta\phi \wedge \delta\phi\, dV \\
&= \int_M d_\mu\Big(\eta^{\mu\nu}\,\delta(\partial_\nu\phi) \wedge \delta\phi\Big) dV - \int_M \eta^{\mu\nu}\,\delta(\partial_\nu\phi) \wedge \delta(\partial_\mu\phi)\, dV \\
&= \int_{\partial M} \eta^{\mu\nu}\,\delta(\partial_\nu\phi) \wedge \delta\phi\, n_\mu\, dA - \int_M \eta^{\mu\nu}\,\delta(\partial_\nu\phi) \wedge \delta(\partial_\mu\phi)\, dV \\
&= \int_{\partial M} \delta(\partial_n\phi) \wedge \delta\phi\, dA \\
&= (-)|_{\partial M}^{*}\omega.
\end{aligned}$$

Note that this implies that the restriction map in this case is of the form $(\phi, \phi^\ddagger) \to (\phi|_{\partial M}, \partial_n\phi|_{\partial M})$.

4. Gauge Symmetries

The formalism defined above is not complete for fields with infinitesimal (gauge) symmetries. Not only do gauge symmetries make the Hamiltonian Cauchy surface view impossible, they also induce relations between equations of motion which violates the regularity assumption of the Koszul complex. In this chapter, we will introduce the notion of a gauge symmetries of a field theory and the algebra required to extend the derived geometric view defined in the previous chapter to appropriately quotient by these symmetries as a non-negatively graded complex as mentioned above. We mainly follow [Sch] and [HT92].

4.1 Symmetries

We start with a discussion of symmetries of local partial differential equations.

Definition 37 *A system of n local PDEs of order k is a map $f : J^kE \to F$ from the k-th order jet bundle of the n-dimensional bundle $\pi_E : E \to M$ to the n-dimensional bundle $\pi_F : F \to M$ over a manifold M, locally denoted as*

$$f_i(x, \phi_1, \ldots, \phi_n, \phi_{1,\mu_1}, \ldots, \phi_{n,\mu_1}, \phi_{1,\mu_1\mu_2}, \ldots, \phi_{n,\mu_1\mu_2}, \phi_{1,\mu_1\ldots\mu_k} \ldots, \phi_{n,\mu_1\ldots\mu_k})_{i=1,\ldots,n} = 0.$$

If we impose the condition that the function f is regular at 0; that is, the corresponding linearized function df has maximal rank everywhere the equation is satisfied, then the level surface $C = f^{-1}(0)$, called the shell, is a manifold by the implicit function theorem.

Definition 38 *A solution of the above system of equations is a section of E, $\phi : M \to E$, such that the jet prolongation $j_k\phi$ satisfies the equation defined by f,*

$$f(x, \phi_1, \ldots, \phi_n, \frac{\partial \phi_1}{\partial x^\mu}, \ldots) = 0.$$

Equivalently, that the image $\mathrm{im}(j_k\phi) \subset J^kE$ of the section $j_k\phi : M \to J^kE$ is contained in the shell C.

Definition 39 *A symmetry of a system of local PDEs defined by f is a local diffeomorphism $p : E \to E$ whose jet prolongation preserves the shell of f, $j_k p(C) \subset C$.*

The converse is not necessarily true for all local PDEs (for example, PDEs with no solutions but a non-trivial shell) but is recovered under the weak assumption that the equation f is locally solvable. [Olv93]

Definition 40 *A vertical symmetry p_v of a system local PDEs is a symmetry whose projection to M recovers the identity on M, $\pi_E \circ p_v = \pi_E$, transforming the fiber coordinates while leaving the manifold coordinates intact.*

Definition 41 *A one-parameter group U_ξ of maps $U_\xi : E \to E$ is a group of symmetries of a system of local PDEs f if every (prolongation of) U_ξ preserves the shell $j_k U_\xi C \subset C$.*

Taking the derivative at $\xi = 0$, the generator $j_{k*}H$ (where $H \in \Gamma TE$ represents the Lie-algebra generator of U_ξ) is tangent to the shell, $(j_{k*}H)_y \in T_y C$ for every $y \in C$. Since 0 is a regular value, the tangent space of the shell is perpendicular to the gradient of f, $T_y C = \ker df_y$, so tangency is equivalent to $df_y(j_{k*}H) = 0$. Using $df(X) = X(f)$, this is

$$(j_{k*}H)f = 0 \text{ on } C.$$

Definition 42 *An infinitesimal symmetry is a vector field H whose prolongation is tangent to the shell C.*

Definition 43 *If an infinitesimal symmetry H has components only along the field coordinates ∂_{ϕ^i}, it is called an evolutionary symmetry.*

In the case of field theory, the shell C is defined by the Euler-Lagrange equations $\delta_{EL}\mathcal{L} = 0$, given in components as $\mathrm{EL}_{i,\mathrm{jet}} = 0$. Infinitesimal symmetries of the Euler-Lagrange equations are vector fields $\eta \in \Gamma TE$ such that

$$(j_{k*}\eta)\mathrm{EL}_{i,\mathrm{jet}} = 0 \text{ on } C.$$

An important subclass (Theorem 5.53 in [Olv93]) of symmetries of the shell are the so-called variational symmetries of the system, also called the infinitesimal symmetries of the Lagrangian density; they are characterized as being the infinitesimal symmetries which leave the Lagrangian density invariant in cohomology (up to a horizontal exact form).

Definition 44 *An infinitesimal symmetry η is a variational symmetry of the system defined by a Lagrangian density $\mathcal{L}$ if and only if*

$$\mathcal{L}_{j_{k*}\eta}\mathcal{L} = d_H \hat{J}.$$

These are exactly the symmetries which correspond to conserved quantities, via Noether's theorem.

Theorem 5.52 of [Olv93] states that restricting to evolutionary symmetries yields the same conserved quantities, which will be done in the following for simplicity.

Expanding the variational symmetry formula, the left-hand side becomes

$$\begin{aligned}\mathcal{L}_{j_{k*}\eta}\mathcal{L} &= \iota_{j_{k*}\eta} d\mathcal{L} + d\iota_{j_{k*}\eta} L \\ &= \iota_{j_{k*}\eta}\delta\mathcal{L} \\ &= \iota_{j_{k*}\eta}\delta_{EL}\mathcal{L} + d_H \iota_{j_{k*}\eta}\Theta\end{aligned}$$

Plugging the result back into the variational symmetry formula,

$$\iota_{j_{k*}\eta}\delta_{EL}\mathcal{L} + d_H \iota_{j_{k*}\eta}\Theta = d_H \hat{J},$$

then

$$d_H(\hat{J} - \iota_{j_{k*}\eta}\Theta) = \iota_{j_{k*}\eta}\delta_{EL}\mathcal{L},$$

which implies that the integral of the differential form $J = \hat{J} - \iota_\eta \Theta \in \Omega^{n-1,0}$ along any two non-intersecting (possibly Cauchy) hypersurfaces is the same on-shell, by Stokes' theorem. Therefore, the system defined by the Lagrangian $\mathcal{L}$ admits a conserved quantity $Q_\eta = \int_\Sigma J$ called the conserved charge.

4.2 Gauge Symmetries

Gauge field theories are field theories which admit a representation of a group G, called a gauge group, on fibers, which locally assemble to produce vertical symmetries given a spacetime-dependent function $\epsilon^a : M \to G$.

Definition 45 *An infinitesimal gauge symmetry is an evolutionary symmetry assembled by the representation of the lie algebra $\mathfrak{g}$ of G on fibers given a spacetime-dependent function $\epsilon^a : M \to \mathfrak{g}$. Assuming the bundle E is trivial, given generators c^a of $\mathfrak{g}$, an arbitrary infinitesimal gauge symmetry R is given by*

$$R = \sum_{l=0}^{s} R^{i\mu_1\cdots\mu_l}_{\alpha}\, \epsilon^{\alpha}_{,\mu_1\cdots\mu_l} \partial_{\phi^i} = (R^i_\alpha \epsilon^\alpha(x) + R^{i\mu}_\alpha \epsilon^\alpha_{,\mu}(x) + \ldots + R^{i\mu_1,\ldots,\mu_s}\epsilon^\alpha_{,\mu_1\ldots\mu_s}(x))\partial_{\phi^i}$$

where the components $R^{i\mu_1\cdots}_\alpha$ are functions on the jet bundle J^kE.

Note that the algebra of such infinitesimal gauge symmetries is infinite-dimensional.

To show that a collection of infinitesimal gauge transformations R are variational symmetries of the Lagrangian $\mathcal{L}$, (the prolongation of) R must satisfy the variational symmetry formula

$$\mathcal{L}_R L = d_H \hat{J},$$

equivalently,

$$d_H J = \iota_R \delta_{EL} L.$$

Locally,

$$\iota_R \delta_{EL} L = (R^i_\alpha \epsilon^\alpha + R^{i\mu}_\alpha \epsilon^\alpha_{,\mu} + \ldots + R^{i\mu_1,\ldots,\mu_s} \epsilon^\alpha_{,\mu_1\ldots\mu_s}) \mathrm{EL}_{i,\mathrm{jet}} dV$$

$$= \epsilon^\alpha (R^i_\alpha \mathrm{EL}_{i,\mathrm{jet}} - \partial_\mu (R^{i\mu}_\alpha \mathrm{EL}_{i,\mathrm{jet}}) + \ldots) dV + d_H \underbrace{((R^{i\mu}_\alpha \mathrm{EL}_{i,\mathrm{jet}} \epsilon^a + \ldots) dS)}_{K}$$

$$= \epsilon^\alpha \sum_{l=0}^{s} (-1)^l \, \partial_{\mu_1} \cdots \partial_{\mu_l} \Big(R^{i\mu_1\cdots\mu_l}_\alpha E_i \Big) dV + d_H K$$

where

$$K = \sum_{l=1}^{s} \sum_{p=0}^{l-1} (-1)^p \left(\partial_{\mu_1} \cdots \partial_{\mu_p} \left(R^{i\mu\nu_1\cdots\nu_p\rho_1\cdots\rho_{l-1-p}}_\alpha E_i \right) \right) \epsilon^\alpha_{,\rho_1\cdots\rho_{l-1-p}} dS$$

is the boundary term obtained by cohomological integration by parts as in the derivation of the Euler-Lagrange equations.

Since the first term on the right is not a total differential for arbitrary ϵ^α, it must vanish identically for the identity to hold,

$$\sum_{l=0}^{s} (-1)^l \, \partial_{\mu_1} \cdots \partial_{\mu_l} \Big(R^{i\mu_1\cdots\mu_l}_\alpha E_i \Big) = 0,$$

these are called the Noether identities which apply globally. The conserved charges associated to these gauge symmetries are then given by K

$$Q = \int_\Sigma K \approx 0 \quad \text{on } C$$

since K is proportional to the equations of motion and their derivatives. However, K is only specified up to a horizontally exact form since for any $(n-2)$-form I, $\hat{K} = K + d_H I$, $d_H \hat{K} = d_H(K + d_H I) = d_H K = \iota_R \delta_{EL} \mathcal{L}$. Then, if the infinitesimal gauge symmetries are not compactly supported, conserved charges can be obtained either on the boundary, or asymptotically by conformal compactification. We will only consider compactly supported infinitesimal gauge symmetries in the following, more information on asymptotic charges in [WZ00].

It can be seen that not all gauge symmetries produce Noether identities. If the evolutionary vector field R is of the form

$$R = \kappa^{ij}_\alpha \mathrm{EL}_{j,\mathrm{jet}} \partial_{\phi^i}$$

for some anti-symmetric $\kappa^{ij}_\alpha = -\kappa^{ji}_\alpha$, the corresponding Noether identity is trivial,

$$\sum_{l=0}^{s} (-1)^l \, \partial_{\mu_1} \cdots \partial_{\mu_l} \Big(R^{i\mu_1\cdots\mu_l}_\alpha E_i \Big) = 0 \implies \kappa^{ij}_\alpha \mathrm{EL}_{j,\mathrm{jet}} \mathrm{EL}_{i,\mathrm{jet}} = 0.$$

4.3 The Koszul-Tate Resolution

The above proves that when non-trivial gauge symmetries exist, the equations of motion are not independent and do not form a regular sequence. The Koszul complex does not form a resolution and the lower degree cohomology groups are non-zero, and capture these gauge symmetries.

Recall the form of the Koszul complex,

$$\cdots \xrightarrow{\iota_s} \Gamma(\Lambda^2 VJ^kE) \xrightarrow{\iota_s} \Gamma(VJ^kE) \xrightarrow{\iota_s} C^\infty(J^kE) \longrightarrow 0, \qquad \iota_s = (\mathcal{L}, -),$$

with antifield generators $\phi^\ddagger_a = \frac{\delta}{\delta\phi^a}$ in degree -1 and differential the contraction with $s = \sum_a \mathrm{EL}_{a,\mathrm{jet}}\, \delta\phi^a$, $\mathrm{EL}_{a,\mathrm{jet}} = \frac{\delta_{EL} L}{\delta\phi^a}$.

The zeroth cohomology group $H^0(\iota_s) = C^\infty(J^kE)/(\mathrm{EL}_{a,\mathrm{jet}})$ is the algebra of functions on the shell, as intended. In degree -1, the closed elements are the vertical vectors $Z = \sum_{|I|} Z^{i,I}\phi^\ddagger_{i,I}$ with $\iota_s Z = 0$. Using $\iota_s\phi^\ddagger_{i,I} = \partial_I \mathrm{EL}_{i,\mathrm{jet}}$ and the fact that ι_s commutes with the total derivative d_H, this condition reads $\sum_{i,I} Z^{i,I}\partial_I\mathrm{EL}_{i,\mathrm{jet}} = 0$, which is of the form of a Noether identity. By the above, each non-trivial infinitesimal symmetry R_α provides such a Noether identity,

$$Z_\alpha = \sum_{l=0}^{s}(-1)^l\, \partial_{\mu_1}\cdots\partial_{\mu_l}\Big(R_\alpha^{i\mu_1\cdots\mu_l}\,\phi^\ddagger_i\Big), \qquad \iota_s Z_\alpha = 0.$$

The exact elements entering from degree -2 are the trivial infinitesimal gauge symmetries,

$$\iota_s\Big(\kappa^{ij}\phi^\ddagger_i\phi^\ddagger_j\Big) = \kappa^{ij}\Big(\mathrm{EL}_{i,\mathrm{jet}}\,\phi^\ddagger_j - \mathrm{EL}_{j,\mathrm{jet}}\,\phi^\ddagger_i\Big).$$

Therefore

$$H^{-1}(\iota_s) \cong \frac{\{\text{gauge symmetries}\}}{\{\text{trivial gauge symmetries}\}}.$$

This group is non-zero exactly when the field theory possesses non-trivial gauge symmetries. We call a set of infinitesimal gauge symmetries R_α, defined by their coefficients $R_\alpha^{i\mu_1\ldots\mu_l}$, a generating set if they generate all the Noether identities without redundancy.

A generating set is not in general closed under the commutator of evolutionary vector fields; the bracket $[R_\alpha, R_\beta]$ is again an infinitesimal gauge symmetry, but expressing it through the generators may require coefficients that depend on the fields and a term that vanishes on the shell,

$$[R_\alpha, R_\beta] = -C^\gamma_{\alpha\beta}\, R_\gamma + M^{ij}_{\alpha\beta}\, E_j\, \partial_{\phi^i}, \qquad M^{ij}_{\alpha\beta} = -M^{ji}_{\alpha\beta},$$

where the structure functions $-C^\gamma_{\alpha\beta}$ are functions on the jet bundle and the last term is a trivial gauge symmetry of the form discussed above.

We use the convention where the structure constants of the Lie algebra are $-C^\alpha_{\beta\gamma}$ so that the longitudinal differential on ghost generators (defined below) is positive.

When $M^{ij}_{\alpha\beta} = 0$ and the structure constants are constant, the generators close off-shell and $C^{\gamma}_{\alpha\beta}$ obey the Jacobi identity, so the R_α span a Lie algebra of evolutionary vector fields. We assume that all generating sets are closed in what follows.

It is possible to modify the Koszul complex to recover a resolution by adjoining one generator for each infinitesimal gauge symmetry in a generating set. For each generating R_α introduce a so-called antighost generator $c^\ddagger_\alpha$ in degree -2 and extend the differential by

$$\iota_s c^\ddagger_\alpha \;=\; -Z_\alpha.$$

Since $\iota_s Z_\alpha = 0$ the extended differential still squares to zero, and by construction Z_α is now exact, it no longer appears in the -1 cohomology group H^{-1}. The new generators contribute no homology of their own and H^{-2} and below again vanish. The result is called the Koszul-Tate resolution. If the generating set of infinitesimal gauge symmetries is reducible, higher anti-ghosts need to be adjoined.

4.4 The Longitudinal Complex

The existence of compactly supported gauge symmetries is still problematic after dealing with Noether identities since they obstruct the existence of a Cauchy surface. A compact gauge symmetry supported on a subset of spacetime away from a Cauchy surface transforms the field values in a manner not determined by the initial conditions specified on the Cauchy surface, obstructing any isomorphism between the fields and the jet bundle restricted to the Cauchy surface. Then, a Cauchy surface only determines fields up to compact gauge transformation, which suggests its existence in the quotient space.

The naive quotient space of fields by the infinite-dimensional group of gauge symmetries is not well-behaved, and for gauge field theories with $G = SU(N)$, picking a single representative for each gauge orbit is impossible, see [Sin78]. Instead, we define a differential along the gauge orbits and compute the quotient homologically by introducing the algebra of differential forms along the gauge orbits, called longitudinal forms, following [HT92].

As usual, the differential forms along all directions form the de Rham complex

$$\Omega^\bullet(\Gamma E) \;=\; C^\infty\Big(T[1]\Gamma E\Big),$$

the algebra of functions on the shifted tangent bundle $T[1]\Gamma E$.

This is a non-negative-graded differential graded algebra with degree-$(+1)$ generators the one-forms dx^i whose differential is the de Rham differential, which represents a non-positively-graded dg-manifold $T[1]\Gamma E$. This $T[1]\Gamma E$ is the negative-degree counterpart of the positive-degree shifted cotangent $T^*[-1]\Gamma E$ of the previous chapter; instead of polyvector fields, there are differential forms.

Restricting to the gauge orbit directions, we keep only the forms along the distribution spanned by the infinitesimal gauge symmetries. At a field $\phi \in \Gamma E$ the gauge directions are the values of the generating set of infinitesimal gauge transformations R_α ranging over choices of $\epsilon^\alpha : M \to \mathfrak{g}$, and as $\phi \in \Gamma E$ ranges over the field space these span the gauge distribution $\Delta \subseteq T\Gamma E$, whose integral leaves are the gauge orbits.

A longitudinal p-form then takes in p vectors in the distribution Δ at a certain point, interpreted as the tangent vectors to the gauge orbits, and the differential is the de Rham differential of ΓE restricted to differentiation along Δ, called the exterior longitudinal differential d_L. By the assumption that the generating set closes, the Frobenius theorem states that the exterior longitudinal differential is nilpotent $d_L^2 = 0$, so the algebra of longitudinal differential forms is a differential graded algebra whose degree-zero cohomology

$$H^0(d_L) = \{\, F : d_L F = 0 \,\}$$

is the algebra of gauge-invariant functions. Note that this is not a resolution of the algebra of gauge-invariant functions, since the higher cohomology groups are in general non-zero and capture the failure to construct the naive quotient.

To make this algebra explicit, we form a basis for the algebra of longitudinal forms. The generators R_α span Δ, so let c^α be the longitudinal one-forms dual to this frame,

$$c^\alpha(R_\beta) = \delta^\alpha_\beta,$$

which are typically called the ghost generators, or ghost fields.

Recall that an infinitesimal gauge transformation acts on the fields by flow along R,

$$R = (R^i_\alpha\, \epsilon^\alpha + R^{i\mu}_\alpha\, \epsilon^\alpha_{,\mu} + \cdots)\partial_{\phi^i} = (\sum_{l=0}^{s} R^{i\mu_1\cdots\mu_l}_\alpha\, \epsilon^\alpha_{,\mu_1\cdots\mu_l})\partial_{\phi^i},$$

with ϵ^α the gauge parameter. The longitudinal differential is obtained by replacing the parameter ϵ^α with the ghost c^α, so that $d_L\phi^i$ is the infinitesimal gauge transformation with the parameter ϵ^α promoted to the ghost generators c^α,

$$d_L\, \phi^i = \sum_{l=0}^{s} R^{i\mu_1\cdots\mu_l}_\alpha\, c^\alpha_{,\mu_1\cdots\mu_l} = R^i_\alpha c^\alpha + R^{i\mu}_\alpha c^\alpha_{,\mu} + \cdots .$$

On the ghosts themselves d_L is fixed by the closure of the generating set, $[R_\alpha, R_\beta] = -C^\gamma_{\alpha\beta} R_\gamma$, to be

$$d_L\, c^\alpha = \tfrac{1}{2}\, C^\alpha_{\beta\gamma}\, c^\beta c^\gamma.$$

Notice that the algebra of longitudinal differential forms is then the differential graded algebra of functions on the differential graded manifold $\Gamma E \times \mathfrak{g}[1]$.

The equations above imply that the algebra of longitudinal differential forms form a special algebra called the Chevalley-Eilenberg algebra of the action of gauge symmetries on the field space, where the equation for $d_L c^\alpha$ is called the Maurer-Cartan equation.

4.4.1 The Chevalley-Eilenberg Algebra

Definition 46 *The Chevalley-Eilenberg algebra of a Lie algebra $\mathfrak{g}$ is the exterior algebra $\Lambda^\bullet \mathfrak{g}^*$ on its dual, viewed as the dg-commutative algebra of functions on the shifted vector space $\mathfrak{g}[1]$. Choosing a basis T_α of $\mathfrak{g}$ with $[T_\beta, T_\gamma] = -C^\alpha_{\beta\gamma} T_\alpha$, the algebra is freely generated by the degree-one coordinates c^α dual to the T_α, and it is equipped with a degree-+1 differential d_{CE} defined on these generators by*

$$d_{\mathrm{CE}}\, c^\alpha = \tfrac{1}{2}\, C^\alpha_{\beta\gamma}\, c^\beta c^\gamma$$

and extended to all of $\Lambda^\bullet \mathfrak{g}^*$ as a graded derivation. Nilpotency, $d^2_{\mathrm{CE}} = 0$, holds because the structure constants obey the Jacobi identity, $C^\alpha_{\beta[\gamma} C^\beta_{\delta\epsilon]} = 0$, so the Maurer-Cartan equation states that the Chevalley-Eilenberg algebra is the dual of the Lie bracket.

In the following, we will follow the literature and use condensed DeWitt notation where instead of writing $\sum_{l=0}^{s} R^{i\mu_1\cdots\mu_l}_{\alpha}\, c^\alpha_{,\mu_1\cdots\mu_l}$ we will simply write $R^i_\alpha c^\alpha$.

Definition 47 *An action of a Lie algebra $\mathfrak{g}$ on a manifold X by infinitesimal diffeomorphisms is a Lie homomorphism $\rho : \mathfrak{g} \to \Gamma(TX)$; in coordinates $R = c^\alpha R^a_\alpha\, \partial_{\phi^a}$. The algebra of the action of a Lie algebra, denoted $X/\mathfrak{g}$ is the abstract form of the longitudinal complex above; the dg-algebra given by the exterior algebra of $C^\infty(X) \otimes \mathfrak{g}^*$ over $C^\infty(X)$ with a differential of the same form as the exterior longitudinal differential mentioned above*

$$CE(X/\mathfrak{g}) = \Big(\wedge^\bullet_{C^\infty(X)} (C^\infty(X) \otimes \mathfrak{g}^*),\ d_{CE} \Big), \quad d_{CE}\, \phi^a = c^\alpha R^a_\alpha, \quad d_{CE}\, c^\alpha = \tfrac{1}{2} C^\alpha{}_{\beta\gamma} c^\beta c^\gamma,$$

which can be seen as the algebra of functions on the graded manifold $X \times \mathfrak{g}[1]$.

On the degree-0 generators ϕ^a,

$$\begin{aligned} d^2_{CE}\, \phi^a = d_{CE}(c^\alpha R^a_\alpha) = (d_{CE} c^\alpha)\, R^a_\alpha - c^\alpha\, c^\beta R^b_\beta\, \partial_b R^a_\alpha \\ = \tfrac{1}{2}\, c^\beta c^\gamma \Big(\underbrace{R^b_\beta \partial_b R^a_\gamma - R^b_\gamma \partial_b R^a_\beta}_{[R_\beta, R_\gamma]^a} + C^\alpha{}_{\beta\gamma} R^a_\alpha \Big), \end{aligned}$$

so $d^2_{CE} \phi^a = 0$ is equivalent to $[R_\beta, R_\gamma] = -C^\alpha{}_{\beta\gamma} R_\alpha$, i.e. to ρ being a Lie-algebra homomorphism. On the degree-1 generators c^α,

$$d^2_{CE}\, c^\alpha = C^\alpha{}_{\beta\gamma}\, (d_{CE} c^\beta)\, c^\gamma = \tfrac{1}{2}\, C^\alpha{}_{\beta\gamma}\, C^\beta{}_{\delta\epsilon}\, c^\delta c^\epsilon c^\gamma = \tfrac{1}{2}\, C^\alpha{}_{\beta[\gamma}\, C^\beta{}_{\delta\epsilon]}\, c^\delta c^\epsilon c^\gamma.$$

This vanishes when ${C^\alpha}_{\beta[\gamma}{C^\beta}_{\delta\epsilon]} = 0$, which is the Jacobi identity for $\mathfrak{g}$. Therefore, $d_{CE}^2 = 0$ enforces the condition that the map ρ is a Lie-algebra homomorphism ($\mathfrak{g}$ acts on the space X), and that $\mathfrak{g}$ satisfies the Jacobi identity and hence is indeed a Lie algebra.

It is instructive to look at the simplest case where the manifold $X = *$ consists of a single point. In this case of $*/\mathfrak{g}$, the Chevalley-Eilenberg algebra of the action of $\mathfrak{g}$ reduces to the Chevalley-Eilenberg algebra of $\mathfrak{g}$ itself,

$$CE(*/\mathfrak{g}) = \Big(\Lambda^\bullet \mathfrak{g}^*,\ d_{CE}\, c^\alpha = \tfrac{1}{2}\, C^\alpha_{\beta\gamma}\, c^\beta c^\gamma\Big).$$

In degree zero, $H^0(d_{CE}) = \mathbb{R}$ is the algebra of functions on a point, in agreement with the set-theoretic quotient, but the higher cohomology groups are in general non-zero; for abelian $\mathfrak{g}$ the differential vanishes identically and $H^\bullet(d_{CE}) = \Lambda^\bullet \mathfrak{g}^*$. Therefore the graded manifold $\mathfrak{g}[1]$ is not equivalent to a point, and the homological quotient retains the symmetries even when they act trivially on the underlying space. This is the infinitesimal analogue of the classifying stack $BG = */G$.

4.4.2 L_∞-Algebras

The above action by Lie algebras is typically not enough for general gauge field theories. If the gauge parameters ϵ^α admit gauge transformations of their own, the parameter space must itself be quotiented for the Cauchy surface to exist, producing the ghosts-of-ghosts in degree two, and so on.

As an example, let the space of fields be two-forms $B \in \Omega^2(M)$ with Lagrangian built from the curvature $H = dB$, so that the gauge transformations are

$$B \ \to\ B + d\lambda_1, \qquad \lambda_1 \in \Omega^1(M),$$

and the generating set of infinitesimal gauge symmetries is the family of vector fields $R_{\lambda_1} = (d\lambda_1)^i\, \partial_{B^i}$. This generating set is reducible, since by $d^2 = 0$ the parameters λ_1 and $\lambda_1 + d\lambda_0$ with $\lambda_0 \in \Omega^0(M)$ generate the same transformation,

$$R_{\lambda_1 + d\lambda_0} = R_{\lambda_1},$$

which means the parameter space itself must be quotiented, which forms the two-term complex

$$\mathfrak{g} \ = \ \Big(\Omega^0(M) \xrightarrow{d} \Omega^1(M)\Big),$$

with $\Omega^1(M)$ in Lie algebra degree 0 and $\Omega^0(M)$ in Lie algebra degree -1, the differential being the de Rham differential.

The longitudinal complex can then be constructed as above, as the algebra of functions on $\Omega^2(M) \times \mathfrak{g}[1]$

$$\Omega^2(M) \times \mathfrak{g}[1] = \left(\Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M)\right)$$

with differential d_L the dual of d, and $d_L^2 = 0$ holds by $d^2 = 0$. This algebra is seen to be the Chevalley-Eilenberg algebra of a more general algebraic object, since the differential d_L is clearly not the dual of a bracket but of a unary operator that is nilpotent. The general algebraic object with n-ary operators that satisfy generalized Jacobi identities is called an L_∞-algebra, whose operations are read off from the coefficients of the differential by ghost degree denoted by l_n, where in the above case $l_1 = d$ and $l_k = 0$ for all $k > 1$. Note that below the action of an L_∞ algebra is shifted up by one compared to the above example, placing the fields in degree $+1$, to match convention.

Definition 48 *An L_∞-algebra is a graded vector space $\mathfrak{g}$ equipped with n-ary anti-symmetric multilinear operators $\ell_k : \mathfrak{g}^{\otimes k} \to \mathfrak{g}$ of degree $2 - k$, satisfying a generalized Jacobi identity.*

The Jacobi identity of an L_∞-algebra is easier understood in terms of its Chevalley-Eilenberg algebra.

Definition 49 *The Chevalley-Eilenberg algebra of an L_∞-algebra as a graded vector space $\mathfrak{g}$ with basis $\{t_\alpha\}$ and with shifted dual generators $\{c^\alpha\}$ so that $\deg c^\alpha = 1 - \deg t_\alpha$ is the free graded commutative algebra $\mathrm{Sym}(\mathfrak{g}[1]^*)$ generated by the c^α, equipped with degree $+1$ derivation d_{CE} of the form*

$$d_{CE}\, c^\alpha = \sum_{k \geq 1} \frac{1}{k!} C^\alpha{}_{\beta_1 \cdots \beta_k} c^{\beta_1} \cdots c^{\beta_k}.$$

Note the sign convention, the dual c^α is the negative of the grading of t^α, which is then shifted up by one. This means that a non-negatively graded Chevalley-Eilenberg algebra is associated with a L_∞ algebra concentrated in degrees less than or equal to one. The Chevalley-Eilenberg algebra is the non-negative differential-graded manifold we are interested in.

For $\mathfrak{g}$ concentrated in degree 0 and d_{CE} quadratic this reduces to the Chevalley-Eilenberg algebra of an ordinary Lie algebra $(C^\infty(\mathfrak{g}[1]^*), d_{\mathrm{CE}})$, and the condition $d_{CE}^2 = 0$ reduces to the Jacobi identity.

To obtain the description in terms of operations l_k mentioned above, pass to the dual and note that the coefficient of order k in d_{CE} is the graded antisymmetric multilinear operation

$$l_k : \ \mathfrak{g}^{\wedge k} \longrightarrow \mathfrak{g}, \qquad \deg l_k = 2 - k,$$

and the single equation $d_{CE}^2 = 0$ becomes generalized Jacobi identities on the l_k.

Definition 50 *A Lie n-algebra is an L_∞-algebra concentrated in degrees $1-n,\dots,0$. A Lie 1-algebra is a plain Lie algebra, and a dg-Lie algebra is an L_∞-algebra with $l_k = 0$ for $k > 2$. The Maurer-Cartan equation generalizes to*

$$\sum_{k\geq 1} \frac{1}{k!}\, l_k(a,\dots,a) = 0, \qquad \deg a = 1.$$

Definition 51 *The algebra of the action of a Lie ∞-algebra $X/\mathfrak{g}$ is the algebra of functions on $X \times \mathfrak{g}[1]$ represented by the following Chevalley-Eilenberg algebra,*

$$CE(X/\mathfrak{g}) = \Big(\mathrm{Sym}_{C^\infty(X)}\Big(C^\infty(X) \otimes \mathfrak{g}[1]^*\Big),\ d_{CE}\Big),$$

with the differential on generators of the form

$$d_{CE}\,\phi^a = c^\alpha R^a_\alpha(\phi) + \dots, \qquad d_{CE}\, c^\alpha = \ Z^\alpha{}_A\, c^A + \tfrac{1}{2}\, C^\alpha{}_{\beta\gamma}(\phi)\, c^\beta c^\gamma - \cdots\ ,$$

where now both the brackets and the action coefficients may depend on the point of X and contributions of every polynomial order are admitted.

The algebra of the action of a Lie n-algebra is such a dg-manifold concentrated in degrees 1 through $1-n$, the action of a Lie 0-algebra being a plain manifold and the action of a Lie 1-algebra simply the action of a Lie algebra.

This is the most general form for the longitudinal complex for gauge field theories. For a reducible generating set with ghosts-of-ghosts up to degree n, the algebra of longitudinal differential forms is the Chevalley-Eilenberg algebra of the action of a Lie n-algebra $\Gamma E/\mathfrak{g}$, with $\mathfrak{g}$ the Lie n-algebra of gauge parameters. For the B-field, $X = \Omega^2(M)$, $\mathfrak{g} = (\Omega^0(M) \to \Omega^1(M))$, and the differential $d_L B = dc$, $d_L c = dc'$, the dual of the exterior derivative on $\mathfrak{g}$ exhibited above is exactly d_{CE} of $\Omega^2(M)/\mathfrak{g}$,

$$CE(\Omega^2(M)/\mathfrak{g}) = (\mathrm{Sym}_{C^\infty(\Omega^2(M))}(C^\infty(\Omega^2(M)) \times \mathfrak{g}[1]^*), d_{CE}).$$

The zero-degree cohomology group in this case is

$$\begin{aligned} H^0_{CE} \ &= \ \ker\Big(d_{CE} : C^\infty(\Omega^2(M)) \to C^\infty(\Omega^2(M)) \otimes_{C^\infty} \Omega^1(M)^*\Big) \\ &= \ \{F \in C^\infty(\Omega^2(M)) \mid d_{CE}F = (dc)^i \partial_{B^i} F = 0\} = C^\infty(\Omega^2(M)\big/ d\Omega^1(M)), \end{aligned}$$

which are the functions on 2-forms defined up to the addition of an exact 2-form, which here are the gauge-invariant functions of the fields as expected.

It can be seen that the Chevalley-Eilenberg algebra is the correct (unique up to quasi-isomorphism) notion of a homological quotient as a result of a more general mathematical theory of structures under symmetry called formal deformation theory, following [KS].

4.5 Formal Deformation Theory

Formal deformation theory is the study of infinitesimal families of mathematical objects through the use of formal geometry. This is done by studying infinitesimal neighborhoods on the space of all such objects, called the moduli space.

4.5.1 Infinitesimal Neighborhoods

Definition 52 (Theorem 3.22 of [KS17]) *The infinitesimal disk of order k at a point $x \in M$ of a bundle $E \to M$ is the C^∞ space $\mathbb{D}^k_x$ whose maps into E form the jet bundle of order k over the point x,*

$$Hom_{/M}(\mathbb{D}^k_x, E) = (J^k E)_x;$$

that is, $\mathbb{D}^k_x$ is the C^∞ space that represents the operation of taking jet fibers at x, and should be understood as a probe for infinitesimal information of E.

The notation $\mathrm{Hom}_{/M}(\mathbb{D}^k_x, E)$ denotes the maps that respect the projection to M. The disk sits over x via its inclusion $\iota : \mathbb{D}^k_x \to M$, and an admissible map $f : \mathbb{D}^k_x \to E$ is required to project back to that inclusion under $p : E \to M$, $p \circ f = \iota$; the composite $\mathbb{D}^k_x \xrightarrow{f} E \xrightarrow{p} M$ must equal the inclusion of the disk to x.

For example, $D^0_x = \{x\}$ simply probes the fiber of E,

$$\mathrm{Hom}_{/M}(\mathbb{D}^0_x, E) = (J^0 E)_x = E_x.$$

Then, assuming E is a vector bundle, D^1_x probes the first-order infinitesimal information as follows

$$\mathrm{Hom}_{/M}(\mathbb{D}^1_x, E) = (J^1 E)_x = E_x \oplus (E_x \otimes T^*_x M),$$

and in general,

$$\mathrm{Hom}_{/M}(\mathbb{D}^k_x, E) = (J^k E)_x = E_x \otimes \bigoplus_{j=0}^{k} S^j T^*_x M,$$

where S^j is the j-th symmetric power.

Its algebra of functions is obtained by applying the above formula to $E = M \times \mathbb{R}$,

$$\begin{aligned} C^\infty(\mathbb{D}^k_x) &= \mathrm{Hom}_{/M}(\mathbb{D}^k_x, \mathbb{R}) = \mathrm{Hom}_{/M}(\mathbb{D}^k_x, M \times \mathbb{R}) \\ &= (J^k(M \times \mathbb{R}))_x = \bigoplus_{j=0}^{k} S^j T^*_x M = \frac{\mathrm{Sym}(T^*M)}{\mathrm{Sym}^{>k}(T^*M)} \end{aligned}$$

which identifies $C^\infty(\mathbb{D}^k_x)$ with the symmetric algebra of polynomials on T^*M up to order k, which is identified with algebras generated by nilpotent elements

$$C^\infty(\mathbb{D}^k_x) = \mathbb{R}[\epsilon_1, \dots, \epsilon_n]/(\text{Monomials of degree } k+1),$$

one for every dimension of M. Note that the infinitesimal disk $\mathbb{D}^k$ also maps to vector spaces $V \cong \mathbb{R}^n$ by considering the bundle $E = \mathbb{R} \times V$,

$$\begin{aligned}\mathrm{Hom}_{/\mathbb{R}}(\mathbb{D}^k_0, E) &= \mathrm{Hom}_{/\mathbb{R}}(\mathbb{D}^k_0, \mathbb{R} \times \mathbb{R}^n)\\ &= \mathrm{Hom}_{/\mathbb{R}}(\mathbb{D}^k_0, \mathbb{R} \times \mathbb{R})^n = C^\infty(\mathbb{D}^k_0) \otimes V = V[\epsilon]/(\epsilon^{k+1}),\end{aligned}$$

which augments V by a nilpotent element ϵ of order k.

The algebras of functions of the infinitesimal disks form a subset of the Weil algebras, which correspond to the algebras of functions of general infinitesimal neighborhoods. (Typically, Artinian local algebras over an arbitrary base are used, but we will always work over the base field $\mathbb{R}$.)

Definition 53 *A Weil algebra A over $\mathbb{R}$ is a finite-dimensional, commutative, local $\mathbb{R}$-algebra with a unique maximal ideal $\mathfrak{m}$ that is nilpotent*

$$\mathfrak{m}^{r+1} = 0$$

for some $r \geq 0$ and satisfies

$$A/\mathfrak{m} \cong \mathbb{R}.$$

The infinitesimal neighborhood that the Weil algebra defines is denoted as $\mathrm{Spec}(A)$.

The infinitesimal disks do behave like disks, in that every infinitesimal neighbourhood defined by a Weil algebra is contained in some infinitesimal disk $\mathbb{D}^k_x$, and they are all contained in the full infinitesimal disk $\mathbb{D}_x$ defined in terms of the infinite-dimensional jet bundle $(J^\infty E)_x$, or equivalently as the limit of the algebras of functions of the lower-order infinitesimal disks. Therefore, all the infinitesimal information near a point $x \in E$ is probed by the infinitesimal disk $\mathbb{D}_x$.

4.5.2 The Deformation Functor

An instructive example of formal deformation theory is given by the deformation of the differential d of a graded vector space V. A deformation of d is a degree-one endomorphism $\delta \in \mathrm{End}^1(V)$ of V such that the deformed map $D = d + \delta$ remains a differential, $D^2 = 0$. Expanding and using $d^2 = 0$ gives the condition $d\delta + \delta d + \delta^2 = 0$.

To obtain the infinitesimal picture, map an infinitesimal disk $\mathbb{D}^k_d$ into the vector space of degree-one differentials $\mathrm{End}^1(V)$ on V at the point d to obtain the infinitesimal deformations of order k, $\mathrm{End}^1(V)[\epsilon]/(\epsilon^{k+1})$. For the first-order disk $\mathbb{D}^1$, the infinitesimal deformations are $\mathrm{End}^1(V)[\epsilon]/(\epsilon^2)$ with arbitrary element $D = d + \epsilon\delta$. Then, $D^2 = 0$ becomes

$$D^2 = d^2 + \epsilon(d\delta + \delta d) + \epsilon^2\delta^2 = \epsilon(d\delta + \delta d),$$

since $d^2 = 0$ and the nilpotency $\epsilon^2 = 0$ of the generator of $C^\infty(\mathbb{D}^1)$ kills the δ^2 term.

Notice that the graded vector space $\mathrm{End}(V)$ can be endowed with a Lie algebra structure via the commutator. Then, the above requirement reduces to

$$[d, \delta] = 0.$$

If the adjoint action of d, $[d, -]$, is taken to be the differential on $\mathrm{End}(V)$, this is a statement that δ is a closed element in $\mathrm{End}^1(V)$.

The exact elements would then be the endomorphisms of the form $\delta = d\beta - \beta d$ for degree-zero β. These are the first-order parts of the conjugations of d by the invertible maps $e^\beta \in GL(V)$, which produce equivalent differentials and count as trivial deformations. Two first-order deformations are equivalent when they differ by such an exact element, and the first-order deformations up to equivalence are the elements of the cohomology $H^1(\mathrm{End}(V), [d, -])$, which can be thought of as a tangent space at d of the moduli space of differentials on V.

Passing to the higher-order disks $\mathbb{D}^k$ with $C^\infty(\mathbb{D}^k) = \mathbb{R}[\epsilon]/(\epsilon^{k+1})$ extends the deformation to $D = d + \epsilon\delta_1 + \epsilon^2\delta_2 + \cdots + \epsilon^k\delta_k$, and now $D^2 = 0$ holds modulo ϵ^{k+1}. Collecting the coefficient of each power of ϵ gives a tower of equations, with the order-ϵ equation $[d, \delta_1] = 0$ recovering the closure condition and the order-ϵ^2 equation $[d, \delta_2] + \delta_1^2 = 0$ exhibiting δ_1^2 as an exact element. When this equation holds, it is possible to extend a first-order deformation δ_1 to second order δ_2, thus the obstructions are exhibited by the existence of classes δ_1^2 in $H^2(\mathrm{End}(V), [d, -])$. Taking the limit over k to the full infinitesimal disk $\mathbb{D}$ recovers the full equation $d\delta + \delta d + \delta^2 = [d, \delta] + \frac{1}{2}[\delta, \delta] = 0$, which is the Maurer-Cartan equation of $\mathrm{End}(V)$.

Let the functor MC denote the action of taking Maurer-Cartan elements,

$$\mathrm{MC}\Big(\mathrm{End}(V)\Big) = \Big\{\delta \in \mathrm{End}^1(V) \mid \partial\delta + \tfrac{1}{2}[\delta, \delta] = 0\Big\},$$

this is functorial since Maurer-Cartan elements are preserved under Lie algebra homomorphisms.

Now let the parameter range over all infinitesimal neighbourhoods, not only the disks $\mathbb{D}^k$. Replace $C^\infty(\mathbb{D}^k) = \mathbb{R}[\epsilon]/(\epsilon^{k+1})$ by the Weil algebra $(A, \mathfrak{m})$. The ideal $\mathfrak{m}$ is nilpotent, $\mathfrak{m}^{N+1} = 0$, and replaces the ideal (ϵ). A deformation over A is

$$D = d + \delta, \qquad \delta \in \mathrm{End}^1(V) \otimes \mathfrak{m},$$

and $D^2 = 0$ is the Maurer-Cartan equation in $\mathrm{End}(V) \otimes \mathfrak{m}$,

$$\mathrm{MC}\Big(\mathrm{End}(V) \otimes \mathfrak{m}\Big) = \Big\{\delta \in \mathrm{End}^1(V) \otimes \mathfrak{m} \mid \partial\delta + \tfrac{1}{2}[\delta, \delta] = 0\Big\}$$

just as in the case of the infinitesimal disks.

Over A the conjugating maps are e^{β} for $\beta \in \mathrm{End}^0(V) \otimes \mathfrak{m}$. They act by

$$\delta \longmapsto e^{\beta}(d+\delta)e^{-\beta} - d = \delta - \partial\beta + [\beta, \delta] + \cdots ,$$

with first-order part $-\partial\beta$, recovering the exact elements $[d, \beta]$. Write $\exp\Big(\mathrm{End}^0(V) \otimes \mathfrak{m}\Big)$ for this group.

The deformation functor of $\mathrm{End}(V)$ is the quotient

$$\mathrm{Def}_{\mathrm{End}(V)}(A) = \mathrm{MC}\Big(\mathrm{End}(V) \otimes \mathfrak{m}_A\Big) \Big/ \exp\Big(\mathrm{End}^0(V) \otimes \mathfrak{m}_A\Big),$$

a functor from Weil algebras to sets. Note that this quotient can also be taken homologically, in which case it becomes a functor from Weil algebras to chain complexes. It sends A to the deformations of d over A up to equivalence. Its value on the first-order disk is the tangent space,

$$\mathrm{Def}_{\mathrm{End}(V)}\Big(\mathbb{R}[\epsilon]/\epsilon^2\Big) = H^1\Big(\mathrm{End}(V), [d, -]\Big).$$

The group $H^2(\mathrm{End}(V), [d, -])$ is again interpreted as the obstruction space.

4.5.3 Formal Moduli Problems

We call such a functor describing the infinitesimal disk of mathematical objects on the moduli space of objects a formal moduli problem. A theorem of Lurie and Pridham [Lur11, Pri10] states that over a field of characteristic zero every formal moduli problem arises in this way, as Def_L above for some L_∞-algebra, unique up to quasi-isomorphism.

Since the algebra that controls the deformation problem is an L_∞ algebra, the construction of the Chevalley-Eilenberg algebra applies.

The dg-Lie algebra $\mathrm{End}(V)$ with $l_1 = [d, -]$ and l_2 the commutator, and $l_k = 0$ for $k > 2$, is an L_∞-algebra. Choosing a basis t_α of $\mathrm{End}(V)$ and dual generators c^α of negated degree of the corresponding t_α shifted up by one, its Chevalley-Eilenberg algebra is

$$CE(\mathrm{End}(V)) = \Big(\mathrm{Sym}(\mathrm{End}(V)[1]^*),\, d_{CE}\Big), \qquad d_{CE}\, c^\alpha = C^\alpha{}_\beta\, c^\beta + \tfrac{1}{2}\, C^\alpha{}_{\beta\gamma}\, c^\beta c^\gamma,$$

with the linear coefficient $C^\alpha{}_\beta$ dual to $l_1 = [d, -]$ and the quadratic coefficient $C^\alpha{}_{\beta\gamma}$ dual to l_2.

A degree-one element $\delta \in \mathrm{End}^1(V) \otimes \mathfrak{m}$ over a Weil algebra $(A, \mathfrak{m})$ is the same as an algebra map $\mathrm{Sym}(\mathrm{End}(V)[1]^*) \to A$ sending each generator c^α of degree zero (due to the shift by one) to its component δ^α, and this map intertwines d_{CE} with the zero differential on A exactly when $d_{CE} c^\alpha = C^\alpha{}_\beta\, c^\beta + \frac{1}{2}\, C^\alpha{}_{\beta\gamma}\, c^\beta c^\gamma = 0 \iff \partial\delta + \frac{1}{2}[\delta, \delta] = 0$. Therefore

$$\mathrm{MC}\,\Big(\mathrm{End}(V) \otimes \mathfrak{m}\Big) = \mathrm{Hom}_{\mathrm{dgMan}}(\mathrm{Spec}(A), \mathrm{End}(V)[1]) = \mathrm{Hom}_{\mathrm{CDGA}}\,\Big(CE(\mathrm{End}(V)), A\Big),$$

the Maurer-Cartan elements over A are the maps out of $CE(\mathrm{End}(V))$, and $CE(\mathrm{End}(V))$ is the algebra of functions of the space $\mathrm{End}(V)[1]$ as expected. The conjugating group $\exp(\mathrm{End}^0(V) \otimes \mathfrak{m})$ acts on these maps and the quotient by it is $\mathrm{Def}_{\mathrm{End}(V)}(A)$.

The degree-zero generators of $CE(\mathrm{End}(V))$ are dual to $\mathrm{End}^1(V)$, and the linear part of d_{CE} on them is dual to $l_1 = [d, -]$, so the cohomology of d_{CE} on the linear generators in degree zero is dual to the cohomology of $[d, -]$ on $\mathrm{End}^1(V)$, which is the dual to the tangent space to the moduli space

$$\mathrm{Def}_{\mathrm{End}(V)}\Big(\mathbb{R}[\epsilon]/\epsilon^2\Big)^* = H^1\Big(\mathrm{End}(V), [d, -]\Big)^* = H^0(d_{CE,\mathrm{lin}}).$$

The full algebra $H^0(d_{CE})$ is the algebra of gauge-invariant functions on elements that satisfy the Maurer-Cartan equation on $\mathrm{End}^1(V)$.

The longitudinal complex is of the same form. Its L_∞-algebra is the action of the gauge symmetries on the fields $\Gamma E/\mathfrak{g} = \Gamma E \times \mathfrak{g}[1]$, with l_1 on $\mathfrak{g}$ the differential of the complex of parameters, parameters-of-parameters, and so on, and the higher l_k the brackets and action coefficients, and $CE(\Gamma E/\mathfrak{g})$ with $d_{CE}\phi^a = c^\alpha R^a_\alpha + \cdots$ and $d_{CE}c^\alpha = Z^\alpha{}_A c^A + \frac{1}{2}C^\alpha{}_{\beta\gamma}c^\beta c^\gamma - \cdots$ is its (shifted) algebra of functions, in the way $CE(\mathrm{End}(V))$ was the algebra of functions on $\mathrm{End}(V)[1]$.

A field configuration deformed along the ghosts by $d_{CE}\phi^a$ is a Maurer-Cartan element of $\Gamma E/\mathfrak{g}$ in the sense that δ was a Maurer-Cartan element of $\mathrm{End}(V)$, and the gauge transformations are the conjugating group. By the theorem of Lurie and Pridham $\Gamma E/\mathfrak{g}$ controls a formal moduli problem $\mathrm{Def}_{\Gamma E/\mathfrak{g}}$, unique up to quasi-isomorphism, whose function algebra is $CE(\Gamma E/\mathfrak{g})$, so the homological quotient of fields by gauge symmetries is the formal moduli problem of the field theory and the longitudinal complex is its unique-up-to-quasi-isomorphism function algebra. The degree-zero cohomology $H^0(d_{CE})$ is the algebra of gauge-invariant functionals, the functions on the naive quotient as computed for $*/\mathfrak{g}$ and $\Omega^2(M)/\mathfrak{g}$ above.

As usual, we pass to the local picture using the jet bundle. The action of the Lie algebra $J^k(E)/\mathfrak{g}$ has as underlying graded manifold the jet bundle $J^k(E \times \mathfrak{g}[1])$, as expected. Since the BRST differential commutes with the exterior differential, the differential naturally extends to jets of field and ghost variables and the local BRST complex transgresses to the global space of fields exactly as in the previous chapter.

5. The BV-BRST Complex

In this chapter, we will develop the full BV-BRST complex by combining the methods of both the non-positively graded Koszul-Tate resolution of the shell and the non-negative graded complex of longitudinal differential forms to form a complex in both positive and negative degrees of ghosts, anti-ghosts, fields, and anti-fields using deformation theory. This allows us to obtain the algebra of gauge-invariant on-shell observables homologically for the field theory under consideration, and equivalently a phase space for field theories with gauge symmetries using this enlarged space.

5.1 Deformation

The previous two chapters produced two complexes that each resolved a separate obstruction to the existence of phase space in gauge field theories, specified by a bundle E and Lagrangian $\mathcal{L}$, with a closed generating set of gauge symmetries R_α. We will be working in the global space of fields ΓE; however, the discussion in this chapter straightforwardly descends to the jet bundle since the full BV differential s_{BV} commutes with the exterior derivative, see [BBH00] for more details.

The Koszul-Tate complex is non-positively graded with differential ι_s acting on the antifield generators $\phi^\ddagger_a$ in degree -1 and the antighost generators $c^\ddagger_\alpha$ in degree -2, with higher antighosts in lower degree for a reducible generating set. On generators it is

$$\iota_s \phi^\ddagger_a = \mathrm{EL}_a, \qquad \iota_s c^\ddagger_\alpha = -\sum_{l\geq 0} (-1)^l\, \partial_{\mu_1} \cdots \partial_{\mu_l} \Big(R^{a\mu_1\cdots\mu_l}_\alpha\, \phi^\ddagger_a \Big),$$

extended as a derivation, with $\iota_s \phi^a = 0$. It has degree $+1$ and resolves the functions on the shell,

$$H^0(\iota_s) = \frac{C^\infty(\Gamma E)}{(\mathrm{EL}_a)}, \qquad H^{-n}(\iota_s) = 0, \quad n > 0.$$

The non-negatively graded complex of longitudinal differential forms with differential d_{CE} acts on the field generators ϕ^a and the ghost generators c^α in degree $+1$. On generators it is

$$d_{CE}\phi^a = R^a_\alpha c^\alpha + \cdots\,, \qquad d_{CE} c^\alpha = \tfrac{1}{2} C^\alpha_{\beta\gamma}\, c^\beta c^\gamma + \cdots\,,$$

the Chevalley-Eilenberg differential of the L_∞-algebra $\Gamma E/\mathfrak{g}$ of the previous section. It has degree $+1$ and its degree-zero cohomology $H^0(d_{CE})$ is the gauge-invariant functions. It is not a resolution and its higher cohomology is in general nonzero.

5.1.1 The Derived Critical Locus

The goal is to combine both complexes above to obtain the algebra of gauge-invariant functions on the shell, which is the algebra of observables of a gauge theory.

Since the first is the homotopy intersection with the shell, and the second is the homotopy quotient by the gauge symmetries, it is reasonable to assume that performing both geometric operations will yield the combined complex automatically. This turns out to be the case, and we take the shell inside the homotopy quotient. That is, the derived critical locus of S inside the algebra of the action of the L_∞-algebra $\mathfrak{g}$, $\Gamma E/\mathfrak{g}$,

$$(\Gamma E/\mathfrak{g})_{dS\simeq 0} = \Gamma_0 \cap^h_{T^*(\Gamma E/\mathfrak{g})} \Gamma_{dS},$$

the derived intersection of the section dS along the zero section of the cotangent bundle of $\Gamma E/\mathfrak{g}$.

5.1.2 The Shifted Cotangent Bundle

Theorem 3 (Section 3.3 [Gra22]) *For the action of an L_∞ algebra $\mathfrak{a}$ with a function S, the underlying graded manifold of the derived critical locus $\mathfrak{a}_{dS\simeq 0}$ is given by the shifted cotangent bundle $T^*[-1]\mathfrak{a}$, with algebra of functions the symmetric algebra* $\mathrm{Sym}(\Gamma(T\mathfrak{a})[1])$ *on the shifted vector fields.*

This is the same construction used for the shell without gauge symmetry in the chapter on classical field theory, where we adjoined the anti-field generators $\phi^\ddagger_a = \frac{\delta}{\delta\phi_a}$, applied now with $\mathfrak{a} = \Gamma E/\mathfrak{g}$ in place of ΓE. This shifted cotangent bundle is also an L_∞-algebra, whose structure will become clear after describing its Chevalley-Eilenberg algebra of functions.

Applied to $\mathfrak{a} = \Gamma E/\mathfrak{g}$, whose graded manifold is $\mathfrak{g}[1] \times \Gamma E$ with field generators ϕ^a in degree 0 and ghost generators c^α in degree $+1$, the shifted cotangent bundle $T^*[-1](\Gamma E/\mathfrak{g})$ adjoins to each generator z of degree p a conjugate generator $z^\ddagger$ of degree $-1-p$. This produces exactly the antifield generators of the Koszul-Tate complex,

$$\phi^a \text{ (degree } 0) \;\to\; \phi^\ddagger_a \text{ (degree } -1), \qquad c^\alpha \text{ (degree } +1) \;\to\; c^\ddagger_\alpha \text{ (degree } -2),$$

with reducibility generators producing the lower antighosts. That every ghost c^α of the BRST complex produces a corresponding anti-ghost $c^\ddagger_\alpha$ of the Koszul-Tate complex is expected, since every independent infinitesimal gauge symmetry produces a corresponding Noether identity, and hence a relation in the equations of motion which becomes a relation between the anti-fields. Note that the corresponding generators in the L_∞-algebra are negated then shifted up by 1 compared to the above description, since the Chevalley-Eilenberg algebra shifts down by 1 and takes the dual.

Recall that the tangent space at a point d of the moduli space $\mathcal{M}$ is defined as

$$T_d\mathcal{M} = \mathrm{Def}_{\mathfrak{g}}(\mathbb{R}[\epsilon]/(\epsilon^2)) = H^0(d_{CE,\mathrm{lin}})^* = H^1(\mathfrak{g}, \ell_1).$$

Definition 54 *Let the tangent complex $\mathbb{T}_x\mathcal{M}$ denote the derived refinement of the tangent space $T_x\mathcal{M}$, defined as the linearization of the whole formal deformation problem. This complex is defined as*

$$\mathbb{T}_x\mathcal{M} = (\mathfrak{g}[1], \ell_1)$$

which is equivalently the complex of $Def_{\mathfrak{g}}$ restricted to the square-zero extensions $\mathbb{R} \oplus \mathbb{R}[n]$.

Note that the above puts fields in degree zero. The ordinary infinitesimal disk $\mathbb{R}[\epsilon]/\epsilon^2 = \mathbb{R} \oplus \mathbb{R}[0]$ is the case $n = 0$ and recovers only the classical tangent space $H^0(\mathbb{T}_x\mathcal{M}) = H^1(\mathfrak{g}, \ell_1)$.

A symplectic form of degree k on the formal moduli problem is seen as an element of $\wedge^2\mathbb{T}^*\mathcal{M}[k]$, which using the identifications given above becomes $\wedge^2(\mathfrak{g}[1])[k]^* = \mathrm{Sym}^2(\mathfrak{g}^*)[k-2]$, equivalent to a non-degenerate symmetric bilinear pairing of degree $k-2$ on the L_∞-algebra $\mathfrak{g}$, in the sense of [CG21].

The shifted cotangent bundle $T^*[-1]\mathfrak{a}$ carries a canonical (-1)-shifted symplectic structure, which a derived critical locus inherits, hence the L_∞ algebra in this case has a -3-shifted pairing. This pairing $\langle -, - \rangle \colon \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{R}$ has degree -3, so it is nonzero only on $\mathfrak{g}_i \otimes \mathfrak{g}_{3-i}$. With the ghosts in degree 0, the fields in degree 1, the antifields in degree 2 and the antighosts in degree 3 of $\mathfrak{g}$, the only nonzero components pair a field with an antifield and a ghost with an antighost, as expected. Nondegeneracy identifies $\mathfrak{g}_i \cong (\mathfrak{g}_{3-i})^\vee$, so each field and ghost is paired with exactly one antifield or antighost, its conjugate, and the pairing is the evaluation of a generator against its conjugate.

The inverse of the (-1)-shifted symplectic form is then the 1-shifted BV (global) antibracket of the previous chapter that pairs each generator with its conjugate,

$$(\phi^a, \phi^\ddagger_b) = \delta^a_b\,, \qquad (c^\alpha, c^\ddagger_\beta) = \delta^\alpha_\beta\,.$$

Under this pairing $\phi^\ddagger_a$ acts as $\delta/\delta\phi^a$ and $c^\ddagger_\alpha$ acts as $\delta/\delta c^\alpha$.

5.1.3 The BV Differential

The differential of this complex is obtained using deformation theory. The Koszul-Tate differential ι_s and the Chevalley-Eilenberg differential of $\Gamma E/\mathfrak{g}$ both extend to act on all the generators by their definition as the contraction with dS and the longitudinal exterior derivative, as follows.

The longitudinal differential acts on the base generators as the exterior derivative along the gauge orbits, $d_{CE}f = (X_\alpha f)\, c^\alpha$ with $X_\alpha = R^a_\alpha\, \delta/\delta\phi^a$ the gauge vector field,

$$d_{CE}\phi^a = R^a_\alpha c^\alpha, \qquad d_{CE}c^\alpha = \tfrac{1}{2}C^\alpha_{\beta\gamma}c^\beta c^\gamma.$$

On the shifted cotangent generators, which represent the tangent vectors to the field and ghost generators, the longitudinal differential acts as the Lie derivative $L_{d_{\mathrm{CE}}} = [d_{\mathrm{CE}}, -]$ which gives the action

$$d_{CE}\phi^\ddagger_a = -\phi^\ddagger_b\,\frac{\delta(R^b_\alpha c^\alpha)}{\delta\phi^a} + (\text{terms in } c^\ddagger), \qquad d_{CE}c^\ddagger_\alpha = -C^\gamma_{\alpha\beta}\,c^\ddagger_\gamma c^\beta + (\text{terms in } \phi^\ddagger),$$

the last terms appearing when the structure constants depend on the fields or the R^a_α are not closed under bracket of evolutionary vector fields, respectively.

The Koszul-Tate differential acts on the generators as the contraction with the one-form $dS = \mathrm{EL}_a\, d\phi^a$. On the antifield $\phi^\ddagger_a = \frac{\partial}{\partial\phi^a}$ it evaluates to

$$\iota_s\phi^\ddagger_a = \iota_{dS}\frac{\partial}{\partial\phi^a} = \frac{\delta S}{\delta\phi^a} = \mathrm{EL}_a,$$

and vanishes on the field and ghost generators,

$$\iota_s\phi^a = 0, \qquad \iota_s c^\alpha = 0.$$

The antighost $c^\ddagger_\alpha$ is the Tate generator corresponding to the Noether identity $R^a_\alpha\phi^\ddagger_a$, $\iota_s(R^a_\alpha\phi^\ddagger_a) = R^a_\alpha\mathrm{EL}_a = 0$. Then, we set

$$\iota_s c^\ddagger_\alpha = -R^a_\alpha\phi^\ddagger_a.$$

We denote the grading of the complex of longitudinal differential forms by the ghost number gh and the grading of the Koszul-Tate complex by the negative of the anti-field number $-$af.

The naive approach of simply combining the differentials $\iota_s + d_{CE}$, with total grading deg $=$ gh $-$ af, does not usually lead to a differential on the full complex; $[\iota_s, d_{CE}]$ does not vanish. Instead of requiring that the strict sum of both differentials leads to a differential of the complex, allow higher correction terms which deformation theory can be used to find. The resulting differential s_{BV} must have overall degree $+1$, which means the correction terms must have anti-field degree k and ghost degree $k+1$.

Take ι_s as the differential to be deformed and write

$$s_{\mathrm{BV}} = \iota_s + t, \qquad t = d_{CE} + s_1 + s_2 + \cdots, \qquad \mathrm{af}(s_k) = k,$$

where s_k raises the antifield number by k. The condition $s_{\mathrm{BV}}^2 = 0$ is the Maurer-Cartan equation

$$[\iota_s, t] + \tfrac{1}{2}[t, t] = 0$$

in the graded Lie algebra of derivations of the combined algebra, with differential $[\iota_s, -]$ and bracket the graded commutator. This is the equation $\partial\delta + \frac{1}{2}[\delta, \delta] = 0$ of the previous chapter, with ι_s in place of $[d, -]$ and the antifield number in place of the order in the nilpotent parameter. Refer to [HT92] for the detailed recursive construction. When the generating set is closed and irreducible the series terminates at $\iota_s + \gamma$. The induced differential on $H(\iota_s)$ is γ, thus

$$H^0(s_{\mathrm{BV}}) = H^0\Big(\gamma,\ H^0(\iota_s)\Big) = \{F \in C^\infty(\Gamma E)/(\mathrm{EL}_a) \mid \gamma F = 0\},$$

the gauge-invariant on-shell functions. This is the algebra of observables of the gauge theory, obtained as a single degree-zero cohomology.

The L_∞-algebra structure of $T^*[-1](\Gamma E/\mathfrak{g})$ is now immediate, l_1 is the linear part of the differential s_{BV}, l_2 is dual to its quadratic part, and higher brackets l_n of degree $2-n$ dual to the terms of higher polynomial order. The L_∞-algebra Jacobi identity is then equivalent to $s_{\mathrm{BV}}^2 = 0$.

5.1.4 The Master Action

In this formalism, the full differential s_{BV} can be expressed in terms of the anti-bracket. Explicitly, take the master action for a closed irreducible generating set,

$$S_{\mathrm{BV}} = \int_M dV\, \Big(\mathcal{L} - \phi^\ddagger_a\, R^a_\alpha c^\alpha - \tfrac{1}{2}\, c^\ddagger_\alpha\, C^\alpha_{\beta\gamma}\, c^\beta c^\gamma\Big) = S_0 + S_1 + S_2, \qquad \mathrm{af}(S_k) = k.$$

On the four generators,

$$\begin{aligned}
s_{\mathrm{BV}}\phi^a &= (S_{\mathrm{BV}}, \phi^a) = -\frac{\delta S_{\mathrm{BV}}}{\delta \phi^\ddagger_a} = R^a_\alpha c^\alpha,\\
s_{\mathrm{BV}}c^\alpha &= (S_{\mathrm{BV}}, c^\alpha) = -\frac{\delta S_{\mathrm{BV}}}{\delta c^\ddagger_\alpha} = \tfrac{1}{2}\, C^\alpha_{\beta\gamma} c^\beta c^\gamma,\\
s_{\mathrm{BV}}\phi^\ddagger_a &= (S_{\mathrm{BV}}, \phi^\ddagger_a) = \frac{\delta S_{\mathrm{BV}}}{\delta \phi^a} = \underbrace{\mathrm{EL}_a}_{\text{from } S_0} - \underbrace{\phi^\ddagger_b\, \frac{\delta(R^b_\alpha c^\alpha)}{\delta \phi^a}}_{\text{from } S_1},\\
s_{\mathrm{BV}}c^\ddagger_\alpha &= (S_{\mathrm{BV}}, c^\ddagger_\alpha) = \frac{\delta S_{\mathrm{BV}}}{\delta c^\alpha} = -\underbrace{R^a_\alpha \phi^\ddagger_a}_{\text{from } S_1} - \underbrace{C^\gamma_{\alpha\beta}\, c^\ddagger_\gamma c^\beta}_{\text{from } S_2}.
\end{aligned}$$

Sorting each result by antifield number splits $s_{\mathrm{BV}} = \iota_s + \gamma$ into the part that lowers af by one and the part that preserves it:

$$\begin{aligned}
&\iota_s\phi^a = 0, &&\iota_s c^\alpha = 0, &&\iota_s\phi^\ddagger_a = \mathrm{EL}_a, &&\iota_s c^\ddagger_\alpha = -R^a_\alpha \phi^\ddagger_a,\\
&\gamma\phi^a = R^a_\alpha c^\alpha, &&\gamma c^\alpha = \tfrac{1}{2} C^\alpha_{\beta\gamma} c^\beta c^\gamma, &&\gamma\phi^\ddagger_a = -\phi^\ddagger_b \frac{\delta(R^b_\alpha c^\alpha)}{\delta\phi^a}, &&\gamma c^\ddagger_\alpha = -C^\gamma_{\alpha\beta} c^\ddagger_\gamma c^\beta,
\end{aligned}$$

which are the differentials written above.

For general gauge field theories, the homological perturbation can be performed on the level of action functionals. Writing the antifield-number expansion

$$S_{\mathrm{BV}} = S_0 + S_1 + S_2 + \cdots, \qquad \mathrm{af}(S_k) = k,$$

the components $(S_k, \cdot)$ are exactly the operators $\iota_s, \gamma, s_1, \ldots$ above. The nilpotency $s_{\mathrm{BV}}^2 = 0$ becomes

$$s_{\mathrm{BV}}^2 = \tfrac{1}{2}((S_{\mathrm{BV}}, S_{\mathrm{BV}}), \cdot) = 0,$$

so up to the centre of the antibracket it is equivalent to the classical master equation

$$(S_{\mathrm{BV}}, S_{\mathrm{BV}}) = 0.$$

This procedure will lead to the master action functional, which in terms of the dual to polynomial degree k elements of s_{BV}, ℓ_k, can be written as

$$S_{\mathrm{BV}}(a) = \sum_{n\geq 1} \frac{1}{(n+1)!} \Big\langle a, \, l_n(a, \ldots, a) \Big\rangle.$$

and it is straightforward to see that its classical master equation $(S_{\mathrm{BV}}, S_{\mathrm{BV}}) = 0$ encodes the generalized Jacobi identities of the L_∞-algebra. This can be used to give a concise general definition of a classical field theory as an L_∞-algebra, as done in [Section 4.4 [CG21]].

5.2 The BV-BFV Correspondence

Now, it is possible to construct a phase space for our gauge field theory, assuming spacetime M is globally hyperbolic with spacelike Cauchy surface $\Sigma \subset M$, and the linearized equations of motion are Green hyperbolic, following [CMR14]. Note that if the equations of motion are not Green hyperbolic, it is sometimes sufficient to move to a quasi-isomorphic L_∞ algebra, typically called gauge fixing.

Taking a subspace of spacetime with boundary Σ, the same variational identity that gave the local relation in the classical field theory chapter now holds on the full BV complex.

Writing $\Omega_{\mathrm{BV}} = \delta\phi^\ddagger_a \wedge \delta\phi^a + \delta c^\ddagger_\alpha \wedge \delta c^\alpha + \ldots$ for the BV presymplectic density of the conjugate pairs, the second variation of the master Lagrangian gives, exactly as before,

$$d_H \Omega = \iota_{s_{\mathrm{BV}}} \Omega_{\mathrm{BV}},$$

the local BV-BFV relation, now with Ω the presymplectic current of the extended fields. The only change from the non-gauge case is that Ω_{BV} includes higher ghost-antighost pairs, so the right-hand side carries the longitudinal as well as the Koszul-Tate data.

Transgressing to M exactly as in the previous chapter,

$$\tau_M(d_H\Omega) = \int_M d_M \operatorname{ev}_k^*\Omega = (-)|_\Sigma^* \int_\Sigma \operatorname{ev}_k^*\Omega = (-)|_\Sigma^*\omega_{\mathrm{BFV}}, \qquad \tau_M(\iota_{s_{\mathrm{BV}}}\Omega_{\mathrm{BV}}) = L_Q\,\omega_{\mathrm{BV}},$$

so that

$$(-)|_\Sigma^*\omega_{\mathrm{BFV}} = L_Q\,\omega_{\mathrm{BV}}.$$

The boundary symplectic form is again the BV differential of the bulk BV symplectic form, but now ω_{BFV} is the graded symplectic form on the boundary BV-BFV complex. Its degree-zero part is the Hamiltonian symplectic form $\omega|_\Sigma$.

Unlike the case without gauge symmetry, the differential Q restricts to a differential Q_∂ on the boundary which determines the appropriate gauge-invariant algebra of functions on the phase space in homology in degree zero, since the phase space is now a derived space, and the boundary action S_{BFV} is determined by this differential as its corresponding Hamiltonian function $\iota_{Q_\partial}\omega_{\mathrm{BFV}} = \delta S_{\mathrm{BFV}}$.

For a region bounded by two Cauchy surfaces Σ_{in} and Σ_{out},

$$L_Q\,\omega_{\mathrm{BV}} = \omega_{\mathrm{BFV}}|_{\Sigma_{\mathrm{in}}} - \omega_{\mathrm{BFV}}|_{\Sigma_{\mathrm{out}}},$$

so $L_Q\omega_{\mathrm{BV}}$ is again a potential for the boundary symplectic form, and the phase space is independent of the choice of Cauchy surface in Q-cohomology.

5.2.1 Electrodynamics

To demonstrate how the formalism above is applied, we work out the example of free electrodynamics on Minkowski space.

Let spacetime be Minkowski space $M = (\mathbb{R}^{3,1}, \eta)$ with Minkowski metric $\eta = \operatorname{diag}(-1,1,1,1)$ and field bundle the cotangent bundle $E = T^*M \to M$ whose sections are the 1-form potentials $A = A_\mu dx^\mu$. On the jet bundle with coordinates $(x^\mu, A_\nu, A_{\nu,\mu}, \ldots)$, define the field strength as the exterior differential of the 1-form potential

$$F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu},$$

and the Maxwell Lagrangian density

$$\mathcal{L} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}\,dV \in \Omega^{4,0}.$$

Its variation is computed exactly as in the free scalar case,

$$\delta\mathcal{L} = -F^{\mu\nu}\,\delta A_{\nu,\mu} \wedge dV = \underbrace{\partial_\mu F^{\mu\nu}\,\delta A_\nu \wedge dV}_{\delta_{EL}\mathcal{L}} - d_H\,\underbrace{\left(F^{\mu\nu}\,\delta A_\nu \wedge \iota_{\partial_\mu} dV\right)}_{\Theta},$$

so the components of the Euler-Lagrange operator, the pre-symplectic form, and the pre-symplectic current are are

$$\mathrm{EL}^\nu = \partial_\mu F^{\mu\nu}, \qquad \Theta = F^{\mu\nu}\,\delta A_\nu \wedge \iota_{\partial_\mu} dV, \qquad \Omega = \delta\Theta = \delta F^{\mu\nu} \wedge \delta A_\nu \wedge \iota_{\partial_\mu} dV.$$

Defining the electric field E^μ magnetic field B^μ and vector field u^μ as sections of the tangent bundle $TM \to M$ where $\eta_{\mu\nu}u^\mu u^\nu = -1$ and $\eta_{\mu\nu}u^\mu E^\nu = \eta_{\mu\nu}u^\mu B^\nu = 0$, it is possible to decompose $F^{\mu\nu}$ as

$$F^{\mu\nu} = u^\mu E^\nu - E^\mu u^\nu + \epsilon^{\mu\nu\eta\xi}u_\eta B_\xi$$

where $\epsilon^{\mu\nu\eta\xi}$ is the totally anti-symmetric tensor with $\epsilon^{0123} = -1$. Define $h_{\mu\nu} = \eta_{\mu\nu} + u_\mu u_\nu$ as the projection orthogonal to u^μ, then the shell $\mathrm{EL}^\nu = \partial_\mu F^{\mu\nu} = 0$ consists of the vacuum Maxwell equations as follows.

The equation $\partial_\mu F^{\mu\nu} = 0$ decomposes as $u_\nu \partial_\mu F^{\mu\nu} = 0$ and $h^\xi_\nu \partial_\mu F^{\mu\nu} = 0$ which become

$$u_\nu \partial_\mu F^{\mu\nu} = \partial_\mu u_\nu F^{\mu\nu} = \partial_\mu E^\mu = 0 \iff \nabla \cdot E = 0.$$

and

$$h^\sigma_\nu \partial_\mu F^{\mu\nu} = h^\sigma_\nu \partial_\mu (u^\mu E^\nu + \epsilon^{\mu\nu\eta\xi}u_\eta B_\xi) = 0 \iff \frac{\partial E}{\partial t} = \nabla \times B;$$

the remaining half of Maxwell's equations $\partial_{[\lambda}F_{\mu\nu]} = 0$ (obtained by swapping the role of E and B up to a sign) holds identically on the jet bundle by the definition of F as a horizontal exterior derivative.

The gauge symmetries are the evolutionary vector fields which shift by a total derivative

$$R_\epsilon = \epsilon_{,\nu}\,\partial_{A_\nu}$$

since A is a 1-form. The corresponding Noether identity,

$$\sum_l (-1)^l \partial_{\mu_1} \cdots \partial_{\mu_l}\Big(R_\nu{}^{\mu_1\cdots\mu_l}\,\mathrm{EL}^\nu\Big) = -\partial_\mu \mathrm{EL}^\mu = -\partial_\mu\partial_\nu F^{\nu\mu} = 0,$$

holds identically by antisymmetry of $F^{\nu\mu}$. The gauge Lie algebra is $\mathfrak{g} = C^\infty_c(M)$, acting on $\Gamma E = \Omega^1(M)$ by $A \mapsto A + d\epsilon$.

Since the generating set is closed and irreducible, the BV complex is the shifted cotangent bundle $T^*[-1](\Gamma E/\mathfrak{g})$, with generators A_μ, c, $A^{\ddagger\mu}$, $c^\ddagger$ in ghost numbers $0, +1, -1, -2$ and the antibracket pairing each generator with its conjugate. The homological perturbation series terminates, $s_{\mathrm{BV}} = \iota_s + \gamma$, and the master action is

$$S_{\mathrm{BV}} = \int_M \Big(-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} - A^{\ddagger\mu}\,\partial_\mu c\Big)\,dV,$$

with $s_{\mathrm{BV}} = (S_{\mathrm{BV}}, \cdot)$ acting as

$$\gamma A_\mu = \partial_\mu c, \qquad \iota_s A^{\ddagger\mu} = \partial_\nu F^{\nu\mu}, \qquad \iota_s c^\ddagger = \partial_\mu A^{\ddagger\mu},$$

and vanishing on the remaining generators, since the gauge generators are field-independent and the structure functions vanish. Both the nilpotency of the Koszul–Tate part and the classical master equation

$$(S_{\mathrm{BV}}, S_{\mathrm{BV}}) = 2\int_M \partial_\mu F^{\mu\nu}\,\partial_\nu c\, dV = -2\int_M \Big(\partial_\nu\partial_\mu F^{\mu\nu}\Big)\, c\, dV = 0$$

reduce to the Noether identity. The antighost $c^\ddagger$ is the Tate generator killing the degree -1 element $\partial_\mu A^{\ddagger\mu}$, so $H^{-n}(\iota_s) = 0$ for $n > 0$ and

$$H^0(s_{\mathrm{BV}}) = H^0\Big(\gamma, H^0(\iota_s)\Big) = \Big\{ f \in C^\infty(\Gamma E)/(\partial_\mu F^{\mu\nu}) \;\Big|\; \gamma f = (dc)^\mu \partial_{A^\mu} f = 0 \Big\},$$

the functions of A^μ invariant under the addition of total differentials to the A^μ quotiented by the equations of motion, the gauge-invariant on-shell observables. The transgressions $\tau_M(f\, dV)$ give the local observables of electrodynamics.

Since the linearized Euler-Lagrange operator is not Green hyperbolic, the theory is gauge-fixed by passing to a quasi-isomorphic L_∞-algebra in two steps, neither of which changes $H^0(s_{\mathrm{BV}})$. This gauge-fixing procedure is the homological equivalent of gauge-fixing in classical electrodynamics, where below we will enforce the Lorenz gauge condition $\partial_\mu A^\mu = 0$.

First, adjoin field coordinates $(\bar{c}, b)$ in ghost numbers $(-1, 0)$ with their antifields, extending the master action by $-\int_M \bar{c}^\ddagger b\, dV$ so that $s_{\mathrm{BV}}\bar{c} = b$ and $s_{\mathrm{BV}} b = 0$.

Second, apply the canonical transformation generated by the gauge-fixing fermion $\Psi = -\int_M \bar{c}\,\partial^\mu A_\mu\, dV$, which shifts $A^{\ddagger\mu} \mapsto A^{\ddagger\mu} - \partial^\mu \bar{c}$ and $\bar{c}^\ddagger \mapsto \bar{c}^\ddagger + \partial^\mu A_\mu$ and preserves the antibracket. This allows transforming to vanishing antifields which yields the Lorenz-gauge action

$$S_\Psi\Big|_{\phi^\ddagger=0} = \int_M \Big(-\tfrac{1}{4} F_{\mu\nu}F^{\mu\nu} + b\,\partial^\mu A_\mu + \partial^\mu \bar{c}\,\partial_\mu c \Big)\, dV.$$

Note the Lagrange multiplier term $b\partial^\mu A_\mu$ which enforces the Lorenz gauge. The equations of motion combine to $\Box A^\nu = \partial^\nu b$, $\Box b = \Box c = \Box \bar{c} = 0$, which are Green hyperbolic, which means every space-like hyperplane Σ_t is a Cauchy surface and the Peierls bracket is defined through the causal propagator of $\Box$.

We suppress the trivial pair $(\bar{c}$,$b)$ in what follows. The BV pre-symplectic current and symplectic form are then

$$\Omega_{\mathrm{BV}} = \Big(\delta A^{\ddagger\mu} \wedge \delta A_\mu + \delta c^\ddagger \wedge \delta c\Big)\, dV, \qquad \omega_{\mathrm{BV}} = \tau_M(\Omega_{\mathrm{BV}}).$$

Let M have a boundary given by a spacelike Cauchy surface, and compute the BV-BFV relation $(-)|^*_{\partial M}\, \omega_{\mathrm{BFV}} = L_Q\, \omega_{\mathrm{BV}}$. The field-antifield coordinates give

$$\int_M \delta(\partial_\nu F^{\nu\mu}) \wedge \delta A_\mu\, dV = \int_M d_\nu\Big(\delta F^{\nu\mu} \wedge \delta A_\mu\Big)\, dV - \int_M \delta F^{\nu\mu} \wedge \delta A_{\mu,\nu}\, dV,$$

and the volume term vanishes since

$$\delta F^{\nu\mu} \wedge \delta A_{\mu,\nu} = \tfrac{1}{2}\, \delta F^{\nu\mu} \wedge \Big(\delta A_{\mu,\nu} - \delta A_{\nu,\mu}\Big) = \tfrac{1}{2}\, \delta F^{\nu\mu} \wedge \delta F_{\nu\mu} = 0.$$

The ghost-antighost coordinates give

$$\int_M \Big(\partial_\mu \delta A^{\ddagger\mu} \wedge \delta c + \delta A^{\ddagger\mu} \wedge \partial_\mu \delta c\Big)\, dV = \int_M d_\mu \Big(\delta A^{\ddagger\mu} \wedge \delta c\Big)\, dV.$$

Applying Stokes' theorem to both boundary terms,

$$L_Q\, \omega_{\mathrm{BV}} = \int_{\partial M} \Big(\delta F^{\nu\mu} \wedge \delta A_\mu + \delta A^{\ddagger\nu} \wedge \delta c\Big)\, n_\nu\, dA = (-)|^*_{\partial M}\, \omega_{\mathrm{BFV}}.$$

Specializing to the Cauchy surface $\Sigma = \{(x, y, z, 0)\}$ with unit normal ∂_0, the above implies that the boundary fields are

$$A_i\Big|_\Sigma, \qquad E^i := F^{0i}\Big|_\Sigma, \qquad c\Big|_\Sigma, \qquad c^\ddagger_\Sigma := A^{\ddagger 0}\Big|_\Sigma$$

with the boundary symplectic form

$$\omega_{\mathrm{BFV}} = \int_\Sigma \Big(\delta A_i \wedge \delta E^i + \delta A^{\ddagger 0} \wedge \delta c\Big)\, dA.$$

and restriction map

$$\Big(A_\mu,\, c,\, A^{\ddagger\mu},\, c^\ddagger\Big) \;\longmapsto\; \Big(A_i|_\Sigma,\; E^i = -F^{i0}|_\Sigma,\; c|_\Sigma,\; c^\ddagger_\Sigma = A^{\ddagger 0}|_\Sigma\Big),$$

generalizing the map $(\phi, \phi^\ddagger) \mapsto (\phi|_{\partial M}, \partial_n \phi|_{\partial M})$ of the scalar field. The degree-zero part of ω_{BFV} is exactly the Hamiltonian symplectic form of electrodynamics as expected.

The boundary differential Q_∂ is induced by restriction of the bulk differential to the boundary fields,

$$Q_\partial A_i = \partial_i c, \qquad Q_\partial E^i = 0, \qquad Q_\partial c = 0, \qquad Q_\partial c^\ddagger_\Sigma = \partial_\nu F^{\nu 0}\Big|_\Sigma = -\partial_i E^i.$$

The vector field Q_∂ is Hamiltonian for ω_{BFV}, $\iota_{Q_\partial}\omega_{\mathrm{BFV}} = \delta S_{\mathrm{BFV}}$, with BFV action

$$S_{\mathrm{BFV}} = \int_\Sigma c\, \partial_i E^i\, dA,$$

and satisfies the boundary master equation $\{S_{\mathrm{BFV}}, S_{\mathrm{BFV}}\} = 0$ trivially. The derived manifold (Σ_t, Q_∂) with symplectic form ω_{BFV} is then the phase space of classical electrodynamics.

The algebra of observables is then its degree-zero cohomology,

$$H^0(Q_\partial) = \Big\{ f \in C^\infty \Big(\Big\{ (A_i, E^i) \mid \partial_i E^i = 0 \Big\} \Big) \;\Big|\; (dc)^i \partial_{A^i} f = 0 \Big\}.$$

Recall that the pair (A_i, E^i) with the Gauss's law constraint $\partial_i E^i = 0$ is initial data for electrodynamics; briefly, the spatial part A_i of the 1-form potential determines (up to a gauge transformation) the magnetic field strength, hence we obtain the initial electric and magnetic field configuration (B^i, E^i); the purely spatial constraint $u_\nu \mathrm{EL}^\nu = \partial_i E^i = \nabla \cdot E = 0$ must be satisfied on the Cauchy surface Σ_t, and is propagated for all other times. The algebra of observables of the BFV boundary theory is then the expected algebra of functions on the phase space of electrodynamics, therefore the equivalence between the algebra of observables in the BFV boundary theory and the algebra of observables in the BV theory follows by pullback of the solution map, as in the case of classical mechanics.

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