Weight polynomials of $\overline{\mathrm{Jac}}$ and link invariants

§1. Compactified Jacobians & affine Springer fibers

§2. Orbital integrals

§3. Elliptic Hall algebra

§4. Shalika germs

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§1. $\overline{\mathrm{Jac}}$, Sp_γ, M_a

Let X/k be smooth projective, D effective divisor $\deg D > 2g_X$. $G = GL_n$ throughout, $k = \overline{\mathbb{F}_q}$ or $\mathbb{C}$

<u>Def.</u> A Higgs bundle is a pair

$(\mathcal{E}, \varphi : \mathcal{E} \longrightarrow \mathcal{E} \otimes \mathcal{O}(D))$, $\mathcal{E}$ rk n vector bundle.

Note: $\mathrm{tr}\,\varphi : \mathcal{O} \to \mathrm{End}(\mathcal{E}) \to \mathcal{O}(D) \in H^0(\mathcal{O}(D))$

similarly, $\mathrm{tr} \Lambda^i \varphi \in H^0(\mathcal{O}(iD))$

Define $\mathrm{char}_\varphi(X) = X^n - \mathrm{tr}(\varphi) X^{n-1} + \dots + (-1)^n \mathrm{tr}(\Lambda^n \varphi)$

Def. The Hitchin moduli space is

$$\mathcal{M} = \{(\mathcal{E}, \varphi)\}, \text{ and the Hitchin fibration}$$

is

$$\begin{array}{c} \mathcal{M} \\ \downarrow \pi \\ B = \bigoplus_{i=1}^{n} H(\mathcal{O}(iD)) \end{array}$$

with $\pi(\mathcal{E}, \varphi) = \mathrm{Char}_\varphi$.

A <u>Hitchin fiber</u> is $\mathcal{M}_a := \pi^{-1}(a)$, $a \in B$.

If $k = \overline{\mathbb{F}_q}$, have adèlic description as follows.

Def $\mathbb{A}_F$ = adèles of function field $F = k(X)$.

$$\begin{array}{l} \cup \\ \mathcal{O} = \prod_{x \in |X|} \underbrace{\hat{\mathcal{O}}_{X,x}}_{\cong k[[t_p]]} \end{array}$$

Divisor $D = \sum n_p [p] \rightsquigarrow \varpi_D = (t_p^{n_p}) \in \mathbb{A}_F^\times$

Let $\mathcal{H}_a \subset G(\mathbb{A}_F)/G(\mathcal{O}) \times \mathfrak{g}(F)$

be all (g, γ) with

$\mathrm{Char}_\gamma(x) = a \in B$, $g^{-1}\gamma g \in \varpi_D^{-1} \mathfrak{g}(\mathcal{O})$.

This has left $G(F)$-action, and

Prop $\left[G(F) \backslash \mathcal{H}_a\right] \cong \mathcal{M}_a$ as groupoids

(Ngô, Laumon)

Let $\gamma \in B^{n.s.s.}$. Its centralizer is

$T_\gamma(F) \subset G(F)$ some max torus (not split in general)

$\rightsquigarrow$ $[G(F)\backslash \mathcal{H}_a] = [T_\gamma(F)\backslash \mathcal{H}_\gamma]$ where

$$\mathcal{H}_\gamma = \{ g \in G(\mathbb{A}_F)/G(\mathcal{O}) \mid g^{-1}\gamma g \in \varpi_D^{-1}\mathfrak{g}(\mathcal{O}) \}$$

<u>Lemma</u>

$$\mathcal{H}_\gamma = \prod_{x \in |X|}' Sp_{\gamma_x}(k), \text{ where}$$

$$Sp_{\gamma_x} = \{ g \in G(F_{x,x})/G(\mathcal{O}_{x,x}) \mid g^{-1}\gamma g \in \mathfrak{g}(\mathcal{O}_{x,x}) \}$$

is an <u>affine Springer fiber</u>.

On the other hand, let

$$Tot(\mathcal{O}(D)) = \mathrm{Spec}\left(\bigoplus_{i=0}^{\infty} \mathcal{O}(-iD)x^i\right) \longrightarrow X$$

$a \in B$ defines a closed curve in $Tot(\mathcal{O}(D))$ given by $\{a(x)=0\}$.

This is the spectral curve, $Y_a \longrightarrow X$ finite ramified cover.

<u>Theorem</u> (Hitchin – BNR)

$$\mathcal{M}_a \cong \left\{ \begin{array}{c} \text{Rank 1 torsion-free coherent sheaves} \\ \text{on } Y_a \end{array} \right\}$$

$:= \overline{\mathrm{Pic}}(Y_a) =$ the Compactified Picard of Altman Kleiman Iarrobino

<u>Remark 1</u> If we take SL_n, and require $\deg(\det(\gamma)) = 0$ above, get $\overline{\mathrm{Jac}}(C)$.

<u>Remark 2</u> For good properties, need $a \in B^{ell}$, i.e. Y_a irreducible.

§2. Orbital integrals

<u>Prop</u> (Ngô) (Notes by Chaudouard)

$a \in \mathcal{B}^{\mathrm{ell}}$

The number of $\mathbb{F}_q$-points on $\mathcal{M}_a^{SL_n}$ equals (up to a normalization factor)

$$\underbrace{\prod_{x \in |X|} \int_{T_{\gamma_x}(F_x)\backslash G(F_x)} \mathbb{1}_{\mathfrak{g}(\mathcal{O}_x)}(g^{-1}\gamma_x g)\, dg_x}$$

 $\subset \mathrm{Tot}(\Omega(D))$

If # of pts is finite, only finitely many factors here are $\neq 1$.

They are $\neq 1$ iff $Sp_{\gamma_x}/T(F_x) \neq pt$.

Thus, we will be interested in

$$\int_{\substack{G(K) \\ \left\{ T(K)\backslash G(K) \right. \\ \simeq G(K)\gamma}} \mathbb{1}_{\mathfrak{g}(\mathcal{O})}(g^{-1}\gamma g)\, dg \;, \quad K = k((t)) \;; \text{ i.e.}$$

<u>local</u> orbital integrals. Variants:

Let $L \subset \mathfrak{g}(K)$ be free full rank $\mathcal{O}$-submodule i.e. a <u>lattice</u>. For example $L = \mathrm{Lie}(I) =$ Iwahori subalgebra, etc. $\rightsquigarrow$ $\mathbb{1}_L$ = characteristic fn.

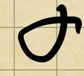
In general, if $f \in C^\infty(\mathfrak{g}(K))$,

Def $I_\gamma(f) := \int_{T(K)\backslash G(K)} f(g^{-1}\gamma g)\,dg$. This is an invariant distribution on $\mathfrak{g}(K)$ = orbital integral of γ

We will further normalize things so that $X_*(T) \subset T(K)$ has covolume 1; this gives

$$I_\gamma(\mathbb{1}_{\mathfrak{g}(\mathcal{O})}) = \#\{gG(\mathcal{O}) \mid g^{-1}\gamma g \in \operatorname{Lie} G(\mathcal{O})\}/X_*(T)$$

Ex. $\gamma = \begin{pmatrix} t & \\ & -t \end{pmatrix}$ $\quad Sp_\gamma =$  $\circlearrowleft \mathbb{Z}^2 = \Lambda = X_*(T)$

$\#$pts on $Sp_\gamma/\Lambda =$  $= q+1-1 = q$

Weight polynomials: Spreading out:

If $k = \mathbb{C}$, can find for every $\gamma \in \mathfrak{g}(\mathbb{C}((t)))$ a finitely generated field over $\mathbb{Q}$ s.t. γ defined over this. Take its ring of integers R, and some large prime therein; work modulo p.

In other words, Sp_γ becomes a scheme over $\operatorname{Spec} R$, and its reduction over $\operatorname{Spec} \mathbb{F}_q$.

Counting points recovers the weight filtn. on $\pi_* \overline{\mathbb{Q}}_\ell$,

$$Sp_\gamma \xrightarrow{\pi} \operatorname{Spec} R.$$

Hence, we are computing weight polynomials of local $\operatorname{Jac}(C)$.

More on invariant distributions:

$\mathrm{Dist}_G(\mathfrak{g})$ = G-invariant distributions on $\mathfrak{g} = \mathrm{Lie}(G)$ (over $k((t))$)

<u>Thm</u> (Harish-Chandra, Howe, Hales, Waldspurger, de Backer ...)

$\dim_{\mathbb{C}} \mathrm{Dist}_G(\mathfrak{g}) < \infty$

basis given by $I_\lambda(\cdot) :=$ nilpotent orbital integral

$\lambda \vdash n$, $I_\lambda(f) = \int_{Z_G(\lambda) \backslash G(F)} f(g^{-1}\lambda g)\, dg$

<u>Corollary</u> $\gamma \in \mathfrak{g}(k((t)))^{r.s.}$

$$I_\gamma(f) = \sum_\lambda \underbrace{\Gamma_\lambda(\gamma)}_{\text{don't depend on } f!} I_\lambda(f)$$

$\Gamma_\lambda(\gamma) =:$ Shalika germs of γ

Goal: Compute $I_\gamma(\mathbb{1}_{\mathfrak{g}(\mathcal{O})})$ using this knowledge.

§3. Elliptic Hall algebra. $k = \mathbb{C}$

Motivation: $0 \in Y_a$ spectral curve, singular at 0

$$\mathrm{Link}_0(Y_a) := S^3_{0,\varepsilon} \cap Y_a$$

ex. $\prec\ x^3 - y^2 \rightsquigarrow$ trefoil knot in S^3.

<u>Conjecture</u> (ORS) (+ Maulik-Yun, Migliorini-Shende)

$$\frac{\sum q^i \, gr_p^i \, H^*(\overline{\mathrm{Jac}}(Y_a))}{1-q} = \underbrace{HHH}_{\text{triply graded KR homology}}{}^{a=0}(\mathrm{Link}(Y_a))$$

$\mathbb{Z}^2$-graded v.s.

<u>Conjecture</u> Poincaré poly of $HHH^{a=0}$ can be computed using "superpolynomials"

Let $\ddot{\mathcal{U}}_{q_1 q_2 q_3}(gl_1) = \mathcal{E}_{q_1, q_2, q_3}$, $q_1 q_2 q_3 = 1$

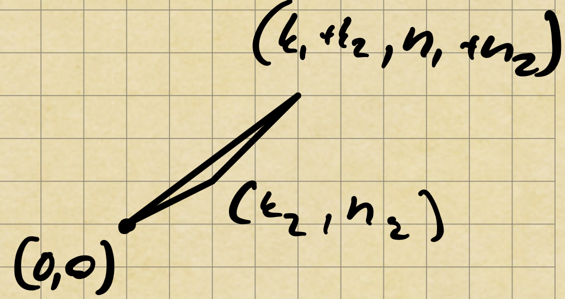
be generated by e_i, f_i, $i \in \mathbb{Z}$, $\psi_i^{\pm}$, $i \in \mathbb{Z}_{\geq 0}$, mod...

Fact 1. $\mathcal{E} = \underbrace{\mathcal{E}^{>0}}_{\text{we will be only looking at this part.}} \otimes \mathcal{E}^0 \otimes \mathcal{E}^{<0}$

One can prove $\mathcal{E}^{>0}$ is generated by operators $P_{k,n}$, $k, n \in \mathbb{Z}_{\geq 0}$.

These satisfy the relations

1) $\left[P_{k_1,n_1}, P_{k_2,n_2}\right] = \dfrac{\Theta_{k_1+k_2,n_1+n_2}}{\alpha_1}$ if

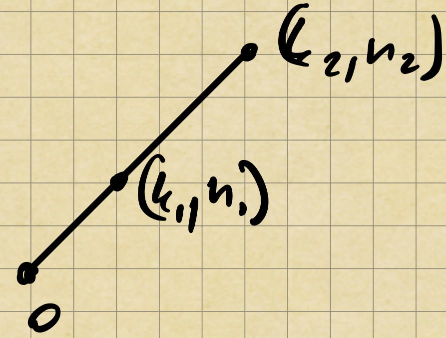


contains no lattice pts

where $\alpha_1 = \dfrac{(q_1^n - 1)(q_2^n - 1)(q^{-n} - 1)}{n}$ and

$$\sum_{n \geq 0} x^n \Theta_{na,nb} = \exp\left(\sum_{n=1}^{\infty} \alpha_n x^n P_{na,nb} \right)$$

2) $\left[P_{k_1,n_1}, P_{k_2,n_2}\right] = 0$ if



Thm (Burban-Schiffmann, Feigin-Tsymbaliuk)

$$\mathcal{E}_{q,t,\frac{1}{qt}} \curvearrowright K^{(\mathbb{C}^*)^2}(\mathrm{Hilb}^\bullet(\mathbb{C}^2)) \cong \mathrm{Sym}_{q,t}$$

"Fock space representation"

<u>Thm</u> (Negut, Gorsky-Negut)

In the basis of Macdonald polynomials, the matrix coefficients of $P_{km,kn}$ are

$$\langle H_\mu | P_{m,n} | H_\lambda \rangle = \frac{\chi^{kn}}{[k]} \cdot \frac{g_\lambda}{g_\mu} \cdot$$

$$\sum_{\mu = \lambda + \square_1 + \dots + \square_{kn}}^{\mathrm{SYT}} \left(\sum_{j=0}^{k-1} \left(\frac{q}{t}\right)^j \frac{x_{n(k-1)+1}\, x_{n(k-2)+1} \cdots x_{n(k-j)+1}}{x_{n(k-1)}\, x_{n(k-2)} \cdots x_{n(k-j)}} \right) \cdot$$

$$\frac{\prod_{i=1}^{kn} x_i^{\lfloor \frac{im}{n} \rfloor - \lfloor \frac{(i-1)m}{n} \rfloor} (q + x_i - 1)}{\left(1 - \frac{q}{t} + \frac{x_2}{x_1}\right) \cdots \left(1 - \frac{q}{t} + \frac{x_{kn}}{x_{kn-1}}\right)} \prod_{1 \le i < j \le kn} \omega^{-1}\left(\frac{x_j}{x_i}\right) \prod_{1 \le i \le kn}^{\mathrm{Del}} \omega^{-1}\left(\frac{\chi(\square)}{x_i}\right)$$

<u>Remark</u>. In the above, pairing wrt

$\langle \, , \, \rangle_{q,t}$. Orthogonal for $\tilde{H}_\lambda$.

<u>Corollary</u> (Gorsky-Negut, Cherednik)

Have explicit expression for $P_{m,n} \cdot 1 \in \mathrm{Sym}_{q,t}$.
as sum of Macdonald polynomials.
(= superpolynomials of torus knots)

Degeneration as $\frac{q}{t} \longrightarrow 1$.

Thm. (Tsymbaliuk)

$$\ddot{\mathcal{U}}_{t,1,\frac{q}{t}}(\mathfrak{gl}_1)^{\wedge} \cong \mathcal{U}\left(\widetilde{\mathrm{Diff}}_{\frac{q}{t}=e^{\hbar}}(\mathbb{C}^*)\right)$$

At $\frac{q}{t}=1$, $P_{m,n}\cdot 1$ simplifies a **lot**.

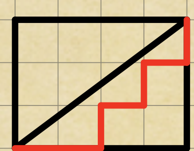
In particular, have

Prop (Gorsky, F. Bergeron, ...)

$$P_{m,n}\cdot 1 \underset{\frac{q}{t}=1}{=} \sum_{\pi\in \mathrm{Dyck}_{m,n}} t^{\mathrm{area}(\pi)} \underset{\sim}{e_\pi}$$

Elementary symmetric function.

Dyck paths:

$\pi =$  $\in \mathrm{Dyck}_{3,4}$

$\mathrm{area}\,\pi = 0$

$e_2 e_1^2 = e_{\boxplus}$

Cherednik, Danilenko, Gorsky

torus knots (m,n) $\rightsquigarrow$ iterated torus knots $(m_1,n_1),(m_2,n_2),\dots,(m_r,n_r)$

$(\frac{q}{t}=1)$ $P_{m,n}\cdot 1$

expand $P_{m_r,n_r}\cdot 1$ in elementary sym. fns, and replace each e_k by $P_{km_{r-1},kn_{r-1}}$, apply to 1, etc. until you reach (m_1,n_1)

§4. Shalika germs ($G = GL_n$)

Def $\mathrm{Dist}_G(\mathfrak{g})$ = G-invariant distributions on $\mathfrak{g}$ over $K = \mathbb{F}_q((t))$.

Thm (Many people)

$\mathrm{Dist}_G(\mathfrak{g})$ is finite-dimensional.

It is spanned by the nilpotent orbital integrals, in particular we have the Shalika germ expansion: For $f \in C_c^\infty(\mathfrak{g})$,

$$I_\gamma(f) = \sum_{\lambda \in \mathcal{N}_G} \Gamma_\lambda(\gamma) I_\lambda(f) .$$

The functions $\Gamma_\lambda(\gamma)$ are defined in a neighborhood of $0 \in \mathfrak{g}$, and are called Shalika germs.

Goal: Find nice test functions for which $I_\gamma(f)$, $I_\lambda(f)$ can be computed.

Inverting matrix of enough such functions yields $\Gamma_\lambda(\gamma)$.

<u>Thm</u> (Waldspurger '89)

Let $\mathbb{1}_\mu$ be the characteristic function of $\mathrm{Lie}(P_\mu)$, where P_μ is a standard parahoric.

Then $$I_\lambda(\mathbb{1}_\mu) = \langle \underset{\text{elementary symmetric fn}}{e_\lambda}, \underset{\text{homogeneous symmetric fns.}}{h_\mu} \rangle_{+}$$

One can upgrade this to more complicated characteristic functions, for which

$$I_\gamma(\mathbb{1}_\alpha) = \int_{T\backslash G^{<d}} \mathbb{1}_\alpha(g^{-1}\gamma^{<d} g)\,dg \,,$$

an integral on a <u>smaller group</u>.

After several steps, Waldspurger gives a recursive algorithm for $\underbrace{\Gamma^{st}_\lambda(\gamma)}$; by writing

$$\gamma = \underset{\text{depth}}{\gamma_{d_1} + \cdots + \gamma_{d_r}}$$

slight modification.

Bernstein Zelevinsky $\left\{\mathcal{X}(\mathrm{Ind}_\lambda\, st)\right\}$

Span $\underset{\sim\sim}{\mathrm{Dist}_G(\mathfrak{g})}$

$$I_\gamma(F) = \Sigma_\lambda\, \mathcal{X}_{\mathrm{Ind}_\lambda st}(F) \cdot \Gamma^{st}_\lambda(\gamma)$$

but commenting that it seems to him very hard to compute anything.

More precisely, write $F=\sum_\lambda h_\lambda \Gamma^{st}_\lambda(\gamma)$.

Then **Thm** (Waldspurger, ~'91) $\left(\begin{smallmatrix} d=m_{r-1}\\ e=n_{r-1}\end{smallmatrix}\right)$

$$\sum \Gamma^{st}_\lambda(\gamma)\, h_\lambda = \sum_{\lambda' \vdash \frac{n}{e}} \Gamma^{st}_{\lambda'}(\gamma')\, E^<[\lambda', d, e]$$

$\gamma' = \gamma - \gamma_{d_r}$ $= \gamma^{<d}$, $E^<[\lambda', d, e] =$ Coefficient of a certain series

$$h_n = \sum_{\alpha \vDash n} (-1)^{\ell(\alpha)} e_\alpha$$

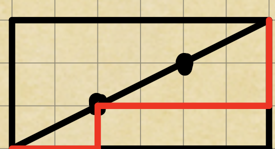
Thm (K.-Tsai)

$g = \sum e_\lambda\, \sigma_\lambda(\gamma)$

$(g=F)$

$$\sum e_\lambda\, \sigma_\lambda(\gamma) = \sum_{\lambda' \vdash \frac{n}{e}} \sigma_{\lambda'}\, E[\lambda', d, e]$$

Moreover, $E[\lambda', d, e] = e_{\lambda'}$ where we replace each $e_{\lambda'_i}$ by $\sum_{\pi \in \mathrm{Dyck}_{\lambda'_i d, \lambda'_i e}} t^{\mathrm{area}(\pi)} e_\pi$



Rmk Waldspurger's $E^<[\lambda', e, d] =$

$$\sum_{\pi \in \mathrm{Dyck}^<_{\lambda'_i d, e\lambda'_i}} t^{\mathrm{area}(\pi)} e_\pi$$

Weight polynomial:

$\langle F, e_n \rangle = \# M_d(\mathbb{F}_q)$

Note that we need to change basis

$$h_\lambda \rightsquigarrow \tilde{h}_\lambda = \lim_{q \to 1} \tilde{H}_\lambda, \text{ to get Shalika germs } (*)$$

$$\rightsquigarrow e_\lambda, \text{ to get} \ldots$$

TODO: Find combinatorial formula for expansion of f in (*).

§5. Bonus: Hilbert schemes, q-deformation

Recent work w/ Gorsky–Oblomkov

$$\begin{array}{l} \gamma \in \mathfrak{g}((t))^{r.s.,\,ell} \\ \big\downarrow \\ \widetilde{\mathcal{F}}_\gamma \in QCoh^{\mathbb{C}^\times}(\mathrm{Hilb}_0(\mathbb{C}^2)) \end{array}$$

Expectation 1:

$$\begin{tikzcd} & \gamma \arrow[r, squiggly] \arrow[dl] & \beta,\ \bar{\beta} = \mathrm{Link}_0(\gamma_a) \arrow[d, squiggly, "\mathrm{OR},\ \mathrm{GNR}"] \\ \mathcal{F}_\gamma & & \mathcal{G}_\gamma \in D^b Coh^{\mathbb{C}^\times \times \mathbb{C}^\times}(\mathrm{Hilb}(\mathbb{C}^2)) \end{tikzcd}$$

in K-theory (at $q=1$), $[\mathcal{F}_\gamma] = [\mathcal{G}_\gamma]$

Expectation 2:

$$[\mathcal{G}_\gamma] \in K_{\mathbb{C}^\times \times \mathbb{C}^\times}(\mathrm{Hilb}^\cdot(\mathbb{C}^2)) \cong \mathrm{Sym}_{q,t}$$

$\quad\ \ \| $

$F =$ "master" symmetric function for γ.

These expectations would imply that Shalika germs $\Gamma_\lambda(\gamma) =$ localization weights of $\widetilde{\mathcal{F}}_\gamma$ at $q=1$.

This also suggests an upgrade of $\Gamma_\lambda(\gamma)$ to a q-deformed version in p-adic integration ($\mathrm{Dist}_G(\mathfrak{g})$, valued in $\mathbb{C}[q]$)

$$\sum_{n\geq 0} \int \mathbb{1}_{g(\mathcal{O})}(g)\, \mathbb{1}_{\mathfrak{g}(\mathcal{O})}(g^{-1}\gamma g)\, q^{\det(g)=n}\, dg = \sum q^n \, \# \mathrm{Hilb}^n(Y_{a_\gamma})$$

$\tilde{\gamma} = (\gamma, v) \in \mathfrak{gl}_n(K) \oplus K^n$

$$\Big(G(K).\tilde{\gamma} \cap \mathfrak{gl}_n(\mathcal{O}) \oplus \mathcal{O}^n\Big) \Big/ GL_n(\mathcal{O})$$

Garner-K.

$\cong \mathrm{Hilb}^{\bullet}(Y_{a_\gamma})$

$|\det(g)|^s$, $t^s = q$

$\varpi_D\, \mathfrak{g}(\mathcal{O})$ $\rightsquigarrow$ replace $\mathfrak{g}(\mathcal{O}_x)$ by Iwahori subgroup

$$\int \mathbb{1}_{\mathrm{Lie}(I)}(g^{-1}\gamma g)\, dg = \# \text{ points on } \tilde{Sp}_\gamma = \{ g \in G(K)/I \mid g^{-1}\gamma g \in \mathrm{Lie}\, I \}$$

$\qquad \| \qquad\qquad\qquad\qquad\qquad \cong \mu_a^{par}$

$\langle f, \underset{\sim}{e_{(1)^n}} \rangle$

repl. by λ for general parahoric.

Shende-Treumann-(Williams)-Zaslow

Minh-Tam Trinh

Casals-Gorsky-Gorsky-Simental) "braid varieties"

Mellit