Maulik's proof of the OS/DHS conjecture

§1. Framed stable pairs + Hilbert schemes on singular curves

§2. Flop identities from motivic Hall alg.

§3. Colored HOMFLY polynomials

§4. The proof

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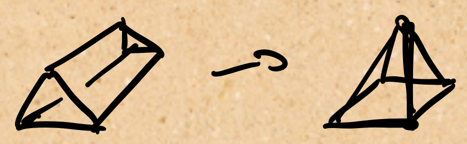
§1. Let $Y = \mathrm{Tot}(\mathcal{O}(-1) \oplus \mathcal{O}(-1)) \xrightarrow{\pi} \mathbb{P}^1$

be the resolved conifold.

Resolves $\{xy - zw = 0\} \subset \mathbb{C}^4$ with exceptional $E \cong \mathbb{P}^1$ = zero secn.



Fix $\pi^{-1}(0) \cong \mathbb{C}^2$

Def. A stable pair on Y is

$(\mathcal{F}, \sigma)$, $\mathcal{F}$ pure 1d sheaf

$$\sigma : \mathcal{O}_Y \longrightarrow \mathcal{F} \to \underbrace{Q}_{\dim 0} \to 0$$

Def $(\mathcal{F}, \sigma)$ is C-framed if

No cous OUTSIDE E set-theoretically

$$(\mathcal{F}|_{Y-E}, \sigma|_{Y-E}) \cong (\mathcal{O}_{Y-E}, \mathcal{O}/I_C|_{Y-E})$$

Want:	Moduli space of these. But there's an
Issue:	Y not projective. (Cf. Francesca's talk.)
Fix:	Compactify. If X projective 3fold, Thm (Le Potier, PT) $\exists$ Moduli of stable pairs $PT_{\beta,n}(X)$.

Def. above ok for X. Given $\overline{C}$-framed stable pair, define

$$\beta = [\mathrm{Supp}(\mathcal{F})] - [\overline{C}] \in \mathbb{Z}\cdot[E] \subset H_2(X).$$

$$\chi = \chi(Q) - \chi(I_{\overline{C}, \mathrm{Supp}\,\mathcal{F}})$$

Took this to be exceptional,

Could be anything
Meaning, frame on other Z

Corollary (Maulik)

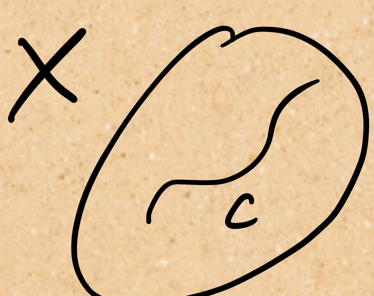
$$P(X, \bar{C}, E)_{\beta,\chi} \text{ is representable, projective.}$$

Corollary $P(Y, C, E)_{\beta,\chi}$ is representable, projective

(In fact, $P(Y, C, E) \cong P(\bar{X}, \bar{C}, E)$, $\mathrm{Supp}(Q) \subset E$)

Remark $P(X, C, E)$ only depends on formal completion of X along $E \cup C$.

What do these look like? Example:



$\mathrm{supp}\, \mathcal{F} = C$, (i.e. $\beta = 0$)

$$\mathcal{O}_X \twoheadrightarrow \mathcal{O}_C \to 0$$

or $$\mathcal{O}_X \twoheadrightarrow \mathcal{O}_C \xrightarrow{\sigma} L \to \underbrace{Q}_{\text{zeros of } \sigma} \to 0$$

In general, dualizing $\mathcal{O}_C \xrightarrow{\sigma} \mathcal{F}$ on Gorenstein curve by $R\mathcal{H}om(-,\mathcal{O}_C)$

<u>Thm</u> (P-T)

$$0 \to F^* \to \mathcal{O}_C \to \mathcal{E}xt^1(Q,\mathcal{O}_C) \to 0$$

↖ Ideal sheaf $\in \mathrm{Hilb}_{\mathrm{len}(Q)}(C)$ (Ilaria's talk)

More generally, get for CM curves

<u>Thm</u> (Bejleri-K., folklore)

Take $R\mathcal{H}om(-,\omega_C)$; get

$$0 \to F^* \to \omega_C \to \underbrace{\mathcal{E}xt^1(Q,\omega_C)}_{\text{Finite length quotient of } \omega_C \,\in\, \mathrm{Quot}_{\mathrm{len}(Q)}(\omega_C)} \to 0$$

local to C

More generally, when $[\mathrm{Supp}(\mathcal{F})] = [C] + r[E]$ get relative flag Hilbert/Quot schemes

(DHS, BK): $P(X, C, r, n) \cong Q^{[\ell, r]}(C) \to \mathrm{Quot}^{\ell}(\omega_C)$

<u>Ex.</u> $r = 0,1$, $C = \{x^3 - y^2\}$ $r = 2$, $\ell \geq 2$

Empty fiber over Quot^{ℓ}

Fiber at $m_{(0,0)}$ is lines in m/m^2

$r > 2$ empty.

(Actually, this is only the locus where
$\mathrm{Supp}\, Q = O = E \cap C$. The rest
is easy to describe using a stratification)

Framed invariants:

Def $$Z(X, C, q, a) = \sum_{r,n} a^r q^n \chi_{top}\left(P(X,E,C)_{rE,n-\chi(C)}\right)$$

Any const. fn OK. Motive?

So far, no knots. Now let's introduce the precise setup we work in.

We could take $X = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(-1) \oplus \mathcal{O}(-1))$
but it's not CY. Morrison-Vafa '96:
Let X^- be smooth elliptic fibration
(Almost all fibers $\textcircled{\scriptsize 0}$) over $\mathbb{F}_1 = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(-1))$
with section s, $E^- \subset \mathbb{F}_1 \overset{s}{\hookrightarrow} X^-$ is a
(-1)-curve on $\mathbb{F}_1$. Then it has normal
bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ and contracts

$$\begin{array}{ccc} X^- & & \\ \uparrow \text{flop} & \searrow & \\ & & X_0 = \text{singular CY3 with unique conifold sing.} \end{array}$$

Have also $\overset{\cup\!\!\shortmid}{X}{}^+ \nearrow$ another crepant resolution with $\mathbb{P}^2$ and there's an $E^+ \subset X^+$ as the exceptional locus, with $E^+ \cap \mathbb{P}^2 = pt.$
(also a $(-1,-1)$ curve)

Formal completion of Y along $E \cup \pi^{-1}(0)$
$\cong$ open subset of formal cpltn of X^+ along $E^+ \cup \mathbb{P}^2$.

Note: $\mathbb{F}_1 \cong \mathrm{Bl}_0(\mathbb{P}^2)$ and the flop realizes this blowup/blowdown.

The curves: Fix $C \subset \mathbb{C}^2$ with singularity at 0, and if $\{f_1 \cdots f_m = 0\} = C$, fix m partitions $(\lambda^{(1)}, \ldots, \lambda^{(m)}) = \vec{\lambda}$ of arbitrary sizes.

Fix an affine chart of Y at $(0,0,0) \in E \cup \pi^{-1}(0)$ with coordinates (x, y, z) normal direction

and define

$$C_{\vec{\lambda}} = \bigcup_{i=1}^{m} C_{i,\lambda^{(i)}} \quad \text{where}$$

$C_{i,\lambda^{(i)}}$ is cut out by

$$\left.\begin{aligned} f^{\lambda_1^{(i)}} &= 0 \\ z f^{\lambda_2^{(i)}} &= 0 \\ &\vdots \\ z^{\ell(\lambda^{(i)})-1} f &= 0 \\ z^{\ell(\lambda^{(i)})} &= 0 \end{aligned}\right\}$$

Nonreduced CM curve in $\mathbb{C}^3$

<u>Ex.</u> $f = x$, $\lambda =$  , $C_\lambda = \left\{\begin{aligned} x^3 &= 0 \\ x^2 z &= 0 \\ z^2 x &= 0 \\ z^3 &= 0 \end{aligned}\right\}$

Nice torus action,

$\mathrm{Quot}(\omega_C)^{\mathbb{C}^\times} \longleftrightarrow$ "PT-vertex with one leg"

Denote $Z(Y, C_{\vec{\mu}}, q, a)$ PT function as before, and let $\mathcal{L} = \mathrm{Link}(C^{red}_{\vec{\mu}})$ be link of C at origin. Let

$W(\mathcal{L}, \vec{\mu}, v, s)$ be colored HOMFLY from Ilaria's talk (Defined using cabling by Q_λ-diagrams in HOMFLY skeins of  and )

Thm (Maulik, conjectured by DHS and for $\vec{\mu} = \emptyset$ by OS)

$$\frac{Z(Y, C_{\vec{\mu}}, q, a)}{\prod_{k \geq 0} (1 + \frac{q^k}{t} a)^k} \overset{!}{=} \alpha W(\mathcal{L}, \vec{\mu}, v, s)$$

meaning they agree up to $\frac{q}{t} \longmapsto s^2$

$a \longmapsto -v^2$

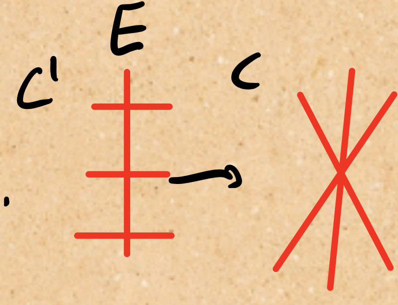
and an explicit monomial shift (expressed in terms of $\mathcal{L}, \vec{\mu}$)

§2. Flop identity

Consider $E^- \subset \mathrm{Bl}_0(\mathbb{P}^2) \longrightarrow \mathbb{P}^2$

$$\begin{array}{ccc} & \cup & & \cup \\ \text{strict tfm} & \overline{C}' & \longrightarrow & \overline{C} \end{array}$$

$\overline{C}' \cap E^- = \{p_1, \dots, p_e\}$ ex.



Add colors: If $C_1, C_2, \dots, C_m$ are cpts, $\mu^{(1)}, \dots, \mu^{(m)}$ partitions, define the curve $C_{\vec{\mu}}$ as before and $C'_{\vec{\mu}}$ as $\overline{\phi(C_{\vec{\mu}})}$ where $\phi : Y \setminus E^+ \longrightarrow Y^- \setminus E^-$.

__Prop.__ Flop identity (Maulik, based on Bridgeland, Calabrese, Toda, Stoppa-Thomas...)

$$q^{|\vec{\mu}|} Z'(Y, C_{\vec{\mu}}, q, a^{-1}) = q^{\delta(\vec{\mu})} Z'(Y^-, C'_{\vec{\mu}}, q, a)$$

Replace by X^+, X^- if you want.

__Proof.__ I'll not prove this in detail, but the key steps are:

1) Perverse coherent sheaves $\mathrm{Per}(Y/X) = \{ E \in D^b(\mathrm{Coh}(Y))$
s.t. $E \simeq [E_{-1} \to E_0]$
$R^1 f_* \mathcal{H}^0(E) = 0$
$R^0 f_* \mathcal{H}^{-1}(E) = 0$
$Rf_* \mathcal{H}om_Y(\mathcal{H}^0 E, \mathcal{F}) = 0 \quad \forall \mathcal{F} \in \mathrm{Coh}(Y)$ with $Rf_* \mathcal{F} = 0 \}$

(Francesca's talk)

Here $Y \xrightarrow{f}$ conifold Y', $Y^- \nearrow$

Examples: $\mathcal{O}_Y$, $\mathcal{O}_{\mathbb{P}^1}(-m)[1]$, $m \leq 1$

$\mathcal{O}_{pt}$

Below, $F \in {}^p\mathcal{F}$ have $\mathrm{supp} F$ $E^! \supset$

<u>Remark</u> This is ${}^{-1}\mathrm{Per}(Y/Y')$, obtained by tilting $D^b \mathrm{Coh}(Y)$ wrt torsion pair

$({}^{-1}\mathcal{T}, {}^{-1}\mathcal{F})$, ${}^{-1}\mathcal{T} = \left\{ T \in \mathrm{Coh}\, Y \,\middle|\, \begin{matrix} R^1 f_* T = 0 \\ \mathrm{Hom}(T, \mathcal{C}) = 0 \end{matrix} \right\}$

${}^{-1}\mathcal{F} = \{ F \in \mathrm{Coh}\, Y \mid f_* F = 0 \}$

could also define

${}^0\mathrm{Per}(Y/Y')$ = tilt wrt ${}^0\mathcal{T} = \{ T \in \mathrm{Coh}\, Y \mid R^1 f_* T = 0 \}$

${}^0\mathcal{F} = \{ F \in \mathrm{Coh}\, Y \mid f_* F = 0,\ \mathrm{Hom}(\mathcal{C}, F) = 0 \}$

Bridgeland: $\Phi : D^b \mathrm{Coh}\, Y \xrightarrow{\sim} D^b \mathrm{Coh}\, Y^-$
$\cup$
${}^{-1}\mathrm{Per}(Y/Y') \longrightarrow {}^0\mathrm{Per}(Y/Y')$

$= {}^{-1}\mathrm{Per}(Y^-/Y')$

Again, work with X, X^- instead of Y, Y^- for technical reasons.

2) Perverse Hilbert scheme

${}^p\mathrm{Hilb}(X)$: Recall $\mathrm{Hilb}(X) = \{\underbrace{\mathcal{O}_X \twoheadrightarrow \mathcal{F}}_{\text{in } \mathrm{Coh}(X)}\}$

The p-version is $\{\underbrace{\mathcal{O}_X \twoheadrightarrow \mathcal{F}}\}$

but now in ${}^p\mathrm{Per}(X/X')$, $p = 0, -1$.

Before, fixed n, β to get a component of $\mathrm{Hilb}^{\text{linear Hilbert poly}}(X) \rightsquigarrow DT_{\beta,n} = \chi(\mathrm{Hilb}(X)_{\beta,n})$

Euler char, Behrend fn...

Similarly, define

$${}^pDT_{\beta,n} = \chi({}^p\mathrm{Hilb}(X)_{\beta,n})$$

These are NOT the ones we care about, though.

Given $I \in \mathrm{Hilb}(X)_{\beta,n}$ or $I \in {}^p\mathrm{Hilb}(X)_{\beta,n}$,

<u>Def</u> I is $C_{\vec{\mu}}$-framed if

$$0 \to I \to \mathcal{O}_X \twoheadrightarrow \mathcal{F} \to 0$$

restricts to

$$0 \to I_{C_{\vec{\mu}}-\epsilon} \to \mathcal{O}_{X-\epsilon} \to \mathcal{O}_{C_{\vec{\mu}}-\epsilon} \to 0$$

(At least in usual Hilb. It's a bit unclear what Maulik means in the beginning of Sec 2.5)

Note: $\phi : X \overset{\text{flop}}{\dashrightarrow} X^-$

gives $H_2(X) \xrightarrow[\cong]{\phi_*} H_2(X^-)$.

In particular, ${}^{p}DT(X, C_{\vec{\mu}}, q, a) :=$

$$\sum_{n,\beta} \chi\left({}^{p}\mathrm{Hilb}(X, C_{\vec{\mu}}\text{-framed})_{\beta,n}\right) q^n a^\beta$$

satisfies

$${}^{0}DT(X^-, C'_{\vec{\mu}}, q, a) = \underbrace{\phi_*}_{\substack{\text{applied as} \\ \phi_*(a^\beta) = a^{\phi_*(\beta)}}}\left({}^{-1}DT(X, C_{\vec{\mu}}, q, a)\right)$$

(Maulik and Calabrese claim this is obvious. It is not obvious to me...)

3) Motivic Hall algebras (Tanguy's talk)

Let ${}^{p}\mathrm{Per}(X/X') =: A^p$.

Object supp. in dim ≤ 1 give category

$\mathcal{A}^p_{\leq 1} \rightsquigarrow$ MHA $H(\mathcal{A}^p_{\leq 1})$ with

associative product $*$. Elements of it are "stack functions" $[R \xrightarrow{f} \mathcal{M}]$ where $\mathcal{M}$ = moduli stack of complexes of $\mathcal{A}^p_{\leq 1}$. (This is Bridgeland's adaptation of Joyce-Song)

${}^p\mathrm{Hilb}(C_{\vec{\mu}}\text{-framed})$ and $\mathrm{Hilb}(-)$ give elements of $H(\mathcal{A}^p_{\leq 1})$, as does the moduli stack of complexes in ${}^p\tilde{\mathcal{F}}[1] \subset \mathcal{A}^p_{\leq 1}$.

<u>Thm</u> (Calabrese + Maulik)

the above with section

$$\mathrm{Hilb}^p(C_{\vec{\mu}}\text{-framed}) * 1_{{}^p\tilde{\mathcal{F}}[1]} = 1^{\mathcal{O}}_{{}^p\tilde{\mathcal{F}}[1]} * \mathrm{Hilb}(-).$$

4) Combining the theorems above gives a relation with usual framed DT:

$$\frac{DT(X^-, C'_{\vec{\mu}}, q, a)}{Z(q, a^{[\epsilon^-]})} = \phi_*\left(\frac{DT(X, C_{\vec{\mu}}, q, a)}{Z(q, a^{[\epsilon_+]})}\right)$$

5) Finally, we cross a wall to PT, using Stoppa - Thomas:

Thm

$$\frac{DT(X, C_{\tilde{\mu}}, q, a)}{\left(\prod_{k\geq 0} \frac{1}{(1-q^k)^k}\right)^2} = Z(X, C_{\tilde{\mu}}, q, a)$$

$\chi_{top}(E_+)$

Next, we "localize" the flop identity. Working on Y, Y^-, take $\mathbb{C}^*$-action dilating normal bundle of $\mathbb{A}^2$ or its strict transform under flop. It acts on stable pair moduli as well.

Since

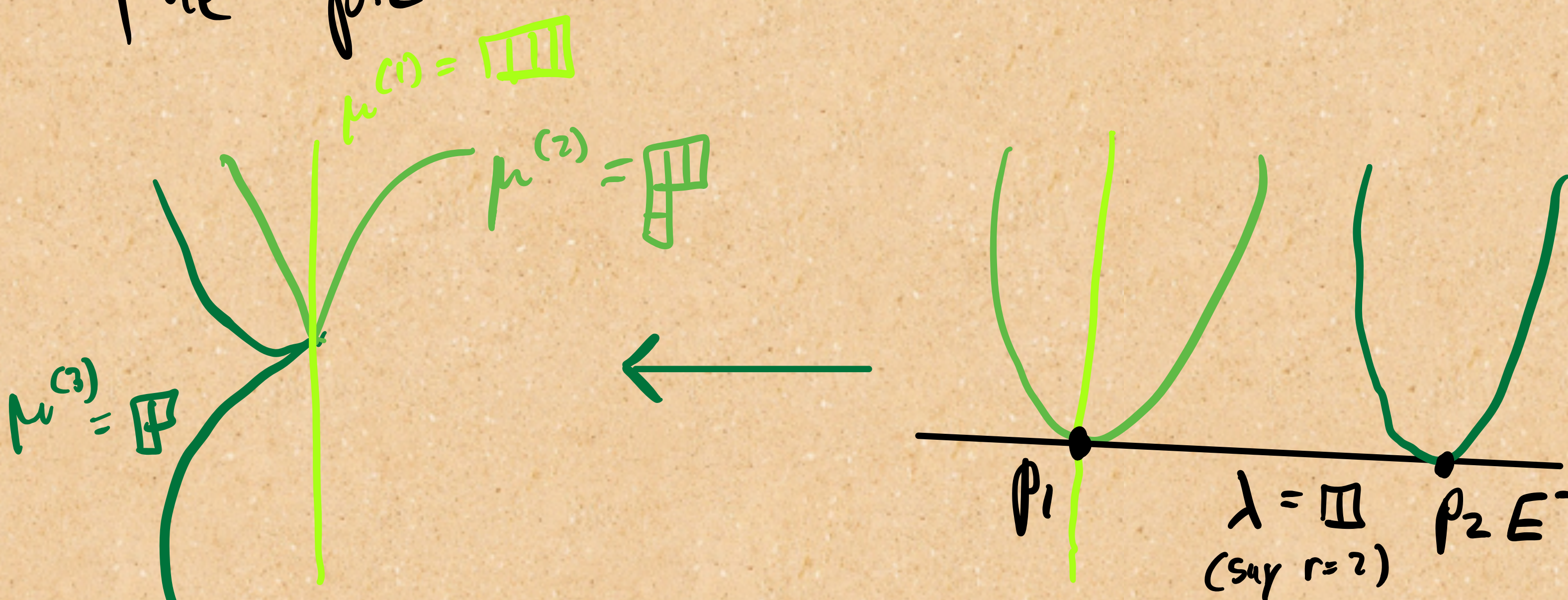
$$\chi\big(P(Y, C_{\vec{\mu}}, r, n)\big) = \chi\big(P(Y, C_{\vec{\mu}}, r, n)^{\mathbb{C}^*}\big),$$

we reduce to $C_{\vec{\mu}}$-framed stable pairs supported on $\mathbb{C}^*$-invariant CM curves with class $[C_{\vec{\mu}}] + r[E]$.

this is fixed by $\mathbb{C}^*$

The nonreduced structure here is a partition $\lambda \vdash r$.

The picture is like this:

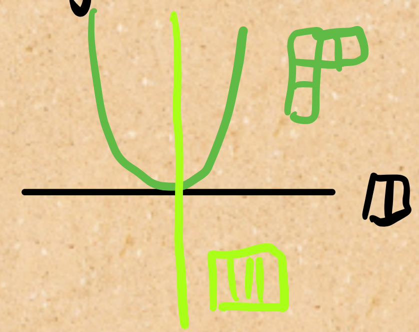


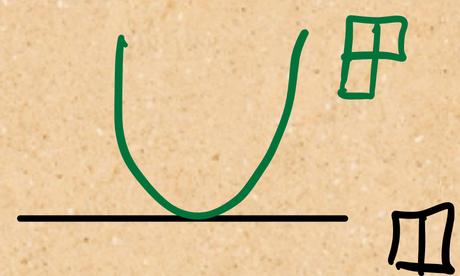
$$C = C_1 \cup C_2 \cup C_3, \quad C' = C_1' \cup C_2' \cup C_3'$$

Define $D_1^{red}, \dots, D_k^{red}, D_1, \dots, D_k$

as $\left(\bigcup_{C_i' \text{ meeting } p_1} C_i'\right) \cup E^-, \dots, \left(\bigcup_{C_i' \text{ meeting } p_k} C_i'\right) \cup E^-,$

and without "red" by coloring with $(\vec{\mu}, \lambda)$.

e.g. $D_1 =$ 

$D_2 =$ 

The punctual contribution (remember that everything's stratified by $\operatorname{supp}(Q)$) at p_k equals $Z(Y, D_k, q, a)$ at $a = 0$, by definition.

The cokernel can have support elsewhere

on E^-; the contribution there is $\underbrace{H_\lambda(q)}_{\text{"1-leg PT vertex" or "principal specialization of Schur function"}}^{2-k}$.

Using this, RHS of flop identity becomes

$$\sum_{r,\,\lambda\in P(r)} \left(a^{|\lambda|} q^{\overbrace{\theta(\lambda,\vec{\mu})+\delta}^{=\chi(\mathcal{O}_{\overline{D}_\lambda})-\chi(\mathcal{O}_{C_{\vec{\mu}}})+|\vec{\mu}|}} H_\lambda(q)^{2-k} \prod_{i=1}^{k} Z'(Y, D_i, q, a=0) \right)$$

Next, we write terms in this link-theoretically, and prove the main thm inductively.

§ 3. Colored HOMFLY.

Recall from Ilaria's talk that given a framed link $\mathcal{L} = \mathcal{L}_1 \cup \dots \cup \mathcal{L}_m$, partitions $\mu^{(1)}, \dots, \mu^{(m)}$, we form

$$\mathcal{L} * \left(Q_{\mu^{(1)}}, \dots, Q_{\mu^{(m)}} \right)$$

by thickening each $\mathcal{L}_i$ to annulus, and "placing" the "diagram" of $Q_{\mu^{(i)}}$ on this annulus. Embedding the annulus to $\mathbb{R}^2$, we get

expressed with writhe, contents and sizes.

$$W(\mathcal{L}, \vec{\mu}, v, s) = \left\langle \mathcal{L} * Q_{\vec{\mu}} \right\rangle \cdot v^{f(\mathcal{L}, \vec{\mu})} s^{g(\mathcal{L}, \vec{\mu})}$$

It is a link invariant (but harder to compute than usual HOMFLY).

(1,2):

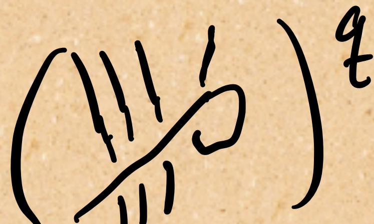
Algebraic links: $C = \{\prod_{i=1}^{m} f_i = 0\}$

Each branch has Puiseux series

$$y = x^{\frac{q_1}{p_1}}\left(a_1 + x^{\frac{q_2}{p_1 p_2}}\left(\cdots \right)\right)$$

And the link is constructed using

$T_p^q = \widehat{\beta_p^q}$ = closure of



p strands.

<u>Thm</u> C irred $\Rightarrow$

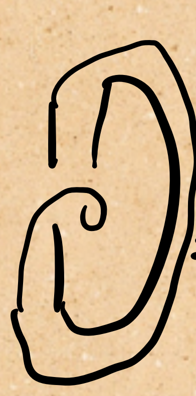
$$\mathrm{Link}_{C,0} \cong T_{p_1}^{q_1} * \cdots * T_{p_s}^{q_s}(O)$$

<u>Ex.</u> $f(x,y) = y^4 - 2x^3y^2 - 4x^5y + x^6 - x^7$

$$y = x^{\frac{7}{4}} + x^{\frac{6}{4}} = x^{\frac{3}{2}}\left(1 + x^{\frac{1}{4}}\right)$$

$\Rightarrow (q_1, p_1) = (3, 2)$

$(q_2, p_2) = (1, 2)$

$T_{p_2}^{q_2} =$  $= O$

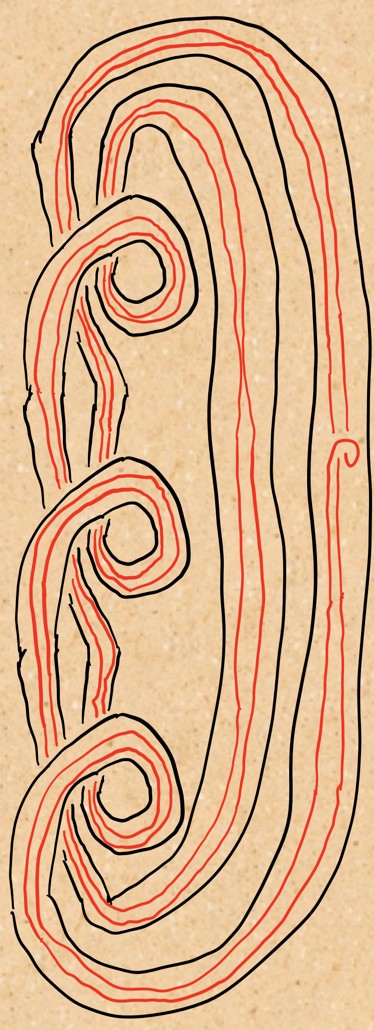
Thicken to solid torus and place link on it.

$$T_{p_1}^{q_1} * T_{p_2}^{q_2} = (2,13)\text{-}\underline{\text{cable}} \text{ of trefoil.}$$

$$13 = 1 + 2\cdot 2\cdot 3$$

$$q_2 + p_1 \cdot p_2 \cdot q_2$$

(Eisenbud-Neumann)



Blowing up:

$$y = x^{\frac{q_0 - p_0}{p_0}}\left(a_1 + x^{\frac{q_1}{p_0 p_1}}\left(\cdots\right)\right)$$

$$\rightsquigarrow \mathrm{Link}_{C',0} = T_{p_0}^{q_0 - p_0} * \cdots * T_{p_s}^{q_s}(0)$$

Note: Iterated torus knot given by <u>cabling</u> $(a_1, b_1), \dots, (a_s, b_s)$ torus knots is algebraic iff

$$a_i > 0, \quad b_i > a_i a_{i-1} b_{i-1} \quad \forall i.$$

What if we have many branches?

$$\sigma_p^q = \Big($$  $$\Big)^q$$

$p+1$

$$S_p^q = \widehat{\sigma_p^q}$$

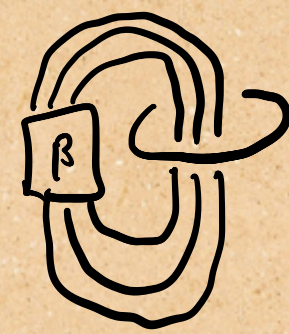
<u>Thm</u> $\mathrm{Link}_{C,0} =$

$$S_{p^{(1)}}^{q^{(1)}} * \Big(L_{\alpha_1}, S_{p^{(2)}}^{q^{(2)}} * \Big(L_{\alpha_2}, \dots, S_{p^{(k-1)}}^{q^{(k-1)}} * \Big(L_{\alpha_k}, T_{p^{(k)}}^{q^{(k)}} * L_{\alpha_k} \Big) \cdots \Big)$$

↑ all concatenation of links with first Puiseux pair α_1

Blowing up separates some branches,
get $\mathrm{Link}_{C_i', p_i} = \mathrm{Link}_{C_i, 0}$ with full twist removed.

$\mathrm{Link}^{red}_{D_i, p_i}$ = Union of links of C_i' meeting at p_i and a meridian (from E^-)



Using above equations, and some Skein computations, can show

$$[\mathrm{Link}_C] = T_1' * [\mathrm{Link}_{C'}] .$$

Next, we write

$$[L_C] = \sum_{\gamma \vdash n} c_\gamma(v,s)\, Q_\gamma ,$$

or more generally for X in skein() — Form a basis of Skein of annulus

<u>**Theorem**</u>

lowest v-degree

$$(-v)^n \langle X \rangle = \frac{\sum_\lambda (-v^{-2})^{|\lambda|} s^{-k} \langle Q_\lambda \rangle^{low} \langle M_\lambda \Phi^{-1} X \rangle^{low}}{\prod_k (1 - s^{2k} v^{-2})^k}$$

§4. The proof.

To finish, we identify the

$$(*) \quad \langle M_\lambda \Phi^{-1}(L_c * Q_{\vec{\mu}}) \rangle^{low} \langle Q_\lambda \rangle^{low}$$

terms with the terms in the RHS of the flop identity. The base cases are colored Hopf + unlink, which we know from 1+2-leg PT vertex.

In (*), $\langle M_\lambda \prod x_i \rangle^{low}$

$$= \langle \Omega_\lambda \rangle^{low\; 2-n} \prod \langle M_\lambda x_i \rangle^{low}$$

$$(L_{D_k}) = M_\lambda (L_{B_k})$$