Lecture notes:
Combinatorics, Aalto, Fall 2014

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Preface

These notes are from a course in Combinatorics at Aalto University taught during the first quarter of the school year 14-15. The intended structure is five separate chapters on topics that are fairly independent. The choice of topics could have been done in many other ways, and we don't claim the included ones to be in any way more important than others. There is another course on combinatorics at Aalto, towards computer science. Hence, we have selected topics that go more towards pure mathematics, to reduce the overlap. A particular feature about all of the topics is that there are active and interesting research going on in them, and some of the theorems we present are not usually mentioned at the undergraduate level.

We should end with a warning: These are lecture notes. There are surely many errors and lack of references, but we have tried to eliminate these. Please ask if there is any incoherence, and feel free to point out outright errors. References to better and more comprehensive texts are given in the course of the text.

CHAPTER 1

Posets

Definition 1.1. A *poset* (or *partially ordered set*) is a set P with a binary relation $\leq \subseteq P \times P$ that is

(i) reflexive: $p \leq p$ for all $p \in P$;
(ii) antisymmetric: if $p \leq q$ and $q \leq p$, then $p = q$;
(iii) transitive: if $p \leq q$ and $q \leq r$, then $p \leq r$

Definition 1.2. We say the elements $p, q \in P$ are *comparable*, if $p \leq q$ or $q \leq p$. Otherwise p and q are *incomparable*, denoted $x||y$. An element $p \in P$ is *larger than* q, denoted $p > q$, if $p \geq q$ (ie. $q \leq p$) and $p \neq q$. The element p *covers* r, denoted $p \succ r$, if $p > r$ and there does not exist q such that $p > q > r$.

Definition 1.3. A *Hasse diagram* is a picture of a poset with the elements represented by dots and covering relations represented by lines such that larger elements are drawn above the smaller ones.

Example 1.1. Hasse diagrams for the integers from 2 to 5 in increasing order, and the subsets of $\{a, b, c\}$ ordered by inclusion, as shown in Figure 1.

Definition 1.4. A *poset map* $\phi : P \to Q$ is a map of sets (by abuse of notation the posets and corresponding sets are identified with the same letter) $P \to Q$ satisfying $\phi(p) \geq \phi(q)$ in Q if $p \geq q$ in P.

Definition 1.5. A poset map is called *bijective* resp. *injective* resp. *surjective* if the corresponding map of sets is bijective resp. injective resp. surjective. Two posets are called *isomorphic*, if there exist bijective poset maps (*isomorphisms*) ϕ, ψ between them that are inverses of each other.

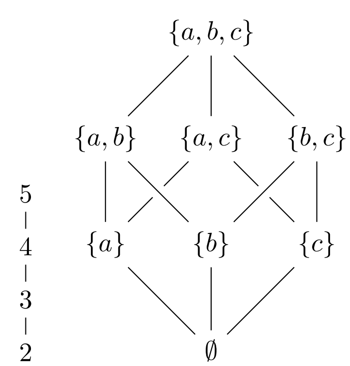


FIGURE 1. Hasse diagrams of $2 < 3 < 4 < 5$ and B_3.

Definition 1.6. If P is a poset and Q is a subset of P, then Q becomes a *subposet* of P by inheriting all relations from P between elements in Q.

Proposition 1.1. *A subposet is a poset.*

PROOF. It is an easy exercise that by inheritance, the reflexivity, antisymmetry and transitivity of the induced order $\leq$ are satisfied. □

Definition 1.7. Let $p, q \in P$. If there exists a well-defined least upper bound for p and q, then it is called the *join* of p and q, and is denoted by $p \vee q$. If there exists a well-defined greatest lower bound of p and q, it is called the *meet* of p and q, and is denoted by $p \wedge q$.

Definition 1.8. If both the join and meet exists for all $p, q \in L$, then L is called a *lattice*.

Proposition 1.2. *Let S be a set and P the poset of all subsets of S ordered by inclusion. Then P is a lattice.*

PROOF. We have $p \wedge q = p \cap q$ and $p \vee q = p \cup q$. □

Definition 1.9. Any poset isomorphic to one constructed as in Proposition 1.2 is called a *boolean lattice*.

Definition 1.10. A sequence $p_1 < p_2 < \cdots < p_k$ of elements in a poset is called a *chain* of *length* k. It is *saturated* if each $<$ is a covering relation.

Definition 1.11. A poset is called *bounded* if the lengths chains are bounded.

Definition 1.12. A unique maximal element of a poset is denoted $\hat{1}$ and a unique minimal element is denoted $\hat{0}$. Adjoining them to a poset P yields a new poset $\hat{P}$.

Proposition 1.3. *Every bounded lattice L has a $\hat{0}$ and a $\hat{1}$.*

PROOF. Take a chain $p_1 < \cdots < p_l$ of maximal length in L. Suppose $p_l \neq \hat{1}$. Then there is another maximal element p' in L, with $p_l < p_l \vee p'$, hence l cannot be the maximal length, a contradiction. □

Example 1.2. The positive integers ordered by divisibility (that is, $i \leq j$ if and only if $i|j$) form a lattice, with join and meet given by the least common multiple and greatest common divisor, respectively.

Definition 1.13. Let G be a group and $L = L(G)$ be the set of subgroups of G ordered by inclusion. Then L is a lattice with $H_1 \wedge H_2 = H_1 \cap H_2$ and $H_1 \vee H_2 = \langle H_1, H_2 \rangle$. This is called the *subgroup lattice* of G.

Definition 1.14. A subposet of a lattice that is also a lattice is called a *sublattice*.

Example 1.3. The normal subgroups $N(G)$ form a sublattice of $L(G)$.

Example 1.4. The symmetry group of a hexagon is the dihedral group on 12 elements, and denoted by D_{12}. With generators and relations, it can be expressed as $\langle r, f | r^6 = f^2 = e, f^{-1}rf = r^{-1} \rangle$. The subgroup lattice and the generators are given in Figure 2.

Definition 1.15. Let L, M be lattices and $\phi : L \to M$ a poset map. If $\phi(a \wedge b) = \phi(a) \wedge \phi(b)$ and $\phi(a \vee b) = \phi(a) \vee \phi(b)$, then ϕ is called a *lattice homomorphism*. If both L and M have $\hat{0}$ and $\hat{1}$, and in addition we have $\phi(\hat{0}) = \hat{0}$, $\phi(\hat{1}) = \hat{1}$, then ϕ is called a $\hat{0}\hat{1}$–lattice homomorphism.

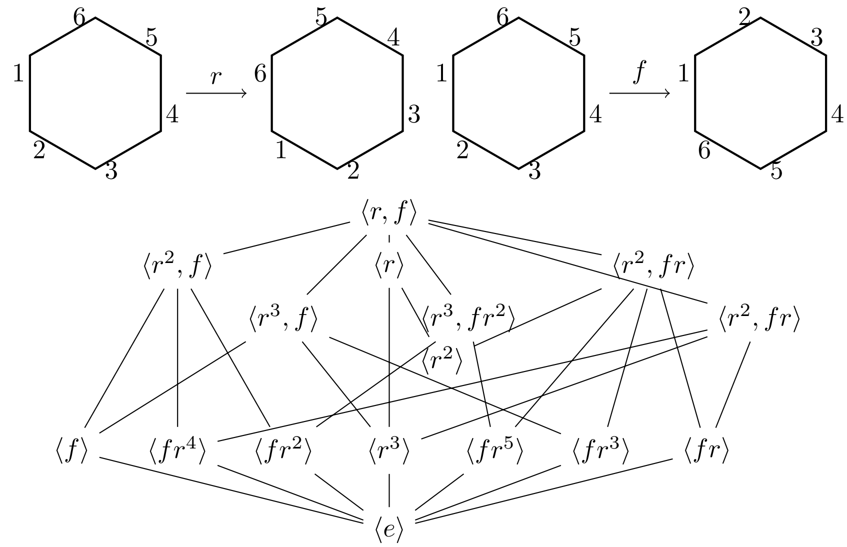


FIGURE 2. The generators and subgroup lattice of D_{12}.

Example 1.5. Consider the subgroup $\langle f\rangle \subseteq D^{12}$ (cf. Example 1.4). Then there is a poset map $L(D_{12}) \to L(\langle f\rangle)$ given by $\phi(H) = H \cap \langle f\rangle$, and which is a lattice homomorphism.

Definition 1.16. A poset map $g : P \to \mathbb{Z}$ satisfying $p \succ q \Rightarrow g(p) \succ g(q)$ is a *grading* of P, and P is called *graded*.

Example 1.6. All posets are not graded, as Figure 3 shows.

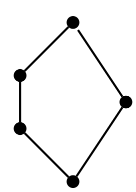


FIGURE 3. A non-graded poset.

Example 1.7. There is a grading of the divisibility lattice of positive integers by $g(\prod_i p_i^{e_i}) = \sum_i e_i$. Consider $1, 2, 3, 6$ to see that g is *not* a lattice homomorphism.

Example 1.8. Let O be the sublattice of odd integers in the divisibility lattice of positive integers. Then the poset map that drops all factors of two is a lattice homomorphism.

Definition 1.17. A subset of a poset consisting of mutually incomparable elements is called an *anti-chain*.

Example 1.9. The integers $6, 49, 55$ form an anti-chain in the divisibility lattice.

Definition 1.18. A subset I of a poset P is called a *lower ideal* in P if $p \leq q \in I \Rightarrow p \in I$. The *transitive closure* or lower ideal generated by a subset of a poset is the smallest lower ideal containing it.

Definition 1.19. The set of lower ideals of a poset is denoted $J(P)$. It has a natural poset structure given by the inclusion order.

Proposition 1.4. *The poset of lower ideals is a lattice.*

PROOF. As in the boolean lattice case, join is union and meet is intersection. □

Remark 1.1. A lower ideal is also a poset.

Example 1.10. *Simplicial complexes* are basic building blocks in topology. In Figure 4, the lower ideal on the left would, by a topologist, be represented by the simplicial complex on the right.

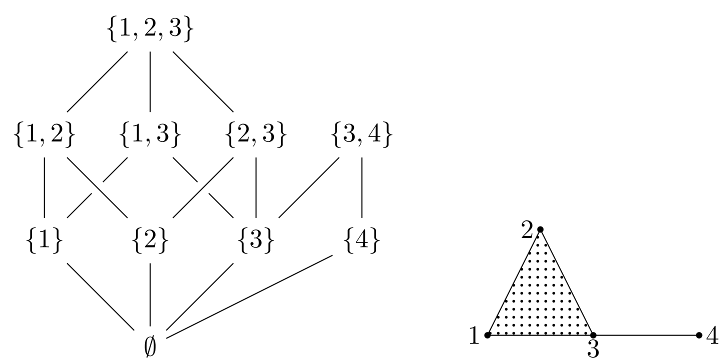


FIGURE 4. A poset and the corresponding simplicial complex.

Proposition 1.5. *There is a bijection between the anti-chains and the lower ideals of a bounded poset.*

PROOF. We can get a lower ideal from an anti-chain by taking the transitive closure. On the other hand, selecting the maximal elements in a lower ideal gives an anti-chain. It is left as an exercise for the reader to verify these operations are well-defined and inverses of each other. □

Example 1.11. Consider the poset of lower ideals $J(P)$ for the poset P in Figure 5. Labeling the lower ideals as in Figure 6, and so on, gives the poset of lower ideals in Figure 7.

Definition 1.20. A lattice D is called *distributive* if $p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$ for all $p, q, r \in D$.

Proposition 1.6. *In Definition 1.20, it is equivalent to require*

$$p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r).$$

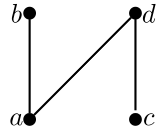


FIGURE 5. The poset P.

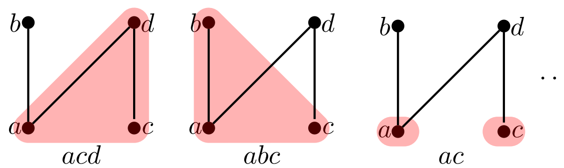


FIGURE 6. Some of the lower ideals of P.

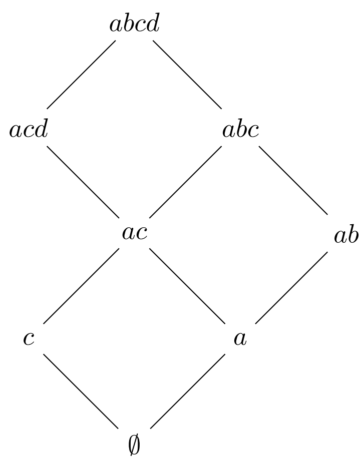


FIGURE 7. The poset of lower ideals of P.

PROOF. Assume $p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$. Then

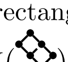
$$(p \vee q) \wedge (p \vee r) = ((p \vee q) \wedge p) \vee ((p \vee q) \wedge r) = p \vee ((p \vee q) \wedge r) =$$
$$p \vee ((q \wedge r) \vee (q \wedge r)) = (p \vee (p \wedge q)) \vee (q \wedge r) = p \vee (q \wedge r).$$

The converse is similar. □

Proposition 1.7. *For any poset P the lattice $J(P)$ is distributive.*

PROOF. Since the join is union and the meet is intersection, the proposition follows from basic distributivity of these in set theory. □

Remark 1.2. Many interesting sets in combinatorics form distributive lattices if the order is given the "right" way, which in turn often makes analysis easier.

Example 1.12. The north/east paths in a rectangle form a lattice structure as in Figure 8. The lattice here is isomorphic to $J($$)$.

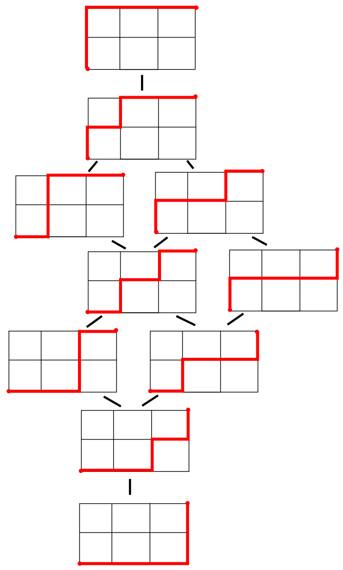


Figure 8. The lattice of N-E paths.

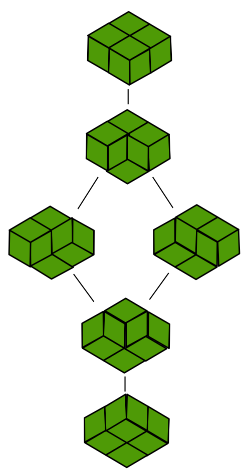


Figure 9. A subdivision of the rectangle into "diamonds".

Example 1.13. There is a distributive lattice structure to subdivisions of hexagons as shown in Figure 9. This lattice is isomorphic to $J($$)$.

Remark 1.3. The large scale behaviour of patterns like this are strongly related to random matrix theory and physics. Andrei Okounkov got his 2006 Fields medal in part for studying how big patterns have a "frozen" and a "liquid" region, as hinted by Figure 10, generated by David Wilson.

Theorem 1.1 (The Fundamental Theorem of Finite Distributive Lattices)**.** *Any finite distributive lattice can be represented as $L \cong J(P)$ for some poset P.*

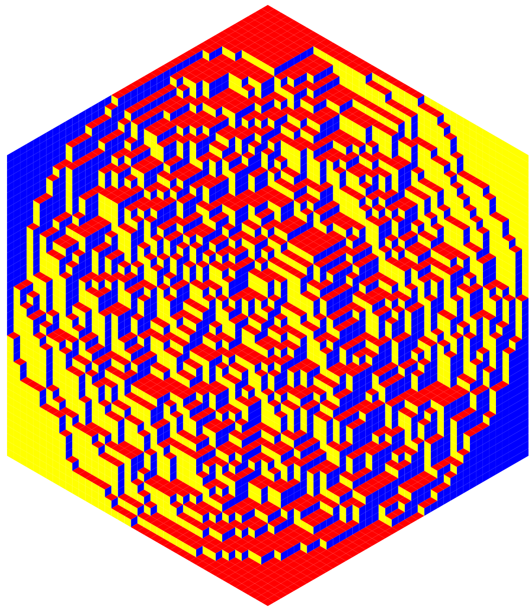


FIGURE 10. Solid and liquid regions in a random tiling.

PROOF. We will sketch a proof after introducing a few concepts. □

Definition 1.21. A lower ideal $I \in J(P)$ is called *principal* if it is of the form

$$\{p \in P | p \leq q\}$$

for some q. Note that there is a copy of P in $J(P)$ given by the principal lower ideals.

Definition 1.22. An element p of a poset is called *join-irreducible* if $p = q \vee r$ forces $p = q$ or $p = r$.

Remark 1.4. In $J(P)$, the principal and join-irreducible elements coincide. To find a P with $L \cong J(P)$, one takes the join-irreducible elements of L and creates P. Elementary further considerations confirm that this gives the representation of L.

Remark 1.5. We have already seen lattices coming from groups. The following is a classical result by Ore [**O**].

Theorem 1.2. *The lattice of subgroups of G is distributive if and only if G is* locally cyclic*.*

Definition 1.23. A group G is *locally cyclic* if every finitely generated subgroup is cyclic.

Remark 1.6. In particular, the finite locally cyclic groups are the cyclic groups. The rational numbers equipped with addition form an infinite locally cyclic group. By only considering the lattice of normal subgroups of G, we often find distributive behaviour, however not always. In the other direction, we have the following result by Silcock [**Si**].

Theorem 1.3. *Every finite distributive lattice is isomorphic to the lattice of normal subgroups of a group.*

Definition 1.24. An injective poset map $t : P \to \mathbb{R}$ (here $\mathbb{R}$ is taken with its usual linear order) is called a *total ordering* of P.

Proposition 1.8. *For any finite poset P there is a total ordering $t : P \to \{1 < 2 < \cdots < |P|\}$.*

PROOF. Induction on $|P|$. As a base case, this is clear for $\emptyset$ with the trivial order. Assume then the proposition holds for all cardinalities up to $|P| - 1$. Take a maximal element p in P and set $t(p) = |P|$. By induction, pick a total ordering on $P - p$. Clearly T is a total ordering of P. $\square$

Remark 1.7. Most of the time we simply relabel the elements of P by $1 < 2 < \ldots < |P|$, instead of carrying the map t around.

Definition 1.25. The *incidence matrix* M_P of P has rows and columns indexed by P and

$$M_P(x,y) = \begin{cases} 1 & x \leq y, \\ 0 & x \not\leq y. \end{cases}$$

Definition 1.26. The *incidence algebra* of a poset P and over a field k (that shall in our case be the field $\mathbb{C}$)

$$I(P;k) = \bigoplus_{x \leq y} k$$

is spanned by the basis $e_{x\leq y}$. We define a multiplication for $f = \sum f_{x\leq y} e_{x\leq y}, g = \sum g_{x\leq y} e_{x\leq y}$ as the convolution

$$(fg)_{x\leq y} = \sum_{x\leq z\leq y} f_{x\leq z} g_{z\leq y}$$

with the identity $\delta = \sum \delta_{x\leq y} e_{x\leq y}$, where

$$\delta_{x\leq y} = \begin{cases} 1 & \text{if } x = y, \\ 0 & \text{if } x < y. \end{cases}$$

Remark 1.8. For finite P, relabeled according to a total ordering, the incidence matrix M_P is upper triangular and $I(P;k)$ is isomorphic to the matrix algebra with nonzero k-entries allowed at nonzero positions of M_P.

Definition 1.27. The *zeta function* of a poset P is defined as $\zeta(x,y) = 1$ for all $x \leq y$ in P.

Remark 1.9. Tabulated as a matrix, the zeta function gives the incidence matrix.

Definition 1.28. For a finite poset P, define the *Möbius function* $\mu(x,y)$ for $x \leq y$ as

$$\mu(x,y) = \begin{cases} 1 & \text{if } x = y, \\ -\sum_{x \leq z < y} \mu(x,z) & \text{if } x < y \text{ in } P. \end{cases}$$

Remark 1.10. As matrices, μ is given by M_P^{-1}.

Proposition 1.9 (Möbius inversion)**.** *If $f, g : P \to \mathbb{C}$, then*

$$g(x) = \sum_{y \leq x} f(y)$$

for all $x \in P$ if and only if

$$f(x) = \sum_{y \leq x} g(y)\mu(y,x)$$

for all $x \in P$.

Example 1.14. For the divisibility lattice, we have

$$\mu(a,b) = \begin{cases} (-1)^t & \text{if } \frac{b}{a} \text{ is a product of } t \text{ distinct primes,} \\ 0 & \text{otherwise.} \end{cases}$$

This recovers the "classical" number-theoretic Möbius function.

Definition 1.29. For a poset P, we may associate a poset $\Delta(P)$, called its *order complex*. It is the poset of chains in P ordered by refinement.

Remark 1.11. By forgetting the order within the chains in Definition 1.29, $\Delta(P)$ becomes an abstract simplicial complex.

Definition 1.30. The *reduced Euler characteristic* of an abstract simplicial complex Δ is

$$\tilde{\chi}(\Delta) = -\sum_{\sigma \in \Delta} (-1)^{|\sigma|}.$$

Proposition 1.10. *Let P be a finite poset. Then*

$$\mu_{\hat{P}}(\hat{0}, \hat{1}) = \tilde{\chi}(\Delta(P)).$$

Exercise 1.1. An *poset map* between posets (or simply a *homomorphism*), $f : P \to Q$, is a function satisfying $a \leq_P b \Rightarrow f(a) \leq_Q f(b)$. Let $f : P \to P$ be an order preserving map from a finite poset to itself.

a) If P has a largest element $\hat{1}$, show that f has a fixed point.
b) Suppose P has a central element, ie. a $c \in P$ such that $\forall x \in P$, $x \leq c$ or $c \leq x$. Show that f has a fixed point.

Exercise 1.2. (Hard) Denote by D_n the poset of positive integer divisors of n, that is, with a partial order defined by $i \leq j \Leftrightarrow i|j$. Draw the Hasse diagram of this poset for $n = 60$. For $n = \prod_{i=1}^m p_i^{a_i}$, where the p_i are primes and $a_i \in \mathbb{Z}$, prove that $D_n \cong \prod_{i=1}^m [a_i]$, where $[k]$ is the linear poset on $0, \ldots, k$. Here isomorphism means an order-preserving bijection. (Here $\prod$ denotes the cartesian product of posets, with order given by $(s,t) \leq (s',t')$ iff $s \leq s'$ and $t \leq t'$.)

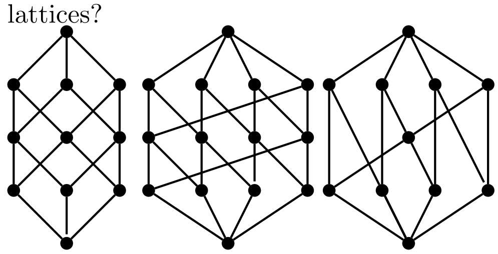
Exercise 1.3. A *chain* in P is a linear subposet. An *antichain* is a subset of P in which no two elements are comparable. True or false: "If all chains and antichains of P are finite, P is finite."

Exercise 1.4. Calculate $\mu(1, n)$ in the divisibility lattice of (positive) integers (the Möbius function can be shown to be defined locally, ie. depending only on the interval in question, so you may just think about D_n). How about $\mu(m, n)$? If you have taken a course in elementary number theory, this might look familiar.

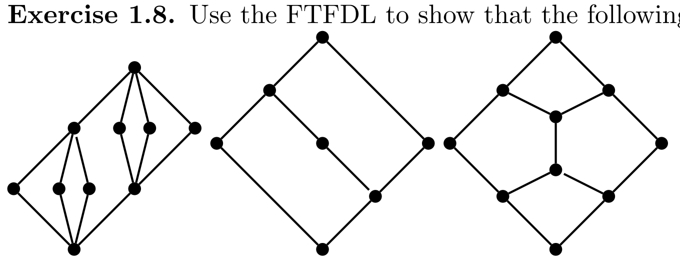
Exercise 1.5. Let $f : \mathbb{Z}_{>0} \to \mathbb{Z}$ be a function, and define $g : \mathbb{Z}_{>0} \to \mathbb{Z}$ by $g(n) = \sum_{d|n} f(d)$. Using the previous exercise, recover f from g using Möbius inversion. If $\phi(n)$ is Euler's ϕ-function, defined as the number of d coprime to n, and we know that $\sum_{d|n} \phi(d) = n$ (do you see why?), find $\phi(n)$.

Exercise 1.6. (Hard) Calculate the Möbius function of the poset P on $\{0, 1\}$ with $0 \leq 1$. Identify the Boolean lattice B_n as an n-fold product of P, and deduce the Möbius function of B_n, using the fact that $\mu_{P\times Q}((s, t), (s', t')) = \mu_P(s, s')\mu_Q(t, t')$.

Exercise 1.7. Which of the following Hasse diagrams describe posets that are lattices?



Exercise 1.8. Use the FTFDL to show that the following are not distributive:



Exercise 1.9. (Hard) Let L be the divisibility lattice of the (positive) integers, with meet and join given by the least common multiple and greatest common divisor. Draw Hasse diagrams of the principal ideals $\langle 4\rangle$ and $\langle 12\rangle$. Show that for every $k \geq 1$ there exists $n_k \in \mathbb{N}$ so that $\langle n_k\rangle \cong B_k$. Deduce that every finite distributive lattice can be embedded into L, given that every finite distributive lattice occurs as a sublattice of some B_m. Give a counterexample for the countable case.

Exercise 1.10. (Hard) Give an example of a meet-semilattice L (that is, a poset with a well-defined meet) with a *largest* element $\hat{1}$ such that L is not a lattice.

CHAPTER 2

Extremal combinatorics

In combinatorics, it is common to study how large you can make a structure, while avoiding certain "forbidden" substructures. The maximal such structures are called *extremal* and a first step in extremal combinatorics is always to try to characterize them.

The second step, is to understand that the extremal structures are very few, and that smaller admissible structures avoiding the forbidden substructures, can be well described as substructures of the extremal ones.

All interesting complications and phenomena that could arise already do it for graphs. After a short recap and intro on graphs, we will proceed to study the extremal questions.

Definition 2.1. A *graph* $G = (V_G, E_G)$ is a set of vertices V_G and edges $E_G \subseteq \binom{V_G}{2}$ (meaning the two–element subsets of V_G).

Example 2.1. Figure 2.1 shows examples of graphs.

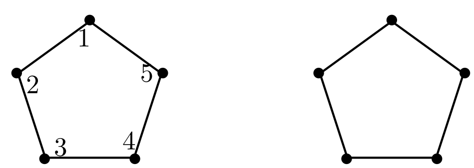


FIGURE 1. A graph with labels and a graph without labels.

Definition 2.2. A *graph homomorphism* φ from $G = (V_G, E_G)$ to $H = (V_H, E_H)$ is a function $\varphi : V_G \to V_H$ preserving edges: $\{u, v\} \in E_G \Rightarrow \{\varphi(u), \varphi(v)\} = E_H$.

Remark 2.1. From now on, an edge $\{u, v\}$ will be denoted uv.

Definition 2.3. A graph $G = (V_G, E_G)$ is a subgraph of $H = (V_H, E_H)$ if $V_G \subseteq V_H$ and $E_G \subseteq E_H$. The graph G is an induced subgraph, if $E_G = E_H \cap \binom{V_G}{2}$.

Example 2.2. Some common graphs are shown in Figure 2.2.

Definition 2.4. A subset $I \subseteq V_G$ is called *independent* if $e \not\subseteq I$ for all $e \in E_G$.

Example 2.3. To give a taste of where we are going, let us consider subgraphs of K_5 containing no triangles K_3. The one with a maximal number of edges is shown in Figure 3.

Note that $K_{2,3}$ has 6 edges. There are no other subgraphs to which an edge can be added without forcing a triangle, cf. Figure 4.

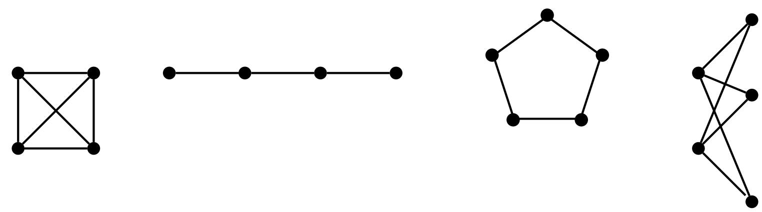


FIGURE 2. From left to right: the complete graph K_4, the path P_4, the cycle C_5 and the complete bipartite graph $K_{2,3}$.

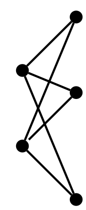


FIGURE 3. $K_{2,3}$

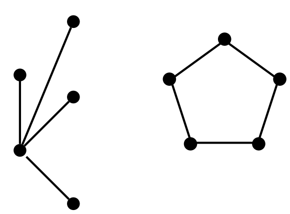


FIGURE 4. $K_{1,4}$ and C_5.

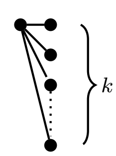


FIGURE 5. The star graph $K_{1,k}$.

Definition 2.5. The maximal number of edges in an n-vertex graph G not containing a copy of H is denoted $\mathrm{ex}(n; H)$.

Exercise 2.1. If H is the star graph in Figure 5, find, using elementary considerations, a tight asymptotic expression for $\mathrm{ex}(n; H)$ as $n \to \infty$.

Exercise 2.2. Draw the Hasse diagram of all nonlabeled subgraphs of K_4 on four vertices not containing a subgraph isomorphic to C_4.

Example 2.4. Many times it is informative to create graphs without forbidden substructures by first taking a random graph and then removing edges to kill forbidden substructures. In this example, we will do this in a very crude way.

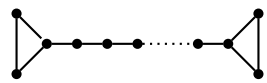


FIGURE 6.

Let us consider a random graph G on n vertices given by including edges independently with probability p. We want to kill triangles by removing an edge from each triangle in G. Later it will be demonstrated that this is indeed far too much.

The expected number of edges is $\binom{n}{2}p$ and the expected number of triangles is $\binom{n}{3}p^3$. After removing one edge per triangle, we might run out of triangles, but a lower bound for the number of remaining edges is $\binom{n}{2}p - \binom{n}{3}p^3$. This, as a real-valued function of p, has a critical point at $p = \frac{1}{\sqrt{n-2}}$, as can be seen by differentiating. Inserted into the previous expression, this yields roughly $\frac{n\sqrt{n}}{3}$ edges. But one can do much better! A complete bipartite graph with around $n/2$ vertices has no triangles, and $n^2/4$ edges in total.

Example 2.5. A better way to avoid K_t in an n-vertex graph G is to partition V_G into $t-1$ partitions of uniform size. This gives

$$\frac{t-2}{t-1}\binom{n}{2}$$

edges as $n \to \infty$, which is optimal. To see this, we will need some more definitions.

Remark 2.2. For a finite graph G on n vertices, there is a graph homomorphism $G \to K_n$ by any bijection $V_G \to \{1, \ldots, n\}$.

Definition 2.6. For a finite graph G, let $\chi(G)$ denote the *chromatic number*, ie. the smallest integer for which there exists a graph homomorphism $G \to K_{\chi(G)}$.

Exercise 2.3. (Easy) Show that $\chi(K_n) = n$ and $\chi(P_n) = 2$.

Exercise 2.4. Let H_t be the graph whose vertices correspond to independent sets of cardinality t in C_5. Two vertices are adjacent if the intersection of the corresponding independent sets is empty. Determine $\chi(H_t)$ for $0 \leq t \leq 2$.

Exercise 2.5. (Hard) Repeat Exercise 2.4 with C_7 and $0 \leq t \leq 3$.

Remark 2.3. The following theorem is most conveniently proved using the *regularity lemma*, which is proved and used extensively in the graph theory course.

Theorem 2.1 (Corollary of Erdös-Stone '46)**.** *Let H be a graph with at least one edge. Then*

$$\lim_{n\to\infty} ex(n,H)\binom{n}{2}^{-1} = \frac{\chi(H)-2}{\chi(H)-1}.$$

Exercise 2.6. Let H be the graph in Figure 6. Show that

$$\lim_{n\to\infty} \frac{\mathrm{ex}(n;H)}{n(n-1)} = 1.$$

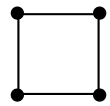


FIGURE 7. C_4.

Remark 2.4. For a graph G with $\mathrm{ex}(n;H)$ edges and n vertices that avoids H, we get $2^{\mathrm{ex}(n;H)}$ graphs without H, by deleting edges in different ways. If all subgraphs of G without H were subgraphs of an extremal graph on $\mathrm{ex}(n;H)$ edges, there would be exactly $2^{\mathrm{ex}(n;H)}$ graphs without H. In fact, that is almost true. There are not substantially many more of them, as we will see.

We will now establish a theorem to that effect, but we shall first introduce the concept of hypergraphs.

Definition 2.7. A *hypergraph* H is a set of vertices V_H and a set of edges $E_H \subseteq 2^{V_H}$ (the set of all subsets of V_H). If all edges are of the same cardinality l, that is $E_H \subseteq \binom{V_H}{l}$, then H is called an l-graph.

Remark 2.5. A 2-graph is an ordinary graph.

Remark 2.6. An l-graph is an anti-chain in a boolean lattice.

Definition 2.8. The concept of (induced) *subgraph* is defined in an analogous way to graphs. In addition, a *graph homomorphism* from a 2-graph to an l-graph is defined in the natural way.

Example 2.6. The complete l-graph on $\{1, \ldots, r\} := [r]$ has the edge set $\binom{[r]}{l}$.

Definition 2.9. A hypergraph H is called *vertex-edge transitive* if there is a group Γ of permutations of V_H such that for every $v_1 \in e_1$ and $v_2 \in e_2$, where $v_1, v_2 \in V_H$ and $e_1, e_2 \in E_H$, there is a permutation $\pi \in \Gamma$ such that $\pi(v_1) = v_2$ and $\pi(e_1) := \{\pi(u) | u \in e_1\} = e_2$.

Example 2.7. The 2-graph in Figure 7 is vertex-edge transitive with the action of the dihedral group D_8.

Exercise 2.7. (Easy) Construct a vertex-edge transitive 3-graph on 4 vertices and 4 edges. Specify the group acting on it.

Exercise 2.8. (Hard) Construct a vertex-edge transitive 3-graph on 6 vertices and 8 edges. Specify the group acting on it.

Remark 2.7. For those who consider the previous exercises too easy, here is a harder one: Find a vertex-edge transitive 3-graph with 12 vertices and 24 edges, such that for each edge $\{u, v, w\}$, there are exactly two edges containing $\{u, v\}$ and the group acting on the graph is $S_3 \times S_4$.

Definition 2.10. For an l-graph H, let $\mathrm{ex}(n, H)$ denote the maximal number of edges in an l-graph on n vertices without H as a subgraph.

Definition 2.11. Define

$$\pi(H) := \lim_{n\to\infty} \mathrm{ex}(n, H)\binom{n}{l}^{-1}.$$

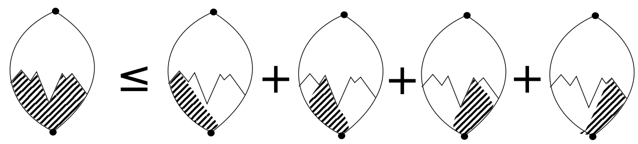


FIGURE 8. The "anti-chain inequality".

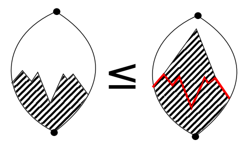


FIGURE 9. An upper bound using a principal ideal.

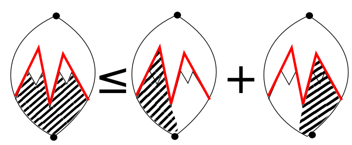


FIGURE 10. The container is in red.

Remark 2.8. We should need to prove that the previous limit exists, but let us leave this out. For 2-graphs $\pi(H)$ can be calculated as stated earlier, but for higher hypergraphs there are no known explicit formulas.

Remark 2.9. The following theorem was at first proved for 2-graphs by Erdös, Frankl and Rödl [**EFR**]. Later on, the l-graph case was settled by Nagle, Rödl and Schact [**NRS**]. There is now a conceptual proof using so called hypergraph containers by Saxton and Thomason [**ST**], which we will discuss later on.

Theorem 2.2. *Let H be an l-graph. The number of H-free l-graphs on the vertex set $\{1, \ldots, n\}$ is $2^{(\pi(H)+o(1))\binom{n}{l}}$.*

Remark 2.10. The l-graphs on $[n]$ is a boolean lattice of subsets of $\binom{[n]}{l}$ ordered by inclusion. Those without H is a lower ideal, and Theorem 2.2 estimates its size. Every principal ideal is of 2-power size, and one option to bound the size of a lower ideal from above, would be to add up the sizes of the principal ideals given by its anti-chain representation, as shown in Figure 2.10.

another option would be to bound it by one principal ideal, as in Figure 2.10.

A mixed strategy to find a container, that is, a lower ideal I satisfying

(1) I contains the lower ideal we want to estimate the size of;
(2) I has few maximal elements, equivalently I is given by a small anti-chain;
(3) every principal ideal given by a maximal element of the container I contains few elements above the lower ideal to be estimated.

An example of a container is in Figure 2.10.

Saxton and Thomason [**ST**] prove a very general theorem for lower ideals of independent sets of hypergraphs and provide an algorithm for finding the good containers. Hypergraphs without forbidden substructures are then reformulated as a problem in containers. Using this method, some of the hardest theorems in additive number theory can also be proven.

Exercise 2.9. (Easy) Let I be the lower ideal $\langle\{1\},\{2\},\{3\}\rangle$ in the boolean lattice on $\{1,2,3\}$. If J_1 is a principal ideal containing I, how small can $|J_1|-|I|$ be? If J_2 is a lower ideal with two maximal elements containing I, how small can $|J_2|-|I|$ be?

Exercise 2.10. Let I be the lower ideal of independent sets of C_5. Consider the containers $J \supseteq I$ with exactly two maximal elements i and j. Find the minimal value of

$$|\{p \leq i\}| + |\{p \leq j\}| - |I|.$$

CHAPTER 3

Chromatic polynomials

Definition 3.1. A *coloring* of a graph G by n colors is a graph homomorphism $\varphi : G \to K_n$.

Remark 3.1. Recall that the chromatic number $\chi(G)$ is the smallest number such that there exists a coloring $\varphi : G \to K_{\chi(G)}$.

Definition 3.2. The set of homomorphisms $\varphi : G \to H$ is denoted $\text{Hom}(G, H)$.

Proposition 3.1. *A graph homomorphism $\varphi \in Hom(H_1, H_2)$ induces a set map $\Phi : Hom(G, H_1) \to Hom(G, H_2)$ by $\Phi(\alpha) = \varphi\alpha$.*

PROOF. A composition of the graph homomorphisms $\alpha \in \text{Hom}(G, H_1)$ and $\varphi \in \text{Hom}(H_1, H_2)$ is a graph homomorphism. □

Proposition 3.2. *A graph homomorphism $\varphi \in Hom(G_1, G_2)$ induces a set map $\Phi : Hom(G_2, H) \to Hom(G_1, H)$ by $\Phi(\alpha) = \alpha\varphi$.*

PROOF. A composition of the graph homomorphisms $\varphi \in \text{Hom}(G_1, G_2)$ and $\alpha \in \text{Hom}(G_2, H)$ is a graph homomorphism. □

Exercise 3.1. Show that for $m \geq n$,

$$|\text{Hom}(K_n, K_m)| = \frac{m!}{(m-n)!}.$$

Exercise 3.2. Show that for $m \geq n = |G|$,

$$|\text{Hom}(G, K_m)| \geq \binom{m}{n} |\{\alpha \in \text{Hom}(G, K_n) | |\alpha(V_G)| = n\}| .$$

Remark 3.2. Note that for a fixed n, $\frac{m!}{(m-n)!}$ is a polynomial in m. We have a polynomial lower bound for $|\text{Hom}(G, K_m)|$ by Exercise 3.2. For the bound to make sense, we need $|\text{Hom}(G, K_n)| > 0$, which is achieved when $n = |G|$. Our lower bound in m is of degree n.

Exercise 3.3. Let G be a graph on n vertices and 0 the graph on n isolated vertices. Use a graph homomorphism $0 \to G$ to prove that $|\text{Hom}(G, K_m)| \leq m^n$ for $m \geq n$.

Remark 3.3. We have seen that $|\text{Hom}(G, K_m)|$ is confined between two degree n polynomials in m. This makes the following theorem plausible. We will give several proofs of it.

Theorem 3.1. *The function $P_G(m) = |Hom(G, K_m)|$ is a polynomial in m of degree $|G|$.*

Exercise 3.4. Find the chromatic polynomials for the families C_n, K_n, P_n. How about wheels on k spokes (a k-cycle coned over a vertex)?

Remark 3.4. Our argument for the plausibility of this theorem, and its first proof, are easy to generalize to a huge context, because counting problems turning polynomial is a very general phenomenon. The general reason for this is captured in commutative algebra by *Hilbert polynomials*.

Remark 3.5. The first proof requires some basic geometry.

Definition 3.3. A hyperplane in $\mathbb{R}^n$ is a subspace cut out by linear equations.

Proposition 3.3. *There is a bijection between Hom(G, K_m) and*

$$X = (\mathbb{Z}^n \cap [0, m-1]^n) \setminus \bigcup_{ij \in E_G} \{x_i = x_j\},$$

where $|G| = n$*.*

Proof. First we construct a unique point in X for every $\varphi \in \mathrm{Hom}(G, K_m)$. Label the vertices of K_m by $0, \ldots, m-1$ and the vertices of G by $0, \ldots, n-1$. Associate the point

$$x^\varphi = (\varphi(0), \ldots, \varphi(n-1)) \in \mathbb{Z}^n \cap [0, m-1]^n$$

to φ. It is by construction unique, but we should check that it is in X. If $x^\varphi \notin X$, then there exists an edge $ij \in E_G$ with $x_i^\varphi = x_j^\varphi$, or equivalently $\varphi(i) = \varphi(j)$, which would contradict that $\varphi \in \mathrm{Hom}(G, K_m)$. In the other direction, take $x = (x_0, \ldots, x_{n-1}) \in X$. Then construct a $\varphi^x : V_G \to [0, \ldots, m-1]$ by $\varphi^x(i) = x_i$. This is unique and $\varphi^x \in \mathrm{Hom}(G, K_m)$, because $\varphi^x(i) \neq \varphi^x(j)$ for any edge $ij \in E_G$, since $\{x_i = x_j\} \cap X = \emptyset$. □

Proposition 3.4. *There are m^{n-k+1} points in*

$$(\mathbb{Z}^n \cap [0, m-1]^n) \cap \{x_1 = \ldots = x_k\}.$$

Proof. Consider the bijection to

$$\mathbb{Z}^{n-k+1} \cap [0, m-1]^{n-k+1}$$

given by removing the first $k-1$ coordinates. □

Proof of Theorem 3.1. Count the number of points in X given in Proposition 3.3 by inclusion-exclusion using Proposition 3.4, and taking complements for each piece. □

Remark 3.6. A far reaching generalization is given by the following small modification of Proposition 3.4. Instead of hyperplanes of the type $\{x_1 = \ldots = x_k\}$, we could consider for example $\{x_1 = 2x_2\}$. The number of points in

$$\mathbb{Z}^2 \cap [0, m-1]^2 \cap \{x_1 = 2x_2\}$$

is

m	1	2	3	4	5	6	...
# points	1	1	2	2	3	3	...

or

$$f(m) = \begin{cases} m/2, & m \equiv 0(2), \\ (m+1)/2, & m \equiv 1(2). \end{cases}$$

Remark 3.7. A function that is a polynomial in every second term, third term, and so on, is called a *quasi-polynomial*. More intricate counting problems that are not polynomial, are quasi-polynomial instead.

Exercise 3.5. What is the number of points in

$$\mathbb{Z}^3 \cap [0, m-1]^3 \backslash (\{x_1 = 2x_2\} \cup \{x_1 = 3x_3\})$$

as a function of m?

Remark 3.8. The next proof of Theorem 3.1 is more elementary, and hard-codes the combinatorics of the first proof.

Definition 3.4. Let G be a graph with an edge $e = ij$. The *deletion* $G \backslash e$ is given by deleting e. The *contraction* G/e is given by replacing the vertices i and j of G by a new vertex that is adjacent to all vertices that are adjacent to i or j in G.

Remark 3.9. When contracting an edge of a triangle, it might be more natural to get two vertices with a double edge rather than an edge as in our definition above. That could be achieved by taking the edge set to be a multiset, but we will not take on that path on this course.

Proposition 3.5. *Let $e \in E_G$ for a graph G. Then there is a bijection*

$$Hom(G \backslash e, K_m) \to Hom(G, K_m) \sqcup Hom(G/e, K_m).$$

Proof. Let $e = ij$. Take a $\varphi \in \mathrm{Hom}(G \backslash e, K_m)$. If $\varphi(i) \neq \varphi(j)$, then $\varphi \in \mathrm{Hom}(G, K_m)$. If $\varphi(i) = \varphi(j)$, construct $\varphi \in \mathrm{Hom}(G/e, K_m)$ by sending all "old" vertices of G to wherever φ sends them, and the vertex of the contracted edge to $\varphi(i) = \varphi(j)$. This map is a bijection, as seen by constructing the inverses from $\mathrm{Hom}(G, K_m)$ and $\mathrm{Hom}(G/e, K_m)$. □

Lemma 3.1. *Let G be a graph with an edge e. Then $P_G(m) = P_{G \backslash e}(m) - P_{G/e}(m)$.*

Proof. Use the previous proposition. □

Lemma 3.2. *Let 0 be the graph on n vertices and no edges. Then $P_0(m) = m^n$.*

Proof. This is clear. □

Proof of Theorem 3.1. The proof goes by induction on the number of edges. If there are none, use Lemma 3.2. Otherwise, use Lemma 3.1 and induction. □

Remark 3.10. Originally, the chromatic polynomial was invented to attack the four coloring conjecture, which states that every planar graph can be colored with four colors. The attack failed, but some curious results were obtained, in the case of chromatic polynomials evaluated at noninteger values.

Definition 3.5. A *triangulation* is an edge-maximal planar graph.

Definition 3.6. Let $\xi = \frac{1+\sqrt{5}}{2}$ be the golden ratio.

Proposition 3.6 ([**T**]). *Let G be a triangulation. Then*

$$|P_G(\xi + 1)| \leq \xi^{5-|G|}.$$

Example 3.1. We have the following table in Figure 1. Note that all of the graphs are planar, and triangulations except for the dodecahedron (it has pentagons as faces).

Graph G	$\lvert G\rvert$	$P_G(\xi+1)$
Tetrahedron K_4	4	-1
Octahedron	6	$-4+2\sqrt{5}\approx 0.47$
Icosahedron	12	$-4575+2046\sqrt{5}\approx -0.0049$
Dodecahedron	20	$\frac{14667-32815\sqrt{5}}{2}\approx -9.30$

FIGURE 1. Some values of $P_G(\xi+1)$ for planar graphs.

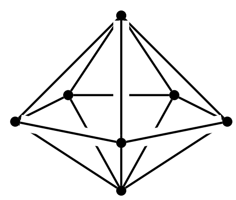


FIGURE 2. The graph T_5.

Exercise 3.6. (Hard) Let T_n be the triangulated graph formed from a cycle by "suspending" it from two vertices, ie. so that each of the two vertices is adjacent to each vertex in the cycle. For an example, see Figure 2. Prove that $P_{T_n}(\xi+1) = (-1)^n\frac{-a_n+b_n\sqrt{5}}{2}$, where $a_3 = 3, a_4 = 8, a_5 = 25$, $a_6 = 69$, and $a_n = 4a_{n-1} - 3a_{n-2} - 2a_{n-3} + a_{n-4}$, for $n > 6$. Also $b_3 = 1$, $b_4 = 4$, $b_5 = 11$, $b_6 = 31$, and $b_n = 4b_{n-1} - 3b_{n-2} - 2b_{n-3} + b_{n-4}$, for $n > 6$.

Remark 3.11. Another weird fact about polynomials enumerating combinatorial structures, is that inserting negative numbers many times has an explicit combinatorial interpretation.

Definition 3.7. An *orientation* of a graph is an assignment of direction to each edge, ie. •—• becomes •→• or •←•. An orientation is called *acyclic*, if there are no directed cycles.

Theorem 3.2 ([**St73**]). *Let G be a graph. Then there are $(-1)^{|G|}P_G(-1)$ acyclic orientations for G.*

Exercise 3.7. Prove Theorem 3.2 using deletion-contraction.

Exercise 3.8. Continuing with notations as in the previous theorem, let us generalize the above result and consider $P_G(-x)$ for $x > 0$.

a) For a (proper) coloring $f \in \mathrm{Hom}(G, K_n)$, define an orientation ρ as follows:
$$\rho := \{\{i,j\} \in E | i < j, f(i) > f(j)\},$$
ie. direct the edges (i,j) if $e \notin \rho$, and (j,i) if $e \in \rho$. Here we have enumerated the vertices as $1,\ldots,|G|$ and $1,\ldots,n$. Prove that the associated digraph G' is acyclic.
b) Call a coloring f and an orientation ρ strictly compatible if $f(i) > f(j)$ for all $(i,j) \in G'$. Prove that for a strictly compatible pair (f,ρ), the coloring f is proper and ρ is an acyclic orientation.
c) Prove that $(-1)^n P_G(-x)$ is the number of compatible pairs (f,ρ) with f a proper x-coloring and ρ an acyclic orientation. You should use a similar deletion-contraction argument as in the previous exercise.

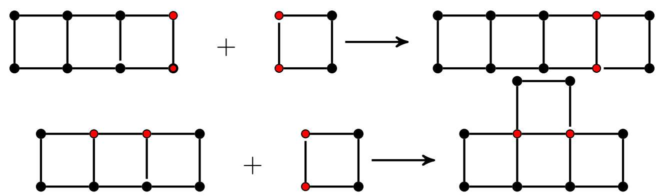


FIGURE 3. Examples of gluing along K_2 to get nonisomorphic graphs with the same chromatic polynomials.

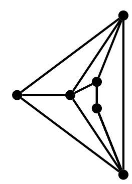


FIGURE 4. A graph that we cannot decompose the chromatic polynomial of.

Remark 3.12. It is easy to produce nonisomorphic graphs with the same chromatic polynomial, using the following proposition.

Proposition 3.7. *Let G_1, G_2 be two disjoint graphs, each containing a copy of K_r. Let G be the graph given by gluing G_1, G_2 along this copy. Then*

$$P_G(t) = \frac{P_{G_1}(t)P_{G_2}(t)}{P_{K_r}(t)}.$$

Example 3.2. Some examples of gluing along complete subgraphs are shown in Figure 3.

Example 3.3. There exist graphs with equal chromatic polynomials, not coming from the previous trick on gluing. For example, consider the wheel on 5 spokes, ie. C_5 with an additional vertex adjoined to each vertex of the cycle, and

Exercise 3.9. A graph is called *chromatically unique*, if it is the only one with its chromatic polynomial. For example, the empty, complete, cyclic, and complete bipartite graphs with parts of equal order are all unique. Can you find a chromatically unique graph on four vertices which is connected and does not come from the aforementioned classes?

Definition 3.8. A sequence $a_0, \dots, a_n$ of real numbers is *unimodal*, if $a_0 \leq a_1 \leq \dots \leq a_i \geq \dots \geq a_n$ for some i, and *log-concave*, if $a_{i-1}a_{i+1} \leq a_i^2$.

Proposition 3.8. *Any log-concave sequence of strictly positive real numbers is unimodal.*

Remark 3.13. The study of log-concave and unimodal sequences of polynomial coefficients goes back to Newton, and is a rich source of inequalities. There are many open and solved conjectures regarding it. Let $P_G(t) = a_n t^n - a_{n-1}t^{n-1} + \ldots + (-1)^n a_0$ be the chromatic polynomial of G. One can see that all $a_i \geq 0$. Read conjectured in 1968 [**R**], that the a_i sequence is unimodal. This conjecture, in a more general form, was proven by June Huh in 2011. The proof is by algebraic geometry, and deals with matroids, a generalization of graphs in this context.

Theorem 3.3 ([**H**]). *If $P_G(t) = a_n t^n - a_{n-1} t^{n-1} + \ldots + (-1)^n a_0$ is the chromatic polynomial of G, then the a_i sequence is log-concave.*

Exercise 3.10. Let $a_n, \ldots, a_0$ be the coefficient sequence for the path P_n. When does it have a maximal element? If it does, what is it, and what is its value?

CHAPTER 4

Acyclic matchings on posets

Definition 4.1. A *matching* on a graph G is a set of disjoint edges.

Remark 4.1. There are many questions in graph theory and combinatorics in general that can be formulated as regarding matchings. In applications, they are also important, but we will mostly focus on their relation to topological combinatorics.

Remark 4.2. In inclusion-exclusion arguments, it is common to have expressions like

$$\sum_{a\in A} 1 + \sum_{b\in B} -1$$

to evaluate. One method to compute these is to define a graph on $A \cup B$ with vertices between elements of A and B, and then remove matched vertices. This will keep the sum invariant.

Remark 4.3. We shall do matchings in posets, which requires some more terminology.

Definition 4.2. A *matching* on a poset P is a set of pairs $p \prec q$ in P with no element in more than one pair.

Definition 4.3. A sequence $p_1 \prec q_1, p_2 \prec q_2, \ldots, p_n \prec q_n$ of pairs in a matching on a poset is a *cycle* if $q_1 \geq p_2, q_2 \geq p_3, \ldots, q_n \geq p_1$ and $n > 1$.

Definition 4.4. A matching is *acyclic* or a *Morse matching* if there are no cycles.

Example 4.1. Consider the simplicial complex on $\{1, 2, 3, 4\}$ with maximal elemnts $12, 23, 34, 14$. There is an acyclic matching shown in Figure 1.

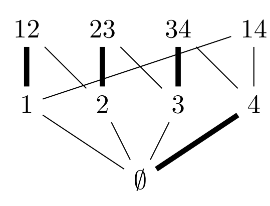


Figure 1. An acyclic matching on a simplicial complex.

That only 14 is not matched in Figure 1 can be given a topological interpretation. Drawing the simplicial complex as in Figure 2, we get one hole, forcing 14 to be unmatched. We will not state nor prove the topological facts related to acyclic matchings but rather work in a completely combinatorial setting. Note that $|\tilde{\chi}| = 1$ is obvious from the matching.

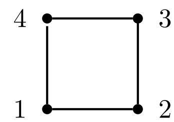


FIGURE 2. A simplicial complex on $\{1,2,3,4\}$.

Exercise 4.1. Let B_n be the boolean lattice on $[n]$. Show, without relying on later technology, that the matching $M = \{a \prec a \cup \{1\} | a \subseteq \{2,\ldots,n\}\}$ is acyclic.

Remark 4.4. The procedure in Exercise 4.1 is called *matching by adding* 1.

Exercise 4.2. (Hard) Construct an acyclic matching on the subposet of the boolean lattice B_{2n} given by sets of order $\leq n$. All elements of order $< n$ should be matched.

Exercise 4.3. (Easy) Construct an acyclic matching for any poset P.

Remark 4.5. It is important to get as many elements matched as possible.

Definition 4.5. The nonmatched elements are called *critical*.

Exercise 4.4. Let L be a bounded lattice with $\hat{0}, \hat{1}$, not equal to each other. Show that there is an acyclic matching on L with $\hat{0}$ not critical.

Remark 4.6. The following lemma is quite powerful in constructing acyclic matchings.

Lemma 4.1. *Let P, Q be posets and $\varphi : P \to Q$ a poset map. If M_q is an acyclic matching on $\varphi^{-1}(q)$ for every $q \in Q$, then*

$$M = \bigcup_{q \in Q} M_q$$

is an acyclic matching on P.

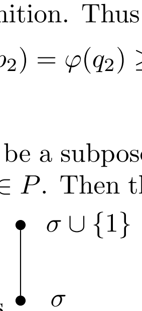
PROOF. Say that $p_1 \prec q_1 \geq p_2 \prec q_2 \geq \ldots \prec q_n \geq p_1$ would be a cycle in P. It cannot be in the same fiber $\varphi^{-1}(q)$, since each M_q is acyclic. Every $p_i \prec q_i$ is in the same fiber by definition. Thus

$$\varphi(p_1) = \varphi(q_1) \geq \varphi(p_2) = \varphi(q_2) \geq \cdots \varphi(q_i) > \varphi(p_{i+1}) = \varphi(q_{i+1}) \geq \cdots \geq \varphi(p_1),$$

a contradiction. □

Example 4.2. Let P be a subposet of the boolean lattice on $\{1,\ldots,n\}$, satisfying $\sigma \backslash \{1\} \in P \Leftrightarrow \sigma \cup \{1\} \in P$. Then there is a poset map $\varphi : P \to Q = \{\sigma \in P | 1 \notin \sigma\}$.

The fiber of $\sigma \in Q$ is



and there is an acyclic matching $M_\sigma = \{\sigma \prec \sigma \cup \{1\}\}$. Together $M = \cup \sigma \in Q M_\sigma$ becomes an acyclic matching on P, without critical elements.

Definition 4.6. An acyclic matching without critical elements is called *complete*.

Remark 4.7. Now we will study acyclic matchings on simplicial complexes coming from combinatorics, and graphs in particular.

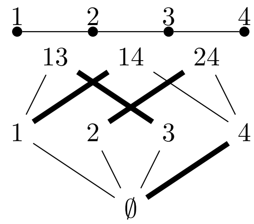


FIGURE 3. P_4 and its independent sets.

Example 4.3. The independent sets of P_4 are shown in Figure 3.

Definition 4.7. The simplicial complex consisting of all independent sets of G is called the *independence complex* of G and denoted by $\mathrm{Ind}(G)$.

Proposition 4.1. *Let G be a graph with an isolated vertex v. Then there is a complete acyclic matching on Ind(G).*

PROOF. Match with v. □

Proposition 4.2. *Let P be a finite poset whose elements can be bipartitioned into "odd" and "even" elements, so that no odd element covers an odd one and no even element covers an even one. If there is an acyclic matching with all critical elements odd or even, then that is the minimal number of critical elements in any acyclic matching of P.*

PROOF. Every matched odd/even pair kills one odd and one even critical cell, so there are at least $|\#\text{odd}-\#\text{even}|$ critical elements. □

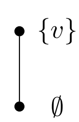
Remark 4.8. Whenever we are in the situation of Proposition 4.2, we say that the acyclic matching is *optimal*.

Exercise 4.5. Let M be an acyclic matching with exactly one critical cell on a finite simplicial complex. Let odd/even be the parity of the cardinality of the elements. Show that the matching is optimal.

Definition 4.8. The *neighborhood* of a vertex v in a graph G, denoted $N(v)$, is the set of vertices adjacent to v.

Lemma 4.2. *Let v be a vertex of G, $G_1 = G\backslash\{v\}$, and $G_2 = G\backslash\{N(v) \cup \{v\}$. If there are acyclic matchings on Ind(G_1) with c_1 critical elements, and on Ind(G_2) with c_2 critical elements, then there is an acyclic matching on Ind(G) with $c_1 + c_2$ elements.*

PROOF. Consider the poset map

$\mathrm{Ind}(G) \rightarrow$ 

given by $\sigma \mapsto \sigma \cap \{v\}$. Then $\varphi^{-1}(\emptyset)$ is $\mathrm{Ind}(G\backslash v)$. Any independent set of G containing v will have no vertices from the neighborhood $N(v)$. This gives a poset isomorphism

$$\alpha : \mathrm{Ind}(G\backslash\{N(v) \cup \{v\}\}) \rightarrow \varphi^{-1}(\{v\})$$

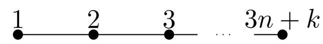


FIGURE 4. A path of length $3n+k$.

given by $\sigma \mapsto \sigma \cup \{v\}$. By Lemma 4.1, any acyclic matchings on $\text{Ind}(G\backslash v)$ and $\text{Ind}(G\backslash\{N(v)\cup\{v\}\})$ give one on $\text{Ind}(G)$. □

Theorem 4.1. *The optimal acyclic matchings on the independence complexes of paths are as follows:*

- *The complexes Ind*(P_{3n}) *have one critical element of cardinality* n*;*
- *The complexes Ind*(P_{3n+1}) *have no critical elements;*
- *The complexes Ind*(P_{3n+2}) *have one critical element of cardinality* $n+1$.

Exercise 4.6. (Easy) Prove Theorem 4.1 for $n = 0$.

PROOF OF THEOREM 4.1. We will use induction on n. The base case was handled in Exercise 4.6. For $n > 0$, we have a path P as in Figure 4. Use vertex 2 and Lemma 4.2 to break the graph into two cases:

- The independence complex of $P\backslash\{2\}$ has a perfect acyclic matching, since 1 is an isolated vertex.
- The independence complex of $P\backslash\{N(2)\cup\{2\}\} = P\backslash\{1,2,3\}$ has one critical element if $k = 0, 2$, and none if $k = 1$, by induction. The potential critical element σ lifts from $P\backslash\{1,2,3\}$ to $\sigma\cup\{2\}$ for P.

□

Exercise 4.7. (Hard) Let S be a star with at least one edge. Show that the optimal acyclic matching on the independence complex has one critical element.

Exercise 4.8. Use Lemma 4.2 to find an optimal acyclic matching for the independence complex of the four-cycle.

Theorem 4.2. *Let* G *be the disjoint union of the graphs* G_1, G_2. *If there are acyclic matchings on their independence complexes with* c_1, c_2 *critical elements respectively, then there is one on the independence complex of* G *with* c_1c_2 *critical elements.*

PROOF. The elements of $\text{Ind}(G)$ are of the form $\sigma_1\cup\sigma_2$, where $\sigma_1 \in\text{Ind}(G_1)$ and $\sigma_2 \in\text{Ind}(G_2)$ are arbitrary independent sets. Construct a matching on $\text{Ind}(G)$ by matching $\sigma_1\cup\sigma_2$ and $\tau_1\cup\sigma_2$ whenever σ_1,τ_1 are matched in $\text{Ind}(G_1)$, and match $\sigma_1\cup\sigma_2$ whenever σ_1 is critical on $\text{Ind}(G_1)$ with σ_2,τ_2 matched in $\text{Ind}(G_2)$. This matching can be demonstrated then to be acyclic by studying the fibers of the map from $\text{Ind}(G)$ into $\text{Ind}(G_1)$ defined by identifying matching elements. The critical cells are exactly those $\sigma_1\cup\sigma_2$ for which σ_1 is critical in $\text{Ind}(G_1)$ and σ_2 is critical in $\text{Ind}(G_2)$. □

Exercise 4.9. Let G be a graph consisting of disjoint paths. Is there an optimal acyclic matching on the independence complex of G?

Exercise 4.10. (Hard) Let us consider the n-gons drawn in the plane with some noncrossing internal edges, seen as a poset with inclusion order. Drawn in Figure 5 are the cases $n = 4, 5$ with optimal acyclic matchings shown. Find an optimal acyclic matching in the case $n = 6$.

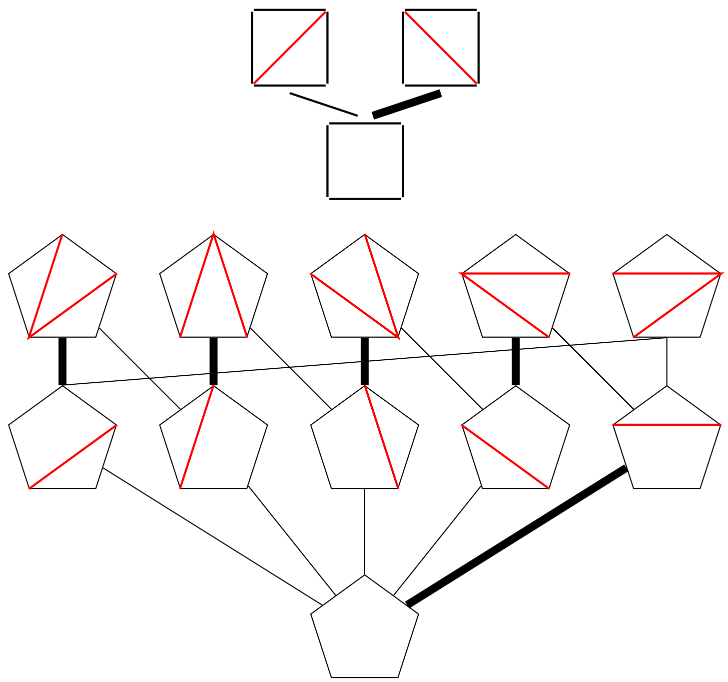


FIGURE 5. Acyclic matchings in polygons.

CHAPTER 5

Complete (perhaps not acyclic) matchings

Remark 5.1. Last week, we saw that more often than not, it is not possible to find complete acyclic matchings. Sometimes this was obstructed by parity reasons with too many odd compared to even elements, rather than the cycles themselves. How about the situation with the odd/even obstruction gone? It will not be enough in itself, but a slightly stronger condition from linear algebra is. The main aim of this chapter is to describe this condition, building on results by Stanley [**St93**].

Remark 5.2. We start with some linear algebra. From physics courses, you might know that a collection of vectors spanning a subspace can be given two orientations, ie. the left/right–handed ones.

Definition 5.1. Let $V = \{v_1, \cdots, v_r\}$ be a finite set and $\Bbbk$ a field. The *exterior algebra* $\Lambda(\Bbbk V)$ has elements of the form $\alpha v_{i_1} \wedge \cdots \wedge v_{i_k}$, with addition defined by

$$\alpha_1 v_{i_1} \wedge \cdots \wedge v_{i_k} + \alpha_2 v_{i_1} \wedge \ldots \wedge v_{i_k} = (\alpha_1 + \alpha_2) v_{i_1} \wedge \cdots \wedge v_{i_k}$$

and multiplication by

$$(\alpha v_{i_1} \wedge \cdots \wedge v_{i_k})(\beta v_{j_1} \wedge \cdots \wedge v_{j_l}) = \alpha\beta v_{i_1} \wedge \cdots \wedge v_{i_k} \wedge v_{j_1} \wedge \cdots \wedge v_{j_l}$$

with the relations $v_i \wedge v_j + v_j \wedge v_i = 0$ for all $1 \leq i, j \leq r$.

Remark 5.3. We denote the empty form by $\mathbb{1}$.

Remark 5.4. Any form can be transformed to have indices unique and in order, and these form a basis for Λ over $\Bbbk$.

Example 5.1. Let $r = 2$. Consider the element $v = v_1 + v_2$. We have that

$$\begin{aligned}
\mathbb{1} \wedge v &= v_1 + v_2, \\
v_1 \wedge (v_1 + v_2) &= v_1 \wedge v_1 + v_1 \wedge v_2 = v_1 \wedge v_2, \\
v_2 \wedge (v_1 + v_2) &= v_2 \wedge v_1 + v_2 \wedge v_2 = -v_1 \wedge v_2, \\
(v_1 \wedge v_2) \wedge (v_1 + v_2) &= v_1 \wedge v_2 \wedge v_1 + v_1 \wedge v_2 \wedge v_2 = 0.
\end{aligned}$$

The right multiplication of v on

$$\alpha_\phi \mathbb{1} + \alpha_1 v_1 + \alpha_2 v_2 + \alpha_{12} v_1 \wedge v_2$$

can be described by a matrix transformation $\phi : \Bbbk^4 \to \Bbbk^4$ as

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} \alpha_\phi \\ \alpha_1 \\ \alpha_2 \\ \alpha_{12} \end{pmatrix}.$$

The image of ϕ, im ϕ and the kernel

$$\ker \phi = \{w | \phi(w) = 0\}$$

are in fact the same:

$$\text{im } \phi = \ker \phi = \{(\alpha_\phi, \alpha_1, \alpha_2, \alpha_{12}) | \alpha_\phi = 0, \alpha_1 = \alpha_2\}.$$

Matrices of this type, ie. ones with equal image and kernel, are crucial for this chapter.

Exercise 5.1. Perform the same analysis as in Example 5.1 but for $r = 3$, with $v = v_1 + v_2 + v_3$. State the matrix explicitly and verify the equality of the image and the kernel.

Remark 5.5. The following elementary theorem of graph theory is surprisingly useful. There are variations of the same theme known as Dilworth's, König's, and Menger's theorems; and the Ford-Fulkerson algorithm of the max-flow min-cut theorem.

Lemma 5.1 (The Marriage theorem.)**.** *Let D be a directed graph and A a subset of vertices such that for every $S \subseteq A$ there are at least $|S|$ vertices of $D \backslash A$ with at least one ordered edge from S to them. Then there is a matching $a \to \rho(a)$ for all $a \in A$ with*

$$\{\rho(a) | a \in A\} \subseteq D \backslash A$$

and

$$|\{\rho(a) | a \in A\}| = |A|.$$

Definition 5.2. A matching using all vertices is *complete*.

Exercise 5.2. Find a graph that demonstrates that the condition $S \subseteq A$ in Lemma 5.1 cannot be weakened.

Remark 5.6. The following could be considered a "quantum" version of Lemma 5.1.

Lemma 5.2. *Let D be a directed graph on n vertices V_D and $\Bbbk$ a field. Let ρ be a linear transformation of the $\Bbbk$–vector space $\Bbbk V_D$ with basis V_D. If $\rho(v) \in span_{\Bbbk}\{w \in V_D | vw \in E_V\}$ and im $\rho = \ker \rho$, then D has a complete matching.*

Proof. By linear algebra, $\dim(\text{im } \rho) = \dim(\ker \rho)$, and $\dim(\text{im } \rho) + \dim(\ker \rho) = n$. Choose a subset A of V_D whose image is a basis of $\Bbbk V_D / \text{im } \rho$. Then $|A| = |V_D \backslash A| = \frac{n}{2}$. We argue by contradiction to apply Lemma 5.1. Say that there would be an $S \subseteq A$ with less than $|S|$ vertices of $V_D \backslash A$ with edges to them from S. Then $\{\rho(s) | s \in S\}$ is a linearly dependent set of vectors in $\Bbbk V_D / \Bbbk A$ since it spans less than $|S|$ dimensions. Take a vector $\sum_{s \in S} \alpha_s \rho(s) = 0$ in $\Bbbk V_D / \Bbbk A$ with not all $\alpha_s = 0$. But this is not possible since $\ker \rho = \text{im } \rho$ and A is a basis of $\Bbbk V_D / \text{im } \rho$, which forces all of the α_s to be zero. ☐

Definition 5.3. Let Δ be a simplicial complex on $V = \{v_1, \ldots, v_r\}$. The exterior face ring $\Lambda[\Delta]$ over $\Bbbk$ is $\Lambda[\Bbbk V]$ with the extra relations $v_{i_1} \wedge \ldots \wedge v_{i_k} = 0$ for $1 \leq i_1 < i_2 < \ldots < i_k \leq r$ with $\{i_1, \ldots, i_k\} \notin \Delta$.

Remark 5.7. The exterior algebra is naturally graded by $\text{gr}(v_{i_1} \wedge \ldots \wedge v_{i_k}) = k$ whenever $i_1 < \ldots < i_k$.

Remark 5.8. Multiplying on the right by $v = v_1 + \ldots + v_r$ increases the grading by one. Thus the matrix defining this transformation as in Example 5.1 and Exercise 5.1 splits into a block matrix and the condition $\ker \rho = \text{im } \rho$ can be checked in each grade separately. In topology this is called the coboundary operator and the failure of $\ker \rho = \text{im } \rho$ is measured by the cohomology.

Example 5.2. Consider the simplicial complex Δ on $\{v_1, v_2, v_3, v_4\}$ with maximal elements $\{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_4\}$. Let $v = v_1 + v_2 + v_3 + v_4$. Then

$$\begin{aligned}
\mathbb{1} \wedge v &= v_1 + v_2 + v_3 + v_4, \\
v_1 \wedge v &= v_1 \wedge v_1 + v_1 \wedge v_2 + v_1 \wedge v_3 + v_1 \wedge v_4 \\
&= v_1 \wedge v_2, \\
v_2 \wedge v &= v_2 \wedge v_1 + v_2 \wedge v_2 + v_2 \wedge v_3 + v_2 \wedge v_4 \\
&= -v_1 \wedge v_2 + v_2 \wedge v_3, \\
v_3 \wedge v &= v_3 \wedge v_1 + v_3 \wedge v_2 + v_3 \wedge v_3 + v_3 \wedge v_4 \\
&= -v_2 \wedge v_3 + v_3 \wedge v_4, \\
v_4 \wedge v &= v_4 \wedge v_1 + v_4 \wedge v_2 + v_4 \wedge v_3 + v_4 \wedge v_3 \\
&= -v_3 \wedge v_4,
\end{aligned}$$

and $v_1 \wedge v_2 \wedge v = v_2 \wedge v_4 \wedge v = v_3 \wedge v_4 \wedge v = 0$, since the maximal cardinality of elements in Δ is two.

The matrix defining the coboundary transformation with rows and columns ordered as $\phi, 1, 2, 3, 4, 12, 23, 34$ becomes

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & -1 & 0 & 0 & 0
\end{pmatrix}$$

Exercise 5.3. Consider the simplicial complex on $\{v_1, \ldots, v_r\}$ with maximal elements $\{v_1, v_2\}, \ldots, \{v_{r-1}, v_r\}$ generalizing Example 5.2. What is the general form of the coboundary matrix?

Definition 5.4. A simplicial complex Δ is called *acyclic* if $\operatorname{im} \rho = \ker \rho$, where ρ is the linear transformation defined by the coboundary operator on the external face ring of Δ.

Remark 5.9. The simplicial complex of all subsets of a set (a *simplex*) is interesting enough with its graded coboundary maps. To get a name, it is called the *Koszul complex*. The applications of the Koszul complex range far outside pure mathematics, for example numerical analysis [**AFW**].

Remark 5.10. We have previously (see Example 1.10) hinted that simplicial complexes can be realized as triangulated subspaces of Euclidean spaces. Every simplicial complex that looks like a ball or a disc is acyclic; so there are many acyclic simplicial complexes. This is proved in any basic course on algebraic topology.

Theorem 5.1 (Stanley)**.** *Let Δ be an acyclic simplicial complex. Then there is a complete matching M on the poset of Δ, and*

$$\Delta' = \{\sigma | \sigma \prec \tau \text{ is in } M\}$$

is a subcomplex of Δ.

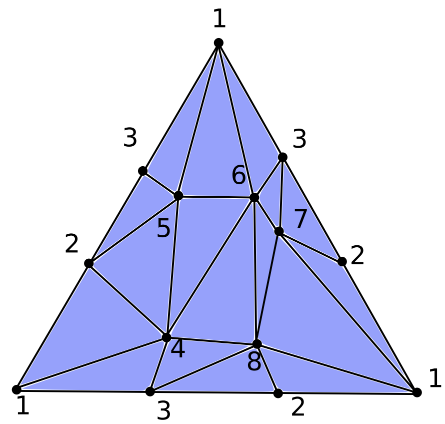


FIGURE 1. The dunce hat.

PROOF. To get a complete matching, we just introduce the coboundary operator, construct a directed graph D whose vertices are all $\sigma \in \Delta$ and $\sigma \to \tau$ whenever $\sigma \prec \tau$, and then apply Lemma 5.2. To get a matching with Δ' a subcomplex, the choice of A in the proof of Lemma 5.2 has to be made more precise: The basis of V_D can be given a lexicographic order. If we insist on A being the subset minimal in that order, among those suitable, then it can be verified that Δ' is a subcomplex. □

Exercise 5.4. This is an exercise only suitable for those skilled with computers, and not mandatory for higher grades. The simplicial complex in Figure 1 is called the *dunce hat*, and it is acyclic. Construct a complete matching on its poset by inserting it in standard software for finding matchings or contemporary versions of Ford–Fulkerson.

Remark 5.11. One can show, using topology, that there does not exist a complete acyclic matching of the poset of the dunce hat. There is an acyclic matching with two critical elements. It can be found using Fourier–Morse matchings, see [**E**].

Exercise 5.5. (Hard) Consider the infinite poset in Figure 2. Could you construct a 2×2 matrix and use it for each level as a linear transformation such that the infinite block matrix satisfies $\operatorname{im} \rho = \ker \rho$, and there is a complete matching supported by the matrix? Note that we have *not* defined things in the general infinite setting. Instead, you are asked to give a reasonable interpretation in this specific case.

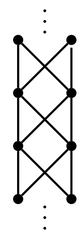


FIGURE 2. The Hasse diagram of an infinite poset.

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