Discrete Time H^{∞} Algebraic Riccati Equations

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Contents

Preface

In this book, we consider various structures associated to the dynamical system, given by the difference equations

$$\begin{cases} x_{j+1} &= Ax_j + Bu_j, \\ y_j &= Cx_j + Du_j, \quad j = 0, \pm 1, \pm 2, \cdots, \end{cases}$$

on Hilbert spaces, and the discrete time algebraic Riccati equation (DARE)

$$\begin{aligned} &A^*PA - P + C^*JC \\ &= (D^*JC + B^*PA)^* (D^*JD + B^*PB)^{-1} (D^*JC + B^*PA), \end{aligned}$$

where J is a self-adjoint, bounded cost operator. Various types of algebraic Riccati equations are connected to a number of fields in mathematics where state space realizations for analytic operator-valued functions are of importance. Typical applications within system theory are feedback control and stabilization of systems, initial state estimation by the Kalman filter, system identification by the spectral factorization of an estimated spectral function, state space H^∞ control theory, and various applications in game theory. Our main interest lies on the operator and analytic function theory aspects of the DARE, such as the spectral factorization, the inner-outer factorization and certain invariant subspace problems.

Our treatment does not require the finite dimensionality of any of the Hilbert spaces, but a certain compactness assumption is often required to reduce the "infinite dimensionality" of the system. For the full results, the cost operator J is required to be nonnegative. State space isomorphism techniques and special realizations of transfer functions are neither considered nor applied, due to time and page limitations.

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Twinkle, twinkle, little-$*$,
how I wonder what you are!

Notation

We use the following notations throughout the paper: $\mathbf{Z}$ is the set of integers. $\mathbf{Z}_+ := \{j \in \mathbf{Z} \mid j \geq 0\}$. $\mathbf{Z}_- := \{j \in \mathbf{Z} \mid j < 0\}$. $\mathbf{T}$ is the unit circle and $\mathbf{D}$ is the open unit disk of the complex plane $\mathbf{C}$. If H is a Hilbert space, then $\mathcal{L}(H)$ denotes the bounded and $\mathcal{LC}(H)$ the compact linear operators on H. If $A \in \mathcal{L}(H)$, then $A^0 := I$, by convention. Elements of a Hilbert space are denoted by lower case letters; for example $u \in U$. Sequences in Hilbert spaces are denoted by $\tilde{u} = \{u_i\}_{i \in I} \subset U$, where I is the index set. Usually $I = \mathbf{Z}$ or $I = \mathbf{Z}_+$. Given a Hilbert space Z, we define the sequence spaces

$$\begin{aligned}
&Seq(Z) := \big\{\{z_i\}_{i\in\mathbf{Z}} \mid z_i \in Z \quad \text{and} \quad \exists I \in \mathbf{Z} \quad \forall i \leq I : z_i = 0\big\},\\
&Seq_+(Z) := \big\{\{z_i\}_{i\in\mathbf{Z}} \mid z_i \in Z \quad \text{and} \qquad \forall i < 0 : z_i = 0\big\},\\
&Seq_-(Z) := \big\{\{z_i\}_{i\in\mathbf{Z}} \in Seq(Z) \mid z_i \in Z \quad \text{and} \qquad \forall i \geq 0 : z_i = 0\big\},\\
&\ell^p(\mathbf{Z}; Z) := \big\{\{z_i\}_{i\in\mathbf{Z}} \subset Z \mid \sum_{i\in\mathbf{Z}} ||z_i||_Z^p < \infty\big\} \quad \text{for} \quad 1 \leq p < \infty,\\
&\ell^p(\mathbf{Z}_+; Z) := \big\{\{z_i\}_{i\in\mathbf{Z}_+} \subset Z \mid \sum_{i\in\mathbf{Z}_+} ||z_i||_Z^p < \infty\big\} \quad \text{for} \quad 1 \leq p < \infty,\\
&\ell^\infty(\mathbf{Z}; Z) := \big\{\{z_i\}_{i\in\mathbf{Z}} \subset Z \mid \sup_{i\in\mathbf{Z}} ||z_i||_Z < \infty\big\}.
\end{aligned}$$

The following linear operators are defined for $\tilde{z} \in Seq(Z)$:

- the projections for $j, k \in \mathbf{Z} \cup \{\pm\infty\}$

$$\begin{aligned}
&\pi_{[j,k]}\tilde{z} := \{w_j\}; \quad w_i = z_i \quad \text{for} \quad j \leq i \leq k, \quad w_i = 0 \quad \text{otherwise},\\
&\pi_j := \pi_{[j,j]}, \quad \pi_+ := \pi_{[1,\infty]}, \quad \pi_- := \pi_{[-\infty,-1]},\\
&\bar{\pi}_+ := \pi_0 + \pi_+, \quad \bar{\pi}_- := \pi_0 + \pi_-,
\end{aligned}$$

- the bilateral forward time shift τ and its inverse, the backward time shift τ^*

$$\begin{aligned}
\tau\tilde{u} &:= \{w_j\} \quad \text{where} \quad w_j = u_{j-1},\\
\tau^*\tilde{u} &:= \{w_j\} \quad \text{where} \quad w_j = u_{j+1}.
\end{aligned}$$

Other notations are introduced when they are needed.

Chapter 1

Discrete Time Linear Systems

1.1 Introduction

In this chapter, we develop the basic system theoretic tools that are required to conveniently describe the behaviour of the dynamical system, given by the difference equations

$$\begin{cases} x_{j+1} &= Ax_j + Bu_j, \\ y_j &= Cx_j + Du_j, \quad j = 0, \pm 1, \pm 2, \cdots, \end{cases} \tag{1.1}$$

where $A \in \mathcal{L}(H)$, $B \in \mathcal{L}(U;H)$, $C \in \mathcal{L}(H;Y)$, $D \in \mathcal{L}(U;Y)$, and U, H and Y are Hilbert spaces. To equations (1.1), we associate a data structure ϕ, called a discrete time linear system (DLS) in difference equation form. The DLS ϕ is simply an ordered quadruple $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix} \right)$ of the bounded linear operators A, B, C and D, appearing in equations (1.1. Basic theory of the DLS ϕ is given in Section 1.2. In Section 1.3, we introduce another data structure Φ, called a DLS in I/O form. Such Φ is a quadruple $\left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix} \right]$ of three linear mappings $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$, together with the family of bounded linear operator $\{A^j\}_{j \geq 0}$. By Theorem 15, these two notions of DLSs give two equivalent formalisms to describe the general class of dynamical systems, given by equations (1.1). Adjoints, compositions and inverses of DLSs are introduced in Section 1.4. In Section 1.5, we develop the state feedback structure for DLSs in difference form, and in Section 1.6 we do this for DLSs in I/O form. It is shown in Lemma 26 that these feedback notions are equivalent.

Until now no stability assumptions or definitions for DLSs have been made, if one does not regard the boundedness of the operators in (1.1), known as the well-posedness of the DLS, as a stability assumption. In Section 1.7, the ℓ^2 norms and inner products are introduced to the input and output sequences. Topological versions of the linear mappings $\mathcal{B}, \mathcal{C}, \mathcal{D}$, together with their domains, are defined. In Definition 32, several stability notions for DLSs are given, such as input stability, output stability, strong H^2 stability and I/O stability. The closed graph property, the density of domain and the boundedness for the (topological) operators $\mathcal{C}$, $\mathcal{D}\pi_0$ and $\mathcal{D}\bar{\pi}_+$ are studied. In Section 1.8, a stronger topology for the state space is introduced. With this new topology, we transform the original strongly H^2 stable DLS into a modified system with the same I/O map and algebraic properties, but with the additional property that the modified system is output stable. In Section 1.9, we return to consider the state feedback structure, but now we assume that the DLSs and feedback have some stability properties. In the final Section 1.10, we consider the transfer functions and nontangential limits of the transfer functions of DLSs. Most of the results of this chapter appeared in [56] (Malinen, 1997).

1.2 Difference equation systems

Let A, B, C and D be bounded linear operators between appropriate Hilbert spaces U, H and Y. In this section, we introduce the basic algebraic notions associated to the system of difference equations

$$\begin{cases} x_{j+1} = Ax_j + Bu_j, \\ y_j = Cx_j + Du_j, \quad j \in \mathbf{Z}, \end{cases} \tag{1.2}$$

where $u_j \in U$, $x_j \in H$ and $y_j \in Y$ for all $j \in \mathbf{Z}$. The index j is regarded as a discrete time parameter. For notational convenience, we associate a data structure to equations (1.2).

Definition 1. *Let U, H and Y be Hilbert spaces. Let the operators $A \in \mathcal{L}(H,H)$, $B \in \mathcal{L}(U,H)$, $C \in \mathcal{L}(H,Y)$, $D \in \mathcal{L}(U,Y)$ be arbitrary.*

(i) The ordered quadruple $\phi := \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ of operators A, B, C and D is the discrete time linear system (DLS) in difference equation form.

(ii) The space U is the input space, H is the state space, and Y is the output space of ϕ.

(iii) The operator A is the semigroup generator of the DLS ϕ. The operator B is the input operator, the operator C is the output operator, and the operator D is the feed-through operator of the DLS ϕ. The operators A, B, C and D are the generating operators of the DLS ϕ.

Let a sequence $\tilde{u} := \{u_j\}_{j\in\mathbf{Z}} \in Seq(U)$ be arbitrary. Then there is the largest $j'(\tilde{u}) \in \mathbf{Z}$, such that $u_j = 0$ for all $j < j'(\tilde{u})$. Furthermore, there exists a unique sequence of states $\{x_j(\tilde{u})\}_{j\in\mathbf{Z}}$, such that

$$x_{j+1}(\tilde{u}) = Ax_j(\tilde{u}) + Bu_j$$

and $x_j(\tilde{u}) = 0$ for all $j \le j'(\tilde{u})$. Define sequence $\tilde{y} := \{y_j\}_{j\in\mathbf{Z}} \in Seq(Y)$ by

$$y_j = Cx_j(\tilde{u}) + Du_j.$$

It can be easily seen that the mapping $Seq(U) \ni \tilde{u} \mapsto \tilde{y} \in Seq(Y)$ is well-defined and linear. This mapping is the I/O (input-output) map of the DLS $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$, associated to the equations (1.2).

Definition 2. *Let* $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ *be a DLS.*

(i) The I/O map $\mathcal{D}_\phi$ *is the linear map* $Seq(U) \ni \tilde{u} \mapsto \tilde{y} \in Seq(Y)$ *associated to the DLS* ϕ *via (1.2), as explained above. The DLS* ϕ *is called a realization of its I/O map* $\mathcal{D}_\phi$*.*

(ii) The sequence $\tilde{u} := \{u_j\}_{j \in \mathbf{Z}}$ *is the input sequence of the DLS* ϕ*. The sequence* $\{x_j\}_{j \in \mathbf{Z}}$ *is a sequence of states, and* $\tilde{y} := \{y_j\}_{j \in \mathbf{Z}}$ *is an output sequence of* ϕ*, if they satisfy the equations (1.2) for some input sequence* $\tilde{u} \in Seq(U)$*.*

It is not difficult to calculate the formula for $\mathcal{D}_\phi$:

Proposition 3. *Let* $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ *be a DLS, and let* $\tilde{u} \in Seq(U)$ *be arbitrary. The I/O map is given componentwise by*

$$(\mathcal{D}_\phi \tilde{u})_j = \sum_{i=0}^{\infty} CA^i B u_{j-i-1} + D u_j \tag{1.3}$$

for all $j \in \mathbf{Z}$*.*

Note that the sum (1.3) is actually finite (and thus well-defined), because we assume that $\tilde{u} \in Seq(U)$. Because $\mathcal{D}_\phi$ depends only on the operators D and CA^iB for $i \geq 0$, but not directly on A, B and C, it follows that several different DLSs can have a common I/O map.

In order to study the time dynamics of various mappings associated to a DLS, we define some projections and a shift operator on the sequence spaces.

Definition 4. *Let* Z *be a Hilbert space, and let* $\tilde{z} \in Seq(Z)$ *be arbitrary. Then we define the following linear mappings in* $Seq(Z)$*:*

(i) the interval projections for $j, k \in \mathbf{Z}$

$$\pi_{[j,k]} \tilde{z} := \{w_j\}; \quad w_i = z_i \quad \textit{for} \quad j \leq i \leq k, \quad 0 \quad \textit{otherwise};$$
$$\pi_j := \pi_{[j,j]},$$

(ii) the future and past projections

$$\pi_+ := \pi_{[1,\infty]}, \quad \pi_- := \pi_{[-\infty,-1]},$$

(iii) the composite projections

$$\bar{\pi}_+ := \pi_0 + \pi_+, \quad \bar{\pi}_- := \pi_0 + \pi_-,$$

(iv) *the bilateral forward time shift* τ *and its (formal) adjoint, backward time shift* τ^*

$$\begin{aligned}\tau\tilde{u} &:= \{w_j\} \quad \textit{where} \quad w_j = u_{j-1}, \\ \tau^*\tilde{u} &:= \{w_j\} \quad \textit{where} \quad w_j = u_{j+1}.\end{aligned}$$

We call τ^* to be the "adjoint" of τ rather than inverse for notational simplicity. At this stage, we do not yet have a Hilbert space structure on the sequence spaces that would make τ^* a true adjoint. However, such structure will be introduced later.

Definition 5. *Let* $\mathcal{D} : Seq(U) \to Seq(Y)$ *be a linear mapping. Then*

(i) $\mathcal{D}$ *is shift-invariant, if* $\tau\mathcal{D} = \mathcal{D}\tau$*,*

(ii) $\mathcal{D}$ *is causal, if it is shift-invariant and* $\bar{\pi}_+\mathcal{D}\pi_- = 0$*,*

(iii) $\mathcal{D}$ *is anticausal, if it is shift-invariant and* $\pi_-\mathcal{D}\bar{\pi}_+ = 0$*,*

(iv) $\mathcal{D}$ *is static, if it is shift-invariant and* $\pi_0\mathcal{D} = \mathcal{D}\pi_0$*.*

Clearly, if $\mathcal{D}$ is static then $\pi_j\mathcal{D} = \mathcal{D}\pi_j$ for all $j \in \mathbf{Z}$. Furthermore, each static mapping is causal and anticausal. Conversely, a causal and anticausal mapping is static.

Proposition 6. *Let* $\mathcal{D} : Seq(U) \to Seq(Y)$ *be a linear mapping. Then the following are equivalent:*

(i) $\mathcal{D}$ *is static.*

(ii) *There is a linear mapping* $D : U \to Y$ *such that*

$$\mathcal{D}\{u_j\}_{j\in\mathbf{Z}} = \{Du_j\}_{j\in\mathbf{Z}}.$$

Proof. Assume (i), and define $D = \pi_0\mathcal{D}\pi_0$. With the obvious identification of range(π_0) with U and Y, it follows from the linearity of $\mathcal{D}$ that $D : U \to Y$ is linear. The mapping D extends to a mapping $\tilde{D} : Seq(U) \to Seq(Y)$ by setting $\tilde{D}\{u_j\}_{j\in\mathbf{Z}} := \{Du_j\}_{j\in\mathbf{Z}}$ for all $\{u_j\}_{j\in\mathbf{Z}} \in Seq(U)$. It is easy to see that $\tilde{D}$ is a static mapping. If we show that $\mathcal{D} = \tilde{D}$ on $Seq(U)$, claim (ii) is established.

For contradiction, assume that $\mathcal{D} \neq \tilde{D}$. Then there is a $\tilde{u} \in Seq(U)$ and $j \in \mathbf{Z}$ such that $\pi_j\mathcal{D}\tilde{u} \neq \pi_j\tilde{D}\tilde{u}$. By shifting $\tilde{u}$, we may assume that $j = 0$. Because $\mathcal{D}$ is static, it follows that $\pi_0\mathcal{D}\tilde{u} = \pi_0 \cdot \pi_0\mathcal{D}\tilde{u} = \pi_0\mathcal{D}\pi_0\tilde{u}$ and similarly $\pi_0\tilde{D}\tilde{u} = \pi_0\tilde{D}\pi_0\tilde{u}$. But both $\pi_0\tilde{D}\pi_0\tilde{u}$ and $\pi_0\tilde{D}\pi_0\tilde{u}$ are identifiable with Du_0, by the definitions of D and $\tilde{D}$. So, the counter assumption $\pi_j\mathcal{D}\tilde{u} \neq \pi_j\tilde{D}\tilde{u}$ leads to a contradiction, and $\mathcal{D} = \tilde{D}$ follows. It is a triviality that (ii) implies (i). □

So, static mappings are identifiable with the linear mappings $D : U \to Y$. In fact, we write from now on

$$D\{u_j\}_{j\in\mathbf{Z}} = \{Du_j\}_{j\in\mathbf{Z}}$$

for all $\{u_j\}_{j\in\mathbf{Z}} \in Seq(U)$. In the following proposition, we deal with certain limits and infinite sums of linear, shift-invariant and causal mappings.

Proposition 7. *Let $\{T_j\}_{j\geq 0} \subset \mathcal{L}(U;Y)$ be a countable family.*

(i) For arbitrary $n \geq 1$, the mapping $\mathcal{D}^{(n)} : Seq(U) \to Seq(Y)$, given by

$$\mathcal{D}^{(n)} := \sum_{i=0}^{n} T_i \tau^i,$$

is a linear, shift-invariant and causal.

(ii) There is a unique linear, shift-invariant and causal mapping $\mathcal{D} : Seq(U) \to Seq(Y)$ such that for all $\tilde{u} \in Seq(U)$ and $j \in \mathbf{Z}$

$$\pi_j \mathcal{D}^{(n)} \tilde{u} \to \pi_j \mathcal{D} \tilde{u} \tag{1.4}$$

(in the norm of Y) as $n \to \infty$. We write $\sum_{i=0}^{\infty} T_i \tau^i := \mathcal{D}$.

Proof. It is a matter of easy computation to see that claim (i) holds. For the converse implication, let $\tilde{u} \in Seq(U)$ and $j \in \mathbf{Z}$ be arbitrary. Then, by the definition of $\mathcal{D}^{(n)}$, we have

$$\pi_j \mathcal{D}^{(n)} \tilde{u} = \sum_{i=0}^{n} T_i \pi_j \tau^i \tilde{u} = \sum_{i=0}^{n} T_i \tau^i \pi_{j-i} \tilde{u}.$$

Because $\tilde{u} \in Seq(U)$, the sum $\sum_{i=0}^{n} T_i \tau^i \pi_{j-i} \tilde{u}$ remains a constant finite sum for all n large enough. So we can define $y_j \in Y$ for all $z \in \mathbf{Z}$ by setting $y_j := \lim_{n\to\infty} \pi_j \mathcal{D}^{(n)} \tilde{u}$ with the obvious identification of spaces Y and range(π_j). Define $\tilde{y} := \{y_j\}_{j\in\mathbf{Z}}$. Clearly $y_j = 0$ if j is small enough, because $\tilde{u} \in Seq(U)$. So we have constructed a mapping $\mathcal{D} : Seq(U) \to Seq(Y)$ by setting $\mathcal{D}\tilde{u} = \tilde{y}$. Clearly $\mathcal{D}$ is linear, and it is the only mapping that can serve as a candidate for the limit of the sequence $\{\mathcal{D}^{(n)}\}$ because equation (1.4) is satisfied, by the construction.

We show that $\mathcal{D}$ is shift-invariant. Let $\tilde{u} \in Seq(U)$ and $j \in \mathbf{Z}$ be arbitrary. Because each $\mathcal{D}^{(n)}$ is shift-invariant, we have

$$\pi_j \mathcal{D}^{(n)} \tau \tilde{u} = \pi_j \tau \mathcal{D}^{(n)} \tilde{u} = \tau \pi_{j-1} \mathcal{D}^{(n)} \tilde{u} \to \tau \pi_{j-1} \mathcal{D} \tilde{u}$$

(in the norm of Y) as $n \to \infty$. But also $\pi_j \mathcal{D}^{(n)} \tau \tilde{u} \to \pi_j \mathcal{D} \tau \tilde{u}$, and the uniqueness of the limit implies that $\pi_j \mathcal{D} \tau \tilde{u} = \pi_j \tau \mathcal{D} \tilde{u}$. Because both $\tilde{u}$ and j are arbitrary, it follows that $\mathcal{D}\tau = \tau\mathcal{D}$ on $Seq(U)$. The causality of $\mathcal{D}$ is proved in a similar way. □

We say that the infinite sum $\sum_{i=0}^{\infty} T_i \tau^i \tilde{u}$ exists in the sense of componentwise convergence. This notion of convergence gives the vector spaces $Seq(U)$ and $Seq(Y)$ the topology of componentwise (pointwise) convergence. We are now prepared to show that linear, shift-invariant and causal mappings on $Seq(U)$ can be represented with the aid of countable combinations of static mappings and the forward shift τ.

Proposition 8. *Let $\mathcal{D} : Seq(U) \to Seq(Y)$ be a linear, shift-invariant and causal mapping, such that $\pi_i \mathcal{D}_\phi \pi_0 \in \mathcal{L}(U;Y)$ with the obvious identification of* range(π_i) *with Y and* range(π_0) *with U.*

Then there is a unique countable family $\{T_i\}_{i \geq 0} \subset \mathcal{L}(U;Y)$ such that $\mathcal{D} = \sum_{i=0}^{\infty} T_i \tau^i$ in the sense of componentwise convergence.

Proof. Define $T_i := \pi_i \mathcal{D} \pi_0$ for $i \geq 0$. By the identification of range(π_i) with Y and range(π_0) with U, it follows that $\{T_i\}_{i \geq 0} \subset \mathcal{L}(U;Y)$. By Proposition ii, there is a unique linear, shift-invariant and causal mapping $\mathcal{D}' : Seq(U) \to Seq(Y)$, such that for all $j \in \mathbf{Z}$ and $\tilde{u} \in Seq(U)$

$$\pi_j \mathcal{D}' \tilde{u} := \sum_{i=0}^{\infty} \pi_j T_i \pi_j \tau^i \pi_{j-i} \tilde{u}. \tag{1.5}$$

Here T_i is regarded as a static operator on $Seq(U)$, and the sum contains only a finite number of nonzero terms. We proceed to show that $\mathcal{D} = \mathcal{D}'$ on $Seq(U)$. It is enough to show that $\pi_j \mathcal{D} \tilde{u} = \pi_j \mathcal{D}' \tilde{u}$ for all $j \in \mathbf{Z}$ and $\tilde{u} \in Seq(U)$.

By the definition of the static operator $T_i : Seq(U) \to Seq(Y)$, it follows that $\pi_j T_i \pi_j = \tau^{j-i} \pi_i \mathcal{D} \pi_0 \tau^{*j}$. But then

$$\begin{aligned} \pi_j T_i \pi_j \tau^i &= \tau^{j-i} \pi_i \mathcal{D} \pi_0 \tau^{*(j-i)} \\ &= \pi_j \tau^{j-i} \mathcal{D} \tau^{*(j-i)} \pi_{j-i} = \pi_j \mathcal{D} \pi_{j-i}. \end{aligned}$$

By combining this with equation (1.5), we obtain

$$\pi_j \mathcal{D}' \tilde{u} = \sum_{i=0}^{\infty} \pi_j \mathcal{D} \pi_{j-i} \tilde{u} = \pi_j \mathcal{D} \sum_{i=0}^{\infty} \pi_{j-i} \tilde{u} = \pi_j \mathcal{D} \pi_{[-\infty,j]} \tilde{u}, \tag{1.6}$$

where we have used the linearity of $\mathcal{D}$ and the fact that all the sums have only a finite number of nonzero terms, by assumption $\tilde{u} \in Seq(U)$. Because $\mathcal{D}$ is causal, we have $\pi_j \mathcal{D} \pi_{[j+1,\infty]} \tilde{u} = 0$. This, together with equation (1.6), implies that $\pi_j \mathcal{D}' \tilde{u} = \pi_j \mathcal{D} \tilde{u}$. The uniqueness of the representing family $\{T_i\}_{i \geq 0}$ is trivial, and the proof is complete. □

As a combination of Propositions 3 and 8, we can represent the I/O map as the sum

$$\mathcal{D}_\phi \tilde{u} = D\tilde{u} + \sum_{i \geq 0} C A^i B \tau^{i+1} \tilde{u}. \tag{1.7}$$

The sum in formula (1.7) converges componentwise in $Seq(Y)$.

In some subspaces of $Seq(U)$, the shift τ can be realized as a multiplication by a complex variable z. This gives us the transfer function representation for the I/O map. The operator-valued analytic transfer function is given by

$$\mathcal{D}_\phi(z) = D + \sum_{i\geq 0} CA^i B z^{i+1} \quad \text{for} \quad z \in \mathbf{C}, \tag{1.8}$$

where the power series converges in a neighborhood of the origin. For example, this is true for all z satisfying $|z| < ||A||^{-1}$. We return to these questions in Section 1.10

We have now considered the linear, shift-invariant and causal mappings between the vector spaces $Seq(U)$ and $Seq(Y)$. The following realization lemma characterizes the set of I/O maps for DLSs in this larger set.

Lemma 9. *Let $\mathcal{D} : Seq(U) \to Seq(Y)$ be a linear, causal and shift-invariant mapping. Then the following are equivalent*

(i) $\mathcal{D}$ is an I/O map of a DLS.

(ii) The mapping $\mathcal{D}$ has a unique componentwise convergent series representation

$$\mathcal{D} = \sum_{i\geq 0} T_i \, \tau^i, \tag{1.9}$$

where the operators $T_i \in \mathcal{L}(U;Y)$, $i \geq 0$, satisfy the growth bound $||T_i|| < C\, r^i$ for some $C < \infty$ and $r > 0$.

Proof. The implication (i) $\Rightarrow$ (ii) follows trivially from formula (1.7) and the fact that the generating operators A, B, C and D are bounded. The proof of the converse implication (ii) $\Rightarrow$ (i) requires the construction of a DLS whose I/O maps equals $\mathcal{D}$. Let us first show that any linear, causal and shift-invariant mapping $\mathcal{D} : Seq(U) \to Seq(Y)$ can always be written in the form of (1.9) where $T_i \in \mathcal{L}(U;Y)$. For $\tilde{u} \in Seq(U)$ satisfying $\pi_0 \tilde{u} = \tilde{u}$ we have

$$\mathcal{D}\pi_0\tilde{u} = \sum_{i\geq 0} \pi_i \mathcal{D}\pi_0\tilde{u} = \sum_{i\geq 0} \tau^i \, (\tau^{*i}\pi_i\mathcal{D}\pi_0)\tilde{u} = \sum_{i\geq 0} T_i\tau^i\pi_0\tilde{u} \tag{1.10}$$

where $T_i : U \to Y$ is given by $T_i := \tau^{*i}\pi_i\mathcal{D}\pi_0 = \pi_0\tau^{*i}\mathcal{D}\pi_0$ with the obvious identification of spaces. The uniqueness of this representation for the inputs of type $\pi_0\tilde{u}$ is clear, and ever more so for more general inputs. The boundedness of T_i's follows from the assumed growth bound $||T_i|| < C\, r^i$. The assumed shift-invariance, linearity and causality of $\mathcal{D}$ makes it possible to extend equation

(1.10) for all $\tilde{u} \in Seq(U)$, by using the unique formal sum representation $\tilde{u} = \sum_{j>J} \pi_j \tilde{u}$ for elements of $Seq(U)$ and noting that the sum defining $\pi_j \mathcal{D}$ is finite, for all $j \in \mathbf{Z}$. We conclude that formula (1.9) holds, in the sense of componentwise convergence.

To complete the proof, we must find bounded operators A, B, C and D on some Hilbert spaces U, Y and H such that

$$D = T_0, \quad CA^{i-1}B = T_i \quad \text{for} \quad i \geq 1.$$

The choice of D is clear. The input space U and the output spaces Y are fixed by the choice of $\mathcal{D}$, but the state space H will have to be constructed. We first define

$$\begin{aligned}
&A := \bar{\pi}_+ \tau^* : Seq_+(Y) \to Seq_+(Y) \\
&B := [T_1 \, T_2 \, T_3 \cdots]^T : U \to Seq_+(Y) \\
&C := \pi_0 : Seq_+(Y) \to Y
\end{aligned}$$

and the define the state space H to be a certain Hilbert subspace of $Seq_+(Y)$ such that the operators become continuous.

By using the growth bound $||T_i|| < C\, r^i$ assumption, we can choose $r < \infty$ so large that $\sum_{i\geq 0} r^{-i} \, ||T_i||^2 < \infty$. An inner product can be defined in a subset of $Seq_+(Y)$ by

$$\langle \tilde{y}, \tilde{w} \rangle_H := \sum_{i \geq 0} r^{-2i} \, \langle y_i, w_i \rangle_Y .$$

Now the state space $H \subset Seq_+(Y)$ is, by definition, the closure of the finite length sequences in this inner product. For $u \in U$, we have

$$\begin{aligned}
||Bu||_H^2 &= \sum_{i \geq 0} r^{-2i} \, \langle T_i u, T_i u \rangle_Y \\
&= \sum_{i \geq 0} ||r^{-i} T_i u||_Y^2 \leq ||u||_U^2 \sum_{i \geq 0} r^{-i} \, ||T_i||^2 .
\end{aligned}$$

This proves that B maps U into H boundedly. To show the boundedness of A on H, we calculate

$$\begin{aligned}
\frac{||A\tilde{y}||_H^2}{||\tilde{y}||_H^2} &= \frac{\sum_{i\geq 1} r^{-2(i-1)} \, \langle y_i, y_i \rangle_Y}{\sum_{i \geq 0} r^{-2i} \, \langle y_i, y_i \rangle_Y} \\
&= r^2 \, \frac{\sum_{i\geq 1} r^{-2i} \, \langle y_i, y_i \rangle_Y}{\sum_{i \geq 0} r^{-2i} \, \langle y_i, y_i \rangle_Y} \leq r^2 .
\end{aligned}$$

Thus we obtain the norm estimate $||A||_H < r$. The boundedness of $C = \pi_0$ is trivial. This completes the proof. $\square$

The realization that has been constructed in the proof of Lemma 9 is a variant of the shift realization of a H^∞ transfer function. For further information about shift realizations, see [35, Chapter III]. The number

$$\inf \{ r > 0 \mid \exists\, C < \infty : \, ||T_i|| \le C\, r^i \}$$

majorizes the spectral radius $\rho(A)$ of the semigroup generator A. The proof of Lemma 9 implies that given a DLS $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix} \right)$ and an arbitrary $r > \rho(A)$, we can find another DLS ϕ' such that $\mathcal{D}_\phi = \mathcal{D}_{\phi'}$, whose semigroup generator A' satisfies $||A'|| \le r$. Because the generating operators of a DLS ϕ are bounded, it follows that the transfer function $\mathcal{D}_\phi(z)$ is analytic in a neighborhood of the origin. In the continuous time language of [89] and [98], this can be called the well-posedness of the system. Thus DLSs are well-posed discrete time linear systems.

In addition to the I/O map, we also define two other linear mappings — the controllability and observability maps.

Definition 10. *Let $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix} \right)$ be a DLS.*

(i) The controllability map $\mathcal{B}_\phi : Seq(U) \to H$ is the linear mapping defined by

$$\mathcal{B}_\phi \tilde{u} := \sum_{i \ge 0} A^i B u_{-i-1} \tag{1.11}$$

for all $\tilde{u} \in Seq(U)$.

(ii) The observability map $\mathcal{C}_\phi : H \to Seq(Y)$ is the linear mapping defined by

$$(\mathcal{C}_\phi x_0)_j := \begin{cases} C A^j x_0, & \textit{for} \quad j \ge 0, \\ 0, & \textit{for} \quad j < 0, \end{cases} \tag{1.12}$$

for all $x_0 \in H$.

As we shall see in a moment, the controllability map brings data into the DLS. The state space H serves as a "memory" of the DLS, and the semigroup generator A "processes the data" there. Finally, the observability map "reads the memory", and outputs its contents.

It is not always the case that we want to "start" the DLS with initial condition $x_J = 0$ indefinitely far in the past, even though the input sequence space $Seq(U)$ has been designed for this purpose. In the initial value setting, we start at some specific time point (usually chosen to be $j = 0$) with a given initial state $x_0 \in H$. The input sequences $\tilde{u}$ as well as the output sequences $\tilde{y}$ would then lie in the spaces $Seq_+(U)$ and $Seq_+(Y)$, respectively. In practice, we postulate that we have a capability of "loading a state" of our choice into the "memory" of a DLS — or at least being able "reset" the state of a DLS to zero at a given moment.

Proposition 11. *Let $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ be a DLS. Let $x_0 \in H$ be an initial state at time $j = 0$, and $\tilde{u} \in Seq_+(U)$ be an input sequence.*

(i) The state of ϕ at time $j \geq 0$ is denoted by $x_j(x_0, \tilde{u}) \in H$, and given by

$$x_j(x_0, \tilde{u}) = A^j x_0 + \sum_{i=0}^{j-1} A^i B u_{j-i} = A^j x_0 + \mathcal{B}_\phi \tau^{*j} \tilde{u}, \tag{1.13}$$

where τ is the time shift defined in Definition 4.

(ii) The output sequence $\tilde{y}(x_0, \tilde{u}) := \{y_j(x_0, \tilde{u})\}_{j \geq 0} \in Seq_+(Y)$ of ϕ is given by

$$\begin{aligned} y_0(x_0, \tilde{u}) &= Cx_0 + Du_0 = \pi_0(\mathcal{C}_\phi x_0 + \mathcal{D}_\phi \tilde{u}), \\ y_j(x_0, \tilde{u}) &= CA^j x_0 + \sum_{i=0}^{j-1} CA^i B u_{j-i} + Du_j \\ &= \pi_j(\mathcal{C}_\phi x_0 + \mathcal{D}_\phi \tilde{u}) \quad \textit{for} \quad j \geq 1. \end{aligned} \tag{1.14}$$

It is true that the mappings $\mathcal{D}_\phi$, $\mathcal{B}_\phi$ and $\mathcal{C}_\phi$ share an important property with our (classical notion of the) universe, namely the causality. The following proposition collects the results how the I/O map, controllability map and observability map interact with the time projections and shifts.

Lemma 12. *Let $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ be a DLS. Then*

(i) $\mathcal{D}_\phi$, $\mathcal{B}_\phi$ and $\mathcal{C}_\phi$ are causal; i.e. they satisfy

$$\pi_- \mathcal{D}_\phi \bar{\pi}_+ = 0, \quad \mathcal{B}_\phi \bar{\pi}_+ = 0, \quad \pi_- \mathcal{C}_\phi = 0,$$

(ii) $\mathcal{B}_\phi$ satisfies

$$\begin{aligned} \mathcal{B}_\phi \tau^* \tilde{u} &= (A\,\mathcal{B}_\phi + \mathcal{B}_\phi \tau^* \pi_0)\,\tilde{u} \\ &= A\,\mathcal{B}_\phi \tilde{u} + B u_0, \\ \mathcal{B}_\phi \tau^{*j} \tilde{u} &= A^j\,\mathcal{B}_\phi \tilde{u} + \sum_{i=0}^{j-1} A^i B u_{j-i-1}, \end{aligned}$$

for all $j \geq 1$, $\tilde{u} \in Seq(U)$,

(iii) $\mathcal{C}_\phi$ satisfies

$$\bar{\pi}_+ \tau^* \mathcal{C}_\phi = \mathcal{C}_\phi A,$$

(iv) $\mathcal{D}_\phi$ *satisfies*

$$\bar{\pi}_+\mathcal{D}_\phi - \mathcal{D}_\phi\bar{\pi}_+ = \bar{\pi}_+\mathcal{D}_\phi\pi_- = \mathcal{C}_\phi\mathcal{B}_\phi, \quad \mathcal{D}_\phi\tau = \tau\mathcal{D}_\phi, \quad \mathcal{D}_\phi\tau^* = \tau^*\mathcal{D}_\phi.$$

Proof. Claim (i) is a direct consequence of Definition 10. The first part of claim (ii) will be proved by calculating for any $\tilde{u} \in Seq(U)$

$$\mathcal{B}_\phi\tau^*\tilde{u} = \sum_{i\geq 0} A^i B u_{-i} = A\sum_{i\geq 0} A^i B u_{-i-1} + Bu_0 = A\mathcal{B}_\phi\tilde{u} + Bu_0.$$

Quite easily we note that $Bu_0 = \mathcal{B}_\phi\tau^*\bar{\pi}_+\tilde{u}$. This gives the first part of claim (ii). The latter part of claim (ii) follows from the first part by induction. The claim (iii) is an immediate conclusion of Definition 10.

The first equality of claim (iv) is trivial. For the proof of the second equality we proceed as follows ($j \geq 0$)

$$\begin{aligned}
(\mathcal{D}_\phi\pi_-\tilde{u})_j &= \sum_{i=0}^{\infty} CA^i B(\pi_-\tilde{u})_{j-i-1} + D(\pi_-\tilde{u})_j \\
&= \sum_{i\geq j}^{\infty} CA^i B u_{j-i-1} = \sum_{i\geq 0}^{\infty} CA^{i+j} B u_{j-(i+j)-1} \\
&= CA^j \sum_{i\geq 0}^{\infty} A^i B u_{-i-1} = (\mathcal{C}_\phi\mathcal{B}_\phi\tilde{u})_j.
\end{aligned}$$

This proves the former part of claim (iv). The remaining part in claim (iv) is clear. This completes the proof the lemma. $\square$

1.3 DLS in I/O form

In previous Section 1.2, we associated three linear mappings $\mathcal{B}_\phi$, $\mathcal{C}_\phi$, $\mathcal{D}_\phi$ to any DLS $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$. In this section, we forget the generating operators B, C and D for a while and work only with operators A, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ that are postulated to satisfy the properties of A, $\mathcal{B}_\phi$, $\mathcal{C}_\phi$ and $\mathcal{D}_\phi$ as given by Lemma 12. We can, in fact, characterize the DLS starting from such operators A, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$. This will be the main result of this section.

Definition 13. *Let U, Y and H be Hilbert. Let $A \in \mathcal{L}(H)$. Let $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ be linear operators of the following kind:*

(i) $\mathcal{B} : Seq_-(U) \to H$, $\mathcal{C} : H \to Seq_+(Y)$ *and* $\mathcal{D} : Seq(U) \to Seq(Y)$.

(ii) $\mathcal{D}$, $\mathcal{B}$ *and* $\mathcal{C}$ *are causal in the sense of Lemma 12*

$$\pi_- \mathcal{D} \bar{\pi}_+ = 0, \quad \mathcal{B} \bar{\pi}_+ = 0, \quad \pi_- \mathcal{C} = 0.$$

(iii) $\mathcal{B}$ *satisfies*

$$\begin{aligned} &\mathcal{B}\tau^* = A\,\mathcal{B} + \mathcal{B}\tau^* \pi_0, \\ &\mathcal{B}\pi_{-1} \in \mathcal{L}(U, H), \end{aligned}$$

where U is identified with range(π_{-1}) *on $Seq(U)$ in the natural way.*

(iv) $\mathcal{C}$ *satisfies*

$$\begin{aligned} &\bar{\pi}_+ \tau^* \mathcal{C} = \mathcal{C} A, \\ &\pi_0 \mathcal{C} \in \mathcal{L}(H, Y), \end{aligned}$$

where Y is identified with range(π_0) *on $Seq(Y)$ in the natural way.*

(v) $\mathcal{D}$ *satisfies*

$$\begin{aligned} &\bar{\pi}_+ \mathcal{D} \pi_- = \mathcal{C}\mathcal{B}, \\ &\mathcal{D}\tau = \tau\mathcal{D}, \quad \mathcal{D}\tau^* = \tau^*\mathcal{D}, \\ &\pi_0 \mathcal{D} \pi_0 \in \mathcal{L}(U, Y), \end{aligned}$$

where U and Y are identified with range(π_0) *in the natural way.*

Then the ordered quadruple

$$\Phi = \begin{bmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{bmatrix} \tag{1.15}$$

is the discrete time linear system (DLS) in I/O form. The operator A is the semigroup generator of Φ, and the family of the operators $\{A^j\}_{j \geq 0}$ is the (discrete) semigroup of Φ. The mapping $\mathcal{B}$ is the controllability map, the mapping $\mathcal{C}$ is the observability map, and the mapping $\mathcal{D}$ is the I/O map of Φ.

We remind that the DLS $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ in Definition 1 is called a DLS in difference equation form. Lemma 12 associates to each DLS ϕ in difference equation form a unique DLS in I/O form, namely $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}_\phi \tau^{*j} \\ \mathcal{C}_\phi & \mathcal{D}_\phi \end{smallmatrix}\right]$. It appears that also the converse of Lemma 12 holds.

Lemma 14. *Let $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ be a DLS in I/O form. Then there are unique linear operators $B \in \mathcal{L}(U, H)$, $C \in \mathcal{L}(H, Y)$, $D \in \mathcal{L}(U, Y)$ such that $\mathcal{B} = \mathcal{B}_\phi$, $\mathcal{C} = \mathcal{C}_\phi$, $\mathcal{D} = \mathcal{D}_\phi$ for the DLS $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ in difference equation form. The semigroup generator ϕ is the same operator A as the semigroup generator of Φ. Furthermore, we have*

(i) $B := \mathcal{B}\pi_{-1}$ *where* range (π_{-1}) *and U are identified in the natural way,*

(ii) $C := \pi_0 \mathcal{C}$ *where* range (π_0) *and Y are identified in the natural way,*

(iii) $D := \pi_0 \mathcal{D} \pi_0$*, where the range of the right π_0 is identified with U, and the range of the left π_0 is identified with Y in the natural way.*

Proof. We begin the proof by considering the mapping $\mathcal{B}$. Define $B := \mathcal{B}\pi_{-1}$. Then by assumption $B \in \mathcal{L}(U, H)$, and $\mathcal{B}\tau^* \tilde{u} = A\mathcal{B}\tilde{u} + Bu_0$ for all $\tilde{u} \in Seq(U)$. By induction, just as in the proof of the last part of claim (ii) of Lemma 12, we have for all $j \geq 1$ and $\tilde{u} = \{u_j\} \in Seq(U)$

$$\mathcal{B}\tau^{*j}\tilde{u} = A^j \mathcal{B}\tilde{u} + \sum_{i=0}^{j-1} A^i B u_{j-i-1}. \tag{1.16}$$

We have the finite sum representation $\pi_- \tilde{u} = \sum_{j<0} \pi_j \tilde{u}$ for each $\tilde{u} \in Seq(U)$. This and the linearity of $\mathcal{B}$ imply

$$\mathcal{B}\tilde{u} = \mathcal{B}\pi_- \tilde{u} = \sum_{j<0} \mathcal{B}\pi_j \tilde{u} = \sum_{j<0} \mathcal{B}\tau^{*|j|}\pi_0 \tau^{|j|}\tilde{u}. \tag{1.17}$$

By formula (1.16), we have for all $j < 0$

$$\begin{aligned} \mathcal{B}\pi_j \tilde{u} &= \mathcal{B}\tau^{*|j|}\left(\pi_0 \tau^{|j|}\tilde{u}\right) \\ &= A^{|j|}\mathcal{B}(\pi_0 \tau^{|j|}\tilde{u}) + \sum_{i=0}^{j-1} A^i B(\pi_0 \tau^{|j|}\tilde{u})_{|j|-i-1} \\ &= A^{|j|-1} B u_{-|j|} = A^{-j-1} B u_j. \end{aligned} \tag{1.18}$$

Formulae (1.17) and (1.18) together give

$$\mathcal{B}\tilde{u} = \sum_{j<0} A^{-j-1} B u_j = \sum_{j\geq 1} A^{j-1} B u_{-j} = \sum_{j\geq 0} A^j B u_{-j-1}.$$

This proves that $\mathcal{B} = \mathcal{B}_\phi$ for the DLSs of the form $\phi = \left(\begin{smallmatrix} A & B \\ * & * \end{smallmatrix} \right)$, where $*$ stands for an irrelevant entry.

To make a similar analysis for $\mathcal{C}$, we first define $C := \pi_0 \mathcal{C}$. By assumption, $C \in \mathcal{L}(H, Y)$. For any $x_0 \in H$ and $j \geq 0$, a direct calculation gives

$$(\mathcal{C}x_0)_j = (\tau^{*j}\mathcal{C}x_0)_0 = \pi_0 \tau^{*j} \mathcal{C} x_0.$$

But by assumption (iv) and definition of C

$$\pi_0 \tau^{*j} \mathcal{C} x_0 = \pi_0 \mathcal{C} A^j x_0 = C A^j x_0.$$

Thus $(\mathcal{C}x_0)_j = CA^j x_0$, and $\mathcal{C} = \mathcal{C}_\phi$ for all DLSs $\phi = \left(\begin{smallmatrix} A & * \\ C & * \end{smallmatrix} \right)$.

Our final task is to construct an operator a $D \in \mathcal{L}(U, Y)$ such that $\mathcal{D} = \mathcal{D}_\phi$ for the DLS $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix} \right)$, where B and C are as constructed above. Define $D := \pi_0 \mathcal{D} \pi_0$ with the obvious identifications of spaces. Set $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix} \right)$. Clearly $\mathcal{B} = \mathcal{B}_\phi$ and $\mathcal{C} = \mathcal{C}_\phi$, by what we have already proved above. Because the choice of the feed-through operator does not change the controllability and observability maps, it remains to check that the we have $\mathcal{D}_\phi = \mathcal{D}$ on $Seq(U)$. By an application of the representation formula (1.9) of Lemma 9 on the mapping $\mathcal{D}_\phi - \mathcal{D}$, it is enough to show that $\mathcal{D}_\phi \tilde{u} = \mathcal{D}\tilde{u}$ for sequences satisfying $\tilde{u} = \pi_0 \tilde{u}$. But we have in $Seq(U)$

$$(\mathcal{D}_\phi - \mathcal{D})\pi_0 = \pi_-(\mathcal{D}_\phi - \mathcal{D})\pi_0 + \pi_0(\mathcal{D}_\phi - \mathcal{D})\pi_0 + \pi_+(\mathcal{D}_\phi - \mathcal{D})\pi_0.$$

Now, $\pi_-(\mathcal{D}_\phi - \mathcal{D})\bar{\pi}_+ = 0$ because both $\mathcal{D}_\phi$ and $\mathcal{D}$ are causal. Furthermore, $\pi_+(\mathcal{D}_\phi - \mathcal{D})\pi_0 = \pi_+ \tau (\mathcal{D}_\phi - \mathcal{D})\tau^* \pi_0 = \tau \cdot \bar{\pi}_+(\mathcal{D}_\phi - \mathcal{D})\pi_- \cdot \pi_0 \tau^* = \tau(\mathcal{C}_\phi \mathcal{B}_\phi - \mathcal{C}\mathcal{B})\pi_0 \tau^* = 0$, because $\mathcal{B} = \mathcal{B}_\phi$ and $\mathcal{C} = \mathcal{C}_\phi$. We conclude that

$$(\mathcal{D}_\phi - \mathcal{D})\pi_0 = \pi_0(\mathcal{D}_\phi - \mathcal{D})\pi_0 = 0,$$

because $\pi_0 \mathcal{D}_\phi \pi_0 - \pi_0 \mathcal{D} \pi_0 = D - D = 0$, by the choice of D. This proves the last part of the lemma. □

We now summarize an immediate conclusion of Lemmas 12 and 14.

Theorem 15. *There is one-to-one correspondence between DLS in difference equation form and DLS in I/O form. To get the DLS given in difference equation form into the I/O form, the formulae of Lemma 12 are used. To get the DLS given I/O form into difference equation form, the formulae of Lemma 14 are used.*

If the DLSs $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix} \right)$ and $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix} \right]$ are equivalent in the sense of Theorem 15, we write $\phi = \Phi$. We adopt the convention that a symbol in lower case denotes a DLS in difference equation form, and the same symbol in upper case denotes the same DLS in I/O form. For example, $\phi_1^2 = \Phi_1^2$ and so on.

1.4 Adjoint, product and inverse DLS

In this section, we define and study three algebraic operations on DLSs.

Definition 16. *Let* $\phi_1 = \left(\begin{smallmatrix} A_1 & B_1 \\ C_1 & D_1 \end{smallmatrix}\right)$, $\phi_2 = \left(\begin{smallmatrix} A_2 & B_2 \\ C_2 & D_2 \end{smallmatrix}\right)$ *be two DLSs. Assume that the input space of* ϕ_2 *is* U, *the output space of* ϕ_2 *and the input space of* ϕ_1 *is* W, *and the output space of* ϕ_1 *is* Y.

(i) If $D_1^{-1} \in \mathcal{L}(Y;U)$ *exists, then define*

$$\phi_1^{-1} = \begin{pmatrix} A_1 - B_1 D_1^{-1} C_1 & B_1 D_1^{-1} \\ -D_1^{-1} C_1 & D_1^{-1} \end{pmatrix}$$

This DLS is the inverse DLS of ϕ_1.

(ii) Define

$$\phi_1\phi_2 = \left(\begin{matrix} \begin{bmatrix} A_1 & B_1C_2 \\ 0 & A_2 \end{bmatrix} & \begin{bmatrix} B_1D_2 \\ B_2 \end{bmatrix} \\ \begin{bmatrix} C_1 & D_1C_2 \end{bmatrix} & D_1D_2 \end{matrix} \right).$$

This DLS is the product DLS of ϕ_1 *and* ϕ_2.

(iii) Define

$$\widetilde{\phi_1} = \begin{pmatrix} A_1^* & C_1^* \\ B_1^* & D_1^* \end{pmatrix}.$$

This DLS is the adjoint DLS of ϕ_1.

Proposition 17. *Let* ϕ_1, ϕ_2 *be as in Definition 16.*

(i) $\mathcal{D}_{\phi_1} : Seq(W) \to Seq(Y)$ *is invertible and its inverse is a I/O maps of a DLS if and only if* $D_1^{-1} \in \mathcal{L}(Y;W)$ *exists. In this case, the inverse* $\mathcal{D}_{\phi_1}^{-1} : Seq(Y) \to Seq(W)$ *is given by* $\mathcal{D}_{\phi_1}^{-1} = \mathcal{D}_{\phi_1^{-1}}$.

(ii) The composition of the I/O maps $\mathcal{D}_{\phi_1}$ *and* $\mathcal{D}_{\phi_2}$ *satisfies* $\mathcal{D}_{\phi_1}\mathcal{D}_{\phi_2} = \mathcal{D}_{\phi_1\phi_2}$.

(iii) The adjoint DLSs satisfy $\widetilde{(\widetilde{\phi_1})} = \phi_1$, *and* $\widetilde{(\phi_1^{-1})} = (\widetilde{\phi_1})^{-1}$. *Furthermore,* $\mathcal{D}_{\widetilde{\phi_1\phi_2}} = \mathcal{D}_{\widetilde{\phi_2}\widetilde{\phi_1}}$.

Proof. Consider first the "if" part of claim (i). Assume that the feed-through operator of ϕ_1 has a bounded inverse D_1^{-1}. Assume $\tilde{y} := \{y_j\}_{j\in\mathbf{Z}} \in Seq(Y)$ and $\tilde{u} := \{u_j\}_{j\in\mathbf{Z}} \in Seq(W)$ satisfy $\tilde{y} = \mathcal{D}_{\phi_1}\tilde{u}$. By $j'(\tilde{u}) \in \mathbf{Z}$ denote the largest integer such that $u_j = 0$ for all $j < j'(\tilde{u})$. Then, by Definition 2 of the causal I/O map, $y_j = 0$ for all $j < j'(\tilde{u})$. Because the state sequence satisfies $x_j(\tilde{u}) = 0$ for all $j \leq j'(\tilde{u})$ by the definition of the I/O map, it follows that $y_{j'(\tilde{u})} = D_1 u_{j'(\tilde{u})}$. Thus $y_{j'(\tilde{u})} \neq 0$ and $j'(\tilde{y}) = j'(\tilde{u})$. By the shift-invariance of $\mathcal{D}_{\phi_1}$, we may assume without loss of generality that $j'(\tilde{y}) = j'(\tilde{u}) = 0$, $\tilde{u} \in Seq_+(W)$, $\tilde{y} \in Seq_+(Y)$, and the initial condition for the difference equations defining $\mathcal{D}_{\phi_1}$ is given by $x_0 = 0$. Then we have the equivalences

$$\begin{aligned}
&\tilde{y} = \mathcal{D}_{\phi_1}\tilde{u} \\
\Leftrightarrow\quad &\begin{cases} x_{j+1} = A_1 x_j + B_1 u_j, \\ y_j = C_1 x_j + D_1 u_j, \end{cases} \qquad \text{for all} \quad j \geq 0, \\
\Leftrightarrow\quad &\begin{cases} x_{j+1} = A_1 x_j + B_1 u_j, \\ u_j = -D_1^{-1} C_1 x_j + D_1^{-1} y_j, \end{cases} \qquad \text{for all} \quad j \geq 0, \\
\Leftrightarrow\quad &\begin{cases} x_{j+1} = (A_1 - B_1 D_1^{-1} C_1) x_j + B_1 D_1^{-1} y_j, \\ u_j = -D_1^{-1} C_1 x_j + D_1^{-1} y_j, \end{cases} \qquad \text{for all} \quad j \geq 0, \\
\Leftrightarrow\quad &\tilde{u} = \mathcal{D}_{\phi_1^{-1}}\tilde{y}
\end{aligned}$$

We conclude that $\mathcal{D}_{\phi_1^{-1}}\mathcal{D}_{\phi_1} = \mathcal{I}$ on $Seq(U)$. By using $(\phi_1^{-1})^{-1} = \phi_1$, also $\mathcal{D}_{\phi_1}\mathcal{D}_{\phi_1^{-1}} = \mathcal{I}$. So $\mathcal{D}_{\phi_1^{-1}}$ is a two-sided inverse of $\mathcal{D}_{\phi_1}$.

To prove the "only if" part of claim (i), assume that $\mathcal{D}_{\phi_1}^{-1}$ is an I/O map of some DLS ϕ'. Then, because $\mathcal{I} = \mathcal{D}_{\phi_1}^{-1}\mathcal{D}_{\phi_1} = \mathcal{D}_{\phi_1}\mathcal{D}_{\phi_1}^{-1}$, we have $\pi_0 = \pi_0\mathcal{D}_{\phi_1}^{-1}\mathcal{D}_{\phi_1}\pi_0 = \pi_0\mathcal{D}_{\phi_1}^{-1}\pi_0 \cdot \pi_0\mathcal{D}_{\phi_1}\pi_0$, by causality of both $\mathcal{D}_{\phi_1}^{-1}$ and $\mathcal{D}_{\phi_1}$. Now, $\pi_0\mathcal{D}_{\phi_1}\pi_0 = D$, and $I = D'D$, where $D' = \pi_0\mathcal{D}_{\phi_1}^{-1}\pi_0$. Similarly, $I = DD'$. It follows that D is a bounded bijection between Hilbert spaces U, Y. It thus has a bounded inverse $D^{-1} = D'$. This completes the proof of claim (i).

For the second claim (ii), recall formula (1.7) for the I/O map of a DLS. Use this to obtain a formula for $\mathcal{D}_{\phi_1}\,\mathcal{D}_{\phi_2}$

$$\mathcal{D}_{\phi_1}\,\mathcal{D}_{\phi_2} = D_1 D_2 + \sum_{k\geq 1} T_k \tau^k, \tag{1.19}$$

where

$$\begin{aligned}
T_k &:= D_1 C_2 A_2^{k-1} B_2 + C_1 A_1^{k-1} B_1 D_2 \\
&\quad + \sum_{j=1}^{k-1} C_1 A_1^{j-1} B_1 C_2 A_2^{k-j-1} B_2, \quad k \geq 2 \\
T_1 &:= D_1 C_2 B_2 + C_1 B_1 D_2
\end{aligned} \tag{1.20}$$

and all $T_k \in \mathcal{L}(U;Y)$. Sum (1.19) converges in the same sense as formula (1.7) but we omit these details. We calculate the similar formula for the I/O map of the DLS $\phi_1\phi_2$. For this end, note that the powers of an upper triangular (block) matrix can be calculated by

$$\begin{bmatrix} a & c \\ 0 & b \end{bmatrix}^k = \begin{bmatrix} a^k & \sum_{j=0}^{k-1} a^j \, c \, b^{k-j-1} \\ 0 & b^k \end{bmatrix}, \quad k \geq 1.$$

An application of this gives for $k \geq 1$

$$\begin{aligned} C_{\phi_1\phi_2} A^j_{\phi_1\phi_2} B_{\phi_1\phi_2} &= \begin{bmatrix} C_1 & D_1 C_2 \end{bmatrix} \begin{bmatrix} A_1 & B_1 C_2 \\ 0 & A_2 \end{bmatrix}^k \begin{bmatrix} B_1 D_2 \\ B_2 \end{bmatrix} \\ &= \begin{bmatrix} C_1 & D_1 C_2 \end{bmatrix} \begin{bmatrix} A_1^k & \sum_{j=0}^{k-1} A_1^j \, B_1 C_2 \, A_2^{j-i-1} \\ 0 & A_2^k \end{bmatrix} \begin{bmatrix} B_1 D_2 \\ B_2 \end{bmatrix} \\ &= C_1 A_1^k B_1 D_2 + D_1 C_2 A_2^k B_2 + C_1 \left(\sum_{j=0}^{k-1} A_1^j \, B_1 C_2 \, A_2^{j-i-1} \right) B_1. \end{aligned}$$

But this equals T_{k+1} of equation (1.20). The case $k = 0$ gives

$$\begin{bmatrix} C_1 & D_1 C_2 \end{bmatrix} \begin{bmatrix} B_1 D_2 \\ B_2 \end{bmatrix} = C_1 B_1 D_2 + D_1 C_2 B_2 = T_1.$$

Because also the static parts of $\mathcal{D}_{\phi_1} \mathcal{D}_{\phi_2}$ and $\mathcal{D}_{\phi_1\phi_2}$ are both $D_1 D_2$, equations (1.19) and (1.20) give also the I/O map of $\phi_1\phi_2$. The last claim (iii) is immediate. This completes the proof. $\square$

We remark the there is no uniqueness in the product realization $\phi_1\phi_2$ of the shift-invariant causal operator $\mathcal{D}_{\phi_1}\mathcal{D}_{\phi_2}$. Furthermore, generally $\widetilde{\phi_1\phi_2} \neq \widetilde{\phi_2}\widetilde{\phi_1}$ but the state spaces of these product DLSs are unitarily isomorphic. Given an I/O map $\mathcal{D}_\phi$, its adjoint I/O map $\widetilde{\mathcal{D}_\phi}$ is defined by $\widetilde{\mathcal{D}_\phi} := \mathcal{D}_{\widetilde{\phi}}$. It is easy to show, by using formula (1.7), that $\widetilde{\mathcal{D}_\phi}$ is independent of the choice of the realization ϕ.

1.5 State feedback in difference equation form

The state feedback is a basic tool in control theory that is used to change some characteristics of a given system. In this section we study the state feedback of DLSs in difference equation form. In Section 1.6 we carry out the similar work for DLSs in I/O form. It will finally appear that these two feedback notions are equivalent, see Lemma 26. This is essentially a conclusion of Theorem 15.

We first introduce the notion of the (state) feedback pair, originally introduced in [82]. It comprises a pair of such bounded linear operators that can be "coupled" into a given DLS. These operators will serve as an "extra output" that can directly be used as a feedback signal for the original system. Because one of the operators in the feedback pair is allowed to read the whole state space, we speak about state feedback.

Definition 18. *Let $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ be a DLS with input space U, output space Y and state space H.*

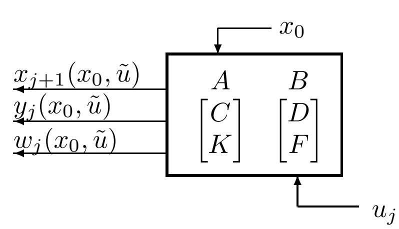
(i) The feedback pair (K, F) (in difference equation form) is an ordered pair of linear operators $K \in \mathcal{L}(H, U)$ and $F \in \mathcal{L}(U, U)$ satisfying $(I - F)^{-1} \in \mathcal{L}(U, U)$. The operator K is the output operator, and the operator F is the feed-through operator of (K, F).

(ii) Let (K, F) a feedback pair. Then

$$\phi^{\text{ext}} := \left(\begin{matrix} A & B \\ \begin{bmatrix} C \\ K \end{bmatrix} & \begin{bmatrix} D \\ F \end{bmatrix} \end{matrix}\right)$$

is the extended DLS (in difference equation form) from ϕ with feedback pair (K, F). We also write $\phi^{\text{ext}} = (\phi, (K, F))$. The input space of ϕ^{ext} is U, the output space is $Y \oplus U$ and the state space is H.

Following the language of [98], we can call the requirement $(I - F)^{-1} \in \mathcal{L}(U, U)$ admissibility of the feedback pair. We shall see in Proposition 22 that this is equivalent with the invertibility on $Seq(U)$ of certain I/O map. The following diagram illustrates the signals for the extended system ϕ^{ext} with a given initial state $x_0 \in H$ at time $j = 0$.



Here the input sequence of ϕ^{ext} is $\tilde{u} = \{u_j\}_{j\geq 0} \in Seq_+(U)$. The state trajectory $x_j = x_j(x_0, \tilde{u})$ and the output sequence $y_j = y_j(x_0, \tilde{u})$ are given by Proposition 11. The new output sequence is defined with the aid of the feedback pair (K, F) by setting

$$w_j(x_0, \tilde{u}) := Kx_j + Fu_j$$

for all $j \geq 0$.

The feedback associated to pair (K, F) arises in a natural way, by using $\tilde{u} = \{u_j\}_{j\geq 0}$ as the input signal for ϕ^{ext}, where

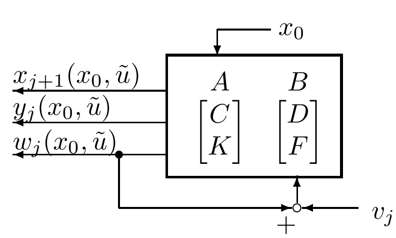
$$u_j := v_j + w_j(x_0, \tilde{u}) \quad \text{for} \quad j \geq 0.$$

Here $\tilde{v} := \{v_j\}_{j\geq 0} \in Seq_+(U)$ is an arbitrary external perturbation signal into the feedback loop. Clearly $\tilde{u} = \tilde{u}(x_0, \tilde{v})$. By the admissibility of the feedback pair (K, F), this is equivalent with

$$u_j = (I - F)^{-1}(Kx_j + v_j), \quad \text{for} \quad j \geq 0. \tag{1.21}$$

Trivially, $\tilde{u} = \tilde{u}(x_0, \tilde{v}) := \{u_j\}_{j\geq 0}$, given by equation (1.21), is an element of $Seq_+(U)$. It can thus be used as an input sequence for the (open loop) DLS ϕ^{ext}. This procedure is referred to as "closing the feedback loop" at time $j = 0$. We could, of course, close the feedback loop at any other time $j \in \mathbf{Z}$, but this would not give us essentially new structure, by the shift-invariance. The initial state $x_0 \in H$ of the closed loop system is either given explicitly, or formed by applying to the open loop DLS ϕ^{ext} an initial state $x_{j'}$ and the past inputs $\{u_j\}_{0>j\leq j'}$ for some $j' < 0$.

In the following diagram, the feedback connection is shown. We call the resulting object the closed loop system. It is not a DLS itself but it nevertheless it defines a linear, shift-invariant and causal mapping $\tilde{v} \mapsto \left[\begin{smallmatrix} \tilde{y} \\ \tilde{w} \end{smallmatrix}\right]$ which is an I/O map of an associated DLS. The input signal $\tilde{u} = \tilde{u}(x_0, \tilde{v})$ to ϕ^{ext} is given by equation (1.21).



Suppose that we are given an initial state $x_0 \in H$ and a perturbation signal $\tilde{v} \in Seq_+(U)$ of the closed loop system, described in the previous diagram. Our task is to compute the state trajectory $\{x_j(x_0, \tilde{u})\}_{j\geq 0}$ and the output sequences $\{y_j(x_0, \tilde{u})\}_{j\geq 0}$ and $\{w_j(x_0, \tilde{u})\}_{j\geq 0}$ in the closed loop. It appears, as a consequence of the admissibility of (K, F), that these equal the state trajectory and the output sequence of another DLS, with initial state x_0 and input $\tilde{v} \in Seq_+(U)$, see Lemma 20. By using equation (1.21), the open loop input signal $\tilde{u}$ can be removed from the difference equations (1.2). As a result of a straightforward computation, obtain a new set of difference equations, that define a new DLS — the closed loop extended DLS $\phi_\diamond^{\mathrm{ext}}$.

Definition 19. *Let $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix} \right)$ be a DLS and (K, F) a feedback pair for ϕ. By $\phi^{\mathrm{ext}} := (\phi, (K, F))$ denote the extended DLS of Definition 18. The the closed loop extended system $\phi_\diamond^{\mathrm{ext}} = (\phi, (K, F))_\diamond$ is a DLS, given by*

$$\begin{aligned} \phi_\diamond^{\mathrm{ext}} &= \begin{pmatrix} A + B(I-F)^{-1}K & B(I-F)^{-1} \\ \begin{bmatrix} C \\ K \end{bmatrix} + \begin{bmatrix} D \\ F \end{bmatrix} (I-F)^{-1}K & \begin{bmatrix} D \\ F \end{bmatrix} (I-F)^{-1} \end{pmatrix} \\ &\equiv \begin{pmatrix} A + B(I-F)^{-1}K & B(I-F)^{-1} \\ \begin{bmatrix} C + D(I-F)^{-1}K \\ (I-F)^{-1}K \end{bmatrix} & \begin{bmatrix} D(I-F)^{-1} \\ (I-F)^{-1} - I \end{bmatrix} \end{pmatrix}. \end{aligned}$$

A straightforward calculation constitutes the proof of the following lemma.

Lemma 20. *Let $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix} \right)$ be a DLS and (K, F) a feedback pair. Let the DLSs $\phi^{\mathrm{ext}} = (\phi, (K, F))$ and $\phi_\diamond^{\mathrm{ext}} = (\phi, (K, F))_\diamond$ be as in Definitions 18 and 19. Let $x_0 \in H$ and $\tilde{v} = \{v_j\}_{j \geq 0} \in Seq_+(U)$ be arbitrary.*

Then the state and output sequences of ϕ^{ext} in the closed loop are given by

$$\begin{aligned} &x_j(x_0, \tilde{u}) = A_\diamond^j x_0 + \mathcal{B}_{\phi_\diamond^{\mathrm{ext}}} \tau^{*j} \tilde{v} \quad \text{for all} \quad j \geq 0, \\ &\begin{bmatrix} \tilde{y}(x_0, \tilde{u}) \\ \tilde{w}(x_0, \tilde{u}) \end{bmatrix} = \mathcal{C}_{\phi_\diamond^{\mathrm{ext}}} x_0 + \mathcal{D}_{\phi_\diamond^{\mathrm{ext}}} \tilde{v}, \end{aligned}$$

where $A_\diamond := A + B(I-F)^{-1}K$ is the semigroup generator of the DLS $\phi_\diamond^{\mathrm{ext}}$, and $\tilde{u} = \tilde{u}(x_0, \tilde{v}) \in Seq_+(U)$ is the input signal to ϕ^{ext}, given by equation (1.21).

The iterated feedbacks behave in an expected way. Given a DLS ϕ, we can define a product in the set of feedback pairs for ϕ by setting

$$(K_2, F_2)\,(K_1, F_1) := ((I - F_1)K_1 + K_2, F_1 + F_2 - F_2 F_1)\,.$$

This gives the set of feedback pairs the structure of a noncommutative group, where the unit element is $(0, 0)$. The iterated feedback is given by the formula

$$((\phi, (K_1, F_1))\,, (K_2, F_2)) = (\phi, (K_1, F_1)(K_2, F_2))\,.$$

The feedback pairs of form $(K, 0)$ are an abelian subgroup of all the feedback pairs. In the literature, it is customary to use just these feedbacks. Clearly, such state feedbacks are a class "general enough" because the left hand column of DLS $\phi_\diamond^{\text{ext}}$ in Definition 19 depends only upon $(I-F)^{-1}K$ but not on the operators F and K separately. It follows that all the feedback pairs (K,F) for which $(I-F)^{-1}K$ is equal, give closed loop DLSs $(\phi,(K,F))_\diamond$ that differ only by a left multiplication by the static operator $(I-F)^{-1}$.

In this chapter, we have chosen to have a nonvanishing feed-through operator F, because then the formulae of the closed loops systems in difference equation form will look like the corresponding formulae in the I/O form, as introduced in Section 1.6. Also, to have a complete one-to-one correspondence between difference equation form feedbacks of this section, and the I/O form feedbacks of the following section 1.6, we have to include $F \neq 0$, corresponding the feed-through part $\pi_0\mathcal{F}\pi_0$ of operator $\mathcal{F}$ in $[\mathcal{K},\mathcal{F}]$ in Definition 21. In later chapters of this book, we use $F=0$ as the feed-through operator of the feedback pairs.

Suppose that a DLS $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ is a model for some physical process. It is possible that the generating operators A, B, C and D correspond to some physically realized and clearly isolated components of this process. After adjoining a feedback pair and closing the feedback loop, both the input-output behavior and the evolution of the state is described by the DLS $\phi_\diamond^{\text{ext}}$, by Lemma 20. It could be difficult (and even impossible) to find the physical counterparts of the closed loop generating operators of $\phi_\diamond^{\text{ext}}$, even if the open loop generating operators are easily identifiable, and conversely. The open loop and closed loop semigroup generators A and $A_\diamond = A+B(I-F)^{-1}K$ are generally very different, even in the special case when they differ by a rank one operator. The reader will find a plenty of such examples in later chapters.

1.6 State feedback in I/O form

In Section 1.5, it was a fairly straightforward task to introduce the notion of state feedback for DLSs in difference equation form. Things get somewhat more complicated when we study the state feedback structure for DLSs in I/O form. We must make lengthy calculations to show that the closed loop "system" is, indeed, a DLS. Again, we start with a basic definition.

Definition 21. *Let $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ be a DLS (in I/O form) whose input space is U, output space is Y and state space is H. Then the feedback pair $[\mathcal{K}, \mathcal{F}]$ (in I/O form) for Φ is an ordered pair of linear operators $\mathcal{K} : H \to Seq_+(U)$ and $\mathcal{F} : Seq(U) \to Seq(U)$ such that*

- *(i)* $\Phi^{\mathrm{fb}} = \left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{K} & \mathcal{F} \end{smallmatrix}\right]$ *is a DLS (in I/O form) with the input space U, the output space U and the state space H.*
- *(ii) The operator $\mathcal{I} - \mathcal{F} : Seq_+(U) \to Seq_+(U)$ is a bijection, and the inverse $(\mathcal{I} - \mathcal{F})^{-1}$ is an I/O map of a DLS.*

The following proposition, together with Definition 18, explains the condition (ii) of Definition 21. It is a direct consequence of Proposition 17.

Proposition 22. *Let $\Phi^{\mathrm{fb}} = \left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{K} & \mathcal{F} \end{smallmatrix}\right]$ be a DLS and $\mathcal{F}$ its I/O map. Define $F := \pi_0 \mathcal{F} \pi_0$, regarded as an operator in $\mathcal{L}(U)$, with the natural identification of spaces* range (π_0) *and U. Then condition (ii) of Definition 21 holds if and only if $I - F$ has a bounded inverse in $\mathcal{L}(U)$. In that case, $\pi_0(\mathcal{I} - \mathcal{F})^{-1}\pi_0 = (I - F)^{-1}$.*

As in Definition 19, we first introduce an extended DLS Φ^{ext} and another object, the closed loop system $\Phi^{\mathrm{ext}}_\diamond$.

Definition 23. *Let $\Phi = \begin{bmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{bmatrix}$ be a DLS and $[\mathcal{K}, \mathcal{F}]$ a feedback pair for Φ.*

- *(i) The DLS*

$$\Phi^{\mathrm{ext}} := \begin{pmatrix} A^j & \mathcal{B}\tau^{*j} \\ \begin{bmatrix} \mathcal{C} \\ \mathcal{K} \end{bmatrix} & \begin{bmatrix} \mathcal{D} \\ \mathcal{F} \end{bmatrix} \end{pmatrix}$$

is the extended DLS (in I/O form) from Φ with feedback pair $[\mathcal{K}, \mathcal{F}]$. The input space of Φ^{ext} is U, the output space is $Y \oplus U$ and the state space is H. For brevity, we write $\Phi^{\mathrm{ext}} = [\Phi, [\mathcal{K}, \mathcal{F}]]$.

(ii) The closed loop extended system $\Phi_\diamond^{\mathrm{ext}}$ *is the* 6*-tuple of linear mappings*

$$\Phi_\diamond^{\mathrm{ext}} := \begin{bmatrix} A^j + \mathcal{B}\tau^{*j}(I-\mathcal{F})^{-1}\mathcal{K} & \mathcal{B}(I-\mathcal{F})^{-1}\tau^{*j} \\ \begin{bmatrix} \mathcal{C} + \mathcal{D}(I-\mathcal{F})^{-1}\mathcal{K} \\ (\mathcal{I}-\mathcal{F})^{-1}\mathcal{K} \end{bmatrix} & \begin{bmatrix} \mathcal{D}(\mathcal{I}-\mathcal{F})^{-1} \\ (\mathcal{I}-\mathcal{F})^{-1} - \mathcal{I} \end{bmatrix} \end{bmatrix}$$
$$=: \begin{bmatrix} A_\diamond(j) & \mathcal{B}_\diamond\tau^{*j} \\ \begin{bmatrix} \mathcal{C}_\diamond \\ \mathcal{K}_\diamond \end{bmatrix} & \begin{bmatrix} \mathcal{D}_\diamond \\ \mathcal{F}_\diamond \end{bmatrix} \end{bmatrix}$$

between appropriate vector spaces $Seq(U)$*,* $Seq(Y)$ *and Hilbert space* H*. For brevity, we write* $\Phi_\diamond^{\mathrm{ext}} = [\Phi, [\mathcal{K}, \mathcal{F}]]_\diamond$*.*

It is a matter of easy checking that all the composite mappings in $\Phi_\diamond^{\mathrm{ext}}$ make sense for all j. Thus $\Phi_\diamond^{\mathrm{ext}}$ is well defined 6-tuple of linear mappings, but we do not yet claim that $\Phi_\diamond^{\mathrm{ext}}$ is a DLS. The fact that $\Phi_\diamond^{\mathrm{ext}}$ is a DLS will be proved in Lemma 24.

Now that we have defined the feedback pairs and related objects, we have to associate a notion of feedback to them. We use the feedback pair $[\mathcal{K}, \mathcal{F}]$ roughly in the same way as the feedback pair (K, F) in equation (1.21). We also use the same symbols for signals as in Section 1.5. Let $x_0 \in H$ and $\tilde{u} \in Seq_+(U)$ be arbitrary. The new output sequence of Φ^{ext}, associated to the feedback pair $[\mathcal{K}, \mathcal{F}]$, is given by

$$\tilde{w}(x_0, \tilde{u}) := \mathcal{K}x_0 + \mathcal{F}\tilde{u} \in Seq_+(U).$$

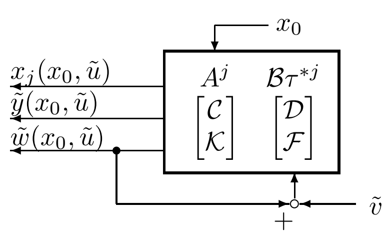
We define the feedback by requiring that the input signal $\tilde{u} = \tilde{u}(x_0, \tilde{v})$ satisfies

$$\tilde{u}(x_0, \tilde{v}) = \tilde{v} + \tilde{w}(x_0, \tilde{u}(x_0, \tilde{v})),$$

where $\tilde{v} \in Seq_+(U)$ is an arbitrary external perturbation signal. By solving $\tilde{u}(x_0, \tilde{v})$ in the previous equation, we get the equivalent formula

$$\tilde{u}(x_0, \tilde{v}) = (\mathcal{I} - \mathcal{F})^{-1}(\mathcal{K}x_0 + \tilde{v}) \tag{1.22}$$

because $\mathcal{I} - \mathcal{F}$ is assumed to be invertible on $Seq(U)$. Now, $\tilde{u}(x_0, \tilde{v}) \in Seq_+(U)$ for any initial state x_0 and external perturbation signal $\tilde{v}$. Thus it is a perfectly valid input sequence for the DLS Φ^{ext}, and the following closed loop connection of signals for Φ^{ext} makes sense.



As in Section 1.5, it is desirable to compute formulae for the state trajectory $\{x_j(x_0, \tilde{u})\}_{j\geq 0}$ and the output sequences $\tilde{y}(x_0, \tilde{u})$ and $\tilde{w}(x_0, \tilde{u})$, where the input signal $\tilde{u} = \tilde{u}(x_0, \tilde{v})$ to Φ^{ext} is given by equation (1.22). By Lemma 20 for a DLS ϕ in difference equation form, these are given by the state and output trajectories of the closed loop DLS $\phi_\diamond^{\mathrm{ext}}$, with the initial state x_0 and the input signal $\tilde{v}$. So as to the DLSs in I/O form, an easy computation, based on equation (1.22), reveals that such a DLS must be $\Phi_\diamond^{\mathrm{ext}}$ of Definition 23, if any DLS at all. It remains to show that the 6-tuple of mappings $\Phi_\diamond^{\mathrm{ext}}$ is, in fact, a DLS in I/O form.

Lemma 24. *The system $\Phi_\diamond^{\mathrm{ext}} = [\Phi, [\mathcal{K}, \mathcal{F}]]$ of Definition 23 is a DLS. The input space of $\Phi_\diamond^{\mathrm{ext}}$ is U, the output space is $Y \oplus U$ and the state space is H.*

Proof. It is sufficient to show that the system

$$\Phi_\diamond = \begin{bmatrix} A_\diamond(j) & \mathcal{B}_\diamond \tau^{*j} \\ \mathcal{C}_\diamond & \mathcal{D}_\diamond \end{bmatrix}$$

is a DLS, where the linear mappings are given by Definition ii. This is because the linear mappings A, $\mathcal{B}$, $\mathcal{K}$ and $\mathcal{F}$ form the DLS Φ^{fb} of Definition 21, and $[\mathcal{K}, \mathcal{F}]$ is a feedback pair for Φ^{fb}, too. In order to consider the lowest row of $\Phi_\diamond^{\mathrm{ext}}$ in Definition 13, we can consider the middle row of the closed loop system $\left[\Phi^{\mathrm{fb}}, [\mathcal{K}, \mathcal{F}]\right]$ instead. We proceed to show that the linear mappings $A_\diamond$, $\mathcal{B}_\diamond$, $C_\diamond$ and $\mathcal{D}_\diamond$ satisfy the conditions of Definition 13.

Because Φ^{fb}, we have $\mathcal{K}\mathcal{B} = \bar{\pi}_+ \mathcal{F} \pi_-$ by property (v) of Definition 13 ($\mathcal{K}$ in place of $\mathcal{C}$). This and the causality of both $\mathcal{F}$ and $(\mathcal{I} - \mathcal{F})^{-1}$ implies the identity

$$\begin{aligned} (1.23) \qquad & (\mathcal{I} - \mathcal{F})^{-1} \mathcal{K} \mathcal{B} (\mathcal{I} - \mathcal{F})^{-1} = (\mathcal{I} - \mathcal{F})^{-1} \bar{\pi}_+ \mathcal{F} \pi_- (I - \mathcal{F})^{-1} \\ & = \bar{\pi}_+ (\mathcal{I} - \mathcal{F})^{-1} \pi_- \end{aligned}$$

that will be used several times in the course of this proof.

We start by showing that the family of linear mappings $\{A^j + \mathcal{B}\tau^{*j}(\mathcal{I} - \mathcal{F})^{-1}\mathcal{K}\}_{j\geq 0}$ is a discrete time semigroup in $\mathcal{L}(H)$. It is a triviality that this family consists of bounded linear operators on H. Furthermore, we have for all $j \geq 1$

$$\begin{aligned} (1.24) \qquad & (A + \mathcal{B}\tau^*(\mathcal{I} - \mathcal{F})^{-1}\mathcal{K})(A^j + \mathcal{B}\tau^{*j}(\mathcal{I} - \mathcal{F})^{-1}\mathcal{K}) \\ & = A^{j+1} + \overbrace{\mathcal{B}\tau^*(\mathcal{I} - \mathcal{F})^{-1}\mathcal{K}A^j}^{(i)} \\ & + \overbrace{A\mathcal{B}\tau^{*j}(\mathcal{I} - \mathcal{F})^{-1}\mathcal{K}}^{(ii)} + \overbrace{\mathcal{B}\tau^*(\mathcal{I} - \mathcal{F})^{-1}\mathcal{K}\mathcal{B}\tau^{*j}(\mathcal{I} - \mathcal{F})^{-1}\mathcal{K}}^{(iii)}. \end{aligned}$$

Now we study the terms (i), (ii) and (iii) of the equation (1.24) separately. Term (i) satisfies

$$(1.25) \qquad \mathcal{B}\tau^*(\mathcal{I} - \mathcal{F})^{-1}\mathcal{K}A^j = \mathcal{B}\tau^*\pi_0(\mathcal{I} - \mathcal{F})^{-1}\bar{\pi}_+\tau^{*j}\mathcal{K},$$

where property (iv) of Definition 13 ($\mathcal{K}$ in place of $\mathcal{C}$) and the causality of $(\mathcal{I}-\mathcal{F})^{-1}$ has been used. Term (ii) satisfies

$$A\mathcal{B}\tau^{*j}(\mathcal{I}-\mathcal{F})^{-1}\mathcal{K} = \mathcal{B}\tau^{*(j+1)}(\mathcal{I}-\mathcal{F})^{-1}\mathcal{K} - \mathcal{B}\tau^{*}\pi_0(\mathcal{I}-\mathcal{F})^{-1}\tau^{*j}\mathcal{K}, \tag{1.26}$$

where the property (iii) of Definition 13 has been used. The last term (iii) requires the most work. Now we have by the shift invariance of $(I-\mathcal{F})^{-1}$ and formula (1.23)

$$\begin{aligned} &\mathcal{B}\tau^{*}(\mathcal{I}-\mathcal{F})^{-1}\mathcal{K}\mathcal{B}\tau^{*j}(\mathcal{I}-\mathcal{F})^{-1}\mathcal{K} \\ &= \mathcal{B}\tau^{*}(\mathcal{I}-\mathcal{F})^{-1}\bar{\pi}_+\mathcal{F}\pi_-(\mathcal{I}-\mathcal{F})^{-1}\tau^{*j}\mathcal{K} \\ &= \mathcal{B}\tau^{*}\bar{\pi}_+(\mathcal{I}-\mathcal{F})^{-1}\pi_-\tau^{*j}\mathcal{K} \\ &= \mathcal{B}\tau^{*}\pi_0(\mathcal{I}-\mathcal{F})^{-1}\pi_-\tau^{*j}\mathcal{K}, \end{aligned} \tag{1.27}$$

where the last equality follows immediately from the definition of $\mathcal{B}$. Now summing up formulae (1.25), (1.26) and (1.27) and combining that with formula (1.24), we obtain

$$\begin{aligned} &(A+\mathcal{B}\tau^{*}(\mathcal{I}-\mathcal{F})^{-1}\mathcal{K})(A^j+\mathcal{B}\tau^{*j}(\mathcal{I}-\mathcal{F})^{-1}\mathcal{K}) \\ &= (A^{j+1}+\mathcal{B}\tau^{*(j+1)}(\mathcal{I}-\mathcal{F})^{-1}\mathcal{K}), \end{aligned}$$

for all $j \geq 1$. By induction, this is equivalent with

$$(A^j+\mathcal{B}\tau^{*j}(\mathcal{I}-\mathcal{F})^{-1}\mathcal{K}) = \left(A+\mathcal{B}\tau^{*}(\mathcal{I}-\mathcal{F})^{-1}\mathcal{K}\right)^j =: A_\diamond^j,$$

where $A_\diamond$ denotes the generator of the closed loop semigroup. This proves the claim about the semigroup.

In order to prove that $\mathcal{B}_\diamond := \mathcal{B}(\mathcal{I}-\mathcal{F})^{-1}$ is a valid controllability map satisfying the conditions of Definition 13, we first check the causality of $\mathcal{B}_\diamond$. We have

$$\begin{aligned} &\mathcal{B}_\diamond\bar{\pi}_+ = \mathcal{B}(\mathcal{I}-\mathcal{F})^{-1}\bar{\pi}_+ = \mathcal{B}\bar{\pi}_+(\mathcal{I}-\mathcal{F})^{-1}\bar{\pi}_+ \\ &= \mathcal{B}\pi_-\bar{\pi}_+(\mathcal{I}-\mathcal{F})^{-1}\bar{\pi}_+ = 0, \end{aligned}$$

where we have used the causality of $(\mathcal{I}-\mathcal{F})^{-1}$. In order to see whether $\mathcal{B}_\diamond$ interacts correctly with the time shift τ^* and the semigroup generator $A_\diamond$, we have to show that $\mathcal{B}_\diamond\tau^* = A_\diamond\mathcal{B}_\diamond + \mathcal{B}_\diamond\tau^*\pi_0$. We have

$$\begin{aligned} &\mathcal{B}_\diamond\tau^* = \mathcal{B}(I-\mathcal{F})^{-1}\tau^* = \mathcal{B}\tau^*(I-\mathcal{F})^{-1} \\ &= A\mathcal{B}(\mathcal{I}-\mathcal{F})^{-1} + \mathcal{B}\tau^*\pi_0(\mathcal{I}-\mathcal{F})^{-1}. \end{aligned} \tag{1.28}$$

On the other hand, we have by the causality of $(I-\mathcal{F})^{-1}$ and equation (1.23)

$$\begin{aligned} &A_\diamond\mathcal{B}_\diamond + \mathcal{B}_\diamond\tau^*\pi_0 \\ &= (A+\mathcal{B}(\mathcal{I}-\mathcal{F})^{-1}\tau^*\mathcal{K})(\mathcal{B}(\mathcal{I}-\mathcal{F})^{-1}) + \mathcal{B}(\mathcal{I}-\mathcal{F})^{-1}\tau^*\pi_0 \\ &= A\mathcal{B}(\mathcal{I}-\mathcal{F})^{-1} + \mathcal{B}\tau^*(\mathcal{I}-\mathcal{F})^{-1}\mathcal{K}\mathcal{B}(\mathcal{I}-\mathcal{F})^{-1} + \mathcal{B}(\mathcal{I}-\mathcal{F})^{-1}\tau^*\pi_0 \\ &= A\mathcal{B}(\mathcal{I}-\mathcal{F})^{-1} + \mathcal{B}\tau^*\bar{\pi}_+(\mathcal{I}-\mathcal{F})^{-1}\pi_- + \mathcal{B}\tau^*(\mathcal{I}-\mathcal{F})^{-1}\pi_0. \end{aligned}$$

The causality of $(\mathcal{I}-\mathcal{F})^{-1}$ and the basic properties of $\mathcal{B}$ now allow us to continue

$$
\begin{aligned}
&= A\mathcal{B}(\mathcal{I}-\mathcal{F})^{-1} + \mathcal{B}\tau^*\pi_0(\mathcal{I}-\mathcal{F})^{-1}\pi_- + \mathcal{B}\tau^*\pi_0(\mathcal{I}-\mathcal{F})^{-1}\tau^*\pi_0 \\
&= A\mathcal{B}(\mathcal{I}-\mathcal{F})^{-1} + \mathcal{B}\tau^*\pi_0(\mathcal{I}-\mathcal{F})^{-1}. \qquad (1.29)
\end{aligned}
$$

Now is it sufficient to compare the right sides of equations (1.28) and (1.29) to see that $\mathcal{B}_\diamond\tau^* = A_\diamond\mathcal{B}_\diamond + \mathcal{B}_\diamond\tau^*\pi_0$. This proves that $\mathcal{B}_\diamond$ is a valid controllability map for any DLS whose semigroup generator is $\mathcal{A}_\diamond$.

Next we check that the mapping $\mathcal{C}_\diamond := \mathcal{C} + \mathcal{D}(\mathcal{I}-\mathcal{F})^{-1}\mathcal{K}$ is a valid observability map for a DLS whose semigroup generator is $A_\diamond$. It is clear that $\mathcal{C}_\diamond$ maps H into $Seq_+(Y)$. To establish $\mathcal{C}_\diamond A_\diamond = \bar{\pi}_+\tau^*\mathcal{C}_\diamond$, we calculate

$$
\begin{aligned}
\mathcal{C}_\diamond A_\diamond &= (\mathcal{C} + \mathcal{D}(\mathcal{I}-\mathcal{F})^{-1}\mathcal{K})(A + \mathcal{B}\tau^*(\mathcal{I}-\mathcal{F})^{-1}\mathcal{K}) \qquad (1.30) \\
&= \overbrace{\mathcal{C}A}^{(i)} + \overbrace{\mathcal{D}(\mathcal{I}-\mathcal{F})^{-1}\mathcal{K}A}^{(ii)} + \overbrace{\mathcal{C}\mathcal{B}\tau^*(\mathcal{I}-\mathcal{F})^{-1}\mathcal{K}}^{(iii)} \\
&+ \overbrace{\mathcal{D}(\mathcal{I}-\mathcal{F})^{-1}\mathcal{K}\mathcal{B}\tau^*(\mathcal{I}-\mathcal{F})^{-1}\mathcal{K}}^{(iv)}.
\end{aligned}
$$

The term (i) clearly equals $\bar{\pi}_+\tau^*\mathcal{C}$ by applying formula (iv) of Definition 13. Term (ii) can be seen to equal $\bar{\pi}_+\mathcal{D}(\mathcal{I}-\mathcal{F})^{-1}\bar{\pi}_+\tau^*\mathcal{K}$ by applying condition (iv) of Definition 13, and noting that $\mathcal{D}$ is causal. So as to term (iii) we note that $\mathcal{C}\mathcal{B} = \bar{\pi}_+\mathcal{D}\pi_-$, by condition (v) of Definition 13. Then term (iii) takes form $\bar{\pi}_+\mathcal{D}\pi_-(\mathcal{I}-\mathcal{F})^{-1}\tau^*\mathcal{K} = \bar{\pi}_+\mathcal{D}\pi_-(\mathcal{I}-\mathcal{F})^{-1}\pi_-\tau^*\mathcal{K}$. The last term (iv) is again of the form of equation (1.23), and equals $\bar{\pi}_+\mathcal{D}\bar{\pi}_+(I-\mathcal{F})^{-1}\pi_-\tau^*\mathcal{K}$, where we have used the causality of $\mathcal{D}$, too. Summing these formulae for all the terms (i) — (iv) of formula (1.30) gives the required identity $\mathcal{C}_\diamond A_\diamond = \bar{\pi}_+\tau^*\mathcal{C}_\diamond$.

So our final task is to check that the I/O map candidate $\mathcal{D}_\diamond$ interacts correctly with the mappings $A_\diamond$, $\mathcal{B}_\diamond$, $\mathcal{C}_\diamond$ and time shifts. Causality of $\mathcal{D}_\diamond$ is again no issue, and neither is the fact $\tau^*\mathcal{D}_\diamond = \mathcal{D}_\diamond\tau^*$. Our work lies in checking that $\bar{\pi}_+\mathcal{D}_\diamond\pi_- = \mathcal{C}_\diamond\mathcal{B}_\diamond$. The proof of this equality goes now in the familiar way by using equation (1.23) and causality of $\mathcal{D}$

$$
\begin{aligned}
\mathcal{C}_\diamond\mathcal{B}_\diamond &= (\mathcal{C} + \mathcal{D}(\mathcal{I}-\mathcal{F})^{-1}\mathcal{K})\mathcal{B}(\mathcal{I}-\mathcal{F})^{-1} \\
&= \mathcal{C}\mathcal{B}(\mathcal{I}-\mathcal{F})^{-1} + \mathcal{D}(\mathcal{I}-\mathcal{F})^{-1}\mathcal{K}\mathcal{B}(\mathcal{I}-\mathcal{F})^{-1} \\
&= \bar{\pi}_+\mathcal{D}\pi_-(\mathcal{I}-\mathcal{F})^{-1} + \mathcal{D}\bar{\pi}_+(\mathcal{I}-\mathcal{F})^{-1}\pi_- \\
&= \bar{\pi}_+\mathcal{D}\pi_-(\mathcal{I}-\mathcal{F})^{-1}\pi_- + \bar{\pi}_+\mathcal{D}\bar{\pi}_+(\mathcal{I}-\mathcal{F})^{-1}\pi_- \\
&= \bar{\pi}_+\mathcal{D}(\mathcal{I}-\mathcal{F})^{-1}\pi_- = \bar{\pi}_+\mathcal{D}_\diamond\pi_-.
\end{aligned}
$$

Now we have proved that the quadruple $\Phi_\diamond = \begin{bmatrix} A_\diamond^j & \mathcal{B}_\diamond\tau^{*j} \\ \mathcal{C}_\diamond & \mathcal{D}_\diamond \end{bmatrix}$ is a DLS. □

Analogues for well-posed linear systems can be found in [98, Theorem 6.1] and [89]. The proof in the latter reference follows these lines, too.

We have proved that the state feedback in I/O form gives a closed loop system, which still is a DLS $\Phi_\diamond^{\text{ext}}$, by Lemma 24. In Theorem 15 we stated that the DLSs in difference form and I/O form have one-to-one correspondence. Then the open loop system Φ has a representation ϕ in difference equation form, and so has the closed loop system $\Phi_\diamond^{\text{ext}}$ a representation ϕ', too. The final question is, whether ϕ' is equal to a closed loop system $(\phi,(K,F))_\diamond$ for some feedback pair (K,F)? And if so, then how how to relate the feedback pairs $[\mathcal{K},\mathcal{F}]$ and (K,F)? The answer to these questions is what one would expect.

Definition 25. *Let $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D}\end{smallmatrix}\right] = \left(\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right) = \phi$ be a DLS. Let $[\mathcal{K},\mathcal{F}]$ be a feedback pair for Φ, and (K,F) for ϕ.*

We say that the feedback pairs $[\mathcal{K},\mathcal{F}]$ and (K,F) correspond to each other if

$$\begin{bmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{K} & \mathcal{F}\end{bmatrix} = \begin{pmatrix} A & B \\ K & F\end{pmatrix}. \tag{1.31}$$

In this case, we write $[\mathcal{K},\mathcal{F}] = (K,F)$.

If $[\mathcal{K},\mathcal{F}] = (K,F)$, it is easy to find formulae connecting the linear mappings $\mathcal{K}$, $\mathcal{F}$, K and F by applying Theorem 15 to equation (1.31).

Lemma 26. *Let $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D}\end{smallmatrix}\right] = \left(\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right) = \phi$ be a DLS. Let $[\mathcal{K},\mathcal{F}]$ be a feedback pair for Φ, and (K,F) a feedback pair for ϕ.*

Then $[\mathcal{K},\mathcal{F}] = (K,F)$ if and only if $[\Phi,[\mathcal{K},\mathcal{F}]]_\diamond = (\phi,(K,F))_\diamond$.

Proof. We have to study when $[\Phi,[\mathcal{K},\mathcal{F}]]_\diamond = (\phi,(K,F))_\diamond$ or equivalently

$$\begin{aligned}
&\begin{bmatrix} A^j + \mathcal{B}\tau^{*j}(\mathcal{I}-\mathcal{F})^{-1}\mathcal{K} & \mathcal{B}(\mathcal{I}-\mathcal{F})^{-1}\tau^{*j} \\ \begin{bmatrix} \mathcal{C}+\mathcal{D}(\mathcal{I}-\mathcal{F})^{-1}\mathcal{K} \\ (\mathcal{I}-\mathcal{F})^{-1}\mathcal{K}\end{bmatrix} & \begin{bmatrix} \mathcal{D}(\mathcal{I}-\mathcal{F})^{-1} \\ (\mathcal{I}-\mathcal{F})^{-1} - I\end{bmatrix}\end{bmatrix} \\
&= \begin{pmatrix} A + B(I-F)^{-1}K & B(I-F)^{-1} \\ \begin{bmatrix} C + D(I-F)^{-1}K \\ (I-F)^{-1}K\end{bmatrix} & \begin{bmatrix} D(I-F)^{-1} \\ (I-F)^{-1} - I\end{bmatrix}\end{pmatrix}
\end{aligned} \tag{1.32}$$

under the assumption that $\Phi = \phi$.

Assume that $[\mathcal{K},\mathcal{F}] = (K,F)$. We are to show that the equality hold in (1.32). We show only that the semigroup generators and observability maps in the right and left sides of formula (1.32) are equal. The other parts are left for the reader. For the semigroup generators we have

$$\begin{aligned}
&A + \mathcal{B}\tau^*(\mathcal{I}-\mathcal{F})^{-1}\mathcal{K} = A + \mathcal{B}\pi_{-1}\tau^*\pi_0(\mathcal{I}-\mathcal{F})^{-1}\bar{\pi}_+\mathcal{K} \\
&= A + \mathcal{B}\pi_{-1}\cdot\tau^*\cdot\pi_0(\mathcal{I}-\mathcal{F})^{-1}\pi_0\cdot\pi_0\mathcal{K},
\end{aligned}$$

where the latter equality holds by the causality of $(I-\mathcal{F})^{-1}$ (see Definition 21). Now, by (1.31), we have $K = \pi_0\mathcal{K}$ and $\pi_0(\mathcal{I}-\mathcal{F})^{-1}\pi_0 = (I-F)^{-1}$ where also Proposition 22 has been used. Also $B = \mathcal{B}\pi_{-1}$, because $\Phi = \phi$. Thus

$$A + \mathcal{B}\tau^*(\mathcal{I}-\mathcal{F})^{-1}\mathcal{K} = A + B(I-F)^{-1}K,$$

where natural identifications of appropriate intermediate spaces have been done.

We check that the observability maps in the right and left sides of formula (1.32) are equal. Because we have already proved the equality of the closed loop semigroup generators, it suffices to consider only the first component mappings of the observability maps. We have

$$\begin{aligned}\pi_0(\mathcal{C} + \mathcal{D}(\mathcal{I}-\mathcal{F})^{-1}\mathcal{K}) &= \pi_0\mathcal{C} + \pi_0\mathcal{D}\bar{\pi}_+(\mathcal{I}-\mathcal{F})^{-1}\pi_+\mathcal{K}\\ &= \pi_0\mathcal{C} + \pi_0\mathcal{D}\pi_0(\mathcal{I}-\mathcal{F})^{-1}\pi_0\mathcal{K},\end{aligned}$$

where the second equality is by the causality of both $\mathcal{D}$ and $(\mathcal{I}-\mathcal{F})^{-1}$. Now, $C = \pi_0\mathcal{C}$ and $D = \pi_0\mathcal{D}\pi_0$, because $\Phi = \phi$. Also $K = \pi_0\mathcal{K}$ and $\pi_0(\mathcal{I}-\mathcal{F})^{-1}\pi_0 = (I-F)^{-1}$ as above, because $[\mathcal{K},\mathcal{F}] = (K,F)$. It follows that

$$\pi_0(\mathcal{C} + \mathcal{D}(\mathcal{I}-\mathcal{F})^{-1}\mathcal{K}) = C + D(I-F)^{-1}K,$$

again with the natural identifications of the intermediate spaces.

For the converse direction, assume that the equality holds in (1.32). By identifying range (π_0) and U, we have $\pi_0((\mathcal{I}-\mathcal{F})^{-1} - I)\pi_0 = (I-F)^{-1} - I)$ and $\pi_0(\mathcal{I}-\mathcal{F})^{-1}\pi_0 = (I-F)^{-1}$. Claim (i) of Proposition 17 implies that

$$\pi_0\left(\mathcal{I}-\mathcal{F}\right)^{-1}\pi_0 = \left(\pi_0(\mathcal{I}-\mathcal{F})\pi_0\right)^{-1} = \left(I - \pi_0\mathcal{F}\pi_0\right)^{-1}$$

But then $(\mathcal{I} - \pi_0\mathcal{F}\pi_0)^{-1} = (I-F)^{-1}$ and thus $\pi_0\mathcal{F}\pi_0 = F$.

By causality and the above proved identity $(I-\pi_0\mathcal{F}\pi_0)^{-1} = (I-F)^{-1}$, we have

$$\pi_0(\mathcal{I}-\mathcal{F})^{-1}\mathcal{K} = \pi_0(\mathcal{I}-\mathcal{F})^{-1}\pi_0 \cdot \pi_0\mathcal{K} = (I-F)^{-1}\cdot\pi_0\mathcal{K} = (I-F)^{-1}K.$$

Because $(I-F)^{-1} \in \mathcal{L}(U)$ has a bounded inverse, it follows $\pi_0\mathcal{K} = K$. Now we have shown that $\pi_0\mathcal{F}\pi_0 = F$ and $\pi_0\mathcal{K} = K$. Thus $[\mathcal{K},\mathcal{F}] = (K,F)$ and the proof is complete. □

In the end of Section 1.5, we considered the group structure of the feedback pairs (K,F) in the difference equation form. We now give a formula for opening a feedback loop by another feedback. Let Φ be a DLS, $[\mathcal{K},\mathcal{F}]$ a feedback pair for it and $\Phi_\diamond^{\mathrm{ext}} := [\Phi, [\mathcal{K},\mathcal{F}]]_\diamond$ the corresponding closed loop DLS. Define the ordered pair of linear mappings $\left[\bar{\mathcal{K}},\bar{\mathcal{F}}\right]$, by setting $\bar{\mathcal{K}} = -(\mathcal{I}-\mathcal{F})^{-1}\mathcal{K}$ and $\bar{\mathcal{F}} = \mathcal{I} - (\mathcal{I}-\mathcal{F})^{-1}$. Clearly, apart from the minus sign, the pair of mappings

$\left[\bar{\mathcal{K}},\bar{\mathcal{F}}\right]$ constitute the lowest row of the DLS $\Phi_\diamond^{\text{ext}}$, given in Definition 23. This implies that $\left[\bar{\mathcal{K}},\bar{\mathcal{F}}\right]$ is a feedback pair for $\Phi_\diamond^{\text{ext}}$. The closed loop extended DLS $(\Phi_\diamond^{\text{ext}})_\diamond^{\text{ext}} := \left[[\Phi,[\mathcal{K},\mathcal{F}]],\left[\bar{\mathcal{K}},\bar{\mathcal{F}}\right]\right]$ is of the form

$$(1.33)\qquad \left(\Phi_\diamond^{\text{ext}}\right)_\diamond^{\text{ext}} = \begin{bmatrix} A^j & \mathcal{B}\tau^{*j} \\ \begin{bmatrix} \mathcal{C} \\ \mathcal{K} \\ -\mathcal{K} \end{bmatrix} & \begin{bmatrix} \mathcal{D} \\ \mathcal{F} \\ -\mathcal{F} \end{bmatrix} \end{bmatrix}$$

as can be seen by a straightforward calculation, based on the identity $(\mathcal{I}-\mathcal{F})^{-1} = \mathcal{I}-\bar{\mathcal{F}}$. We conclude that any state feedback can be undone by another, inverse state feedback.

1.7 Stability notions of DLSs

In this section, we introduce an inner product space structure to certain subspaces of the input and output sequences for a DLS $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$. This gives us the notions of "energy" and "cost" of such input and output sequences of Φ, and various related stability notions for Φ itself, too.

Two kinds of stability notions are considered here. In Definition 27 we review the stability notions of the semigroup generator A of Φ, see Definition 27. The latter kind of stability notions are considered in Definition 32, and they depend on more than one of the generating operators of the DLS Φ. We also study the conditions under which the mappings $\mathcal{C}$, $\mathcal{D}\pi_0$ and $\mathcal{D}\bar{\pi}_+$ of DLS Φ are closed, densely defined and finally bounded.

Definition 27. *Let $A \in \mathcal{L}(H)$. Then*

(i) A is power (or exponentially) stable, if its spectral radius satisfies $\rho(A) < 1$,

(ii) A is strongly ℓ^p stable for $p \in [1, \infty)$, if for all $x \in H$ we have

$$\sum_{j \geq 0} ||A^j x||_H^p < \infty,$$

(iii) A is strongly stable, if $A^j x \to 0$ as $j \to \infty$,

(iv) A is power bounded, if $\sup_{j \geq 0} ||A^j||_H < \infty$.

The semigroup stability notions as related to each other in the following way:

Proposition 28. *Let $A \in \mathcal{L}(H)$. Then, given the following enumeration of propositions:*

(i) $||A||_{\mathcal{L}(H)} < 1$,

(ii) A is power stable,

(iii) $||A^j||_{\mathcal{L}(H)} < M\,\delta^j$ for a constant $M < \infty$ and $0 < \delta < 1$,

(iv) $||A^j x||_H < C(x)\delta_j$, where $C(x) < \infty$ in a set of the second category in H, and $\sum_{j \geq 0} \delta_j < \infty$,

(v) A is strongly ℓ^p stable for some (and then, for all) $p \in [1, \infty)$,

(vi) A is strongly stable,

(vii) *there is an operator* $\tilde{A} \in \mathcal{L}(H)$ *such that* $A^j x \to \tilde{A}x \in H$ *for all* $x \in H$*,*

(viii) A *is power bounded,*

(ix) $\rho(A) \leq 1$*,*

we have the following implications and equivalences:

$$(i) \Rightarrow (ii) \Leftrightarrow (iii) \Leftrightarrow (iv) \Leftrightarrow (v) \Rightarrow (vi) \Rightarrow (vii) \Rightarrow (viii) \Rightarrow (ix)$$

Proof. The first implication (i) $\Rightarrow$ (ii) is the inequality $\rho(A) \leq ||A||$. The implication (ii) $\Rightarrow$ (iii) is trivial, because

$$\rho(A) = \limsup_{j\to\infty} ||A^j||_{\mathcal{L}(H)}^{\frac{1}{j}}, \tag{1.34}$$

and we can define $\delta := \rho(A) + \epsilon < 1$ for $\epsilon > 0$ arbitrarily small. The implication (iii) $\Rightarrow$ (iv) is trivial. The implication (iv) $\Rightarrow$ (ii) is proved as follows. Define the bounded linear operators $T_k(z) := \sum_{j=0}^{k} (zA)^j$ on H for any $|z| \leq 1$ and $k \geq 1$. Then we have for each x for which $C(x) < \infty$ and $m \leq l$

$$|| \left(T_m(z) - T_l(z)\right) x||_H \leq \sum_{j=m}^{l} ||A^j x||_H \leq C(x) \sum_{j=l}^{m} \delta_j.$$

Because $\{\delta_j\}$ is absolutely summable, $\{T_j(z)x\}$ is Cauchy for all x belonging to a set of the second category. The pointwise limit operator $T(z)x := \lim_{j\to\infty} T_j(z)x$ is bounded, [79, Theorem 2.7(b)]. It is easy to check that $T(z)(I - zA) = (I - zA)T(z) = I$ and thus $\frac{1}{z} \notin \sigma(A)$. Because $|z| \leq 1$ was arbitrary, we have $\sigma(A) \subset \mathbf{D}$ and $\rho(A) < 1$.

It is trivial that (iii) $\Rightarrow$ (v). The implication (v) $\Rightarrow$ (ii) is proved by the following argument presented in [96, Proposition 1]. Assume that A is strongly ℓ^p stable. Each of the mappings $T_n : H \to \ell^p(\mathbf{Z}_+; H)$ for $n \geq 1$, given by

$$T_n x = \{x \; Ax \; A^2x \; \cdots \; A^n x \; 0 \; 0 \; 0 \; \cdots\},$$

is bounded, and by the ℓ^p stability assumption, the orbits $\{T_n x\}_{n\geq 0} \subset \ell^p(\mathbf{Z}_+; H)$ are bounded for all $x \in H$. By the Banach–Steinhaus Theorem [79, Theorem 2.5], the set $\{T_n\}_{n\geq 0} \subset \mathcal{L}(H; \ell^p(\mathbf{Z}_+; H))$ is uniformly bounded, and we have a constant $K < \infty$ such that

$$\left(\sum_{j\geq 0} ||A^j x||_H^p\right)^{1/p} \leq K \, ||x||_H. \tag{1.35}$$

In particular, $||A^j||_{\mathcal{L}(H)} \leq K$ for any $j \geq 0$. Then we have

$$\begin{aligned} ||A^j x||_H^p &= \frac{1}{n} \sum_{j=0}^{n-1} ||A^{n-j} A^j x||_H^p \\ &\leq \frac{K^p}{n} \sum_{j=0}^{n-1} ||A^j x||_H^p \leq \frac{K^p}{n} \sum_{j \geq 0} ||A^j x||_H^p. \end{aligned}$$

By equation (1.35), we obtain $||A^j x||_H^p \leq \frac{K^{2p}}{n} ||x||_H^p$ for all $x \in H$, and then $||A^j||_{\mathcal{L}(H)}^p \leq \frac{K^{2p}}{n}$. But this implies that $||A^n|| < 1$ for n large enough, and thus the power stability $\sigma(A^n) \subset \mathbf{D}$. By using the Spectral Mapping Theorem [79, Theorem 10.28], we conclude that $\rho(A) < 1$. This completes the proof of the equivalence part of this proposition.

The implications (v) $\Rightarrow$ (vi) $\Rightarrow$ (vii) are trivial. The implication (vii) $\Rightarrow$ (viii) is an immediate consequence of Banach–Steinhaus Theorem, and the last implication (viii) $\Rightarrow$ (ix) follows from formula (1.34). This completes the proof of the proposition. $\square$

For variants of claim (v) of previous proposition , see [96]. For power bounded operators, see [69] and the references therein. See also [30].

Let $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix} \right]$ be a DLS whose input space is U, state space is H and output space is Y. In Definition 10, the controllability map $\mathcal{B} : Seq_-(U) \to H$ and the observability map $\mathcal{C} : H \to Seq(Y)$ have been introduced by algebraic constructions, without using any topological properties of any of the vector spaces. So as to the I/O map $\mathcal{D} : Seq(U) \to Seq(Y)$, the same comment can be made. In Proposition 8, we have obtained an infinite sum representation for the I/O map of DLS, where we have used the notion (topology) of the componentwise convergence in vector spaces $Seq(U)$ and $Seq(Y)$. In fact, such a result could have been proved for general shift-invariant, causal linear mappings on $Seq(U)$, because all the necessary component mappings are represented by well-defined finite sums by causality. We have required that the generating operators A, B, C and D of the DLS $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix} \right)$ are bounded; this gives us the well-posedness of the DLS. Now we shall require boundedness of the mappings $\mathcal{B}$, $\mathcal{C}$ and several variants of $\mathcal{D}$ in certain Hilbert space norms, to introduce various stability notions for the DLS $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix} \right]$.

In this section, we take the input and output sequences from the inner product spaces $\ell^2(\mathbf{Z}; U) \cap Seq(U)$ and $\ell^2(\mathbf{Z}; Y) \cap Seq(Y)$, respectively. The projections π_+, π_-, π_0, $\bar{\pi}_+$, $\bar{\pi}_-$, $\pi_{[j,k]}$ and the bilateral shift τ of Definition 4 are restricted to these spaces. Then all these projections are orthogonal projections and their operator norms are 1. The shift τ becomes a unitary operation satisfying $\tau^{-1} = \tau^*$. The following definition gives us vector subspaces of H and $Seq(U)$ that will be domains of linear operators.

Definition 29. *Let* $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ *be a DLS. Define the following domain spaces*

$$\operatorname{dom}(\underline{\mathcal{B}}) := Seq_-(U), \tag{1.36}$$

$$\operatorname{dom}(\underline{\mathcal{C}}) := \{x_0 \in H \mid \mathcal{C}x_0 \in \ell^2(\mathbf{Z}_+;Y)\}, \tag{1.37}$$

$$\operatorname{dom}(\underline{\mathcal{D}}) := \{\tilde{u} \in \ell^2(\mathbf{Z};U) \cap Seq(U) \mid \mathcal{D}\tilde{u} \in \ell^2(\mathbf{Z};Y)\}, \tag{1.38}$$

$$\operatorname{dom}(\underline{\mathcal{D}}\bar{\pi}_+) := \{\tilde{u} \in \ell^2(\mathbf{Z}_+;U) \mid \mathcal{D}\bar{\pi}_+\tilde{u} \in \ell^2(\mathbf{Z}_+;Y)\}, \tag{1.39}$$

$$\operatorname{dom}(\underline{\mathcal{D}}\pi_0) := \{\tilde{u} \in \operatorname{range}(\pi_0) \mid \mathcal{D}\pi_0\tilde{u} \in \ell^2(\mathbf{Z}_+;Y)\}, \tag{1.40}$$

$$\operatorname{dom}(\underline{\mathcal{D}}\pi_j) := \tau^j \operatorname{dom}(\underline{\mathcal{D}}\pi_0) \quad \text{for all} \quad j \in \mathbf{Z}\setminus\{0\}, \tag{1.41}$$

where $\ell^2(\mathbf{Z}_+;Y)$ *and* $\bar{\pi}_+\ell^2(\mathbf{Z};Y)$ *are identified in (1.39).*

It easy to see that the sets of Definition 29 are vector spaces. We throughout use the ℓ^2-topology on $\operatorname{dom}(\underline{\mathcal{B}})$, $\operatorname{dom}(\underline{\mathcal{D}})$ and $\operatorname{dom}(\underline{\mathcal{D}}\bar{\pi}_+)$. The set $\operatorname{dom}(\underline{\mathcal{D}}\pi_j)$ has the topology of U. This gives all these spaces an inner product space structure. In $\operatorname{dom}(\underline{\mathcal{C}})$ we use the topology of H, but later we introduce a stronger (inner product) topology there.

Clearly the vector space $\operatorname{dom}(\underline{\mathcal{B}})$ is dense in $\ell^2(\mathbf{Z}_-;U)$, and it does not depend on the structure of the DLS Φ in any way. The other spaces $\operatorname{dom}(\underline{\mathcal{C}})$, $\operatorname{dom}(\underline{\mathcal{D}})$, $\operatorname{dom}(\underline{\mathcal{D}}\bar{\pi}_+)$ and $\operatorname{dom}(\underline{\mathcal{D}}\pi_j)$ need not be dense, and for DLSs "unstable enough" they even might be empty. Trivially, the kernels of $\mathcal{C}$ and $\mathcal{D}$ are always in the respective domains. If there is nothing else, then we say that the domains in question are trivial. The following definition is to be expected.

Definition 30. *Let* $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ *be a DLS. The following restriction mappings are defined*

$$\begin{aligned}
&\underline{\mathcal{B}} := \mathcal{B}|\operatorname{dom}(\underline{\mathcal{B}}), \quad \underline{\mathcal{C}} := \mathcal{C}|\operatorname{dom}(\underline{\mathcal{C}}), \\
&\underline{\mathcal{D}} := \mathcal{D}|\operatorname{dom}(\underline{\mathcal{D}}), \quad \underline{\mathcal{D}}\bar{\pi}_+ := \mathcal{D}|\operatorname{dom}(\underline{\mathcal{D}}\bar{\pi}_+), \\
&\underline{\mathcal{D}}\pi_j := \mathcal{D}|\operatorname{dom}(\underline{\mathcal{D}}\pi_j) \quad \text{for all} \quad j \in \mathbf{Z}.
\end{aligned}$$

These mappings are as follows: $\underline{\mathcal{B}}$ *is the topological controllability map,* $\underline{\mathcal{C}}$ *is the topological observability map and* $\underline{\mathcal{D}}$ *is the topological I/O map of* Φ. $\underline{\mathcal{D}}\bar{\pi}_+$ *is the causal Toeplitz operator of* $\underline{\mathcal{D}}$. *The operators* $\underline{\mathcal{D}}\bar{\pi}_j$ *are the impulse response operator of* Φ. *The sets* $\operatorname{dom}(\underline{\mathcal{B}})$, $\operatorname{dom}(\underline{\mathcal{C}})$, $\operatorname{dom}(\underline{\mathcal{D}})$, $\operatorname{dom}(\underline{\mathcal{D}}\bar{\pi}_+)$ *and* $\operatorname{dom}(\underline{\mathcal{D}}\pi_j)$, *as introduced in Definition 29, are the domains of the respective operators.*

The ranges of these operators are defined in the natural way; e.g. $\operatorname{range}(\underline{\mathcal{B}}) := \underline{\mathcal{B}}\operatorname{dom}(\underline{\mathcal{B}})$, $\operatorname{range}(\underline{\mathcal{D}}) := \underline{\mathcal{D}}\operatorname{dom}(\underline{\mathcal{D}})$, $\operatorname{range}(\underline{\mathcal{D}}\bar{\pi}_+) := \underline{\mathcal{D}}\bar{\pi}_+\operatorname{dom}(\underline{\mathcal{D}}\bar{\pi}_+)$ and so on. We shall make a notational difference between $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$, $\mathcal{D}\bar{\pi}_+$ and $\underline{\mathcal{B}}$, $\underline{\mathcal{C}}$, $\underline{\mathcal{D}}$, $\underline{\mathcal{D}}\bar{\pi}_+$ only in this section of this book.

The bad news is that various topological pathologies can occur as far as a general DLS Φ is concerned. Most of this section handles the cases when we have good news. We start by showing that three of the operators in Definition 30 are closed. In particular, the closed graph property of $\underline{\mathcal{C}}$ will be used in Section 1.8.

Lemma 31. *Let $\Phi = \left[\begin{smallmatrix} \mathcal{A}^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ be a DLS. Then the operators $\underline{\mathcal{C}}$, $\underline{\mathcal{D}}\bar{\pi}_+$ and $\underline{\mathcal{D}}\pi_j$ are closed for all $j \in \mathbf{Z}$.*

Proof. We prove only the claim for the Toeplitz operator $\underline{\mathcal{D}}\bar{\pi}_+$. The proofs of the other two claims are analogous. Let $\mathrm{dom}\,(\underline{\mathcal{D}}\bar{\pi}_+) \ni \tilde{u}_j \to \tilde{u} \in \ell^2(\mathbf{Z}_+;U)$ be a convergent sequence in the norm of $\ell^2(\mathbf{Z};U)$, such that

$$\underline{\mathcal{D}}\bar{\pi}_+\tilde{u}_j \to \tilde{y} \in \ell^2(\mathbf{Z}_+;Y)$$

in the norm of $\ell^2(\mathbf{Z}_+;Y)$. We show that $\tilde{u} \in \mathrm{dom}\,(\underline{\mathcal{D}}\bar{\pi}_+)$ and $\underline{\mathcal{D}}\bar{\pi}_+\tilde{u} = \tilde{y}$, which proves the closed graph property for $\underline{\mathcal{D}}\bar{\pi}_+$.

For each $k \geq 0$ we have

$$\pi_k\underline{\mathcal{D}}\bar{\pi}_+\tilde{u}_j \to \pi_k\tilde{y} \quad \text{as} \quad j \to \infty \tag{1.42}$$

in the norm of Y, with $\mathrm{range}\,(\pi_k)$ and Y identified. On the other hand, we have

$$\begin{aligned} \pi_k\underline{\mathcal{D}}\bar{\pi}_+\tilde{u}_j &= \pi_k\mathcal{D}\bar{\pi}_+\tilde{u}_j \\ &= \pi_k\mathcal{D}\pi_{[0,k]}\tilde{u}_j \to \pi_k\mathcal{D}\pi_{[0,k]}\tilde{u} = \pi_k\mathcal{D}\bar{\pi}_+\tilde{u} \quad \text{as} \quad j \to \infty \end{aligned} \tag{1.43}$$

in the norm of Y, because $\pi_k\mathcal{D}\pi_{[0,k]}$ is a bounded operator on $\ell^2(\mathbf{Z}_+;U)$. The boundedness follows because $\pi_k\mathcal{D}\pi_{[0,k]}\tilde{u}$ is given by the finite sum

$$\pi_k\mathcal{D}\tilde{u} = \sum_{i=0}^{k-1} CA^iBu_{k-i-1} + Du_k$$

for all $j \in \mathbf{Z}$ and $\tilde{u} = \{u_j\}_{j\geq 0} \in Seq_+(U)$, by Proposition 3. Here A, B, C and D are the generating operators of Φ, and $\mathrm{range}\,(\pi_k)$ has been identified with Y.

Now equations (1.42) and (1.43) imply, by the uniqueness of the limit in Y, that $\pi_k\mathcal{D}\bar{\pi}_+\tilde{u} = \pi_k\tilde{y}$ for all $k \geq 0$, or equivalently $\mathcal{D}\bar{\pi}_+\tilde{u} = \tilde{y}$. But then, because $\tilde{y} \in \ell^2(\mathbf{Z}_+;Y)$, we have $\tilde{u} \in \mathrm{dom}\,(\underline{\mathcal{D}})$ and $\tilde{y} = \underline{\mathcal{D}}\bar{\pi}_+\tilde{u}$. This completes the proof of the lemma. □

The fundamental stability notions for DLSs are as follows.

Definition 32. *Let $\Phi = \left[\begin{smallmatrix} \mathcal{A}^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ be a DLS.*

(i) If $\mathrm{dom}\,(\underline{\mathcal{D}}\bar{\pi}_+) = \ell^2(\mathbf{Z}_+;U)$, then Φ is I/O stable.

(ii) If $\mathrm{dom}\,(\underline{\mathcal{D}}\pi_0) = U$, then Φ is strongly H^2 (Hardy 2) stable.

(iii) *If* $\underline{\mathcal{B}} \in \mathcal{L}(\operatorname{dom}(\underline{\mathcal{B}}), H)$*, then* Φ *is input stable.*

(iv) *If* $\operatorname{dom}(\underline{\mathcal{C}}) = H$*, then* Φ *is output stable.*

(v) *If all the above holds, and the semigroup generator* A *of* Φ *is power bounded, then* Φ *is stable.*

(vi) *If* Φ *is stable, and the semigroup generator* A *of* Φ *is strongly stable, then* Φ *is strongly stable.*

Clearly I/O stability implies strong H^2 stability. An I/O stable Toeplitz operator $\underline{\mathcal{D}}\bar{\pi}_+$ is a bounded linear operator from $\operatorname{dom}(\underline{\mathcal{D}}\bar{\pi}_+) = \ell^2(\mathbf{Z}_+; U)$ into $\ell^2(\mathbf{Z}_+; Y)$ because a closed operator with complete (closed) domain is bounded, by the Closed Graph Theorem (see [79, Theorem 2.15]). In this case, the operator norm of $\underline{\mathcal{D}}\bar{\pi}_+$ is given by

$$\begin{aligned}&||\underline{\mathcal{D}}\bar{\pi}_+||_{\ell^2(\mathbf{Z}_+;U)\to\ell^2(\mathbf{Z}_+;Y)}\\ &:= \sup\{||\underline{\mathcal{D}}\bar{\pi}_+\tilde{u}||_{\ell^2(\mathbf{Z}_+;Y)} \mid \tilde{u} \in \ell^2(\mathbf{Z}_+;U)\, ||\tilde{u}||_{\ell^2(\mathbf{Z}_+;U)} = 1\}.\end{aligned}$$

If the $\ell^2(\mathbf{Z}_+; U)$ norm of the input sequence is regarded a a measure of energy, then $||\underline{\mathcal{D}}\bar{\pi}_+||_{\ell^2(\mathbf{Z}_+;U)\to\ell^2(\mathbf{Z}_+;Y)}$ is the energy gain of $\underline{\mathcal{D}}\bar{\pi}_+$. If the Hilbert spaces U and Y are separable, then the I/O stability is equivalent with the requirement that the transfer function $\mathcal{D}(z)$ is bounded in $\mathbf{D}$, see Proposition 55.

By analogous considerations, an H^2 stable DLS has a bounded impulse response operator $\mathcal{D}\pi_0 : U \to \ell^2(\mathbf{Z}_+; U)$, with the natural identification of $\operatorname{range}(\pi_0)$ and U. The basic properties of H^2 stable DLSs are given in the Lemmas 33 and 35.

Lemma 33. *Let* $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D}\end{smallmatrix}\right] = \left(\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right)$ *be a DLS. Enumerate the statements as follows:*

(i) $\sum_{j\geq 0} ||CA^jB||^2 < \infty$,

(ii) Φ *is (strongly)* H^2 *stable,*

(iii) $\ell^1(\mathbf{Z}_+; U) \subset \operatorname{dom}(\underline{\mathcal{D}}\bar{\pi}_+)$ *and* $\underline{\mathcal{D}}\bar{\pi}_+ \in \mathcal{L}(\ell^1(\mathbf{Z}_+; U), \ell^2(\mathbf{Z}_+; Y))$,

(iv) $\underline{\mathcal{D}}\bar{\pi}_+$ *is a densely defined closed operator on* $\ell^2(\mathbf{Z}_+; U)$.

Then (i) $\Rightarrow$ *(ii)* $\Rightarrow$ *(iii)* $\Rightarrow$ *(iv).*

Proof. The first implication (i) $\Rightarrow$ (ii) is trivial because

$$||\mathcal{D}\pi_0\tilde{u}||^2_{\ell^2(\mathbf{Z}_+;U)} = ||Du_0||^2 + \sum_{j\geq 0} ||CA^jBu_0||^2$$

for all $\tilde{u} = \{u_j\}_{j\geq 0} \in Seq_+(U)$. Here A, B, C and D are the generating operators of Φ.

To prove the implication (ii) $\Rightarrow$ (iii), we note that any $\tilde{u} \in \ell^1(\mathbf{Z}_+;U)$ can be written as $\tilde{u} = \sum_{j\geq 0} \pi_j \tilde{u}$, where the sum converges in the norm of $\ell^1(\mathbf{Z}_+;U)$. The triangle inequality gives

$$||\underline{\mathcal{D}}\bar{\pi}_+\tilde{u}||_{\ell^2(\mathbf{Z}_+;Y)} \leq \sum_{j\geq 0} ||\underline{\mathcal{D}}\pi_j\tilde{u}||_{\ell^2(\mathbf{Z}_+;Y)}$$
$$\leq \sum_{j\geq 0} ||\underline{\mathcal{D}}\pi_j||_{U\to\ell^2(\mathbf{Z}_+;Y)} \, ||\pi_j\tilde{u}||_U = ||\underline{\mathcal{D}}\pi_0||_{U\to\ell^2(\mathbf{Z}_+;Y)} \sum_{j\geq 0} ||\pi_j\tilde{u}||_U,$$

where the last equality is by the shift invariance of $\underline{\mathcal{D}}$. Now $\sum_{j\geq 0} ||\pi_j\tilde{u}||_U =: ||\tilde{u}||_{\ell^1(\mathbf{Z}_+;U)}$ and the claim follows. The final implication (iii) $\Rightarrow$ (iv) is trivial because $\ell^1(\mathbf{Z}_+;U)$ is dense in $\ell^2(\mathbf{Z}_+;Y)$. □

The condition (i) of Lemma 33 can be called the uniform H^2 stability. The strong H^2 stability is characterized by the equivalent conditions of Lemma 35. However, we need a preliminary result.

Proposition 34. *Let $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ be a DLS. Then $A \operatorname{dom}(\underline{\mathcal{C}}) \subset \operatorname{dom}(\underline{\mathcal{C}})$.*

Proof. Let $x \in \operatorname{dom}(\underline{\mathcal{C}})$ be arbitrary. Then $\mathcal{C}x \in \ell^2(\mathbf{Z}_+;Y) \subset Seq_+(Y)$ by Definition 29, and $\mathcal{C}Ax = \bar{\pi}_+\tau^*\mathcal{C}x \in Seq_+(Y)$, by claim (iii) of Lemma 12. But now, because both $\bar{\pi}_+$ and τ^* are of norm 1 in $\ell^2(\mathbf{Z};Y)$, it follows that $\mathcal{C}Ax \in \ell^2(\mathbf{Z}_+;Y)$. By Definition 29, $Ax \in \operatorname{dom}(\underline{\mathcal{C}})$, and thus $A \operatorname{dom}(\underline{\mathcal{C}}) \subset \operatorname{dom}(\underline{\mathcal{C}})$. □

Lemma 35. *Let $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right] = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ be a DLS. Then the following are equivalent:*

- *(i)* $BU \subset \operatorname{dom}(\underline{\mathcal{C}})$,
- *(ii)* $\operatorname{range}(\underline{\mathcal{B}}) \subset \operatorname{dom}(\underline{\mathcal{C}})$,
- *(iii) Φ is strongly H^2 stable.*

Proof. Assume now that claim (i) holds. Let $\tilde{u} = \{u_j\}_{j\geq 0} \in Seq_+(U)$ be arbitrary. Denote the corresponding trajectory by $x_j(0,\tilde{u}) = \mathcal{B}\tau^{*j}\tilde{u} \in \operatorname{dom}(\underline{\mathcal{C}})$ for $j \geq 0$. Clearly $x_0(0,\tilde{u}) = 0 \in \operatorname{dom}(\underline{\mathcal{C}})$. Assume that it has already been proved that $x_j(0,\tilde{u}) \in \operatorname{dom}(\underline{\mathcal{C}})$ for some $j \geq 0$. Now $x_{j+1}(0,\tilde{u}) = Ax_j(0,\tilde{u}) + Bu_j$, where $Ax_j(0,\tilde{u}) \in \operatorname{dom}(\underline{\mathcal{C}})$ by Proposition 34. But $Bu_j \in \operatorname{dom}(\underline{\mathcal{C}})$ because claim (i) is assumed to hold. Because $\operatorname{dom}(\underline{\mathcal{C}})$ is a vector space, it follows that $x_{j+1}(0,\tilde{u}) \in \operatorname{dom}(\underline{\mathcal{C}})$. We have now shown that

$$\mathcal{B}\pi_-\tau^{*j}\tilde{u} \in \operatorname{dom}(\underline{\mathcal{C}}),$$

for all $j \geq 0$ and for all $\tilde{u} \in Seq_+(U)$. But clearly

$$\mathrm{dom}\,(\underline{\mathcal{B}}) := Seq_-(U) = \{\pi_- \tau^{*j} \tilde{u} \mid j \geq 0,\ \tilde{u} \in Seq_+(U)\},$$

and claim (ii) follows. It is a triviality that (ii) implies (i). Thus claims (i) and (ii) are equivalent.

Assume that (i) holds, and let $\tilde{u} = \{u_j\}_{j\geq 0} \in Seq(U)$ be arbitrary. Because $BU \subset \mathrm{dom}\,(\mathcal{C})$, then $\mathcal{C}Bu_0 \in \ell^2(\mathbf{Z}_+;Y)$ for all $u_0 \in U$. But then $\mathcal{D}\pi_0\tilde{u} = D\pi_0\tilde{u} + \tau\mathcal{C}Bu_0 \in \ell^2(\mathbf{Z}_+;Y)$ for all $\tilde{u} = \{u_j\}_{j\geq 0}$. Because the component $u_0 \in U$ is arbitrary, it follows that $\mathrm{dom}\,(\mathcal{D}\pi_0) = U$ and claim (iii) follows.

It remains to prove the implication (iii) $\Rightarrow$ (i). Let $\tilde{u} = \{u_j\}_{j\geq 0} \in \ell^2(\mathbf{Z}_+;U)$ be arbitrary. Then

$$\mathcal{D}\pi_0\tilde{u} = D\pi_0\tilde{u} + \tau\mathcal{C}Bu_0. \tag{1.44}$$

Trivially $D\pi_0\tilde{u} \in \ell^2(\mathbf{Z}_+;Y)$, and $\mathcal{D}\pi_0\tilde{u} \in \ell^2(\mathbf{Z}_+;Y)$ by the assumed claim (iii). It follows that $\tau\mathcal{C}Bu_0 \in \ell^2(\mathbf{Z};Y)$ and equivalently $\mathcal{C}Bu_0 \in \ell^2(\mathbf{Z};Y)$. But this implies that $Bu_0 \in \mathrm{dom}\,(\underline{\mathcal{C}})$. Because $u_0 \in U$ is arbitrary, claim (i) follows. □

Now we have dealt with the strong H^2 stability, and we proceed to consider the I/O stable DLSs. Two extensions of the Toeplitz operator $\underline{\mathcal{D}}\bar{\pi}_+$ are given in the following lemma.

Lemma 36. *Let $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ be an I/O stable DLS, and the operators $\underline{\mathcal{D}}$ and $\underline{\mathcal{D}}\bar{\pi}_+$ as in Definition 30.*

(i) The domain of $\underline{\mathcal{D}}$ satisfies

$$\mathrm{dom}\,(\underline{\mathcal{D}}) = \ell^2(\mathbf{Z};U) \cap Seq(U), \tag{1.45}$$

and $\underline{\mathcal{D}}\bar{\pi}_+\tilde{u} = \underline{\mathcal{D}}\tilde{u}$ for all $\tilde{u} \in \ell^2(\mathbf{Z}_+;U)$. The domain $\mathrm{dom}\,(\underline{\mathcal{D}})$ is dense in $\ell^2(\mathbf{Z};U)$, and

$$\underline{\mathcal{D}}\tilde{u} = \lim_{j\to\infty} \tau^{*j}\underline{\mathcal{D}}\bar{\pi}_+\tau^j\tilde{u} \tag{1.46}$$

for all $\tilde{u} \in \mathrm{dom}\,(\underline{\mathcal{D}})$. Furthermore,

$$||\underline{\mathcal{D}}||_{\mathrm{dom}(\underline{\mathcal{D}})\to\ell^2(\mathbf{Z}_+;Y)} = ||\underline{\mathcal{D}}\bar{\pi}_+||_{\ell^2(\mathbf{Z}_+;U)\to\ell^2(\mathbf{Z}_+;Y)}. \tag{1.47}$$

(ii) The topological I/O map $\underline{\mathcal{D}}$ has a unique bounded extension to all of $\ell^2(\mathbf{Z};U)$, denoted by $\bar{\underline{\mathcal{D}}}$. The extension $\bar{\underline{\mathcal{D}}}$ satisfies

$$||\bar{\underline{\mathcal{D}}}||_{\ell^2(\mathbf{Z};U)\to\ell^2(\mathbf{Z};Y)} = ||\underline{\mathcal{D}}\bar{\pi}_+||_{\ell^2(\mathbf{Z}_+;U)\to\ell^2(\mathbf{Z}_+;Y)}.$$

Furthermore, $\bar{\underline{\mathcal{D}}}$ is shift-invariant and causal; i.e. $\bar{\underline{\mathcal{D}}}\tau = \tau\bar{\underline{\mathcal{D}}}$ and $\pi_-\bar{\underline{\mathcal{D}}}\bar{\pi}_+ = 0$.

Proof. It is easy to see that $\ell^2(\mathbf{Z};U) \cap Seq(U) = \{\tau^{*j}\tilde{u} \mid j \geq 0,\ \tilde{u} \in \ell^2(\mathbf{Z}_+;U)\}$. If $\tilde{u} \in \{\tau^{*j}\tilde{u} \mid j \geq 0,\ \tilde{u} \in \ell^2(\mathbf{Z}_+;U)\}$, then there is a $j \geq 0$ such that $\tau^j\tilde{u} \in \ell^2(\mathbf{Z}_+;U)$. Because Φ is I/O stable, it follows that $\tau^j\mathcal{D}\tilde{u} = \mathcal{D}\tau^j\tilde{u} = \underline{\mathcal{D}}\bar{\pi}_+\tau^j\tilde{u} \in \ell^2(\mathbf{Z}_+;Y)$. Because τ^* is unitary, $\mathcal{D}\tilde{u} = \tau^{*j}\underline{\mathcal{D}}\bar{\pi}_+\tau^j\tilde{u} \in \ell^2(\mathbf{Z};Y)$. Because trivially $\tilde{u} \in Seq(U)$, it follows that $\tilde{u} \in \mathrm{dom}\,(\underline{\mathcal{D}})$. We have now shown that $\ell^2(\mathbf{Z};U) \cap Seq(U) \subset \mathrm{dom}\,(\underline{\mathcal{D}})$. The converse inclusion $\mathrm{dom}\,(\underline{\mathcal{D}}) \subset \ell^2(\mathbf{Z};U) \cap Seq(U)$ is a triviality, and equation (1.45) follows, together with the density of domain $\mathrm{dom}\,(\underline{\mathcal{D}})$ in $\ell^2(\mathbf{Z};U)$. Because always $\mathrm{dom}\,(\underline{\mathcal{D}}\bar{\pi}_+) \subset \mathrm{dom}\,(\underline{\mathcal{D}}) \subset Seq(U)$ and the operators $\underline{\mathcal{D}}\bar{\pi}_+$ and $\underline{\mathcal{D}}$ are restrictions of the I/O map $\mathcal{D}$ to the respective domains, it follows that $\underline{\mathcal{D}}\bar{\pi}_+\tilde{u} = \underline{\mathcal{D}}\tilde{u}$ for all $\tilde{u} \in \ell^2(\mathbf{Z}_+;U) = \mathrm{dom}\,(\underline{\mathcal{D}}\bar{\pi}_+)$.

In order to prove equation (1.46), let $\tilde{u} \in \mathrm{dom}\,(\underline{\mathcal{D}}) = \ell^2(\mathbf{Z};U) \cap Seq(U)$ be arbitrary. Then there is a $j' \geq 0$ such that $\tau^{j'}\tilde{u} \in \ell^2(\mathbf{Z}_+;U)$ and

$$\underline{\mathcal{D}}\tilde{u} = \mathcal{D}\tilde{u} = \tau^{*(j'+j)}\mathcal{D}\tau^{j'+j}\tilde{u} = \tau^{*(j'+j)}\underline{\mathcal{D}}\bar{\pi}_+\tau^{j'+j}\tilde{u}$$

for all $j \geq 0$. But then, the $\{\tau^{*(j'+j)}\underline{\mathcal{D}}\bar{\pi}_+\tau^{j'+j}\tilde{u}\}_{j\geq 0}$ is a constant sequence, and equation (1.46) follows. The equality of the norms (1.47) follows from the unitarity of the shift τ and equation (1.46). The unique bounded extension $\bar{\underline{\mathcal{D}}}$ of $\underline{\mathcal{D}}$, of the same norm, exists by [42, Theorem II.3.1]. An easy limit argument is required to prove the causality and shift-invariance of the extension. We consider this lemma to be proved. □

In Lemma 9, it is shown that general shift-invariant causal mappings on $Seq(U)$ can be regarded as I/O maps of DLSs, provided that a certain growth bound, related to the well-posedness of DLSs, is satisfied. The analogous results holds also for the bounded, shift-invariant and causal operators on $\ell^2(\mathbf{Z};U)$.

Lemma 37. *Let $\mathcal{T} : \ell^2(\mathbf{Z};U) \to \ell^2(\mathbf{Z};Y)$ be a bounded operator, satisfying $\mathcal{T}\tau = \tau\mathcal{T}$ and $\pi_-\mathcal{T}\bar{\pi}_+ = 0$. Then there is an I/O stable DLS $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$, such that $\mathcal{T} = \bar{\underline{\mathcal{D}}}$ on $\ell^2(\mathbf{Z};U)$. Here $\bar{\underline{\mathcal{D}}}$ is the (extended topological) I/O map, given in claim (ii) of Lemma 36.*

Proof. Let $\tilde{u} = \{u_j\}_{j\in\mathbf{Z}} \in \ell^2(\mathbf{Z};U)$ be arbitrary. Then, by linearity and causality of $\mathcal{T}$, we have $\mathcal{T}\pi_0\tilde{u} = \{y_j(u_0)\}_{j\geq 0} \in \ell^2(\mathbf{Z}_+;Y)$, where the component mappings

$$T_j : u_0 \mapsto y_j(u_0) \quad \text{for} \quad j \geq 0$$

are linear from U to Y. Now, for arbitrary $\tilde{u}$, we have

$$\begin{aligned} &||T_j u_0||_Y = ||\pi_j\mathcal{T}\pi_0\tilde{u}||_{\ell^2(\mathbf{Z}_+;Y)} \\ &\leq ||\mathcal{T}\pi_0\tilde{u}||_{\ell^2(\mathbf{Z}_+;Y)} \leq ||\mathcal{T}||_{\ell^2(\mathbf{Z};U)\to\ell^2(\mathbf{Z};Y)}||u||_U. \end{aligned}$$

Thus $T_j \in \mathcal{L}(U;Y)$ for all $j \geq 0$, and the family $\{T_j\}_{j\geq 0}$ is uniformly bounded by the norm of $\mathcal{T}$. By claim (ii) of Proposition 7, there is a unique shift-invariant and causal mapping $\mathcal{D} : Seq(U) \to Seq(Y)$ such that for arbitrary $\tilde{u} \in Seq(U)$ we have

$$\pi_j \mathcal{D}^{(n)} \tilde{u} \to \pi_j \mathcal{D} \tilde{u} \quad \text{as} \quad n \to \infty$$

in the norm of Y, where the causal shift-invariant operators are given by $\mathcal{D}^{(n)} := \sum_{j=0}^{n} T_j \tau^j$. By Lemma 9 and the uniform boundedness of the family $\{T_j\}_{j \geq 0}$, the mapping $\mathcal{D}$ is an I/O map of a DLS.

We proceed to show that $\mathcal{T}$ and $\mathcal{D}$ coincide on $\ell^2(\mathbf{Z}; U) \cap Seq(U)$. By the construction of $\mathcal{D}$, $\pi_j \mathcal{T} \pi_0 \tilde{u} = \pi_j \mathcal{D} \pi_0 \tilde{u}$ for any $\tilde{u} \in Seq(U)$ and $j \geq 0$. By the shift-invariance of both $\mathcal{T}$ and $\mathcal{D}$, $\pi_j \mathcal{T} \pi_k \tilde{u} = \pi_j \mathcal{D} \pi_k \tilde{u}$ for any $\tilde{u} \in Seq(U)$ and $j \geq k$. It now follows for any $\tilde{u} \in \ell^2(\mathbf{Z}; U) \cap Seq(U)$ and $j \in \mathbf{Z}$ that

$$\begin{aligned} \pi_j \mathcal{T} \tilde{u} &= \pi_j \mathcal{T} \pi_{[-\infty, j]} \tilde{u} = \pi_j \mathcal{T} \sum_{k \leq j} \pi_k \tilde{u} = \sum_{k \leq j} (\pi_j \mathcal{T} \pi_k \tilde{u}) \\ &= \sum_{k \leq j} (\pi_j \mathcal{D} \pi_k \tilde{u}) = \pi_j \mathcal{D} \tilde{u} \end{aligned}$$

where all the sums are finite. We conclude that $\mathcal{T} \tilde{u} = \mathcal{D} \tilde{u}$ for all $\tilde{u} \in \ell^2(\mathbf{Z}; U) \cap Seq(U)$. Because $\mathcal{T} : \ell^2(\mathbf{Z}; U) \to \ell^2(\mathbf{Z}; Y)$ is bounded, $\mathcal{D} \tilde{u} \in \ell^2(\mathbf{Z}; Y)$ for all $\tilde{u} \in \ell^2(\mathbf{Z}; U) \cap Seq(U)$. In particular, $\mathcal{D} \bar{\pi}_+ \tilde{u} \in \ell^2(\mathbf{Z}_+; Y)$ for all $\tilde{u} \in \ell^2(\mathbf{Z}_+; U)$, and thus $\operatorname{dom}(\underline{\mathcal{D}} \bar{\pi}_+) = \ell^2(\mathbf{Z}_+; U)$. We conclude that $\mathcal{D}$ is an I/O map of an I/O stable DLS.

By claim (i) of Lemma 36, $\operatorname{dom}(\underline{\mathcal{D}}) = \ell^2(\mathbf{Z}; U) \cap Seq(U)$ and $\underline{\mathcal{D}} : \operatorname{dom}(\underline{\mathcal{D}}) \to \ell^2(\mathbf{Z}; Y)$ is bounded. Because $\underline{\mathcal{D}}$ coincides with $\mathcal{T}$ on the dense set $\operatorname{dom}(\underline{\mathcal{D}})$, its unique bounded extension $\bar{\underline{\mathcal{D}}}$ must equal $\mathcal{T}$. This completes the proof. $\square$

By Lemma 37, we can use the expressions "I/O map of an I/O stable DLS" and "bounded shift-invariant causal operator on $\ell^2(\mathbf{Z}; U)$" synonymously.

In this section, we have introduced different notations for different versions of controllability, observability and I/O maps. In order to give precise definitions and rigorous proofs, this has been unavoidable. From now on, we work with considerably lighter notation. We consistently write $\mathcal{C}$ instead of $\underline{\mathcal{C}}$, and $\mathcal{B}$ instead of $\underline{\mathcal{B}}$. For the domains, we write $\operatorname{dom}(\mathcal{C})$ and $\operatorname{dom}(\mathcal{B})$ instead of $\operatorname{dom}(\underline{\mathcal{C}})$ and $\operatorname{dom}(\underline{\mathcal{B}})$. Ranges are defined by $\operatorname{range}(\mathcal{C}) = \mathcal{C}\operatorname{dom}(\mathcal{C})$ and $\operatorname{range}(\mathcal{B}) = \mathcal{C}\operatorname{dom}(\mathcal{B})$; i.e. these refer to the topological versions of $\mathcal{C}$ and $\mathcal{B}$.

The causal Toeplitz operators $\mathcal{D}\bar{\pi}_+ : Seq_+(U) \to Seq_+(Y)$ and $\underline{\mathcal{D}}\bar{\pi}_+ : \operatorname{dom}(\underline{\mathcal{D}}\bar{\pi}_+) \to \ell^2(\mathbf{Z}_+; Y)$ are both denoted by $\mathcal{D}\bar{\pi}_+$. The domain $\operatorname{dom}(\underline{\mathcal{D}}\bar{\pi}_+)$ is denoted by $\operatorname{dom}(\mathcal{D}\bar{\pi}_+)$ and $\operatorname{range}(\mathcal{D}\bar{\pi}_+) = \mathcal{D}\bar{\pi}_+\operatorname{dom}(\mathcal{D}\bar{\pi}_+)$. Analogously, the impulse response operators $\mathcal{D}\pi_0 : \operatorname{range}(\pi_0) = Seq_+(Y)$ and $\underline{\mathcal{D}}\bar{\pi}_0 : \operatorname{range}(\pi_0) \to \ell^2(\mathbf{Z}_+; Y)$ are both denoted by $\mathcal{D}\pi_0$.

If $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}\mathcal{T}^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ is an I/O stable DLS, then both the linear mappings $\mathcal{D} : Seq(U) \to Seq(Y)$ and $\underline{\bar{\mathcal{D}}} : \ell^2(\mathbf{Z};U) \to \ell^2(\mathbf{Z};Y)$ are called I/O maps of Φ and denoted by $\mathcal{D}$. This should not cause confusion, because $\mathcal{D}$ and $\underline{\bar{\mathcal{D}}}$ coincide on $\ell^2(\mathbf{Z};U) \cap Seq(U) = \mathrm{dom}\,(\underline{\mathcal{D}})$, and are thus unique extensions of the restriction: $\mathcal{D} : Seq(U) \to Seq(Y)$ by causality and $\underline{\bar{\mathcal{D}}} : \ell^2(\mathbf{Z};U) \to \ell^2(\mathbf{Z};Y)$ by continuity. Both the operators are also (in their respective spaces unique) shift-invariant and causal extensions of the Toeplitz operator $\mathcal{D}\bar{\pi}_+ : \ell^2(\mathbf{Z}_+;U) \to \ell^2(\mathbf{Z}_+;Y)$.

1.8 Graph topology of the state space

Let $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ be a strongly H^2 stable DLS. In this section, we study certain topologies of the state space H of Φ in detail. If we are only interested in the I/O map of Φ, the vector space $\mathrm{dom}\,(\mathcal{C})$ alone is the essential part of the state space H, see Lemma 35. Clearly, $\mathcal{D}\bar{\pi}_+\tilde{u}$ can be computed for all $\tilde{u} \in Seq_+(U)$, by using the state space realization Φ, without ever referring to any vector in $H \setminus \mathrm{dom}\,(\mathcal{C})$. However, $\mathrm{dom}\,(\mathcal{C})$ need not be closed in the norm of H, and thus it cannot generally be used as a (restricted) state space of a DLS. To deal with this problem, we can do two things. Firstly, we can replace $\mathrm{dom}\,(\mathcal{C})$ by its closure in H. In this case, $\mathcal{C} : \overline{\mathrm{dom}\,(\mathcal{C})} \to \ell^2(\mathbf{Z}_+;U)$ becomes a possibly unbounded, densely defined closed operator which is still an observability map of a DLS. Secondly, we can construct a stronger Hilbert norm into the vector space $\mathrm{dom}\,(\mathcal{C})$ which makes $\mathcal{C} : \mathrm{dom}\,(\mathcal{C}) \to \ell^2(\mathbf{Z}_+;U)$ not only bounded, but an observability map of an output stable DLS ϕ^{g}, too. Moreover, we have the equality of the I/O-maps $\mathcal{D}\bar{\pi}_+ = \mathcal{D}_{\phi^{\mathrm{g}}}\bar{\pi}_+$ on $Seq_+(U)$. The closed graph property of $\mathcal{C}$ is the key in the construction of the new Hilbert space topology, see [42, Chapter 2].

Let us consider the limits of state trajectories. Suppose $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ be a strongly H^2 stable DLS. Because $\mathrm{range}\,(\mathcal{B}) \subset \mathrm{dom}\,(\mathcal{C})$, it follows that $\overline{\mathrm{range}\,(\mathcal{B})} \subset \overline{\mathrm{dom}\,(\mathcal{C})}$, where the closure is taken in the norm of H. Now $\overline{\mathrm{dom}\,(\mathcal{C})}$ is a Hilbert subspace of H, and it is tempting use it as a a state space of a modified version of the DLS Φ. We want to see how the possible limits of trajectories behave under this restriction. Clearly, if $\tilde{u} \in Seq_+(U)$ is such that $x_\infty(0,\tilde{u}) := \lim_{j\to\infty} \mathcal{B}\tau^{*j}\tilde{u}$ exists, then $x_\infty(0,\tilde{u}) \in \overline{\mathrm{dom}\,(\mathcal{C})}$. If, in addition, $\tilde{u} \in \ell^1(\mathbf{Z}_+;U)$, then in fact $x_\infty(0,\tilde{u}) \in \ker\,(\mathcal{C})$. Then we have for all $j \geq 1$

$$\mathcal{C}x_j(0,\tilde{u}) = \tau^{*j}\pi_{[j,\infty]}\mathcal{D}\pi_{[0,j-1]}\tilde{u} = \tau^{*j}\pi_{[j,\infty]}\left(\mathcal{D}\bar{\pi}_+\tilde{u} - \mathcal{D}\pi_{[j,\infty]}\tilde{u}\right).$$

Because $\pi_{[j,\infty]}\tilde{u} \to 0$ in $\ell^1(\mathbf{Z}_+;U)$, it follows that $\mathcal{D}\pi_{[j,\infty]}\tilde{u} \to 0$ in $\ell^2(\mathbf{Z}_+;U)$ because $\mathcal{D}\bar{\pi}_+ : \ell^1(\mathbf{Z}_+;U) \to \ell^2(\mathbf{Z}_+;U)$ is bounded, by claim (iii) of Lemma 33. Because $\mathcal{D}\bar{\pi}_+\tilde{u} \in \ell^2(\mathbf{Z}_+;U)$ by claim (iii) of Lemma 33, also $\pi_{[j,\infty]}\mathcal{D}\bar{\pi}_+\tilde{u} \to 0$. Thus $\lim_{j\to\infty} \mathcal{C}x_j(0,\tilde{u}) = 0$ for all $\tilde{u} \in \ell^1(\mathbf{Z}_+;U)$ for which the limit $x_\infty(0,\tilde{u})$ exists. Because $\mathcal{C}$ is closed by Lemma 31, it follows that $\mathcal{C}x_\infty(0,\tilde{u}) = 0$ and $x_\infty(0,\tilde{u}) \in \ker\,(\mathcal{C})$. Clearly, the same reasoning could have been made for I/O stable Φ and $\tilde{u} \in \ell^2(\mathbf{Z}_+;U)$.

We assume that $\overline{\mathrm{dom}\,(\mathcal{C})} = H$ for the rest of this section. Then $\mathcal{C}$ is a densely defined closed operator in H. Based on the above discussion, there is no great loss of generality. Because Φ is strongly H^2 stable, we have $\mathrm{dom}\,(\mathcal{C}) \neq 0$. If $\mathcal{D} = 0$, then we have $\mathrm{range}\,(\mathcal{B}) \subset \ker\,(\mathcal{C})$. In particular, this is the case when $\mathrm{dom}\,(\mathcal{C}) = \ker\,(\mathcal{C})$ and $\ker\,(\mathcal{C}) = \overline{\mathrm{dom}\,(\mathcal{C})} = H$ because the null space of a closed operator is closed.

We proceed to consider the natural Hilbert space norm of $\operatorname{dom}(\mathcal{C})$. The necessary technical tools are Definition 38 and Lemmas 39, 40 and 41.

Definition 38. *Let* $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ *be a DLS.*

(i) The inner product $\langle\ ,\ \rangle_E$ *in* $\operatorname{dom}(\mathcal{C})$ *is defined by*

$$\langle x, y\rangle_E := \langle x, y\rangle_H + \langle \mathcal{C}x, \mathcal{C}y\rangle_{\ell^2(\mathbf{Z}_+;Y)}.$$

(ii) By E *denote the vector space* $\operatorname{dom}(\mathcal{C})$*, when equipped with the inner product* $\langle,\rangle_E$*.*

It is easy to check that $\langle\ ,\ \rangle_E$ is an inner product in $\operatorname{dom}(\mathcal{C})$. As usual, the inner product of E provides the corresponding norm $||x||_E = \langle x, x\rangle_E$. This norm of E is called the graph norm of the observability map. Clearly,

$$||x||_H \leq ||x||_E \quad \text{for all} \quad x \in E,$$

and the equality holds if and only if $x \in \ker(\mathcal{C})$. The set $\ker(\mathcal{C})$ is a closed subspace of both E and H. The following consequences of the closed graph property of $\mathcal{C}$ are basic.

Lemma 39. *Let* $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ *be a DLS, and let* E *be the inner product space given in Definition 38. Then*

(i) E *is a Hilbert space,*

(ii) $\mathcal{C} \in \mathcal{L}(E; \ell^2(\mathbf{Z}_+; Y))$ *and* $||\mathcal{C}||_{E\to\ell^2(\mathbf{Z}_+;Y)} \leq 1$,

(iii) $C := \pi_0\mathcal{C} \in \mathcal{L}(E; Y)$ *and* $||C||_{E\to Y} \leq 1$.

Proof. In order to show claim (i), it is sufficient to show that E is complete. Let $\{x_j\} \subset \operatorname{dom}(\mathcal{C}) = E$ be a Cauchy sequence in E. Because the norm of E majorizes the norm of H, it follows that $\{x_j\}$ is also a Cauchy sequence in the topology of H. Similarly, the sequence $\{\mathcal{C}x_j\}$ is a Cauchy sequence in $\ell^2(\mathbf{Z}_+; U)$. It follows that the sequence $\{x_j\}$ has a limit $x \in H$ and $\{\mathcal{C}x_j\}$ has a limit $\tilde{y} \in \ell^2(\mathbf{Z}_+; Y)$, by the completeness of both H and $\ell^2(\mathbf{Z}_+; Y)$. Because $\mathcal{C}$ is closed by Lemma 31, it follows that $x \in \operatorname{dom}(\mathcal{C}) = E$ and $y = \mathcal{C}x$. Now we can write

$$||x_j - x||_E^2 = ||x_j - x||_H^2 + ||\mathcal{C}x_j - \mathcal{C}x||_{\ell^2(\mathbf{Z}_+;Y)}^2 \to 0$$

as $j \to \infty$. Thus the arbitrary Cauchy sequence $\{x_j\}$ has a limit $x \in E$, and the completeness of E follows. Claim (ii) follows because

$$||\mathcal{C}x||_{\ell^2(\mathbf{Z}_+;Y)}^2 < ||x||_H^2 + ||\mathcal{C}x||_{\ell^2(\mathbf{Z}_+;Y)}^2 = ||x||_E^2$$

for all $x \in E$, $x \neq 0$. Claim (iii) follows from claim (ii), because π_0 is of norm 1. The proof of this lemma is now complete. □

Under the same assumptions, we can also say several facts about the semigroup generator A when restricted into E.

Lemma 40. *Introduce the same notations as in Definition 38. Then*

(i) *A maps E into itself, and*

(ii) $A|E \in \mathcal{L}(E)$.

(iii) *If A is a power bounded of $\mathcal{L}(H)$, then $A|E$ is a power bounded element of $\mathcal{L}(E)$. Furthermore,*

$$sup_{j>0}||(A|E)^j||_E \leq \max(sup_{j>0}||A^j||_H, 1).$$

(iv) *For all $x \in E$ we have*

$$||A^j x||_H \to 0 \Rightarrow ||A^j x||_E \to 0.$$

If A is strongly stable, then $A|E$ is strongly stable.

Proof. Because $E = \mathrm{dom}\,(\mathcal{C})$ as the algebraic vector space, claim (i) is given by Proposition 34. Claims (ii) and (iii) follow immediately from the calculation

$$\begin{aligned}
\frac{||A^j x||_E^2}{||x||_E^2} &= \frac{||A^j x||_H^2 + ||\mathcal{C}A^j x||^2_{\ell^2(\mathbf{Z}_+;Y)}}{||x||_H^2 + ||\mathcal{C}x||^2_{\ell^2(\mathbf{Z}_+;Y)}} \\
&= \frac{||A^j x||_H^2 + ||\bar{\pi}_+ \tau^{*j}\mathcal{C}x||^2_{\ell^2(\mathbf{Z}_+;Y)}}{||x||_H^2 + ||\mathcal{C}x||^2_{\ell^2(\mathbf{Z}_+;Y)}} \\
&\leq \frac{||A^j||^2_{\mathcal{L}(H)}\,||x||_H^2 + ||\mathcal{C}x||^2_{\ell^2(\mathbf{Z}_+;Y)}}{||x||_H^2 + ||\mathcal{C}x||^2_{\ell^2(\mathbf{Z}_+;Y)}} \leq \max(||A^j||_{\mathcal{L}(H)}, 1).
\end{aligned}$$

To prove claim (iv), note that for arbitrary $x \in E$ we have

$$||A^j x||_E^2 = ||A^j x||_H^2 + ||\bar{\pi}_+ \tau^{*j}\mathcal{C}x||^2_{\ell^2(\mathbf{Z}_+;Y)}.$$

Now the first term of the right hand side approaches zero by assumption. The second term approaches zero, because $\mathcal{C}x \in \ell^2(\mathbf{Z}_+;Y)$ by Definition 29 of $\mathrm{dom}\,(\mathcal{C}) = E$. This completes the proof of the lemma. $\square$

One could regard claim (iv) of the previous lemma as a partial converse to the obvious implication

$$||x_j||_E \to 0 \Rightarrow ||x_j||_H \to 0 \tag{1.48}$$

for all sequences $\{x_j\} \subset E$. If there is an equivalence instead of implication in (1.48), then the bijective inclusion mapping from the Hilbert space $(E, ||\cdot||_E)$

onto the normed vector space $(\mathrm{dom}\,(\mathcal{C})\,, ||\cdot||_H)$ is a bounded with a bounded inverse, by [79, Theorem 1.32]. Furthermore, the norms $||\cdot||_E$ and $||\cdot||_H$ on $\mathrm{dom}\,(\mathcal{C})$ are equivalent, which happens if and only if $\mathrm{dom}\,(\mathcal{C}) = \overline{\mathrm{dom}\,(\mathcal{C})}$ is a closed Hilbert subspace of $(H, ||\cdot||_H)$. If $\overline{\mathrm{dom}\,(\mathcal{C})} = H$ is assumed *a priori*, then the output stability of Φ follows. We conclude that the topology of E is in general genuinely stronger that that inherited from H, and the full converse to formula (1.48) is in general not true. The input operator B and the controllability map $\mathcal{B}$ behave expectedly, too.

Lemma 41. *Let $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ be an H^2 stable DLS, whose input operator is B. Then*

(i) The input operator B maps U into E boundedly. The norm estimate

$$||B||^2_{\mathcal{L}(U;E)} \leq ||B||^2_{\mathcal{L}(U;H)} + ||\mathcal{D}\pi_0||^2_{U\to\ell^2(\mathbf{Z}_+;Y)}$$

holds.

(ii) Assume, in addition, that Φ is I/O stable and input stable. Then $\mathcal{B} \in \mathcal{L}(\mathrm{dom}\,(\mathcal{B})\,; E)$ and

$$||\mathcal{B}||^2_{\mathrm{dom}(\mathcal{B})\to E} \leq ||\mathcal{B}||^2_{\mathrm{dom}(\mathcal{B})\to H} + ||\mathcal{D}||^2_{\mathrm{dom}(\mathcal{D})\to\ell^2(\mathbf{Z};Y)}.$$

Proof. Because $E = \mathrm{dom}\,(\mathcal{C})$, it follows from Lemma 35 that B maps U into E. It follows from Definition 32 and formula (1.44) that

$$||\mathcal{C}Bu_0||_{\ell^2(\mathbf{Z}_+;Y)} \leq ||\mathcal{D}\pi_0\tilde{u}||_{\ell^2(\mathbf{Z}_+;Y)} \leq ||\mathcal{D}\pi_0||_{U\to\ell^2(\mathbf{Z}_+;Y)}||u_0||_U < \infty$$

for any $\tilde{u} = \{u_j\}_{j\geq 0} \in Seq(U)$. Furthermore, for arbitrary $u_0 \in U$ we have

$$\begin{aligned} ||Bu_0||^2_E &= ||Bu_0||^2_H + ||\mathcal{C}Bu_0||^2_{\ell^2(\mathbf{Z}_+;Y)} \\ &\leq \left(||B||^2_{\mathcal{L}(U;H)} + ||\mathcal{D}\pi_0||^2_{U\to\ell^2(\mathbf{Z}_+;Y)}\right)||u_0||^2_U, \end{aligned}$$

and claim (i) follows. In order to prove claim (ii), let $\tilde{u} \in \mathrm{dom}\,(\mathcal{B}) \subset \mathrm{dom}\,(\mathcal{D})$ be arbitrary. Then $||\tilde{u}||^2_{\ell^2(\mathbf{Z};U)} < \infty$ because $\mathrm{dom}\,(\mathcal{B}) = Seq_-(U) \subset \ell^2(\mathbf{Z}_-;U)$, and

$$\begin{aligned} ||\mathcal{B}\tilde{u}||^2_E &= ||\mathcal{B}\tilde{u}||^2_H + ||\mathcal{C}\mathcal{B}\tilde{u}||^2_{\ell^2(\mathbf{Z}_+;U)} \\ &\leq ||\mathcal{B}\tilde{u}||^2_H + ||\bar{\pi}_+\mathcal{D}\pi_-\tilde{u}||^2_{\ell^2(\mathbf{Z}_+;Y)} \\ &\leq \left(||\mathcal{B}||^2_{\mathrm{dom}(\mathcal{B})\to H} + ||\mathcal{D}||^2_{\mathrm{dom}(\mathcal{D})\to\ell^2(\mathbf{Z};Y)}\right)||\tilde{u}||^2_{\ell^2(\mathbf{Z};U)}, \end{aligned}$$

because both $\mathrm{dom}\,(\mathcal{B})$ and $\mathrm{dom}\,(\mathcal{D})$ have the norm of $\ell^2(\mathbf{Z};U)$. This completes the proof of the lemma. □

Actually, the I/O stability is not required to make the conclusion of claim (ii). It would have been sufficient to assume that the Hankel operator $\bar{\pi}_+ \mathcal{D} \pi_- : \ell^2(\mathbf{Z}_-; U) \to \ell^2(\mathbf{Z}_+; U)$ is bounded.

Given a strongly H^2 stable DLS $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix} \right] = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix} \right)$, we have constructed a restricted state space $E = \mathrm{dom}\,(\mathcal{C})$, equipped with the Hilbert space norm $||\cdot||_E$. The restricted generating operators satisfy $A|E \in \mathcal{L}(E)$ and $C|E \in \mathcal{L}(E;Y)$, by Lemmas 39 and 40. Even the input operator satisfies $B \in \mathcal{L}(U;E)$, by Lemma 41. It is now possible to define the DLS ϕ^g whose generating operators we have constructed.

Definition 42. *Let $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix} \right)$ be a strongly H^2 stable DLS, whose input space is U, state space is H and the output space is Y. The graph topology DLS ϕ^g, associated to ϕ, is the DLS*

$$\phi^g := \begin{pmatrix} A|E & B \\ C|E & D \end{pmatrix},$$

where E is the vector space $\mathrm{dom}\,(\mathcal{C})$*, equipped with the Hilbert space inner product of Definition 38. The space U is the input space, Y the output space and E the state space of ϕ^g.*

The basic properties of ϕ^{g} are collected to the following theorem.

Theorem 43. *Let $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix} \right)$ be a strongly H^2 stable DLS.*

(i) The graph norm DLS ϕ^g is given in I/O form by

$$\Phi^g = \begin{bmatrix} (A|E)^j & \mathcal{B}\tau^{*j} \\ \mathcal{C}|E & \mathcal{D} \end{bmatrix}.$$

The DLS ϕ^g is an output stable DLS. ϕ^g is I/O stable if and only if ϕ is.

(ii) If the semigroup generator A of ϕ is power bounded, then so is the semigroup generator $A|E$ of ϕ^g. If A is strongly stable, then so is $A|E$.

(iii) Assume, moreover, that ϕ is I/O stable and input stable. Then ϕ^g is I/O stable and input stable.

Proof. To prove claim (i), we only prove that $\mathcal{D}_{\phi^g} = \mathcal{D}_\phi$ on $Seq(U)$. The cases of observability and controllability maps are similar. By formula (1.7), it is sufficient to show that $CA^jBu = C|E \cdot (A|E)^j \cdot Bu$ for all $u \in U$ and $j \geq 0$. Because $A : E \to E$ by claim (i) of Lemma 40, it follows that $(A|E)^j = A^j|E$ for all $j \geq 0$. Because $B : U \to E$ by Lemma 41, it follows that $(A|E)^j \cdot Bu = A^jBu \in E$ for all $u \in U$. But then $C|E \cdot (A|E)^j \cdot Bu = CA^jBu$, and the equality of I/O maps follows. The output stability of ϕ^{g} is shown in claim (ii) of Lemma 39. Claim (ii) follows directly from claims (iii) and (iv) of Lemma 40. Claim (iii) is shown in claim (ii) of Lemma 41. This completes the proof. □

We complete this section with a discussion of the presented results. Let ϕ be strongly H^2 stable DLS. By Theorem 43, we can change ϕ to an output stable DLS ϕ^{g} by a simple restriction and renorming of the state space H of ϕ. The properties of ϕ^{g} are almost identical to those of the original ϕ, However, the state space is genuinely restricted, except for a trivial case. The drawback is that we cannot consider all initial states $x_0 \in H$ of ϕ with the aid of ϕ^{g}, but only those which give an $\ell^2(\mathbf{Z}_+; Y)$ output with the zero input.

The lack of output stability of ϕ tells us that the state space of ϕ is "too large" or "inconveniently normed", and a better norm should be chosen for the state space. We regard it as a assumption to require that the H^2 stable DLSs are, in addition, output stable. The Riccati equation theory of output stable DLSs, as presented in Chapters 3, 4 and 5, does not require an introduction of new, topologies of the state space. In other words, once the original state space $(H, ||\cdot||_H)$ is replaced by $(E, ||\cdot||_E)$ of Definition 38, the full description of the ϕ and the related Riccati equation can be conveniently done in this fixed topology.

1.9 Stability of the closed loop DLS

In this section we study the feedbacks of DLSs with the additional requirement that the input and output sequences lie in the Hilbert spaces $\ell^2(\mathbf{Z}_+;U)$ and $\ell^2(\mathbf{Z}_+;Y)$. We restrict the notion of feedback pair as presented in Section 1.6 to take these additional requirements into consideration. We study both I/O stable and stable systems, and how the open loop stability is preserved in the closed loop system.

Definition 44. *Let* $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ *be a DLS, and* $[\mathcal{K},\mathcal{F}]$ *a feedback pair for the DLS* Φ *in the sense of Definition 21.*

(i) The feedback pair $[\mathcal{K},\mathcal{F}]$ *is I/O stable if* $\Phi^{\rm fb} := \left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{K} & \mathcal{F} \end{smallmatrix}\right]$ *is an I/O stable DLS and*

$$\operatorname{dom}(\mathcal{C}) \subset \{x_0 \in H \mid \mathcal{K}x_0 \in \ell^2(\mathbf{Z}_+;U)\} =: \operatorname{dom}(\mathcal{K}). \tag{1.49}$$

(ii) The feedback pair $[\mathcal{K},\mathcal{F}]$ *is output stable if* $\operatorname{dom}(\mathcal{K}) = H$*.*

(iii) The feedback pair $[\mathcal{K},\mathcal{F}]$ *is stable if it is I/O stable and output stable.*

(iv) The feedback pair $[\mathcal{K},\mathcal{F}]$ *is outer if* $(\mathcal{I}-\mathcal{F})^{-1}$ *is an I/O map of an I/O stable DLS.*

If the semigroup generator A of Φ is power stable, then all feedback pairs for Φ are stable but only very exceptional of those are outer. Because the mapping $\mathcal{K}$ is the observability map of the DLS $\Phi^{\rm fb}$, and all observability maps are closed, it follows that $\operatorname{dom}(\mathcal{K}) = H$ is equivalent with $\mathcal{K} \in \mathcal{L}(H,\ell^2(\mathbf{Z}_+;U))$. The meaning of inclusion (1.49) is that $\mathcal{K}$ is not allowed to be "more unbounded" than $\mathcal{C}$. It follows that for an output stable DLS, any I/O stable feedback pair is stable. A feedback pair $[\mathcal{K},\mathcal{F}]$ is stable if and only if $\mathcal{K} : H \to \ell^2(\mathbf{Z}_+;U)$ and $\mathcal{F} : \ell^2(\mathbf{Z};U) \to \ell^2(\mathbf{Z};U)$ map boundedly. Note that $\Phi^{\rm fb}$ need not be input stable for a stable feedback pair $[\mathcal{K},\mathcal{F}]$.

Clearly, $[\mathcal{K},\mathcal{F}]$ is I/O stable and outer if and only if the Toeplitz operator of the I/O map $(\mathcal{I}-\mathcal{F})\bar{\pi}_+ : \ell^2(\mathbf{Z}_+;U) \to \ell^2(\mathbf{Z}_+;U)$ is a bounded bijection and inclusion (1.49) holds. Such I/O maps are needed in this section, and we give their basic properties now.

Definition 45. *Let* $\mathcal{X}$ *be an I/O map of an I/O stable DLS.*

(i) If $\overline{\operatorname{range}(\mathcal{X}\bar{\pi}_+)} = \ell^2(\mathbf{Z}_+;U)$*, then* $\mathcal{X}$ *is outer.*

(ii) If, in addition, $\mathcal{X}\bar{\pi}_+$ *is injective and* $\operatorname{range}(\mathcal{X}\bar{\pi}_+) = \ell^2(\mathbf{Z}_+;U)$*, then* $\mathcal{X}$ *is outer with a bounded inverse.*

Between Banach spaces, bounded bijections are exactly those bounded operators that have bounded inverses. Thus, the I/O map $\mathcal{X}$ is outer with a bounded inverse if and only if the Toeplitz operator $\mathcal{X}\bar{\pi}_+$ is a bounded bijection on $\ell^2(\mathbf{Z}_+;U)$.

Proposition 46. *Let $\mathcal{X}$ be an I/O map of an I/O stable DLS. Assume that $\mathcal{X}$ is outer with a bounded inverse.*

(i) The Toeplitz operator $\mathcal{X}\bar{\pi}_+$ has a bounded inverse on $\ell^2(\mathbf{Z}_+;U)$, denoted by $(\mathcal{X}\bar{\pi}_+)^{-1}$. The feed-through operator $X := \pi_0\mathcal{X}\pi_0 \in \mathcal{L}(U)$ has a bounded inverse, and the algebraic inverse $\mathcal{X}^{-1}$ of $\mathcal{X}$ on Seq(U) exists as an I/O map of a DLS. In fact, $(\mathcal{X}\bar{\pi}_+)^{-1} = \mathcal{X}^{-1}\bar{\pi}_+$ on $\ell^2(\mathbf{Z}_+;U)$, and $\mathcal{X}^{-1}$ is I/O stable and outer with a bounded inverse.

(ii) The I/O map $\mathcal{X}$ has a unique bounded extension from $\mathrm{dom}\,(\mathcal{X}) = \ell^2(\mathbf{Z};U)\cap$ *Seq(U) to all of $\ell^2(\mathbf{Z};U)$, denoted by $\bar{\mathcal{X}}$. The operator $\bar{\mathcal{X}} : \ell^2(\mathbf{Z};U) \to \ell^2(\mathbf{Z};U)$ is shift-invariant, causal and a bounded bijection on $\ell^2(\mathbf{Z};U)$. The bounded inverse $\bar{\mathcal{X}}^{-1}$ equals the unique bounded extension of $\mathcal{X}^{-1}$ from* $\mathrm{dom}\left(\mathcal{X}^{-1}\right) = \ell^2(\mathbf{Z};U) \cap$ *Seq(U) to all of $\ell^2(\mathbf{Z};U)$. The operator $\bar{\mathcal{X}}^{-1} : \ell^2(\mathbf{Z};U) \to \ell^2(\mathbf{Z};U)$ is shift-invariant, causal and a bounded bijection on $\ell^2(\mathbf{Z};U)$.*

(iii) The Toeplitz operator $\pi_-\bar{\mathcal{X}}\pi_- : \ell^2(\mathbf{Z}_-;U) \to \ell^2(\mathbf{Z}_-;U)$ is a bounded bijection, and its inverse equals $\pi_-\bar{\mathcal{X}}^{-1}\pi_-$.

Proof. We start with claim (i). We have already stated that $\mathcal{X}\bar{\pi}_+$ has a bounded inverse $(\mathcal{X}\bar{\pi}_+)^{-1}$ on $\ell^2(\mathbf{Z}_+;U)$. We want to conclude that $X := \pi_0\mathcal{X}\pi_0$ has a bounded inverse. We first consider the surjectivity. Let $w_0 \in U$ be arbitrary, and denote $\tilde{w} := \{w_j\}_{j\geq 0} \in \ell^2(\mathbf{Z}_+;U)$ a sequence such that $w_j = 0$ for $j > 0$. Because the Toeplitz operator $\mathcal{X}\bar{\pi}_+$ is surjective, there is a $\tilde{u} \in \ell^2(\mathbf{Z}_+;U)$ such that $\tilde{w} = \mathcal{X}\bar{\pi}_+\tilde{u}$. But then,

$$w_0 = \pi_0\mathcal{X}\bar{\pi}_+\tilde{u} = \pi_0\mathcal{X}\pi_0\tilde{u} = Xu_0,$$

by the causality of $\mathcal{X}$, and the natural identification of $\mathrm{range}\,(\pi_0)$ and U. Because w_0 is arbitrary, it follows that X is surjective. Assume that X is not injective. Then there is a $u_0 \in U$ such that $\pi_0\mathcal{X}\tilde{u} = Xu_0 = 0$, where $\tilde{u} := \{u_j\}_{j\geq 0} \in \ell^2(\mathbf{Z}_+;U)$ is the sequence satisfying $u_j = 0$ for $j > 0$. Because $\mathcal{X}$ is bounded, $\pi_+(\mathcal{X}\bar{\pi}_+\tilde{u}) \in \ell^2(\mathbf{Z}_+;U)$. Because $\mathcal{X}\bar{\pi}_+$ is surjective, there exists a $\tilde{u}' \in \ell^2(\mathbf{Z}_+;U)$ such that

$$\mathcal{X}\bar{\pi}_+\tilde{u}' = -\bar{\pi}_+\tau^*(\mathcal{X}\bar{\pi}_+\tilde{u}) = -\tau^*\pi_+(\mathcal{X}\bar{\pi}_+\tilde{u}).$$

But now

$$\begin{aligned}\mathcal{X}\bar{\pi}_+\left(\tilde{u} + \tau\tilde{u}'\right) &= \mathcal{X}\bar{\pi}_+\tilde{u} + \tau\mathcal{X}\bar{\pi}_+\tilde{u}' = \mathcal{X}\bar{\pi}_+\tilde{u} - \pi_+\mathcal{X}\bar{\pi}_+\tilde{u}\\ &= \pi_0\mathcal{X}\bar{\pi}_+\tilde{u} = \pi_0\mathcal{X}\pi_0\tilde{u} = Xu_0 = 0.\end{aligned}$$

Thus $(\tilde{u} + \tau\tilde{u}') \in \ker(\mathcal{X}\bar{\pi}_+)$ which is contradiction against the injectivity of $\mathcal{X}\bar{\pi}_+$. We have concluded that $X \in \mathcal{L}(U)$ is a bijection. So it has a bounded inverse $X^{-1} \in \mathcal{L}(U)$.

Because X is the feed-through operator of any realization of $\mathcal{X}$ (and such realizations exist), claim (i) of Proposition 17 implies that the inverse mapping $\mathcal{X}^{-1} : Seq(U) \to Seq(U)$ exists. Let $\tilde{u} \in \ell^2(\mathbf{Z}_+; U) \subset Seq(U)$ be arbitrary. Then, by the definition of the inverse operator, $\tilde{u} = \mathcal{X}\bar{\pi}_+(\mathcal{X}\bar{\pi}_+)^{-1}\tilde{u}$. Because $\tilde{u} \in \ell^2(\mathbf{Z}_+; U)$, it follows that

$$\mathcal{X}^{-1}\bar{\pi}_+\tilde{u} = \mathcal{X}^{-1}\tilde{u} = \mathcal{X}^{-1}\mathcal{X}\bar{\pi}_+(\mathcal{X}\bar{\pi}_+)^{-1}\tilde{u} = \bar{\pi}_+(\mathcal{X}\bar{\pi}_+)^{-1}\tilde{u}.$$

Because $\tilde{u}$ is arbitrary and $(\mathcal{X}\bar{\pi}_+)^{-1}\tilde{u} \in \ell^2(\mathbf{Z}_+; U)$, it follows that $\mathcal{X}^{-1}\bar{\pi}_+ = (\mathcal{X}\bar{\pi}_+)^{-1}$, as invertible operators on $\ell^2(\mathbf{Z}_+; U)$. Because $\mathcal{X}\bar{\pi}_+$ is a bounded bijection on $\ell^2(\mathbf{Z}_+; U)$, so is its inverse $(\mathcal{X}\bar{\pi}_+)^{-1}$. But the latter equals the Toeplitz operator $\mathcal{X}^{-1}\bar{\pi}_+$, and thus the I/O map $\mathcal{X}^{-1}$ is outer with a bounded inverse.

We prove claim (ii). From the shift-invariance of $\mathcal{X}$ and boundedness of $\mathcal{X}\bar{\pi}_+$ it follows easily that $\mathcal{X} : \ell^2(\mathbf{Z}; U) \cap Seq(U) \to \ell^2(\mathbf{Z}; U)$ is bounded because the shift τ is unitary $\ell^2(\mathbf{Z}; U)$. The density of $\ell^2(\mathbf{Z}; U) \cap Seq(U)$ in $\ell^2(\mathbf{Z}; U)$ gives the unique bounded linear extension of $\mathcal{X}$ to all of $\ell^2(\mathbf{Z}; U)$, denoted by $\bar{\mathcal{X}}$. It is a matter of a simple limit argument that the extended $\bar{\mathcal{X}}$ is shift-invariant and causal. It remains to check that $\bar{\mathcal{X}} : \ell^2(\mathbf{Z}; U) \to \ell^2(\mathbf{Z}; U)$ is a bounded bijection. We start with its coercivity. Let $\tilde{u} \in \ell^2(\mathbf{Z}; U)$ be arbitrary, and define $\tilde{u}_j := \pi_{[-j,\infty]}\tilde{u}$ for all $j \geq 0$. Then $\{\tilde{u}_j\}_{j\geq 0} \subset \ell^2(\mathbf{Z}; U) \cap Seq(U)$ and $\tilde{u}_j \to \tilde{u}$ in $\ell^2(\mathbf{Z}; U)$. Then, because τ is unitary, we have

$$\begin{aligned} ||\bar{\mathcal{X}}\tilde{u}_j||_{\ell^2(\mathbf{Z};U)} &= ||\mathcal{X}\bar{\pi}_+\tau^j\tilde{u}_j||_{\ell^2(\mathbf{Z}_+;U)} \\ &\geq K \cdot ||\bar{\pi}_+\tau^j\tilde{u}_j||_{\ell^2(\mathbf{Z}_+;U)} = K \cdot ||\tilde{u}_j||_{\ell^2(\mathbf{Z};U)} \end{aligned}$$

where the existence of the constant $K > 0$ follows from the coercivity of the Toeplitz operator $\mathcal{X}\bar{\pi}_+$. Because $\bar{\mathcal{X}}$ is continuous, $||\bar{\mathcal{X}}\tilde{u}_j||_{\ell^2(\mathbf{Z};U)} \to ||\bar{\mathcal{X}}\tilde{u}||_{\ell^2(\mathbf{Z};U)}$ as $j \to \infty$, and it follows that the limits of the norms satisfy $||\bar{\mathcal{X}}\tilde{u}||_{\ell^2(\mathbf{Z};U)} \geq K \cdot ||\tilde{u}||_{\ell^2(\mathbf{Z};U)}$. Thus the extension $\bar{\mathcal{X}}$ is coercive on $\ell^2(\mathbf{Z}_+; U)$. Because range $(\mathcal{X}\bar{\pi}_+)$ $= \ell^2(\mathbf{Z}_+; U)$, the shift-invariance of $\mathcal{X}$ implies that

$$\cup_{j\geq 0}\{\tau^{*j}\ell^2(\mathbf{Z}_+; U)\} \subset \mathrm{range}\,(\mathcal{X}) \subset \mathrm{range}\left(\bar{\mathcal{X}}\right) := \bar{\mathcal{X}}\ell^2(\mathbf{Z}; U).$$

But then range $\left(\bar{\mathcal{X}}\right)$ is dense and, by the coercivity of $\bar{\mathcal{X}}$, it follows that range $\left(\bar{\mathcal{X}}\right)$ $= \ell^2(\mathbf{Z}_+; U)$. It also follows that $\bar{\mathcal{X}}$ is injective, by the coercivity. Thus $\bar{\mathcal{X}}$ is a bounded bijection on $\ell^2(\mathbf{Z}_+; U)$.

We proceed to show that $\bar{\mathcal{X}}^{-1}$ equals the unique bounded extension of the I/O stable I/O map $\mathcal{X}^{-1}$ from $\ell^2(\mathbf{Z}; U) \cap Seq(U)$ to all of $\ell^2(\mathbf{Z}; U)$. Let $\tilde{w} \in \ell^2(\mathbf{Z}; U) \cap Seq(U)$ be arbitrary. Then there is a $\tilde{u} \in \ell^2(\mathbf{Z}; U) \cap Seq(U)$ such that

$\tilde{w} = \mathcal{X}\tilde{u} = \bar{\mathcal{X}}\tilde{u}$. Now

$$\mathcal{X}^{-1}\tilde{w} - \bar{\mathcal{X}}^{-1}\tilde{w} = \mathcal{X}^{-1}\mathcal{X}\tilde{u} - \bar{\mathcal{X}}^{-1}\bar{\mathcal{X}}\tilde{u} = \tilde{u} - \tilde{u} = 0.$$

Thus $\bar{\mathcal{X}}^{-1}$ is an extension of $\mathcal{X}^{-1}$, and claim (ii) follows.

We prove the last claim (iii) by showing that the bounded linear operator $\pi_-\bar{\mathcal{X}}^{-1}\pi_- : \ell^2(\mathbf{Z}_-;U) \to \ell^2(\mathbf{Z}_-;U)$ is the inverse of $\pi_-\bar{\mathcal{X}}\pi_-$. We have

$$\begin{aligned}\pi_-\bar{\mathcal{X}}^{-1}\pi_- \cdot \pi_-\bar{\mathcal{X}}\pi_- &= \pi_-\bar{\mathcal{X}}^{-1}\bar{\mathcal{X}}\pi_- - \pi_-\bar{\mathcal{X}}^{-1}\bar{\pi}_+ \cdot \bar{\mathcal{X}}\pi_- \\ &= \pi_- - \pi_-\bar{\mathcal{X}}^{-1}\bar{\pi}_+ \cdot \bar{\mathcal{X}}\pi_-\end{aligned}$$

where all operators are bounded on $\ell^2(\mathbf{Z};U)$. But because $\bar{\mathcal{X}}^{-1}$ is causal on $\ell^2(\mathbf{Z};U)$, $\pi_-\bar{\mathcal{X}}^{-1}\bar{\pi}_+ = 0$, and $\pi_-\bar{\mathcal{X}}^{-1}\pi_- \cdot \pi_-\bar{\mathcal{X}}\pi_- = \pi_-$ follows. Similarly, $\pi_-\bar{\mathcal{X}}\pi_- \cdot \pi_-\bar{\mathcal{X}}^{-1}\pi_- = \pi_-$, and because π_- is (identifiable with) the identity operator on $\ell^2(\mathbf{Z}_-;U)$, the proof is complete. □

As has been discussed after Definition 32 for general I/O stable I/O maps, we use the same symbol $\mathcal{X}$ for the original I/O map $\mathcal{X} : Seq(U) \to Seq(U)$ and $\bar{\mathcal{X}} : \ell^2(\mathbf{Z};U) \to \ell^2(\mathbf{Z};U)$. An immediate conclusion of claim (i) of previous proposition is the following corollary.

Corollary 47. *Let $\mathcal{X}$ be an I/O map of a DLS. Then $\mathcal{X}$ is outer with a bounded inverse if an only if $\mathcal{X}^{-1} : Seq(U) \to Seq(U)$ exists and is outer with a bounded inverse.*

Now we have made the necessary preparations and proceed to consider the stability of the closed loop DLSs. In Theorem 48 we connect an I/O stable feedback pair to an I/O stable DLS. In Theorem 51 we do the same thing with a stable DLS and a stable feedback pair. Stability properties of the open loop and closed loop semigroup generators are considered in Proposition 49. Throughout this section, we use the following notation for the mapping of the closed loop DLS

$$\begin{aligned}\Phi_\diamond^{\mathrm{ext}} &= \begin{bmatrix} A_\diamond^j & \mathcal{B}_\diamond\tau^{*j} \\ \begin{bmatrix}\mathcal{C}_\diamond \\ \mathcal{K}_\diamond\end{bmatrix} & \begin{bmatrix}\mathcal{D}_\diamond \\ \mathcal{F}_\diamond\end{bmatrix}\end{bmatrix} \\ &= \begin{bmatrix} A^j + \mathcal{B}\tau^{*j}(\mathcal{I}-\mathcal{F})^{-1}\mathcal{K} & \mathcal{B}(I-\mathcal{F})^{-1}\tau^{*j} \\ \begin{bmatrix} \mathcal{C} + \mathcal{D}(I-\mathcal{F})^{-1}\mathcal{K} \\ (\mathcal{I}-\mathcal{F})^{-1}\mathcal{K}\end{bmatrix} & \begin{bmatrix}\mathcal{D}(\mathcal{I}-\mathcal{F})^{-1} \\ (\mathcal{I}-\mathcal{F})^{-1} - \mathcal{I}\end{bmatrix}\end{bmatrix}.\end{aligned} \tag{1.50}$$

Theorem 48. *Let $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D}\end{smallmatrix}\right]$ an I/O stable DLS. Let $[\mathcal{K},\mathcal{F}]$ be an I/O stable feedback pair for Φ. Then the following holds.*

(i) *The (open loop) extended DLS* $\Phi^{\mathrm{ext}} = [\Phi, [\mathcal{K}, \mathcal{F}]]$, *given in Definition 23, is I/O stable, and the domain of its observability map satisfies*

$$\operatorname{dom}\left(\begin{bmatrix} \mathcal{C} \\ \mathcal{K} \end{bmatrix}\right) = \operatorname{dom}(\mathcal{C}). \tag{1.51}$$

(ii) *Assume, in addition, that* $[\mathcal{K}, \mathcal{F}]$ *is outer. Then the closed loop extended DLS* $\Phi^{\mathrm{ext}}_{\diamond} = [\Phi, [\mathcal{K}, \mathcal{F}]]_{\diamond}$, *given in Definition 23, is I/O stable, and the domain of its observability map satisfies*

$$\operatorname{dom}\left(\begin{bmatrix} \mathcal{C}_{\diamond} \\ \mathcal{K}_{\diamond} \end{bmatrix}\right) = \operatorname{dom}(\mathcal{C}_{\diamond}) \cap \operatorname{dom}(\mathcal{K}_{\diamond}) = \operatorname{dom}(\mathcal{C}). \tag{1.52}$$

Proof. The cartesian products of observability maps $\mathcal{C}_1 : \operatorname{dom}(\mathcal{C}_1) \to \ell^2(\mathbf{Z}_+; Y_1)$ and $\mathcal{C}_2 : \operatorname{dom}(\mathcal{C}_2) \to \ell^2(\mathbf{Z}_+; Y_2)$ satisfy

$$\operatorname{dom}\left(\begin{bmatrix} \mathcal{C}_1 \\ \mathcal{C}_2 \end{bmatrix}\right) = \operatorname{dom}(\mathcal{C}_1) \cap \operatorname{dom}(\mathcal{C}_2), \tag{1.53}$$

by Definition 29 and noting that $\ell^2(\mathbf{Z}_+; Y_1) \oplus \ell^2(\mathbf{Z}_+; Y_2) = \ell^2(\mathbf{Z}_+; Y_1 \oplus Y_2)$. Claim (i) follows because $\operatorname{dom}(\mathcal{C}) \subset \operatorname{dom}(\mathcal{K})$ by Definition 44.

We proceed to prove claim (ii). The closed loop mappings in $\Phi^{\mathrm{ext}}_{\diamond}$ are given by

$$\begin{aligned} &\mathcal{C}_{\diamond} := \mathcal{C} + \mathcal{D}(\mathcal{I} - \mathcal{F})^{-1}\mathcal{K}, \quad \mathcal{K}_{\diamond} := (\mathcal{I} - \mathcal{F})^{-1}\mathcal{K}, \\ &\mathcal{D}_{\diamond} := \mathcal{D}(\mathcal{I} - \mathcal{F})^{-1}, \quad \mathcal{F}_{\diamond} := (\mathcal{I} - \mathcal{F})^{-1} - \mathcal{I}, \end{aligned}$$

by formula (1.50). We first show that $\Phi^{\mathrm{ext}}_{\diamond}$ is I/O stable. The Toeplitz operator of its I/O map satisfies on $Seq(U)$

$$\begin{bmatrix} \mathcal{D}_{\diamond} \\ \mathcal{F}_{\diamond} \end{bmatrix} \bar{\pi}_+ = \begin{bmatrix} \mathcal{D}(\mathcal{I} - \mathcal{F})^{-1}\bar{\pi}_+ \\ \left((\mathcal{I} - \mathcal{F})^{-1} - \mathcal{I}\right)\bar{\pi}_+ \end{bmatrix} = \begin{bmatrix} \mathcal{D}\bar{\pi}_+ \left(\mathcal{I} - \mathcal{F}\right)^{-1}\bar{\pi}_+ \\ \mathcal{F}\bar{\pi}_+ \left(\mathcal{I} - \mathcal{F}\right)^{-1}\bar{\pi}_+ \end{bmatrix}$$

by the causality of $\mathcal{D}$ and $(\mathcal{I} - \mathcal{F})^{-1}$. Now, $\mathcal{D}\bar{\pi}_+$ and $\mathcal{F}\bar{\pi}_+$ are bounded on $\ell^2(\mathbf{Z}_+; U)$, by the assumptions that both Φ and $[\mathcal{K}, \mathcal{F}]$ are I/O stable. Because $[\mathcal{K}, \mathcal{F}]$ is outer, $\left(\mathcal{I} - \mathcal{F}\right)^{-1}\bar{\pi}_+$ maps $\ell^2(\mathbf{Z}_+; U)$ onto $\ell^2(\mathbf{Z}_+; U)$ boundedly. By Definition 29, $\operatorname{dom}\left(\left[\begin{smallmatrix} \mathcal{D}_{\diamond} \\ \mathcal{F}_{\diamond} \end{smallmatrix}\right]\bar{\pi}_+\right) = \ell^2(\mathbf{Z}_+; U)$, and the I/O stability of $\Phi^{\mathrm{ext}}_{\diamond}$ follows.

The first equality in formula (1.52) has already been established in formula (1.53). Let $x_0 \in \operatorname{dom}(\mathcal{C})$ be arbitrary. By Definition 44, it follows that $\mathcal{K}x_0 \in \ell^2(\mathbf{Z}_+; U)$. But now

$$\mathcal{K}_{\diamond}x_0 = (\mathcal{I} - \mathcal{F})^{-1}\bar{\pi}_+\mathcal{K}x_0 \in \ell^2(\mathbf{Z}_+; U)$$

because $\left(\mathcal{I} - \mathcal{F}\right)^{-1}\bar{\pi}_+$ maps $\ell^2(\mathbf{Z}_+; U)$ onto $\ell^2(\mathbf{Z}_+; U)$ boundedly. Thus $x_0 \in \operatorname{dom}(\mathcal{K}_{\diamond})$. Because $\mathcal{D}\bar{\pi}_+$ is bounded from $\ell^2(\mathbf{Z}_+; U)$ into $\ell^2(\mathbf{Z}_+; Y)$, it follows

$$\mathcal{C}_{\diamond}x_0 = \mathcal{C}x_0 + \mathcal{D}\bar{\pi}_+\mathcal{K}_{\diamond}x_0 \in \ell^2(\mathbf{Z}_+; Y),$$

and so $x_0 \in \operatorname{dom}(\mathcal{C}_\diamond) \cap \operatorname{dom}(\mathcal{K}_\diamond)$.

For the converse direction, let $x_0 \in \operatorname{dom}(\mathcal{C}_\diamond) \cap \operatorname{dom}(\mathcal{K}_\diamond)$ be arbitrary. Then $\mathcal{K}_\diamond x_0 \in \ell^2(\mathbf{Z}_+; U)$ and $\mathcal{D}\bar{\pi}_+\mathcal{K}_\diamond x_0 \in \ell^2(\mathbf{Z}_+; Y)$ because Φ is I/O stable. Also $\mathcal{C}_\diamond x_0 = \mathcal{C}x_0 + \mathcal{D}\bar{\pi}_+\mathcal{K}_\diamond x_0 \in \ell^2(\mathbf{Z}_+; Y)$. It immediately follows that $\mathcal{C}x_0 \in \ell^2(\mathbf{Z}_+; Y)$ and $x_0 \in \operatorname{dom}(\mathcal{C})$. This completes the proof the theorem. □

Note that the closed loop mapping $\mathcal{F}_\diamond$ is I/O stable if and only if $[\mathcal{K}, \mathcal{F}]$ is outer. In all other cases, perturbations to the closed feedback loop cause instability in the feedback loop. However, the closed loop I/O map $\mathcal{D}_\diamond = \mathcal{D}(\mathcal{I} - \mathcal{F})^{-1}$ can be I/O stable even if neither $\mathcal{D}$ nor $(\mathcal{I} - \mathcal{F})^{-1}$ are I/O stable. In this case, the feedback stabilized the DLS Φ. In Chapter 4, we consider a particular case, associated to nonnegative solutions of a Riccati equation, when a DLS Φ and its feedback pair $[\mathcal{K}, \mathcal{F}]$ are stable, $[\mathcal{K}, \mathcal{F}]$ is generally not outer but nevertheless $\mathcal{D}_\diamond$ is I/O stable.

As has been considered in connection with equation (1.33), any state feedback can be undone by using another, inverse feedback pair. I/O stable and outer feedback pairs behave as follows.

Proposition 49. *Let Φ be a DLS, $[\mathcal{K}, \mathcal{F}]$ a feedback pair for it, and $\Phi_\diamond^{\mathrm{ext}} = [\Phi, [\mathcal{K}, \mathcal{F}]]_\diamond$ the corresponding closed loop DLS. By*

$$[\bar{\mathcal{K}}, \bar{\mathcal{F}}] := [-\mathcal{K}_\diamond,\ -\mathcal{F}_\diamond] = [-(\mathcal{I} - \mathcal{F})^{-1}\mathcal{K},\ \mathcal{I} - (\mathcal{I} - \mathcal{F})^{-1}]$$

denote the inverse feedback pair of $[\mathcal{K}, \mathcal{F}]$.

(i) $[\mathcal{K}, \mathcal{F}]$ is an output stable and outer feedback pair for Φ if and only if $[\bar{\mathcal{K}}, \bar{\mathcal{F}}]$ is a stable feedback pair for $\Phi_\diamond^{\mathrm{ext}}$. $[\bar{\mathcal{K}}, \bar{\mathcal{F}}]$ is an output stable and outer feedback pair for $\Phi_\diamond^{\mathrm{ext}}$ if and only if $[\mathcal{K}, \mathcal{F}]$ is a stable feedback pair for Φ. $[\mathcal{K}, \mathcal{F}]$ is a stable and outer feedback pair for Φ if and only if $[\bar{\mathcal{K}}, \bar{\mathcal{F}}]$ is a stable and outer feedback pair for $\Phi_\diamond^{\mathrm{ext}}$.

(ii) Assume, in addition, that Φ is I/O stable. Then $[\mathcal{K}, \mathcal{F}]$ is an I/O stable and outer feedback pair for Φ if and only if $\left[\bar{\mathcal{K}},\ \bar{\mathcal{F}}\right]$ is an I/O stable and outer feedback pair for $\Phi_\diamond^{\mathrm{ext}}$.

Proof. We have already seen in connection with equation (1.33) that $[\bar{\mathcal{K}}, \bar{\mathcal{F}}]$ is a feedback pair for $\Phi_\diamond^{\mathrm{ext}}$. Claim (i) follows immediately from the formulae connecting $[\mathcal{K}, \mathcal{F}]$ and $[\bar{\mathcal{K}}, \bar{\mathcal{F}}]$. We proceed to prove claim (i). Assume that $[\mathcal{K}, \mathcal{F}]$ is an I/O stable and outer feedback pair for the I/O stable DLS Φ. Because $[\mathcal{K}, \mathcal{F}]$ is outer, the mapping $(\mathcal{I} - \mathcal{F})^{-1}$ is an I/O map of an I/O stable DLS, by Definition 44. Because clearly $(\mathcal{I} - \mathcal{F})^{-1} = \mathcal{I} - \bar{\mathcal{F}}$ holds on $Seq(U)$, it follows that $\bar{\mathcal{F}}$ is an I/O map of an I/O stable DLS. Also, $\left(\mathcal{I} - \bar{\mathcal{F}}\right)^{-1}$ is an

I/O-stable I/O map because it equals $\mathcal{I}-\mathcal{F}$ on $Seq(U)$, and $[\mathcal{K},\mathcal{F}]$ is I/O stable. The domains of the observability maps satisfy

$$\mathrm{dom}\left(\begin{bmatrix} \mathcal{C}_\diamond \\ \mathcal{K}_\diamond \end{bmatrix}\right) = \mathrm{dom}\,(\mathcal{C}) \subset \mathrm{dom}\,(\mathcal{K}) = \mathrm{dom}\left(\bar{\mathcal{K}}\right),$$

where the first equality is by the I/O stability of Φ and claim (ii) of Theorem 48, and the inclusion is by assumption that $[\mathcal{K},\mathcal{F}]$ is an I/O stable feedback pair for the DLS Φ. The final equality follows because $\bar{\mathcal{K}} = -(\mathcal{I}-\mathcal{F})^{-1}\bar{\pi}_+\mathcal{K}$ where $(\mathcal{I}-\mathcal{F})^{-1}\bar{\pi}_+$ is a bounded bijection on $\ell^2(\mathbf{Z}_+;U)$. We have now checked that $[\bar{\mathcal{K}},\bar{\mathcal{F}}]$ is an I/O stable feedback pair for $\Phi_\diamond^{\mathrm{ext}}$.

To prove the converse direction, assume $[\bar{\mathcal{K}},\bar{\mathcal{F}}]$ is an I/O stable feedback pair for $\Phi_\diamond^{\mathrm{ext}}$. We first show that the closed loop DLS $\Phi_\diamond^{\mathrm{ext}}$ itself is I/O stable. It I/O map is $\left[\begin{smallmatrix} \mathcal{D}(\mathcal{I}-\mathcal{F})^{-1} \\ \mathcal{F}(\mathcal{I}-\mathcal{F})^{-1} \end{smallmatrix}\right] : Seq(U) \to Seq(Y \oplus U)$. The I/O map $\mathcal{D} : \ell^2(\mathbf{Z};U) \to \ell^2(\mathbf{Z};Y)$ is bounded because Φ is assumed to be I/O stable. The I/O map $(\mathcal{I}-\mathcal{F})^{-1} : \ell^2(\mathbf{Z};U) \to \ell^2(\mathbf{Z};U)$ is bounded because $(\mathcal{I}-\mathcal{F})^{-1} = \mathcal{I}-\bar{\mathcal{F}}$ and $[\bar{\mathcal{K}},\bar{\mathcal{F}}]$ is I/O stable. Finally, the I/O-map $\mathcal{F} : \ell^2(\mathbf{Z};U) \to \ell^2(\mathbf{Z};U)$ is bounded because $\mathcal{F} = \mathcal{I} - (\mathcal{I}-\bar{\mathcal{F}})^{-1}$ and $[\bar{\mathcal{K}},\bar{\mathcal{F}}]$ is outer. It now follows that $\Phi_\diamond^{\mathrm{ext}}$ is I/O stable.

We can now proceed as in the first direction of this proof, but using the I/O stable DLS $\Phi_\diamond^{\mathrm{ext}}$ in place of Φ and the I/O stable $[\bar{\mathcal{K}},\bar{\mathcal{F}}]$ in place of $[\mathcal{K},\mathcal{F}]$. It follows that the inverse feedback pair $[\mathcal{K}',\mathcal{F}']$ of $[\bar{\mathcal{K}},\bar{\mathcal{F}}]$ is an I/O stable feedback pair for the closed loop extended DLS $(\Phi_\diamond^{\mathrm{ext}})_\diamond^{\mathrm{ext}} := \left[[\Phi,[\mathcal{K},\mathcal{F}]],[\bar{\mathcal{K}},\bar{\mathcal{F}}]\right]$. By formula (1.33), $[\mathcal{K}',\mathcal{F}']$ is also an I/O stable feedback pair for the DLS Φ because the semigroups and controllability maps of these DLSs are equal. It is an immediate consequence of the identity $(\mathcal{I}-\mathcal{F})^{-1} = \mathcal{I}-\bar{\mathcal{F}}$ that $[\mathcal{K}',\mathcal{F}'] = [\mathcal{K},\mathcal{F}]$. Now claim (ii) is proved. To see that claim (i) holds, it is enough to note that $\bar{\mathcal{K}} := -(\mathcal{I}-\mathcal{F})^{-1}\mathcal{K}$ is bounded from H into $\ell^2(\mathbf{Z}_+;U)$ is and only if $\mathcal{K}$ is bounded from H into $\ell^2(\mathbf{Z}_+;U)$, because $(\mathcal{I}-\mathcal{F})^{-1}\bar{\pi}_+$ is a bounded bijection on $\ell^2(\mathbf{Z}_+;U)$. $\square$

The next theorem shows us, how the stability of the semigroup is preserved under the closing of the feedback loop. The role of input stability should be carefully noted.

Theorem 50. *Let $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ be an input stable DLS. Let $[\mathcal{K},\mathcal{F}]$ be a stable and outer feedback pair for Φ. By $A_\diamond$ denote the semigroup generator of $\Phi_\diamond^{\mathrm{ext}} = [\Phi,[\mathcal{K},\mathcal{F}]]$. Then*

(i) A is strongly stable if and only if $A_\diamond$ is strongly stable, and

(ii) A is power bounded if and only if $A_\diamond$ is power bounded.

Proof. We prove the "only if" part of claim (i). Assume that A is strongly stable and $x_0 \in H$ is arbitrary. Then

$$A_\diamond^j x_0 = A^j x_0 + \mathcal{B}\tau^{*j}\mathcal{K}_\diamond x_0.$$

It is enough to estimate the second term on the right hand side and see that it gets small if j is increased. Because $\mathcal{K}_\diamond = (\mathcal{I} - \mathcal{F})^{-1}\mathcal{K}$ and $[\mathcal{K}, \mathcal{F}]$ is stable and outer, it follows that $\mathcal{K}x_0 \in \ell^2(\mathbf{Z}_+; U)$. We have

$$\begin{aligned} ||\mathcal{B}\tau^{*j}\mathcal{K}_\diamond x_0||_H &< ||\mathcal{B}\tau^{*j}\pi_{[0,J]}\mathcal{K}_\diamond x_0||_H \\ &+ ||\mathcal{B}\tau^{*j}||_{\ell^2(\mathbf{Z}_+;U)\to H} \cdot ||\pi_{[J+1,\infty]}\mathcal{K}_\diamond x_0||_{\ell^2(\mathbf{Z}_+;U)}. \end{aligned} \tag{1.54}$$

The second term on the right hand side of equation (1.54) gets small by increasing J because $\mathcal{K}_\diamond x_0 \in \ell^2(\mathbf{Z}_+; U)$ and $\mathcal{B}$ is bounded. We estimate the first term. By claim (ii) of Lemma 12, we have

$$\begin{aligned} ||\mathcal{B}\tau^{*j}\pi_{[0,J]}\tilde{u}||_H &\leq ||A^j\mathcal{B}\pi_- \cdot \pi_{[0,J]}\tilde{u}||_H + ||\sum_{i=0}^{j-1} A^i B(\pi_{[0,J]}\tilde{u})_{j-i-1}||_H \\ = ||\sum_{i=j-J-1}^{j-1} A^i B u_{j-i-1}||_H &= ||A^{j-J-1}\left(\sum_{i=0}^{J} A^i B u_{J-i}\right)||_H \end{aligned}$$

for any $\tilde{u} \in \ell^2(\mathbf{Z}_+; U)$ and $j > J$. Letting $j \to \infty$, the right hand side approaches zero because A is strongly stable. This proves that $A_\diamond$ is strongly stable, thus establishing the "only if" part of claim (i). We outline the proof of the "if" part. By claim (i) of Proposition 49, we see that $[\bar{\mathcal{K}}, \bar{\mathcal{F}}]$ is a stable and outer feedback pair for $\Phi_\diamond^{\mathrm{ext}}$. Also, $\Phi_\diamond^{\mathrm{ext}}$ is input stable because its input operator satisfies $\mathcal{B}_\diamond = \mathcal{B}(\mathcal{I} - \mathcal{F})^{-1}$ and $[\mathcal{K}, \mathcal{F}]$ is outer. Now the semigroup generator of the closed loop extended DLS $(\Phi_\diamond^{\mathrm{ext}})_\diamond^{\mathrm{ext}} := \left[[\Phi, [\mathcal{K}, \mathcal{F}]], [\bar{\mathcal{K}}, \bar{\mathcal{F}}]\right]$ is strongly stable, by the argument presented above. However, it also equals A; the semigroup generator of Φ. Now claim (i) is proved. In order to prove claim (ii), we calculate

$$\begin{aligned} \big|\, ||A_\diamond^j||_{\mathcal{L}(H)} - ||A^j||_{\mathcal{L}(H)} \big| &\leq ||A_\diamond^j - A^j||_{\mathcal{L}(H)} \\ &\leq ||\mathcal{B}||_{\ell^2(\mathbf{Z}_-;U)\to H)} \, ||\mathcal{K}_\diamond||_{H\to\ell^2(\mathbf{Z}_-;U)} < \infty \end{aligned}$$

Thus either both $A_\diamond$ and A are power bounded, or neither are. This completes the proof of the theorem. $\square$

Note that if $[\mathcal{K}, \mathcal{F}]$ is an output stable and outer feedback pair for an input stable Φ with a strongly stable semigroup generator A, then $A_\diamond$ is strongly stable. The proof of the "only if" part in claim (i) of Theorem 50 does not use the I/O stability of Φ. We now consider the case when the feedback pair $[\mathcal{K}, \mathcal{F}]$ is not only I/O stable, but stable and outer. Roughly, outer feedback pairs give stable closed loop DLSs $\Phi_\diamond^{\mathrm{ext}}$.

Theorem 51. *Let $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ be an I/O stable DLS, and $[\mathcal{K},\mathcal{F}]$ an I/O stable and outer feedback pair for it. Define $\Phi^{\text{ext}} := [\Phi,[\mathcal{K},\mathcal{F}]]$ and $\Phi_\diamond^{\text{ext}} := [\Phi,[\mathcal{K},\mathcal{F}]]_\diamond$.*

(i) Both Φ^{ext} and $\Phi_\diamond^{\text{ext}}$ are I/O stable.

(ii) Φ is input stable if and only if Φ^{ext} is input stable if and only if $\Phi_\diamond^{\text{ext}}$ is input stable.

(iii) Φ is output stable and $[\mathcal{K},\mathcal{F}]$ is stable if and only if Φ^{ext} is output stable if and only if $\Phi_\diamond^{\text{ext}}$ is output stable.

(iv) Φ and $[\mathcal{K},\mathcal{F}]$ are stable if and only if Φ^{ext} is stable if and only if $\Phi_\diamond^{\text{ext}}$ is stable.

(v) Φ is strongly stable and $[\mathcal{K},\mathcal{F}]$ is stable if and only if Φ^{ext} is strongly stable if and only if $\Phi_\diamond^{\text{ext}}$ is strongly stable.

Proof. Claim (i) is claim (i) of Theorem 48. We prove claim (ii). Because $[\mathcal{K},\mathcal{F}]$ is I/O stable and outer, the anticausal Toeplitz operator $\pi_-(\mathcal{I}-\mathcal{F})^{-1}\pi_-$ is a bounded bijection with a bounded inverse on $\ell^2(\mathbf{Z}_-;U)$, by claim (iii) of Proposition 46. Claim (ii) follows because $\mathcal{B}_\diamond = \mathcal{B}(\mathcal{I}-\mathcal{F})^{-1} = \mathcal{B}\cdot\pi_-(\mathcal{I}-\mathcal{F})^{-1}\pi_-$, and both the DLSs Φ and Φ^{ext} have the same controllability map.

The first equivalence of claim (iii) is trivial. We prove the "only if" part of the latter equivalence. Assume that Φ is output stable and $[\mathcal{K},\mathcal{F}]$ is stable. Then $\mathcal{K}_\diamond = (\mathcal{I}-\mathcal{F})^{-1}\mathcal{K} : H \to \ell^2(\mathbf{Z}_+;U)$ because $[\mathcal{K},\mathcal{F}]$ is output stable and outer. It follows that $\mathcal{C}_\diamond = \mathcal{C} + \mathcal{D}\mathcal{K}_\diamond : H \to \ell^2(\mathbf{Z}_+;U)$ is bounded because Φ is both I/O stable and output stable. Thus the observability map $\left[\begin{smallmatrix} \mathcal{C}_\diamond \\ \mathcal{K}_\diamond \end{smallmatrix}\right] : H \to \ell^2(\mathbf{Z}_+;U\oplus Y)$ of $\Phi_\diamond^{\text{ext}}$ is bounded, and $\Phi_\diamond^{\text{ext}}$ is output stable. To prove the "if" part, assume that $\Phi_\diamond^{\text{ext}}$ is output stable. Then both $\mathcal{C}_\diamond$ and $\mathcal{K}_\diamond$ are bounded. It follows that $\mathcal{C} = \mathcal{C}_\diamond - \mathcal{D}\mathcal{K}_\diamond$ is bounded because Φ is I/O stable. Thus Φ is output stable. Because $\mathcal{K} = (\mathcal{I}-\mathcal{F})\mathcal{K}_\diamond$ and $[\mathcal{K},\mathcal{F}]$ is I/O stable, it follows that $[\mathcal{K},\mathcal{F}]$ is output stable and hence stable.

In claim (iv) it is trivial that Φ and $[\mathcal{K},\mathcal{F}]$ are stable if and only if Φ^{ext} is stable. Assume that Φ and $[\mathcal{K},\mathcal{F}]$ are stable. Then claims (i), (ii) and (iii) of this theorem imply that $\Phi_\diamond^{\text{ext}}$ is I/O stable, input stable and output stable. Claim (ii) of Theorem 50 implies that $A_\diamond$ is power bounded, and thus $\Phi_\diamond^{\text{ext}}$ is stable. Assume that $\Phi_\diamond^{\text{ext}}$ is stable. Then the I/O stable Φ is input stable and output stable, by claims (ii) and (iii) of this theorem. Claim (iii) also implies that the I/O stable feedback pair $[\mathcal{K},\mathcal{F}]$ is in fact stable. Now claim (ii) of Theorem 50 implies that A is power bounded. Thus Φ is stable and claim (iv) follows.

The proof of claim (v) is analogous to claim (iv), only claim (i) of Theorem 50 must be used instead of claim (ii) of Theorem 50. □

1.10 Transfer functions and boundary traces

In this section, we introduce some necessary tools from the operator-valued analytic function theory, measure theory and harmonic analysis.

Transfer functions

For the I/O map $\mathcal{D}$ of a DLS $\Phi = \phi := \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$, formula (1.7) is given. The bilateral shift operator can be formally replaced by a complex variable z, and the formal sum is obtained

$$D + \sum_{i \geq 0} CA^i B z^{i+1}. \tag{1.55}$$

Because A is bounded by the definition of the DLS, this sum converges for $|z| < ||A^{-1}||^{-1}$, thus defining an analytic $\mathcal{L}(U;Y)$-valued function $\mathcal{D}(z)$ in a neighborhood of the origin. In fact, $\mathcal{D}(z) = D + zC(I - zA)^{-1}B$ for $|z| < ||A^{-1}||^{-1}$. The analytic function $\mathcal{D}(z)$ is called the transfer function of Φ. Because all I/O maps of DLSs have transfer functions analytic in a neighborhood of origin, we say that the the DLS is a well-posed linear system. The well-posedness makes it possible to add and multiply two transfer functions of appropriate type in a common neighborhood of the origin where both are analytic. We remark that the corresponding continuous time notion of well-posedness is deeper, see [89]. Because the power series coefficient (centered at the origin) of an analytic function are unique, we have one-to-one correspondence between the I/O maps of DLSs and operator-valued functions, analytic in a neighborhood of the origin of the complex plane, see Lemma 9.

In the following definition, we consider signals instead of systems.

Definition 52. *Let Z be a Hilbert space.*

(i) The sequence $\tilde{u} = \{u_j\}_{j \in \mathbf{Z}_+} \in Seq_+(Z)$ is well posed, if the power series

$$\tilde{u}(z) := \sum_{j=0}^{\infty} u_j z^j$$

converges to an analytic function in some neighborhood of the origin.

(ii) The mapping $\mathcal{F}_z : \tilde{u} \mapsto \tilde{u}(z)$ is the z-transform.

The set $WSeq_+(Z)$ of well-posed sequences is a vector subspace of $Seq_+(Z)$. It is a matter of taste whether z-transform should be defined to be analytic in a neighborhood of the origin or of the infinity. It seems that in the function theory the former alternative is used, and in the control theory the latter is preferred.

Proposition 53. *Let $\mathcal{D}$ be an I/O map of a DLS, and $\mathcal{D}(z)$ its transfer function. Let $\tilde{u} \in WSeq_+(U)$ and $\tilde{u}(z)$ its z-transform. Let $\tilde{y} \in Seq_+(Y)$. Then the following are equivalent:*

(i) $\tilde{y} = \mathcal{D}\tilde{u}$

(ii) $\tilde{y}$ *is well-posed, and* $\tilde{y}(z) = \mathcal{D}(z)\tilde{u}(z)$ *in some neighborhood of the origin.*

Proof. Assume claim (i). Because both $\mathcal{D}(z)$ and $\tilde{u}(z)$ are analytic in a some common neighborhood of the origin, so is the Y-valued function $f(z) := \mathcal{D}(z)\tilde{u}(z)$. Identify the unilateral shift τ by the multiplication by the complex variable z. By comparing the expression of both $\mathcal{D}\tilde{u}$ and $\mathcal{D}(z)\tilde{u}(z)$, with the aid of formulae (1.7) and (1.55), it is clear that the power series coefficients f_j of f equal the components y_j of $\tilde{y}$. So $\tilde{y} \in WSeq_+(Y)$ is well posed and (ii) follows. The converse direction is similar. □

Corollary 54. *Let ϕ_1 and ϕ_2 be DLSs with compatible input and output spaces. Then $\mathcal{D}_{\phi_1\phi_2}(z) = (\mathcal{D}_{\phi_1}\mathcal{D}_{\phi_2})(z) = \mathcal{D}_{\phi_1}(z)\mathcal{D}_{\phi_2}(z)$.*

Proof. Let $\tilde{u} \in WSeq(U)$ be arbitrary. Then $\mathcal{D}_{\phi_1\phi_2}\tilde{u}$ and $\mathcal{D}_{\phi_2}\tilde{u}$ are well posed sequences by Proposition 53, and

$$\begin{aligned}(\mathcal{D}_{\phi_1\phi_2}\tilde{u})(z) &= \mathcal{D}_{\phi_1\phi_2}(z)\tilde{u}(z) = (\mathcal{D}_{\phi_1}(\mathcal{D}_{\phi_2}\tilde{u}))(z) \\ &= \mathcal{D}_{\phi_1}(z)(\mathcal{D}_{\phi_2}\tilde{u})(z) = \mathcal{D}_{\phi_1}(z)\mathcal{D}_{\phi_2}(z)\tilde{u}(z),\end{aligned}$$

where all the equalities are by Proposition 53, except the second which is by claim (ii) of Proposition 17. Because $\tilde{u}$ is arbitrary, the claim follows. □

We conclude that the algebraic structure of corresponding I/O maps and transfer functions is equivalent, when only the well-posed input sequences are considered. In particular, because I/O map is known if its action on sequences $\tilde{u} \in WSeq(U)$ satisfying $\pi_j\tilde{u} = 0$ for $j \neq 0$, no uniqueness problems can arise if we restrict to well posed inputs. We have $\ell^2(\mathbf{Z}_+; U) \subset WSeq_+(U)$. This is trivially true because $\tilde{u} \in \ell^2(\mathbf{Z}_+; U)$ is a bounded sequence, and thus the power series $\sum_{j\geq 0} u_j z^j$ converge for all $z \in \mathbf{D}$ by a simple argument.

At this point, it is necessary to introduce the Hardy spaces $H^p(\mathbf{D}; \mathcal{L}(U;Y))$ (operator-valued) and $H^p(\mathbf{D}; U)$ (Hilbert space -valued) for each $1 \leq p < \infty$. They are defined as the Banach spaces of analytic functions in $\mathbf{D}$ with the norms

$$||\mathcal{D}(z)||^p_{H^p(\mathbf{D};\mathcal{L}(U;Y))} := \sup_{0<r<1} \frac{1}{2\pi}\int_0^{2\pi} ||\mathcal{D}(re^{i\theta})||^p_{\mathcal{L}(U;Y)}\, d\theta,$$

$$||\tilde{u}(z)||^p_{H^p(\mathbf{D};U))} := \sup_{0<r<1} \frac{1}{2\pi}\int_0^{2\pi} ||\tilde{u}(re^{i\theta})||^p_U\, d\theta.$$

Out of these, the cases $p = 2$ are most important to us. The space $H^2(\mathbf{D};U)$ is Hilbert, with the inner product

$$\langle \tilde{u}(z), \tilde{v}(z)\rangle_{H^2(\mathbf{D};U)} = \lim_{r\to 1-} \frac{1}{2\pi}\int_0^{2\pi} \left\langle \tilde{u}(re^{i\theta}), \tilde{v}(re^{i\theta})\right\rangle_U \, d\theta$$

and the Parseval identity

$$\langle \tilde{u}(z), \tilde{v}(z)\rangle_{H^2(\mathbf{D};U)} = \langle \tilde{u}, \tilde{v}\rangle_{\ell^2(\mathbf{Z}_+;U)}\,. \tag{1.56}$$

The interpretation of equation (1.56) is that the z-transform $\mathcal{F}_z : \tilde{u} \mapsto \tilde{u}(z)$ is an isometric isomorphism of the Hilbert spaces $\ell^2(\mathbf{Z}_+;U)$ and $H^2(\mathbf{D};U)$. For further information, see [77, Section 1.15] and [46, Chapter III].

Now that we have identified the z-transforms of finite energy signals, we identify the transfer functions of I/O stable DLSs. For this end, we meet one more Hardy space, namely the celebrated $H^\infty(\mathbf{D};\mathcal{L}(U;Y))$. We say that $\mathcal{D}(z) \in H^\infty(\mathbf{D};\mathcal{L}(U;Y))$ if it is $\mathcal{L}(U;Y)$-valued analytic function in the whole of $\mathbf{D}$, and

$$||\mathcal{D}(z)||_{H^\infty(\mathbf{D};\mathcal{L}(U;Y))} := \sup_{z\in D} ||\mathcal{D}(z)||_{\mathcal{L}(U;Y)} < \infty.$$

Proposition 55. *Let $\mathcal{D}$ be a I/O map of a DLS, such that all the Hilbert spaces U, H and Y are separable. Then $\mathcal{D}$ is I/O stable if and only if $\mathcal{D}(z) \in H^\infty(\mathbf{D};\mathcal{L}(U;Y))$. Furthermore, $||\mathcal{D}(z)||_{H^\infty(\mathbf{D};\mathcal{L}(U;Y))} = ||\mathcal{D}||_{\ell^2(\mathbf{Z}_+;U)\mapsto\ell^2(\mathbf{Z}_+;Y)}$.*

Proof. This is the contents of [77, Theorem 1.15B]), or [27, Theorem 1.1, Section IX, p. 235], to mention few possible references. In [77], the input and output spaces are written to be the same space. However, by using the Cartesian product Hilbert space $W = U \oplus Y$ as both input and output space, and extending the operators $T \in \mathcal{L}(U;Y)$ to $T' = \left(\begin{smallmatrix} 0 & 0 \\ T & 0 \end{smallmatrix}\right)$, the notational inconvenience is resolved. □

For the representation of bounded causal shift-invariant operators by H^∞ functions, see also [91] and [97]. Related to the operator-valued $H^2(\mathbf{D};\mathcal{L}(U,Y))$-space, another less known variant, called the strong $H^2(\mathbf{D};\mathcal{L}(U,Y))$ is defined as follows:

Definition 56. *The strong $H^2(\mathbf{D};\mathcal{L}(U,Y))$ (briefly: $\mathrm{sH}^2(\mathbf{D};\mathcal{L}(U;Y))$) is the set of $\mathcal{L}(U;Y)$-valued analytic functions $\mathcal{D}(z)$ in $\mathbf{D}$, such that $\mathcal{D}(z)u_0 \in H^2(\mathbf{D};Y)$, for all $u_0 \in U$.*

Clearly $\mathrm{sH}^2(\mathbf{D};\mathcal{L}(U;Y))$ is a vector space. The following proposition gives a hint why $\mathrm{sH}^2(\mathbf{D};\mathcal{L}(U;Y))$ is important to us.

Proposition 57. *The DLS* $\phi := \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ *is strongly* H^2 *stable if and only if the transfer function* $\mathcal{D}_\phi(z) \in \mathrm{sH}^2(\mathbf{D}; \mathcal{L}(U;Y))$.

Proof. Let ϕ be a strongly H^2 stable DLS. We first show that the transfer function $\mathcal{D}_\phi(z)$ is analytic in the whole of $\mathbf{D}$. We have for arbitrary $\tilde{u} := \{u_j\}_{j \geq 0} \in Seq_+(U)$

$$\begin{aligned} &||Du_0||_Y^2 + \sum_{j \geq 0} ||CA^j Bu_0||_Y^2 \\ &= ||\{Du_0\} \cup \{CA^{j-1}Bu_0\}_{j \geq 1}||^2_{\ell^2(\mathbf{Z}_+;Y)} = ||\mathcal{D}_\phi \pi_0 \tilde{u}||^2_{\ell^2(\mathbf{Z}_+;Y)} < \infty. \end{aligned}$$

Because $\tilde{u}$ is arbitrary, we have $\sup_{j \geq 0} ||CA^j Bu_0||_Y < \infty$ for all $u_0 \in H$. Now Banach–Steinhaus Theorem implies that the family $\{CA^j B\}_{j \geq 0}$ is uniformly bounded, and clearly the power series $\sum_{j=0}^\infty CA^j B z^j$ converges for all $z \in \mathbf{D}$. The power series expansion of transfer function $\mathcal{D}_\phi(z)$ is given by

$$\mathcal{D}_\phi(z) = D + \sum_{j \geq 1} CA^{j-1} B z^j, \quad z \in \mathbf{D}.$$

By the strong H^2 stability, $\{CA^j Bu_0\}_{j \geq 0} \subset \ell^2(\mathbf{Z}_+;Y)$ for any $u_0 \in U$. The Parseval identity implies now that $\mathcal{D}_\phi(z)u_0 \in H^2(\mathbf{D};Y)$ for each $u_0 \in U$. So $\mathcal{D}_\phi(z) \in \mathrm{sH}^2(\mathbf{D}; \mathcal{L}(U;Y))$. The converse direction is similar. □

Nontangential limits of transfer functions

We have seen that the I/O maps of DLSs and well-posed signals have a one-to-one correspondence to their transfer functions and z-transforms, respectively. Furthermore, the I/O stability and finite signal energy notions behave well under the z-transform. The following question arises: what essentially new does the replacement of the bilateral shift τ by the complex variable z bring us? A (partial) answer is: point evaluations of the transfer function $\mathcal{D}(z)$ at all points of analyticity z. This gives us the notion of zeroes and poles of the transfer function, at least in the case when all the Hilbert spaces U, H and Y are finite dimensional.

The notions of zeroes and poles are not central in this book, and if it was only for this reason, we would not need to define the transfer functions in the first place. However, there is another reason to introduce transfer functions that is important to us. Namely, there are restricted classes of (transfer) functions $\mathcal{D}(z)$ and (signals) $\tilde{u}(z)$, analytic for $z \in \mathbf{D}$, that can be evaluated in a useful sense at the boundary points $e^{i\theta} \in \mathbf{T} = \partial\mathbf{D}$, too. In these classes, the notion of the nontangential limit functions or, equivalently, boundary traces $\mathcal{D}(e^{i\theta})$ and $\tilde{u}(e^{i\theta})$ can be defined by

$$\mathcal{D}(e^{i\theta})u_0 = \lim_{z_j \to e^{i\theta}} \mathcal{D}(z_j)u_0 \quad \text{for all} \quad u_0 \in U,$$
$$\tilde{u}(e^{i\theta}) = \lim_{z_j \to e^{i\theta}} \tilde{u}(z_j),$$

for all such $e^{i\theta} \in \mathbf{T}$, where the limit exists for all $u_0 \in U$ and all sequences $\mathbf{D} \ni z_j \to e^{i\theta} \in \mathbf{T}$ lying inside some nontangential approach region, as defined in [25, p. 6], [78, Theorem 11.18], or any other book of basic function theory. We remark that the operator limit $\mathcal{D}(e^{i\theta})$ is taken pointwise, in the strong operator topology. If $\mathcal{D}(z)$ is matrix-valued, then the strong nontangential limit is actually a nontangential norm limit, because in a finite dimensional space pointwise convergence implies norm convergence. We proceed to define the classes where boundary traces $\tilde{u}(e^{i\theta})$ and $\mathcal{D}(e^{i\theta})$ are available in a practical sense.

Suppose now that $\tilde{u}(z) \in H^p(\mathbf{D}; U)$ for $1 \le p < \infty$, and $\mathcal{D}(z) \in H^p(\mathbf{D}; \mathcal{L}(U;Y))$ for $1 \le p \le \infty$. By [77, Theorem 4.6A], if U, Y are separable, the nontangential limit functions, denoted by $\tilde{u}(e^{i\theta})$ and $\mathcal{D}(e^{i\theta})$, exist a.e. $e^{i\theta} \in \mathbf{T}$ modulo the Lebesgue measure of the unit circle $\mathbf{T}$. Actually this is true in much larger classes $N(\mathbf{D}; U)$, $N(\mathbf{D}; \mathcal{L}(U;Y))$, $N_+(\mathbf{D}; U)$, $N_+(\mathbf{D}; \mathcal{L}(U;Y))$, defined in the following.

Definition 58. *Let X be U or $\mathcal{L}(U;Y)$.*

(i) Then $N(\mathbf{D}; X)$ is the set of analytic X-valued functions $f(z)$, such that

$$\sup_{0<r<1} \int_0^{2\pi} log_+ ||f(re^{i\theta})||_X \, d\theta < \infty.$$

The set $N(\mathbf{D}; X)$ is called the Nevanlinna class, and its elements are called the functions of bounded type.

(ii) $\mathcal{H}_g(D; X)$ is the set of analytic X-valued functions $f(z)$, such that

$$\sup_{0<r<1} \int_0^{2\pi} g(log_+ ||f(re^{i\theta})||_X) \, d\theta < \infty,$$

where g is a strongly convex function. The space $\mathcal{H}_g(\mathbf{D}; X)$ is called the Hardy-Orlicz class.

(iii) $N_+(\mathbf{D}; X) := \cup \mathcal{H}_g(\mathbf{D}; X)$, where the union is taken over all strongly convex functions g.

A function $g : \mathbf{R} \to \mathbf{R}_+$ is strongly convex (in the sense of [77, p. 135]) if it is convex, nondecreasing, satisfies $\lim_{t\to\infty} g(t)/t = \infty$, and for some $c > 0$ there exists $M \geq 0$ and $a \in \mathbf{R}$ such that $g(t+c) \leq Mg(t)$ for all $t \geq a$. All the sets $\mathcal{H}_g(\mathbf{D}; X), N_+(\mathbf{D}; X), N(\mathbf{D}; X)$ are vector spaces, and $\mathcal{H}_g(\mathbf{D}; X) \subset N_+(\mathbf{D}; X) \subset N(\mathbf{D}; X)$. For additional information, see [77, Chapter 4]. In particular, choosing $g(t) = e^{pt}$ gives the $H^p(\mathbf{D}; X)$ space, for $0 < p < \infty$. Because $H^\infty(\mathbf{D}; X) \subset H^2(\mathbf{D}; X)$, also the bounded analytic functions are of bounded type.

These spaces are introduced because for $f(z) \in N(\mathbf{D}; X)$, the boundary trace function $f(e^{i\theta})$ exists almost everywhere on $\mathbf{T}$. The set of the corresponding boundary traces is denoted, quite naturally, by $N(\mathbf{T}; X)$. The mapping $N(\mathbf{D}; X) \ni f(z) \mapsto f(e^{i\theta}) \in N(\mathbf{T}; X)$ is one-to-one and linear. Furthermore, the operator products of such functions behave expectedly: If $F(e^{i\theta}) \in N(\mathbf{T}; \mathcal{L}(U; Y))$ and $G(e^{i\theta}) \in N(\mathbf{T}; \mathcal{L}(U))$, then $F(e^{i\theta})G(e^{i\theta}) \in N(\mathbf{T}; \mathcal{L}(U; Y))$. If $f(e^{i\theta}) \in N(\mathbf{T}; U)$, then $F(e^{i\theta})f(e^{i\theta}) \in N(\mathbf{T}; \mathcal{L}(Y))$. Not only the sensible products of bounded type functions are of bounded type, but also the boundary trace of the product is always the product of the boundary traces. In the infinite-dimensional cases these are nontrivial facts because the operator multiplication is not continuous in the strong operator topology — in the poetic words of [77, p. 88]: "there is more here than meets the eye". The proofs of these results are based on a powerful representation for the Nevanlinna class functions as a fraction of two H^∞ functions, with a scalar zero-free denominator. The H^∞ case can then be handled more easily. For further information, see [77, Theorem 4.2D and Theorem 4.5A].

Let us return to discuss the special case of $H^p(\mathbf{D}; X)$-spaces and the corresponding boundary trace spaces $H^p(\mathbf{T}; X)$. Ultimately, the spaces $H^p(\mathbf{T}; X)$ are identified with subspaces of certain Lebesgue spaces $L^p(\mathbf{T}; \mathcal{L}(U; Y))$ (operator-valued) and $L^p(\mathbf{T}; U)$ (Hilbert space -valued), for each $1 \leq p \leq \infty$. In order to introduce the operator and vector Lebesgue spaces, it is necessary to remind some notions of measure theory.

Definition 59. *Let U and Y be separable Hilbert spaces. Let the measure space $(\mathbf{T}, \mathcal{B}, d\theta)$ be the usual (Lebesgue completion of the) Borel σ-algebra of the unit circle $\mathbf{T}$, where $d\theta$ denotes the Lebesgue measure of $\mathbf{T}$.*

(i) The U-valued function $f(e^{i\theta})$, defined $d\theta$-almost everywhere on $e^{i\theta} \in \mathbf{T}$, is weakly (Lebesgue) measurable, if for all $u \in U$, the $\mathbf{C}$-valued function $f_u(e^{i\theta}) := \langle f(e^{i\theta}), u \rangle_U$ is $(\mathbf{T}, \mathcal{B}, d\theta)$-measurable.

(ii) The $\mathcal{L}(U; Y)$-valued function $F(e^{i\theta})$, defined $d\theta$-almost everywhere on $e^{i\theta} \in \mathbf{T}$, is weakly (Lebesgue) measurable, if for all $u \in U$, $y \in Y$, the $\mathbf{C}$-valued function $F_{u,y}(e^{i\theta}) := \langle F(e^{i\theta})u, y \rangle_Y$ is $(\mathbf{T}, \mathcal{B}, d\theta)$-measurable.

If $f(e^{i\theta}), g(e^{i\theta}), F(e^{i\theta}), G(e^{i\theta})$ are weakly measurable, then so are $F(e^{i\theta})f(e^{i\theta})$ and $F(e^{i\theta})G(e^{i\theta})$, if the products make sense. Furthermore, the following scalar functions are measurable: $\langle f(e^{i\theta}), g(e^{i\theta})\rangle_U$, $||f(e^{i\theta})||_U$ and $||F(e^{i\theta})||_{\mathcal{L}(U;Y)}$. If $r(e^{i\theta})$ is a measurable scalar function and $u \in U$, $A \in \mathcal{L}(U;Y)$, then $r(e^{i\theta})u$ and $r(e^{i\theta})A$ are weakly measurable, see [24, Part I, Chapter III]), [46, Chapter III, p. 74], [77, comment on p. 81], and [91].

Definition 60. *Let $1 \le p < \infty$. The Lebesgue spaces are defined as follows:*

(i) $L^p(\mathbf{T};U)$ *is the vector space of weakly measurable U-valued functions $f(e^{i\theta})$, defined a.e. $e^{i\theta} \in \mathbf{T}$, such that*

$$||f(e^{i\theta})||^p_{L^p(\mathbf{T};U)} := \frac{1}{2\pi}\int_0^{2\pi} ||f(e^{i\theta})||^p_U \, d\theta < \infty.$$

(ii) $L^p(\mathbf{T};\mathcal{L}(U;Y))$ *is the vector space of weakly measurable $\mathcal{L}(U;Y)$-valued functions $F(e^{i\theta})$, defined a.e. $e^{i\theta} \in \mathbf{T}$, such that*

$$||F(e^{i\theta})||^p_{L^p(\mathbf{T};\mathcal{L}(U;Y))} := \frac{1}{2\pi}\int_0^{2\pi} ||F(e^{i\theta})||^p_{\mathbf{T};\mathcal{L}(U;Y)} \, d\theta < \infty.$$

(iii) $L^\infty(\mathbf{T};\mathcal{L}(U;Y))$ *is the vector space of weakly measurable $\mathcal{L}(U;Y)$-valued functions $F(e^{i\theta})$, such that*

$$||F(e^{i\theta})||_{L^\infty(\mathbf{T};\mathcal{L}(U;Y))} := \mathit{ess\,sup}_{e^{i\theta}\in\mathbf{T}}||F(e^{i\theta}||_{\mathcal{L}(U;Y)} < \infty.$$

Note that the scalar integrals appearing in Definition 60 are well defined, by the assumed weak measurability. All the Lebesgue spaces are Banach spaces. $L^2(\mathbf{T};U)$ is a Hilbert space with the inner product

$$\langle f(e^{i\theta}), g(e^{i\theta})\rangle_{L^2(\mathbf{T};U)} := \frac{1}{2\pi}\int_0^{2\pi} \langle f(e^{i\theta}), g(e^{i\theta})\rangle_U \, d\theta.$$

Because of the nice properties of the weak measurability, much of the scalar Lebesgue space theory can be carried over to the corresponding vector-valued theory, by quite straightforward arguments. For example, because $\mathbf{T}$ is of the finite Lebesgue measure, the Hölder inequality implies that if $1 \le p_1 \le p_2 \le \infty$, then $L^{p_2}(\mathbf{T};X) \subset L^{p_1}(\mathbf{T};X)$.

For $1 \le p \le \infty$, $H^p(\mathbf{T};X)$ can be regarded as a closed subspace of $L^2(\mathbf{T};X)$, such that the Fourier coefficients of $f(e^{i\theta})$ (to be introduced in the next subsection) satisfy $f_j = 0$ for all $j < 0$, see [77, Theorem 4.7C]. Furthermore, $f(z)$ can be recovered from $f(e^{i\theta})$ by both Poisson and Cauchy integrals. Finally, the $H^p(\mathbf{D};X)$-functions $f(z)$ and their boundary traces $f(e^{i\theta}) \in H^p(\mathbf{T};X)$ can be and usually are identified by an isometry, see [77, Theorem 4.7D].

Vector-valued integration and Fourier transform

Let U and Y be separable Hilbert spaces. In order to define the Fourier transform in the Lebesgue spaces $L^p(\mathbf{T};\mathcal{L}(U;Y))$ and $L^p(\mathbf{T};U)$ for $p \geq 1$, we must have an integration theory for these Banach space -valued functions. Note that in previous subsection, only a scalar Lebesgue integration theory, together with a characterization of weakly measurable Banach space -valued functions, was required to define the spaces $L^p(\mathbf{T};\mathcal{L}(U;Y))$ and $L^p(\mathbf{T};U)$. Also recall that if $1 \leq p_1 \leq p_2 \leq \infty$, then $L^{p_2}(\mathbf{T};\mathcal{L}(U;Y)) \subset L^{p_1}(\mathbf{T};\mathcal{L}(U;Y))$ and $L^{p_2}(\mathbf{T};U) \subset L^{p_1}(\mathbf{T};U)$. It is well known that in the largest classes $L^1(\mathbf{T};\mathcal{L}(U;Y))$ and $L^1(\mathbf{T};U)$, a vector-valued integration theory (and in fact many of those) can be developed:

Proposition 61. *Let U and Y be separable Hilbert spaces. Let $f(e^{i\theta}) \in L^1(\mathbf{T};U)$ and $F(e^{i\theta}) \in L^1(\mathbf{T};\mathcal{L}(U;Y))$.*

(i) There is a unique $c \in U$ such that for all $u \in U$

$$\langle c, u\rangle_U = \int_0^{2\pi} \left\langle f(e^{i\theta}), u\right\rangle_U \, d\theta.$$

We call c the weak Lebesgue (Pettis) integral of $f(e^{i\theta})$ and write $\int_0^{2\pi} f(e^{i\theta})\, d\theta := c$.

(ii) There is a unique $C \in \mathcal{L}(U;Y)$ such that for all $u \in U$, $y \in Y$

$$\langle Cu, y\rangle_Y = \int_0^{2\pi} \left\langle F(e^{i\theta})u, y\right\rangle_Y \, d\theta.$$

We call C the weak Lebesgue (Pettis) integral of $F(e^{i\theta})$ and write $\int_0^{2\pi} F(e^{i\theta})\, d\theta := C$.

Proof. For claim (i), see [46, Definition 3.7.1 and Theorem 3.7.1], and note that U, as a Hilbert space, is reflexive. We outline the proof how claim (ii) follows from claim (i). Let $u \in U$. Then $F(e^{i\theta})u$ is a Y-valued weakly measurable function, and by claim (i) there is a unique $c_u \in Y$ such that

$$\langle c_u, y\rangle_Y = \int_0^{2\pi} \left\langle F(e^{i\theta})u, y\right\rangle_Y \, d\theta$$

for all $y \in Y$. It is easy to show that the mapping $U \ni u \mapsto c_u \in Y$ is linear, and we define a linear mapping $C : U \to Y$ by $Cu := c_u$. It remains to be shown that C is bounded. Let now $u \in U$ and $y \in Y$ be arbitrary. Then

$$|\langle Cu, y\rangle_Y| \le \int_0^{2\pi} |\langle F(e^{i\theta})u, y\rangle_Y| \, d\theta \le ||u||_U \cdot ||y||_Y \cdot \int_0^{2\pi} ||F(e^{i\theta})|| \, d\theta,$$

where the first estimate holds by the property of scalar Lebesgue integral, and second by the Schwarz inequality. Because $F(e^{i\theta}) \in L^1(\mathbf{T}; \mathcal{L}(U;Y))$, the integral of its norm is finite, and it the follows that

$$||T||_{\mathcal{L}(U;Y)} := \sup_{||u||_U = ||y||_Y = 1} |\langle Cu, y\rangle_Y| \le ||F(e^{i\theta})||_{L^1(\mathbf{T};\mathcal{L}(U;Y))} < \infty.$$

We regard this proposition as proved. □

Now that we can integrate, we are prepared to introduce the Fourier transforms. Let $f(e^{i\theta}) \in L^1(\mathbf{T}; U)$ and $F(e^{i\theta}) \in L^1(\mathbf{T}; \mathcal{L}(U;Y))$. Trivially, the functions $e^{i\theta} \mapsto e^{ij\theta} f(e^{i\theta}) \in L^1(\mathbf{T}; U)$ and $e^{i\theta} \mapsto e^{ij\theta} F(e^{i\theta}) \in L^1(\mathbf{T}; \mathcal{L}(U;Y))$ for all $j \in \mathbf{Z}$, and we can uniquely define the weak integrals

$$f_j := \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) e^{-ij\theta} \, d\theta \in U, \quad F_j := \frac{1}{2\pi} \int_0^{2\pi} F(e^{i\theta}) e^{-ij\theta} \, d\theta \in \mathcal{L}(U;Y).$$

These integrals are called the Fourier coefficients of the respective functions. We call the formal series

$$f(e^{i\theta}) \sim \sum f_j e^{ij\theta}, \quad F(e^{i\theta}) \sim \sum F_j e^{ij\theta}$$

the Fourier series of the respective functions. Two Fourier series are identical if all their respective coefficients f_j or F_j are identical. The mappings

$$f(e^{i\theta}) \mapsto \{f_j\}_{j \in \mathbf{Z}} \subset U, \quad F(e^{i\theta}) \mapsto \{F_j\}_{j \in \mathbf{Z}} \subset \mathcal{L}(U;Y)$$

are called the Fourier transforms of the respective spaces. It is easy to show that the Fourier transform is a linear mapping, and the Fourier coefficient are uniformly bounded: $||f_j|| \le ||f(e^{i\theta})||_{L^1(\mathbf{T};U)} \le \sqrt{2\pi} ||f(e^{i\theta})||_{L^2(\mathbf{T};U)}$ and $||F_j|| \le ||F(e^{i\theta})||_{L^1(\mathbf{T};\mathcal{L}(U;Y))} \le \sqrt{2\pi} ||F(e^{i\theta})||_{L^2(\mathbf{T};\mathcal{L}(U;Y))}$. The questions of convergence of the Fourier series (in various topologies) are generally highly nontrivial. In this paper, the classes $L^2(\mathbf{T}; U)$ and $L^2(\mathbf{T}; \mathcal{L}(U;Y))$ are of particular interest. The case of the Hilbert space is well known:

Proposition 62. *The Fourier transform $\mathcal{F}_U : f(e^{i\theta}) \mapsto \{f_j\}_{j \in \mathbf{Z}}$ is an isometric isomorphism of the Hilbert space $L^2(\mathbf{T}; U)$ onto the Hilbert space $\ell^2(\mathbf{Z}; U)$. The Fourier series $\sum f_j e^{i\theta}$ converges to $f(e^{i\theta})$ in $L^2(\mathbf{T}; U)$. The Parseval identity holds*

$$\langle f(e^{i\theta}), g(e^{i\theta})\rangle_{L^2(\mathbf{T};U)} = \langle \{f_j\}, \{g_j\}\rangle_{\ell^2(\mathbf{Z};U)}.$$

The Fourier transform intertwines the shift τ and the multiplication operator M_ξ by the function $\xi(e^{i\theta}) = e^{i\theta}$ on $\mathbf{T}$

$$\mathcal{F}_U M_\xi = \tau \mathcal{F}_U.$$

The closed subspace $H^2(U) \subset L^2(\mathbf{T}; U)$ is mapped onto the closed subspace $\ell^2(\mathbf{Z}_+; U) \subset \ell^2(\mathbf{Z}; U)$.

However, we need the following result on the operator-valued $L^2(\mathbf{T}; \mathcal{L}(U;Y))$.

Proposition 63. *Let U and Y be separable Hilbert spaces, and $u \in U$ arbitrary. Let $F(e^{i\theta}) \in L^1(\mathbf{T}; \mathcal{L}(U;Y))$. Define the Y-valued function $F_u(e^{i\theta}) := F(e^{i\theta})u$. Then*

(i) $F_u(e^{i\theta}) \in L^1(\mathbf{T}; Y)$,

(ii) the Fourier coefficients $\{F_j\}_{j\in\mathbf{Z}}$ of $F(e^{i\theta})$ and $\{(F_u)_j\}_{j\in\mathbf{Z}}$ of $F_u(e^{i\theta})$ satisfy

$$F_j u = (F_u)_j \quad \text{for all} \quad j \in \mathbf{Z},$$

(iii) the Fourier series $\sum_{j\in Z} (F_j u)\, e^{ij\theta}$ converges in $L^2(\mathbf{T}; Y)$ to $F(e^{i\theta})u$.

Proof. Claim (i) is trivial. To prove claim (ii), fix $u \in U$, $j \in \mathbf{Z}$, and let $y \in Y$ be arbitrary. By the definition of weak integral, the Fourier coefficient $F_j \in \mathcal{L}(U;Y)$ is an operator such that

$$\langle F_j u, y\rangle_Y = \frac{1}{2\pi}\int_0^{2\pi} \left\langle F(e^{i\theta})\, e^{-ij\theta} u, y\right\rangle_Y \, d\theta \tag{1.57}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \left\langle F(e^{i\theta}) u, y\right\rangle_Y \, e^{-ij\theta}\, d\theta. \tag{1.58}$$

for all $y \in Y$. By the definition of the weak Hilbert space -valued integral, the Fourier coefficient $(F_u)_j \in Y$ is an element such that

$$\langle (F_u)_j, y\rangle_Y = \frac{1}{2\pi}\int_0^{2\pi} \left\langle F_u(e^{i\theta})\, e^{-ij\theta}, y\right\rangle_Y \, d\theta \tag{1.59}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \left\langle F(e^{i\theta}) u, y\right\rangle_Y \, e^{-ij\theta}\, d\theta. \tag{1.60}$$

for all $y \in Y$. Comparing the right hand sides of equations (1.57) and (1.59) implies that $\langle F_j u, y\rangle_Y = \langle (F_u)_j, y\rangle_Y$ for all y, or equivalently $F_j u = (F_u)_j$. Because u and j are arbitrary, this proves claim (ii). The last claim (iii) follows from the previous claim and Proposition 62. $\square$

1.11 Notes and references

In the monograph [44] (Halanay, Ionescu, 1994) the authors state at the beginning of the preface

> *Thus we have often found ourselves in something of a dilemma: on the one hand many facts should be known and on the other hand it is nearly impossible to give an adequate reference to all.*

It is curious that after several decades of intensive research on linear dynamical systems of various kinds, even the field of time-invariant systems appears still to deserve the same comment. Analogous structures have been and are being studied, more or less independently, by several authors under various formalisms. This makes it a rather challenging task for a researcher to obtain even a most humble general understanding of the modern system theory. Nevertheless, such an understanding is quite necessary, as new mathematics should be built upon old mathematics.

Discrete time linear systems and their I/O maps

Let us make a brief and definitely not an exhaustive survey into the literature, with an emphasis on monographs. One of the early books on mathematical system theory is [48] (Kalman, Falb and Arbib, 1969). This book is divided into four parts, of which Parts I and IV, written by Kalman, are most interesting to us. Part I is written in the language of matrix algebra, and basic definitions and results of time-invariant linear discrete time systems are given. The state estimation (Kalman filter) and regulator construction problem are solved for such systems. In Part IV, a purely algebraic theory of discrete time linear systems is developed as a beautiful application of the module theory. The I/O maps of systems are seen as certain module homomorphisms over a polynomial ring. The involved vector spaces can be over any field, also finite. Canonical realizations for finite dimensional systems are given by using the restricted I/O map (corresponding to the Hankel operator) and a certain factorial module as state space. These ideas are presented in the first part of [35] (Fuhrmann, 1981), too. In the final part of [35], the linear systems are considered whose state space is an infinite-dimensional Hilbert space. Realization theoretic results for such I/O maps and systems are given, and the failure of the state space isomorphism techniques is indicated. Special realizations, built around the shift operator, are constructed. Generally speaking, many ideas, modulo natural restrictions, can be carried over from the polynomial models and the algebraic system theory to the operator models and the infinite dimensional system theory on Hilbert spaces, in a quite transparent manner. In fact, [38] (Fuhrmann, 1996) is an exposition of the linear algebra, written by an operator theorist. The article

[37] contains interesting historical notes and an outline of the algebraic system theory. Also [4] (Baras, Brockett and Fuhrmann, 1974), [3] (Baras and Brockett, 1975), [31], [32], [29], [33] and [34] (Fuhrmann) are valuable references, even though much of their contents can be found in the monograph [35].

Monographs on related but more general operator theory and harmonic analysis are [90] (Sz.-Nagy and Foias, 1970), [77] (Rosenblum and Rovnyak, 1985), [70] (Nikolskii, 1986), [27] (Frazho and Foias, 1990) and [28] (Foias, Özbay and Tannenbaum, 1996). These books contain, among other things, descriptions of linear, causal and shift-invariant operators on Hilbert spaces and their applications to a number of system theoretic problems, including the H^∞ control problem. Representations of shift-invariant operators by analytic functions are considered in [97] (G. Weiss, 1991) and [91] (Thomas, 1997). However, these works do not contain an (essential) contribution to the state space realizations of analytic operator-valued functions, i.e. DLSs.

In the monograph [44] (Halanay and Ionescu, 1994), a formalism for time-variant linear discrete time systems is developed, and a number of references and historical remarks are given. Exponentially dichotomic or exponentially stable evolutions are considered, see [44, Chapter 1, Section 3]. The aim of the book is to solve the operator discrete time disturbance attenuation problem, which is a time-variant version of the suboptimal (state space) H^∞ control problem. The systems are assumed to be exponentially stabilizable, and the required closed loop system is exponentially stable, by the definition of the disturbance attenuation problem. Naturally, this time-variant theory can be applied to time-invariant power stabilizable and power stable problems as well.

Continuous time systems and their I/O maps

In continuous time, there is a number of possibilities to develop a linear state space system theory. A classical approach is to consider the dynamical system

$$\begin{cases} x'(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t), \quad t \geq 0 \end{cases} \tag{1.61}$$

where A is a generator of a strongly continuous semigroup, and B, C and D are bounded operators on Hilbert spaces. Such systems, together the associated linear quadratic optimal control and controller synthesis problems, are the subject of monograph [18] (Curtain and Zwart, 1995). Also the matrix-valued Callier–Desoer class of transfer functions is introduced; the original references are [6], [7] (Callier and Desoer, 1978, 1980) and [10] (Callier and Winkin, 1986). We remark that the impulse response of a linear system of type (1.61) is always a strongly continuous function by the strong continuity of the semigroup. See also [18, Lemma 7.3.1] for semigroups having a more general growth bound.

This severely restricts the applicability of the dynamical system (1.61) with B and C bounded.

To be able to cover a larger class of transfer functions, the more general class of well-posed linear systems (WPLS) (abstract linear system in the sense of Salamon and G. Weiss) is defined in roughly the same way as our DLS in I/O form. Three equivalent axiomatizations are used, by Salamon, Staffans and G. Weiss. Our formalism of DLSs has been created, with small modifications, from that appearing in articles [82], [83], [84], [86], [85], [88] and the monograph [89] (Staffans, 1995 – 1999). The general question is, to what extent a WPLS can be written in the form of the differential equations (1.61) for some (possibly unbounded) generating operators A, B, C and D. This is analogous to the relations between DLS in I/O form and in difference equation form, except that a fair amount of extra complication is now present due to the unboundedness of the generating operators and inclusions of various state (vector) subspaces with various topologies.

We first consider the question how to make sense out of the equations (1.61) for unbounded B and C. Let A be a generator of a strongly continuous semigroup. If the input operator B and the output operator C are admissible for A (or the semigroup generated by A) in the sense of [80], [81], (Salamon, 1987, 1989) and [93], [94] (G. Weiss), then controllability and observability maps can be associated to pairs (A, B) and (C, A), respectively. Such a triple of operators (A, B, C) defines a family of (nonstandardly defined) transfer functions $z \mapsto G(z)$, analytic in some right half plane, by setting

$$G(z) = -(s - \beta)C(z - A)^{-1}(\beta - A)^{-1}B + G(\beta),$$

where $z, \beta \in \mathbf{C}$, $z \neq \beta$ are in the resolvent set of A. An extra well-posedness assumption is imposed by requiring that these transfer functions are bounded in some right half plane — such triples (A, B, C) are called well-posed. A well-posed triple (A, B, C) defines a family of WPLSs because now even the I/O map can be defined (but only in a nonstandard way). The nonuniqueness of the WPLS comes from the fact that the bounded feed-through operator D has not been fixed, for the reason that a general WPLS need not have a feed-through operator in the first place, see [21] (Curtain and G. Weiss, 1989).

Conversely, a WPLS defines a well-posed triple of operators (A, B_a, C_a) such that a variant of the differential equation (1.61) holds. It is given by

$$(1.62) \qquad \begin{cases} x'(t) = Ax(t) + B_a u(t) \\ y(t) = C_L \left(x(t) - (\beta - A)^{-1} B_a u(t) \right) + G(\beta)u(t), \end{cases}$$

where C_L is the Lebesgue extension of C_a, $G(z)$ is any well-posed (nonstandardly defined) transfer function associated to the triple (A, B_a, C_a), and $\beta \in \mathbf{C}$ is arbitrary in the resolvent set of A. The operators B_a and C_a are given by representation theorems in [81] (Salamon, 1989), [21] (Curtain and G. Weiss, 1989),

[94] and [93] (G. Weiss, 1989). See also survey [19] (Curtain, 1997). Stability notions for WPLSs can be found in [76] (Rebarber, 1993) and [83] (Staffans, 1997). State feedback and output injection, stabilizability and detectability notions, together with coprime factorizations of the I/O map are considered in [84] (Staffans, 1998). The family of possible transfer functions for WPLSs is described in [98] and [81]; in particular, all H^∞ transfer functions can be realized by WPLSs.

Unfortunately, the dynamical system (1.62) is not of the form of equation (1.62) because we cannot generally write $C_L\left(x(t) - (\beta - A)^{-1}B_a u(t)\right) + G(\beta)u(t) = C_L x(t) + \left(G(\beta) - C_L(\beta - A)^{-1}B_a\right)u(t)$ without getting out of $\operatorname{dom}(C_L)$. Also the transfer functions $G(z)$ are not of the familiar form $D + C(z-A)^{-1}B$. To fix this problem, the generality of the notion of the WPLS has to be reduced. Following [95] (G. Weiss, 1989), the subclass of the regular WPLSs is introduced by requiring the existence of the limit

$$Dv = \frac{1}{\tau}\lim_{\tau\to 0}\int_0^\tau y_v(\sigma)\,d\sigma,$$

defining a bounded feed-through operator D, where $y_v(\sigma)$ is the step response of the I/O map, corresponding to any constant input $v \in U$. The transfer functions of regular WPLSs (in the set of transfer functions of general WPLSs) are characterized by a radial limit condition at $+\infty$ in [99] (G. Weiss, 1994), and the state feedback structure is considered in [98] (G. Weiss, 1994). In Section 1.1 of Chapter 2, we shortly review the optimal control and Riccati equation theories of regular WPLSs.

The Pritchard–Salamon systems are a well-known subclass of the regular WPLSs. Practically all the results of the finite dimensional theory generalize to this class. Basic references are [74], [75] (Pritchard and Salamon, 1985, 1987), [17] (Curtain, Logemann, Townley and H. Zwart, 1994) and [92] (van Keulen, 1993). For the characterization of the I/O maps of (slightly differently defined) Pritchard–Salamon systems, see [52] (Kaashoek, van der Mee and Ran, 1997).

The discrete time and continuous time transfer functions can be mapped to each other by using the Cayley mapping, see [20] (Curtain and Oostveen) and [71] (Ober and Montgomery-Smith, 1990). By this technique, some continuous time results can be converted to discrete time results, and vice versa.

Discussion of the DLS formalism

We conclude this section with a general discussion of the formalism presented in this chapter. We have introduced two equivalent formalisms (DLS in difference equation form and in I/O form) to describe the same class of objects, namely the well-posed, causal shift-invariant linear operators in discrete time. At first sight, this might seem a little superfluous, and we now try to defend ourselves.

We note that all the operators A, B, C, D, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ appearing in quadruples $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ and $\left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ are separate functional blocks, present in any (linear) state space model. The interaction between controllability, observability and I/O maps can be conveniently described because these operators constitute the DLS in I/O form in our formalism. What we have actually done is to collect the operators of the same kind into two different structures: DLS in I/O form and in difference equation form. In applications we use the structure that has less notational overhead. In this framework, the discrete time theory is presented in the analogous manner as the continuous time theory in [89] (Staffans, 1999). Also nonlinear generalizations are admitted.

Our DLS formalism is rather heavy because it is two-fold. Of course, we could use either only the DLSs in difference equation form, or DLSs in I/O form, to obtain an equally powerful theory. If we abandon the difference equation formalism, we would have much trouble in writing down the basic difference equations of systems and the algebraic Riccati equations. If we leave out the I/O formalism, then we would lose the notational analogy to the continuous time WPLSs. Furthermore, we would be compelled to either use the mappings $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ as separate objects, or to represent them by the corresponding transfer functions. In the former choice, we would have lost only the (abstract but useful) notion of the DLS in I/O form, but have the same notational burden. In the latter case, we would end up in notational clumsiness, because the basic operators would be written down as multiplication operators.

For DLS $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ in difference equation form, the bilateral shift τ is an external object in the sense that the four operators defining ϕ have no "dynamical properties". In the same sense, the shift τ is an internal object for the same DLS $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ in I/O form, because the mappings $\mathcal{B}$ and $\mathcal{C}$ intertwine the shift τ to the semigroup generator, and the I/O map $\mathcal{D}$ to the shift on another space. Because the full theory of time-invariant well-posed discrete time systems can be written in two formalisms, it becomes an interesting question to ask how a notion in one formalism is interpreted in the other. In this chapter, we have considered the case of the DLSs itself and the state feedback. The output injection structure, being dual to the state feedback structure, has not been explicitly considered as it has no application in this book. In Chapters 2 and 3, we see that the discrete time algebraic Riccati equation (DARE) (a difference equation form object) is connected to the spectral factorization (an I/O form object). In Chapter 4, the same work is done for the natural order relation of the self-adjoint solutions of the DARE (a difference equation form object), and the partial ordering of inner factors of the I/O map (an I/O form object). This is what a decent DARE theory basically is: identifying corresponding objects under sufficiently but not too restrictive technical assumptions.

Chapter 2

Critical control problem

2.1 Introduction

In this chapter, we present and solve an abstract control problem, associated to an I/O stable DLS $\Phi = \left[\begin{smallmatrix} \mathcal{A}^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ and a self-adjoint, possibly nondefinite cost operator J penalizing the outputs of Φ. We first define a critical control problem with the aid of Φ and J. Then, under proper technical assumptions, we prove the equivalence of

- the solvability of a minimax cost optimization problem, associated to the pair Φ and J,
- the solvability of a certain inner-outer factorization problem for the I/O map $\mathcal{D}$, or equivalently, a spectral factorization problem of the Popov operator $\mathcal{D}^* J \mathcal{D}$, and
- the existence of a special, critical solution of an associated (weak) discrete time algebraic Riccati equation.

We make it a standing hypothesis that $\overline{\mathrm{dom}\,(\mathcal{C})} = H$ throughout this chapter. Because we do not generally assume the DLS Φ to be output stable, we must present the algebraic Riccati equation in such a form that its solutions are conjugate-symmetric sesquilinear forms. Under the output stability assumption, the sesquilinear forms can be replaced by bounded self-adjoint operators on the state space of Φ.

The technical outline of this chapter is as follows. In Section 2.2 we define and prove basic facts about the critical control problem, associated to Φ and

J. Section 2.3 is devoted to the study of (J,S)-inner-outer factorizations of the (extended topological) I/O map $\mathcal{D}$ and S-spectral factorizations of the Popov operator $\mathcal{D}^*J\mathcal{D}$. In Section 2.4 we show that the critical control problem can be solved in state feedback form if and only if $\mathcal{D}$ has a (J,S)-inner-outer factorization, see Theorem 89. Under the same conditions, the sesquilinear form $P_0^{\mathrm{crit}}(\ ,\ )$ of Definition 76 satisfies the discrete time algebraic Riccati equation of Definition 94, as shown in Section 2.5. The converse result is given in Section 2.6. There the existence of a critical solution of the same weak algebraic Riccati equation implies, under stronger technical assumptions, that the equivalent conditions of Theorem 89 hold. Finally, the three equivalent conditions are collected in Theorem 103, the main result of this chapter.

Most of the results of this chapter appeared in [55] (Malinen, 1997). A short version [54] has been presented in the ECC97 conference (Brussels, July, 1997).

2.2 Critical controls and operators

In this section, we associate a minimax control problem to a DLS $\Phi = \left[\begin{smallmatrix} \mathcal{A}^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ and a possibly nondefinite cost functional that evaluates the outputs of Φ.

Let $J \in \mathcal{L}(Y)$ a self-adjoint operator. This operator induces a nonstandard (i.e. not necessarily positive definite) inner product on the output space Y of the DLS Φ. The operator J is called the cost operator, and the associated cost functional is defined as follows.

Definition 64. *Let $\Phi = \left[\begin{smallmatrix} \mathcal{A}^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ be a DLS, and let $J \in \mathcal{L}(Y)$, $R \in \mathcal{L}(U)$ be self-adjoint. Then the nonstandard cost for the output $\tilde{y} = \tilde{y}(x_0, \tilde{u})$ of Φ is*

$$J(x_0, \tilde{u}) := \sum_{j \geq 0} \left(\langle y_j(x_0, \tilde{u}), J y_j(x_0, \tilde{u}) \rangle_Y + \langle u_j, R u_j \rangle_U \right), \tag{2.1}$$

whenever the sum converges either to a finite or infinite limit. Here $\tilde{u} \in \ell^2(\mathbf{Z}_+; U)$ is an input sequence, and $x_0 \in \mathrm{dom}\,(\mathcal{C})$ is the initial state of the DLS at time $j = 0$.

It is a known fact that the control sequence $\tilde{u}$ can always be thought to be "free of charge" because the input can be made visible in the output, by changing the DLS Φ. More precisely, define $C' \in \mathcal{L}(U, Y \times U)$, $D' \in \mathcal{L}(H, Y \times U)$, and $J' \in \mathcal{L}(Y \times U, Y \times U)$ by

$$C' = \begin{pmatrix} C \\ 0 \end{pmatrix}, \quad D' = \begin{pmatrix} D \\ I \end{pmatrix}, \quad J' = \begin{pmatrix} J & 0 \\ 0 & R \end{pmatrix}.$$

Then replace the original DLS $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ by the extended system $\phi' = \left(\begin{smallmatrix} A & B \\ C' & D' \end{smallmatrix}\right)$. Now, if $z_k(x_0, \tilde{u}) := C' x_k + D' u_k$ is the output of ϕ', we get

$$\langle y_j(x_0, \tilde{u}), J y_j(x_0, \tilde{u}) \rangle_Y + \langle u_j, R u_j \rangle_U = \langle z_k(x_0, \tilde{u}), J' z_k(x_0, \tilde{u}) \rangle_{Y \oplus X}.$$

We conclude that there is no loss of generality if we set $R = 0$ in equation (2.1), and this is what we always do. In this case equation (2.1) takes the form

$$J(x_0, \tilde{u}) = \langle \mathcal{C} x_0 + \mathcal{D} \bar{\pi}_+ \tilde{u}, J(\mathcal{C} x_0 + \mathcal{D} \bar{\pi}_+ \tilde{u}) \rangle_{\ell^2(\mathbf{Z}_+; Y)}. \tag{2.2}$$

Note that we use the same letter J for both the self-adjoint operator and for the associated cost functional. Furthermore, in equation (2.2), the cost operator J is extended to a self-adjoint static operator on $\ell^2(\mathbf{Z}_+; Y)$ in a natural way. We shall make this extension throughout this book.

Proposition 65. *Let $J \in \mathcal{L}(Y)$ and $\Phi = \left[\begin{smallmatrix} \mathcal{A}^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ be an I/O stable DLS. Then $|J(x_0, \tilde{u})| < \infty$ for all $x_0 \in \mathrm{dom}\,(\mathcal{C})$ and $\tilde{u} \in \ell^2(\mathbf{Z}_+; U)$.*

Proof. If $x_0 \in \mathrm{dom}\,(\mathcal{C})$ and $\tilde{u} \in \ell^2(\mathbf{Z}_+;U)$, then by the definition of $\mathrm{dom}\,(\mathcal{C})$ and I/O stability, $\mathcal{C}x_0 + \mathcal{D}\tilde{u} \in \ell^2(\mathbf{Z}_+;Y)$. The claim immediately follows. □

If J is nonnegative, then one would be tempted to find an optimal control sequence $\tilde{u}^{\mathrm{opt}}(x_0)$ that minimizes the cost $J(x_0, \tilde{u}^{\mathrm{opt}}(x_0))$ for an arbitrary given initial state x_0. With the nonstandard case, the cost could be made as large or small as we please, just by choosing a suitable input $\tilde{u}$. So, there is not much sense in speaking about minimal or maximal cost. We are led to look for certain control sequences, the critical control sequences in $\ell^2(\mathbf{Z}_+;U)$.

Definition 66. *Let $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ be an I/O stable DLS, and let $x_0 \in \mathrm{dom}\,(\mathcal{C})$ be an initial state. The control $\tilde{u}' \in \ell^2(\mathbf{Z}_+;U)$ is critical at x_0 if the Frechet derivative of the function*

$$\ell^2(\mathbf{Z}_+;U) \ni \tilde{u} \mapsto J(x_0,\tilde{u}) \in \mathbf{R}$$

vanishes at $\tilde{u} = \tilde{u}'$.

So, all the critical control sequences are saddle points of the cost functional $J(x_0,\tilde{u})$. For a fixed $x_0 \in \mathrm{dom}\,(\mathcal{C})$ it is not *a priori* known whether there is a critical control sequence at all, or whether the critical control is unique if it exists. Let us first calculate a necessary and sufficient condition for a control to be critical, and worry the existence and uniqueness questions later.

Lemma 67. *Let $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ be an I/O stable DLS, and let $x_0 \in \mathrm{dom}\,(\mathcal{C})$ be an initial state. Then the control sequence $\tilde{u}' \in \ell^2(\mathbf{Z}_+,U)$ is critical at x_0 if and only if*

$$\bar{\pi}_+\mathcal{D}^*J\mathcal{C}\,x_0 = -\bar{\pi}_+\mathcal{D}^*J\mathcal{D}\bar{\pi}_+\tilde{u}'. \tag{2.3}$$

Furthermore, the corresponding (critical) output sequence $\tilde{y}(x_0,\tilde{u}')$ satisfies

$$\bar{\pi}_+\mathcal{D}^*J\tilde{y}(x_0,\tilde{u}') = 0. \tag{2.4}$$

Proof. We have for all $\tilde{u} \in \ell^2(\mathbf{Z}_+;U)$ and $x_0 \in \mathrm{dom}\,(\mathcal{C})$

$$J(x_0,\tilde{u}) = \langle \mathcal{C}x_0 + \mathcal{D}\bar{\pi}_+\tilde{u}, J\,(\mathcal{C}x_0 + \mathcal{D}\bar{\pi}_+\tilde{u})\rangle_{\ell^2(\mathbf{Z}_+;Y)}\,.$$

The control sequence $\tilde{u}'$ is critical at x_0 if and only if

$$\frac{d}{d\epsilon}J(x_0,\tilde{u}' + \epsilon\tilde{w}) = 0 \quad \text{at} \quad \epsilon = 0$$

for all $\tilde{w} \in \ell^2(\mathbf{Z}_+;U)$. Here ϵ is a real-valued variable. By a simple calculation, we obtain

$$\begin{aligned}&\frac{d}{d\epsilon}J(x_0,\tilde{u}' + \epsilon\tilde{w})|_{\epsilon=0} \\ &= 2Re\,\langle \tilde{w}, \bar{\pi}_+\mathcal{D}^*J\mathcal{C}x_0 + \bar{\pi}_+\mathcal{D}^*J\mathcal{D}\bar{\pi}_+\tilde{u}'\rangle_{\ell^2(\mathbf{Z}_+;Y)} = 0,\end{aligned}$$

which gives equations (2.3) and (2.4). □

We have to comment on the precise meaning of the notation $\bar{\pi}_+\mathcal{D}^*$, appearing in Lemma 67. Because the DLS $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ is assumed to be I/O stable, the Toeplitz operator $\mathcal{D}\bar{\pi}_+ : \ell^2(\mathbf{Z}_+; U) \to \ell^2(\mathbf{Z}_+; Y)$ is bounded, see Definition 32 and the discussion following it. As discussed after Definition 32, $\mathcal{D}\bar{\pi}_+ : \ell^2(\mathbf{Z}_+; U) \to \ell^2(\mathbf{Z}_+; U)$ can be extended (by shift-invariance and boundedness) to a unique bounded, shift-invariant and causal operator $\bar{\mathcal{D}} : \ell^2(\mathbf{Z}; U) \to \ell^2(\mathbf{Z}; Y)$. Clearly, an I/O stable Toeplitz operator $\mathcal{D}\bar{\pi}_+$ can always be regarded as a restriction of $\bar{\mathcal{D}}$ to $\ell^2(\mathbf{Z}_+; U)$. It now follows from the boundedness of all the operators that

$$(\mathcal{D}\bar{\pi}_+)^* = (\bar{\mathcal{D}}\bar{\pi}_+)^* = \bar{\pi}_+\bar{\mathcal{D}}^*.$$

For brevity, we write $\mathcal{D}$ instead of $\bar{\mathcal{D}}$ throughout this book. This gives the precise meaning to expression $\bar{\pi}_+\mathcal{D}^*$. When we identify the cost operator $J \in \mathcal{L}(Y)$ with the unique self-adjoint static operator (I/O map) that it induces on $\ell^2(\mathbf{Z}; Y)$ by Proposition 6, also the expressions $\bar{\pi}_+\mathcal{D}^* J \mathcal{C}$ and $\bar{\pi}_+\mathcal{D}^* J \mathcal{D}\bar{\pi}_+$ appearing in equations (2.3) and (2.4) get a precise meaning. The latter of these operators is important enough to deserve a name of its own, given in the following definition.

Definition 68. *Let $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ be an I/O stable DLS and $J \in \mathcal{L}(Y)$ a cost operator.*

(i) The Toeplitz operator $\bar{\pi}_+\mathcal{D}^* J \mathcal{D}\bar{\pi}_+ \in \mathcal{L}(\ell^2(\mathbf{Z}_+; U))$ is the Popov operator of Φ and J.

(ii) The DLS Φ is J-coercive, if the Toeplitz operator $\bar{\pi}_+\mathcal{D}^* J \mathcal{D}\bar{\pi}_+$ has a bounded inverse on $\ell^2(\mathbf{Z}_+; U)$.

The name "Popov operator" comes from [47]. The Fourier transform of the Popov operator is called the Popov function. In [45], the Popov operator is known as the power spectrum operator. A fair amount of control theory has been written around the Popov operator, see [100], [102] and the references therein. By Lemma 71, the J-coercivity serves as a sufficient condition for the existence of the unique critical control sequence at any $x_0 \in \operatorname{dom}(\mathcal{C})$.

Proposition 69. *Let Φ be an I/O stable and J-coercive DLS. Then $\mathcal{D}\bar{\pi}_+ : \ell^2(\mathbf{Z}_+; U) \to \ell^2(\mathbf{Z}_+; Y)$ is coercive and* range$(\mathcal{D}\bar{\pi}_+)$ *is closed. Similarly, $J\mathcal{D}\bar{\pi}_+ : \ell^2(\mathbf{Z}_+; U) \to \ell^2(\mathbf{Z}_+; Y)$ is coercive and* range$(J\mathcal{D}\bar{\pi}_+)$ *is closed. Furthermore, $\mathcal{D} : \ell^2(\mathbf{Z}; U) \to \ell^2(\mathbf{Z}; Y)$ is coercive.*

Proof. To show the coercivity, assume for contradiction that there is a sequence $\{\tilde{u}_j\} \subset \ell^2(\mathbf{Z}_+; U)$, $||\tilde{u}_j||_{\ell^2(\mathbf{Z}_+;U)} = 1$, such that $\mathcal{D}\bar{\pi}_+\tilde{u}_j \to 0$ as $j \to 0$. Because $\mathcal{D}$ is bounded by the assumed I/O stability, so is $\bar{\pi}_+\mathcal{D}^* J$. But then $\bar{\pi}_+\mathcal{D}^* J \mathcal{D}\bar{\pi}_+\tilde{u}_j \to 0$ as $j \to 0$. This is a contradiction against the J-coercivity of the DLS Φ. The claim involving $J\mathcal{D}\bar{\pi}_+$ is quite analogous. The coercivity of $\mathcal{D}$ follows from its I/O stability, shift-invariance and the coercivity of $\mathcal{D}\bar{\pi}_+$, an in the proof of claim (ii) of Proposition 46. □

Now equation (2.3) calls for the following definition and Lemma 71.

Definition 70. *Let $J \in \mathcal{L}(Y)$ be a cost operator. Let $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ be an I/O stable and J-coercive DLS, such that $\overline{\mathrm{dom}\,(\mathcal{C})} = H$.*

(i) The densely defined linear operator $\mathcal{K}^{\mathrm{crit}} : H \supset \mathrm{dom}\left(\mathcal{K}^{\mathrm{crit}}\right) \to \ell^2(\mathbf{Z}_+; U)$, defined by

$$\mathcal{K}^{\mathrm{crit}} := -(\bar{\pi}_+ \mathcal{D}^* J \mathcal{D} \bar{\pi}_+)^{-1} \bar{\pi}_+ \mathcal{D}^* J \mathcal{C}$$

is the critical (closed loop) feedback map, where $\mathrm{dom}\left(\mathcal{K}^{\mathrm{crit}}\right) := \mathrm{dom}\,(\mathcal{C})$.

(ii) The densely defined linear operator $K^{\mathrm{crit}} : H \supset \mathrm{dom}\left(K^{\mathrm{crit}}\right) \to \ell^2(\mathbf{Z}_+; U)$, defined by

$$K^{\mathrm{crit}} := \pi_0 \mathcal{K}^{\mathrm{crit}}$$

(the spaces $\mathrm{range}\,(\pi_0)$ and U have been identified) is the critical (closed loop) one step feedback operator, where $\mathrm{dom}\left(K^{\mathrm{crit}}\right) := \mathrm{dom}\,(\mathcal{C})$.

(iii) The densely defined linear operator $\mathcal{C}^{\mathrm{crit}} : H \supset \mathrm{dom}\left(\mathcal{C}^{\mathrm{crit}}\right) \to \ell^2(\mathbf{Z}_+; Y)$, defined by

$$\mathcal{C}^{\mathrm{crit}} := \mathcal{C} + \mathcal{D}\mathcal{K}^{\mathrm{crit}},$$

is the critical (closed loop) observability map, where $\mathrm{dom}\left(\mathcal{C}^{\mathrm{crit}}\right) := \mathrm{dom}\,(\mathcal{C})$.

The domains are dense because our standing assumption that $\mathrm{dom}\,(\mathcal{C})$ is dense in H. We shall not state this explicitly from now on. It is easy to see that the above operators are well defined. If K^{crit} is bounded, we can identify it with its continuous extension to the whole of H. By a simple manipulation, we see that

$$\mathcal{C}^{\mathrm{crit}} = (\bar{\pi}_+ - \bar{\pi}_+ \mathcal{D} \bar{\pi}_+ (\bar{\pi}_+ \mathcal{D}^* J \mathcal{D} \bar{\pi}_+)^{-1} \bar{\pi}_+ \mathcal{D}^* J)\, \mathcal{C} =: \Pi\, \mathcal{C},$$

where Π is a bounded projection (by I/O stability and J-coercivity) in $\ell^2(\mathbf{Z}_+; U)$ that commutes with the cost operator J. The following lemma is a consequence of Definitions 68 and 70, Lemma 67 and basic properties of DLSs.

Lemma 71. *Let $J \in \mathcal{L}(Y)$ be a cost operator. Let $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ be an I/O stable and J-coercive DLS.*

(i) For each $x_0 \in \mathrm{dom}\,(\mathcal{C})$ there exists a unique critical control sequence $\tilde{u}^{\mathrm{crit}}(x_0)$ satisfying equation (2.3),

(ii) The critical control sequence $\tilde{u}^{\mathrm{crit}}(x_0)$ is given by

$$\tilde{u}^{\mathrm{crit}}(x_0) = \mathcal{K}^{\mathrm{crit}} x_0.$$

The critical state trajectory is given by

$$x_j^{\mathrm{crit}}(x_0) := A^j x_0 + \mathcal{B}\tau^{*j}\tilde{u}^{\mathrm{crit}}(x_0) = A^{\mathrm{crit}}(j)x_0$$

where $A^{\mathrm{crit}}(j) := A^j + \mathcal{B}\tau^{*j}\mathcal{K}^{\mathrm{crit}}$ *are linear mappings on* $\mathrm{dom}\,(\mathcal{C})$ *for all* $j \geq 0$*. The critical output sequence satisfies*

$$\tilde{y}^{\mathrm{crit}}(x_0) := \mathcal{C}x_0 + \mathcal{D}\tilde{u}^{\mathrm{crit}}(x_0) = \mathcal{C}^{\mathrm{crit}}x_0.$$

By our convention, $A^{\mathrm{crit}}(0) = A^0 = I$, even if A is not invertible. The family of linear mappings $\{A^{\mathrm{crit}}(j)\}_{j\geq 0})$ is in fact a semigroup on $\mathrm{dom}\,(\mathcal{C})$. This is the subject of the following lemma.

Lemma 72. *Let* $J \in \mathcal{L}(Y)$ *be a cost operator. Let* $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ *be an I/O stable and J-coercive DLS.*

(i) The linear mappings $A^{\mathrm{crit}}(j) := A^j + \mathcal{B}\tau^{*j}\mathcal{K}^{\mathrm{crit}} : \mathrm{dom}\,(\mathcal{C}) \to H$ *for* $j \geq 1$ *satisfy*

$$A^{\mathrm{crit}}(j)\,\mathrm{dom}\,(\mathcal{C}) \subset \mathrm{dom}\,(\mathcal{C})\,.$$

(ii) The family $\{A^{\mathrm{crit}}(j)\}_{j\geq 0}$ *of linear mappings on* $\mathrm{dom}\,(\mathcal{C})$ *is a semigroup; i.e.*

$$\begin{aligned} A^{\mathrm{crit}}(j) &= (A^{\mathrm{crit}})^j \quad \text{for all} \quad j \geq 0, \\ A^{\mathrm{crit}}(0) &= A^0 = I, \end{aligned} \tag{2.5}$$

where $A^{\mathrm{crit}} := A^{\mathrm{crit}}(1)$*.*

(iii) The critical state trajectory $\{x_j^{\mathrm{crit}}(x_0)\}_{j\geq 0}$ *at the initial value* $x_0 \in \mathrm{dom}\,(\mathcal{C})$ *is given by*

$$x_j^{\mathrm{crit}}(x_0) = (A^{\mathrm{crit}})^j x_0. \tag{2.6}$$

Proof. The proof of claim (i) is a consequence of the fact that the I/O stable DLS Φ is strongly H^2 stable. By Lemma 35, $\mathrm{range}\,(\mathcal{B}) := \mathcal{B}\,\mathrm{dom}\,(\mathcal{B}) \subset \mathrm{dom}\,(\mathcal{C})$. Because always $\pi_-\tau^{*j}\mathcal{K}^{\mathrm{crit}}x_0 \in \mathrm{dom}\,(\mathcal{B}) := Seq_-(U)$, claim (i) immediately follows. To prove claim (ii) we use a same kind of approach as in the proof of Lemma 67. Fix $x_0 \in \mathrm{dom}\left(\mathcal{K}^{\mathrm{crit}}\right) = \mathrm{dom}\,(\mathcal{C})$ and $j \geq 1$. Let $\epsilon > 0$ and $\tilde{w} \in \ell^2(\mathbf{Z}_+; U)$ be arbitrary. By $\tilde{u}^{\mathrm{crit}}(x_0) := \mathcal{K}^{\mathrm{crit}}x_0$ denote the unique critical control sequence, given by Lemma 71. Then we have

$$\begin{aligned} &J(x_0, \tilde{u}^{\mathrm{crit}}(x_0) + \epsilon\tau^j\tilde{w}) \\ &= \left\langle \pi_{[0,j-1]}\left[\mathcal{C}x_0 + \mathcal{D}\tilde{u}^{\mathrm{crit}}(x_0)\right], J(-,,-)\right\rangle_{\ell^2(\mathbf{Z}_+;Y)} \\ &+ \left\langle \pi_{[j,\infty]}\left[\mathcal{C}x_0 + \mathcal{D}(\tilde{u}^{\mathrm{crit}}(x_0) + \epsilon\tau^j\tilde{w})\right], J(-,,-)\right\rangle_{\ell^2(\mathbf{Z}_+;Y)}, \end{aligned}$$

because $\pi_{[0,j-1]}\mathcal{D}(\epsilon\tau^j\tilde{w}) = \pi_{[0,j-1]}\tau^j\bar{\pi}_+\mathcal{D}(\epsilon\tilde{w}) = 0$ $\mathcal{D}$ is causal and $\tilde{w} \in \ell^2(\mathbf{Z}_+;U)$. A simple calculation, together with part (iii) of Definition 70, allows us to continue

$$\begin{aligned}
(2.7)\qquad & J(x_0, \tilde{u}^{\mathrm{crit}}(x_0) + \epsilon\tau^j\tilde{w}) \\
&= \left\langle \pi_{[0,j-1]}\mathcal{C}^{\mathrm{crit}}x_0, J\mathcal{C}^{\mathrm{crit}} \right\rangle_{\ell^2(\mathbf{Z}_+;Y)} \\
&+ \left\langle \pi_{[j,\infty]}\left[\mathcal{C}^{\mathrm{crit}}x_0 + \epsilon\tau^j\mathcal{D}\tilde{w}\right], J\left[\mathcal{C}^{\mathrm{crit}}x_0 + \epsilon\tau^j\mathcal{D}\tilde{w}\right] \right\rangle_{\ell^2(\mathbf{Z}_+;Y)} \\
&= \left\langle \mathcal{C}^{\mathrm{crit}}x_0, J\mathcal{C}^{\mathrm{crit}} \right\rangle_{\ell^2(\mathbf{Z}_+;Y)} + 2\epsilon\, Re \left\langle \pi_{[j,\infty]}\mathcal{C}^{\mathrm{crit}}x_0, J\tau^j\mathcal{D}\tilde{w} \right\rangle_{\ell^2(\mathbf{Z}_+;Y)} \\
&+ \epsilon^2 \left\langle \mathcal{D}^*J\mathcal{D}\tilde{w}, \tilde{w} \right\rangle_{\ell^2(\mathbf{Z}_+;Y)}.
\end{aligned}$$

Now because $\tilde{u}^{\mathrm{crit}}(x_0)$ is critical, we must have $\frac{d}{d\epsilon}J(x_0, \tilde{u}^{\mathrm{crit}}(x_0) + \epsilon\tau^j\tilde{w})) = 0$ at $\epsilon = 0$ for all $\tilde{w} \in \ell^2(\mathbf{Z};U)$, $j \geq 0$. It follows that

$$Re \left\langle \pi_{[j,\infty]}\mathcal{C}^{\mathrm{crit}}x_0, J\tau^j\mathcal{D}\tilde{w} \right\rangle_{\ell^2(\mathbf{Z}_+;Y)} = 0$$

for all $\tilde{w}$. But then we have for all $j \geq 0$ and $x_0 \in \mathrm{dom}\,(\mathcal{C})$

$$\bar{\pi}_+\mathcal{D}^*J\bar{\pi}_+\tau^{*j}\mathcal{C}^{\mathrm{crit}}x_0 = \bar{\pi}_+\mathcal{D}^*J\bar{\pi}_+\tau^{*j}(\mathcal{C} + \mathcal{D}\mathcal{K}^{\mathrm{crit}})x_0 = 0$$

and

$$\begin{aligned}
&\bar{\pi}_+\mathcal{D}^*J\mathcal{C}A^jx_0 = \bar{\pi}_+\mathcal{D}^*J\bar{\pi}_+\tau^{*j}\mathcal{C}x_0 = -\bar{\pi}_+\mathcal{D}^*J\bar{\pi}_+\mathcal{D}\tau^{*j}\mathcal{K}^{\mathrm{crit}}x_0 \\
&= -(\bar{\pi}_+\mathcal{D}^*J\mathcal{D}\bar{\pi}_+)\tau^{*j}\mathcal{K}^{\mathrm{crit}}x_0 - \bar{\pi}_+\mathcal{D}^*J(\bar{\pi}_+\mathcal{D}\pi_-)\tau^{*j}\mathcal{K}^{\mathrm{crit}}x_0.
\end{aligned}$$

Apply $\bar{\pi}_+\mathcal{D}\pi_- = \mathcal{C}\mathcal{B}$ to the last term on the right hand side. This gives $\bar{\pi}_+\mathcal{D}^*J\mathcal{C}(A^j + \mathcal{B}\tau^{*j}\mathcal{K}^{\mathrm{crit}})x_0 = -(\bar{\pi}_+\mathcal{D}^*J\mathcal{D}\bar{\pi}_+)\tau^{*j}\mathcal{K}^{\mathrm{crit}}x_0$ for $x_0 \in \mathrm{dom}\,(\mathcal{C})$ and $j \geq 1$. This implies by part (i) of Definition 70

$$(2.8)\qquad \bar{\pi}_+\tau^{*j}\mathcal{K}^{\mathrm{crit}}x_0 = \mathcal{K}^{\mathrm{crit}}A^{\mathrm{crit}}(j)x_0$$

because $A^{\mathrm{crit}}(j) := A^j + \mathcal{B}\tau^{*j}\mathcal{K}^{\mathrm{crit}}$. The rest of the proof is now a calculation. For $k \geq 0$, $j \geq 1$ we have by Lemma 71

$$\begin{aligned}
(2.9)\qquad A^{\mathrm{crit}}(k)A^{\mathrm{crit}}(j)x_0 &= A^kx_j^{\mathrm{crit}}(x_0) + \mathcal{B}\tau^{*k}\mathcal{K}^{\mathrm{crit}}A^{\mathrm{crit}}(j)x_0 \\
&= A^kx_j^{\mathrm{crit}}(x_0) + \mathcal{B}\tau^{*k}\bar{\pi}_+\tau^{*j}\mathcal{K}^{\mathrm{crit}}x_0,
\end{aligned}$$

where the last equality is by equation (2.8). The former part in the right of (2.9) can be decomposed as

$$\begin{aligned}
(2.10)\qquad A^kx_j^{\mathrm{crit}}(x_0) &= A^{k+j}x_0 + A^k\mathcal{B}\tau^{*j}\mathcal{K}^{\mathrm{crit}}x_0 \\
&= A^{k+j}x_0 + \mathcal{B}\tau^{*(k+j)}\pi_{[0,j-1]}\mathcal{K}^{\mathrm{crit}}x_0.
\end{aligned}$$

The latter part in the right of (2.9) can be decomposed as

$$(2.11)\qquad \mathcal{B}\tau^{*k}\bar{\pi}_+\tau^{*j}\mathcal{K}^{\mathrm{crit}}x_0 = \mathcal{B}\tau^{*(k+j)}\mathcal{K}^{\mathrm{crit}}x_0 - \mathcal{B}\tau^{*(k+j)}\pi_{[0,j-1]}\mathcal{K}^{\mathrm{crit}}x_0.$$

Formulae (2.9), (2.10) and (2.11) together show that

$$A^{\mathrm{crit}}(k)A^{\mathrm{crit}}(j)x_0 = A^{\mathrm{crit}}(k+j)x_0$$

for all $x_0 \in \mathrm{dom}\,(\mathcal{C})$, and the proof of claim (ii) is compete. Claim (iii) is quite clear, too. □

Definition 73. *Let the DLS Φ, the cost operator J and $A^{\mathrm{crit}}(j)$ be as in Lemma 72. The densely defined linear operator $A^{\mathrm{crit}} : H \supset \mathrm{dom}\,(\mathcal{C}) \to H$, defined by $A^{\mathrm{crit}} := A^{\mathrm{crit}}(1)$ is the critical (closed loop) semigroup generator of Φ. The family of operators $\{(A^{\mathrm{crit}})^j\}_{j\geq 0}$ is the critical (closed loop) semigroup.*

If we write $K^{\mathrm{crit}} := \pi_0\mathcal{K}^{\mathrm{crit}}$, then trivially $A^{\mathrm{crit}} = A + BK^{\mathrm{crit}}$. We also define the critical output operator $C^{\mathrm{crit}} := \pi_0\mathcal{C}^{\mathrm{crit}} : \mathrm{dom}\,(\mathcal{C}) \to Y$. Clearly, $C^{\mathrm{crit}} = C + DK^{\mathrm{crit}}$. The following lemma describes the common algebraic structure of operators A^{crit}, $\mathcal{C}^{\mathrm{crit}}$ and $\mathcal{K}^{\mathrm{crit}}$.

Lemma 74. *Let $J \in \mathcal{L}(Y)$ be a cost operator. Let $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D}\end{smallmatrix}\right]$ be an I/O stable and J-coercive DLS. Then the following equations*

$$\mathcal{K}^{\mathrm{crit}}A^{\mathrm{crit}} = \bar{\pi}_+\tau^*\mathcal{K}^{\mathrm{crit}} \tag{2.12}$$

$$\mathcal{C}^{\mathrm{crit}}A^{\mathrm{crit}} = \bar{\pi}_+\tau^*\mathcal{C}^{\mathrm{crit}} \tag{2.13}$$

are valid on $\mathrm{dom}\,(\mathcal{C})$.

Proof. The proof of equation (2.12) is given in the proof of Lemma 72. To verify claim (2.13), we calculate

$$\begin{aligned}\mathcal{C}^{\mathrm{crit}}A^{\mathrm{crit}} &= \left(\mathcal{C} + \mathcal{D}\mathcal{K}^{\mathrm{crit}}\right)A^{\mathrm{crit}} = \mathcal{C}A + \mathcal{C}\mathcal{B}\tau^*\mathcal{K}^{\mathrm{crit}} + \mathcal{D}\bar{\pi}_+\tau^*\mathcal{K}^{\mathrm{crit}} \\ &= \bar{\pi}_+\tau^*\mathcal{C} + \bar{\pi}_+\mathcal{D}\pi_-\tau^*\mathcal{K}^{\mathrm{crit}} + \bar{\pi}_+\mathcal{D}\bar{\pi}_+\tau^*\mathcal{K}^{\mathrm{crit}} \\ &= \bar{\pi}_+\tau^*\mathcal{C} + \bar{\pi}_+\tau^*\mathcal{D}\mathcal{K}^{\mathrm{crit}} = \bar{\pi}_+\tau^*\mathcal{C}^{\mathrm{crit}},\end{aligned}$$

where the identity $\bar{\pi}_+\mathcal{D}\pi_- = \mathcal{C}\mathcal{B}$ has been used. □

We have now given the algebraic properties of operators A^{crit}, $\mathcal{C}^{\mathrm{crit}}$ and $\mathcal{K}^{\mathrm{crit}}$ as possibly unbounded linear mappings on the vector space $\mathrm{dom}\,(\mathcal{C})$. We remark that $\mathcal{C}^{\mathrm{crit}}$ and $\mathcal{K}^{\mathrm{crit}}$ are valid observability maps for a DLS whose semigroup generator is A^{crit} and state space $\overline{\mathrm{dom}\,(\mathcal{C})} = H$, provided that certain continuity requirements of these operators, associated to well-posedness of the DLS, are satisfied. In particular, A^{crit} should be continuous in the norm of H. Generally this is not the case. Basic stability conditions for closed loop semigroup generator A^{crit} are given in the following lemma.

Lemma 75. *Let $J \in \mathcal{L}(Y)$ be a cost operator. Let $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D}\end{smallmatrix}\right]$ be an I/O stable and J-coercive DLS.*

(i) *Φ is output stable* $\Rightarrow \mathcal{K}^{\mathrm{crit}} \in \mathcal{L}(H;\ell^2(\mathbf{Z}_+;U)) \Rightarrow K^{\mathrm{crit}} := \pi_0\mathcal{K}^{\mathrm{crit}} \in \mathcal{L}(\mathrm{dom}\,(\mathcal{C})\,;U) \Rightarrow BK^{\mathrm{crit}} \in \mathcal{L}(\mathrm{dom}\,(\mathcal{C})) \Leftrightarrow A^{\mathrm{crit}} \in \mathcal{L}(\mathrm{dom}\,(\mathcal{C}))$, *where* $\mathrm{dom}\,(\mathcal{C})$ *is given the norm of* H*.*

(ii) *If Φ is stable, then the critical semigroup satisfies $\{A^{\mathrm{crit}}(j)\}_{j\geq 0} \subset \mathcal{L}(H)$, and there is a constant $C < \infty$ such that*

$$||(A^{\mathrm{crit}})^j||_{\mathcal{L}(H)} \leq C \quad \forall j \geq 1,$$

i.e. A^{crit} is power bounded.

(iii) *If Φ is strongly stable, then*

$$(A^{\mathrm{crit}})^j x_0 \to 0 \quad \forall x_0 \in H,$$

i.e. A^{crit} is strongly stable.

Proof. The only not completely trivial part of (i) is the equivalence. This follows because on $\mathrm{dom}\,(\mathcal{C})$ we have

$$A^{\mathrm{crit}} = A + \mathcal{B}\tau^*\mathcal{K}^{\mathrm{crit}} = A + \mathcal{B}\pi_-\tau^*\bar{\pi}_+\mathcal{K}^{\mathrm{crit}} = A + B\pi_0\mathcal{K}^{\mathrm{crit}},$$

where B is the input operator of Φ and range (π_0) and U have been identified. The proofs of claims (ii) and (iii) are analogous to the proof of Theorem 50. Note that the stability and J-coercivity of Φ imply the boundedness of $\mathcal{K}^{\mathrm{crit}} : H \to \ell^2(\mathbf{Z}_+;U)$. Then proceed as in the proof of Theorem 50, by using $\mathcal{K}^{\mathrm{crit}}$ in place of $\mathcal{K}_\diamond$. □

We remark that if $K^{\mathrm{crit}} := \pi_0\mathcal{K}^{\mathrm{crit}} \in \mathcal{L}(\mathrm{dom}\,(\mathcal{C})\,;U)$ in claim (i) of previous lemma, then K^{crit} has a unique bounded extension to $\overline{\mathrm{dom}\,(\mathcal{C})} = H$. We denote this extension by K^{crit}, too. Under the same conditions, also A^{crit} can be extended to all of H, and the extension is denoted by A^{crit}. The requirement that $K^{\mathrm{crit}} \in \mathcal{L}(H;U)$ is central in this work. It is sufficient but not necessary to make A^{crit} bounded. On the other hand, it is a necessary condition for the critical closed loop DLS Φ^{ext} of equation (2.29) to be a (well-posed) DLS, because the output operator and the semigroup generator of a DLS has to be bounded. A trivial sufficient condition for $K^{\mathrm{crit}} \in \mathcal{L}(H;U)$ is that $J\mathcal{C} \in \mathcal{L}(H, \ell^2(\mathbf{Z}_+;Y))$. Weaker sufficient conditions are not easy to give.

We end this section by introducing a conjugate symmetric sesquilinear form $P_0^{\mathrm{crit}}(\,,\,)$ on $\mathrm{dom}\,(\mathcal{C}) \times \mathrm{dom}\,(\mathcal{C}) \subset H \oplus H$ whose diagonal values give the critical cost of the initial state x_0.

Definition 76. *Let $J \in \mathcal{L}(Y)$ be a cost operator. Let $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ be an I/O stable J-coercive DLS. The conjugate symmetric sesquilinear form $P_0^{\mathrm{crit}}(\,,\,)$ in* $\mathrm{dom}\,(\mathcal{C}) \times \mathrm{dom}\,(\mathcal{C})$ *given by*

$$P_0^{\mathrm{crit}}(x_0, x_1) := \left\langle \mathcal{C}^{\mathrm{crit}}x_0, J\,\mathcal{C}^{\mathrm{crit}}x_1 \right\rangle_{\ell^2(\mathbf{Z}_+;Y)}$$

is the critical sesquilinear form, associated to the DLS Φ and the cost operator J.

The sesquilinear form $P_0^{\mathrm{crit}}(\,,\,)$ is a solution of an algebraic Riccati equation, see Sections 2.5 and 2.6. The I/O stability of Φ has an effect to the limit behavior of the diagonal evaluation of $P_0^{\mathrm{crit}}(\,,\,)$ along the state trajectories.

Proposition 77. *Let $J \in \mathcal{L}(Y)$ be a cost operator. Let Φ be an I/O stable and J-coercive DLS. Then for all $x_0 \in \operatorname{dom}(\mathcal{C})$ and $\tilde{u} \in \ell^2(\mathbf{Z}_+; U)$*

$$P_0^{\mathrm{crit}}(x_j(x_0,\tilde{u}), x_j(x_0,\tilde{u})) \to 0 \quad \textit{as} \quad j \to \infty.$$

Proof. Fix $\tilde{u} \in \ell^2(\mathbf{Z}_+; U)$ and $x_0 \in \operatorname{dom}(\mathcal{C})$. We first remark that

$$\begin{aligned}|P_0^{\mathrm{crit}}(x_j(x_0,\tilde{u}), x_j(x_0,\tilde{u}))| &\le ||J|| \cdot ||\mathcal{C}^{\mathrm{crit}} x_j(x_0,\tilde{u})||^2 \\ &\le ||J|| \cdot ||\Pi||^2 \cdot ||\mathcal{C} x_j(x_0,\tilde{u})||^2,\end{aligned}$$

where Π is the bounded projection that has been introduced after Definition 70. It suffices to show that $\mathcal{C}x_j(x_0,\tilde{u}) \to 0$. We have

$$\begin{aligned}\mathcal{C}x_j(x_0,\tilde{u}) &= \mathcal{C}\left(A^j x_0 + \mathcal{B}\tau^{*j}\bar{\pi}_+\tilde{u}\right) \\ &= \bar{\pi}_+\tau^{*j}\mathcal{C}x_0 + \mathcal{C}\mathcal{B}\tau^{*j}\bar{\pi}_+\tilde{u} = \tau^{*j}\pi_{[j,\infty]}\mathcal{C}x_0 + \bar{\pi}_+\mathcal{D}\pi_-\tau^{*j}\bar{\pi}_+\tilde{u}.\end{aligned} \tag{2.14}$$

The first part of equation (2.14) approaches zero, because $\mathcal{C}x_0 \in \ell^2(\mathbf{Z}_+; Y)$. For the second part, we decompose

$$\begin{aligned}&||(\bar{\pi}_+\mathcal{D}\pi_-\tau^{*j})\,\bar{\pi}_+\tilde{u}||_{\ell^2(\mathbf{Z}_+;U)} \\ &\le ||(\bar{\pi}_+\mathcal{D}\pi_-\tau^{*j})\pi_{[0,N]}\tilde{u}||_{\ell^2(\mathbf{Z}_+;U)} + ||(\bar{\pi}_+\mathcal{D}\pi_-\tau^{*j})\pi_{[N+1,\infty]}\tilde{u}||_{\ell^2(\mathbf{Z}_+;U)}.\end{aligned} \tag{2.15}$$

Let $\epsilon > 0$ be arbitrary. Fix $N > 0$ so large that

$$||\pi_{[N+1,\infty]}\tilde{u}||_{\ell^2(\mathbf{Z}_+;U)} < \epsilon/(2||\mathcal{D}||_{\ell^2(\mathbf{Z};U)\to\ell^2(\mathbf{Z};Y)}).$$

Because the shift τ is unitary, we get the estimate $||(\bar{\pi}_+\mathcal{D}\pi_-\tau^{*j})\pi_{[N+1,\infty]}\tilde{u}|| < \epsilon/2$ for the second term in equation (2.15) for all j. We have for $j > N$

$$\bar{\pi}_+\mathcal{D}\pi_-\tau^{*j}\pi_{[0,N]}\tilde{u} = \bar{\pi}_+\mathcal{D}\tau^{*j}\pi_{[-\infty,j-1]} \cdot \pi_{[0,N]}\tilde{u} = \bar{\pi}_+\tau^{*j}\,\left(\mathcal{D}\pi_{[0,N]}\tilde{u}\right).$$

By the I/O stability of Φ, $\mathcal{D}\pi_{[0,N]} \in \ell^2(\mathbf{Z}_+; Y)$ and the the first term in (2.15) can be made less that $\epsilon/2$ by increasing j. It follows that the second term in (2.14) approaches zero when j increases, and the proof is complete. □

In the following proposition, the last one of this section, we separate the cost of input into two parts, the first of which does not depend on the input sequence $\tilde{u}$ we are applying, but only on the initial value x_0. The second part of the cost depends only on the deviation from the criticality of the applied input sequence $\tilde{u}$.

Proposition 78. *Let $J \in \mathcal{L}(Y)$ be a cost operator. Let $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}_{\mathcal{T}}^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ be an I/O stable and J-coercive DLS. Then the cost functional has the decomposition*

$$J(x_0, \tilde{u}) = J(x_0, \tilde{u}^{\text{crit}}(x_0)) + J(0, \tilde{u} - \tilde{u}^{\text{crit}}(x_0)) \tag{2.16}$$

for all input functions $\tilde{w} \in \ell^2(\mathbf{Z}; U)$ where $\tilde{u}^{\text{crit}}(x_0) = \mathcal{K}^{\text{crit}} x_0$. Moreover, we have

$$P_0^{\text{crit}}(x_0, x_0) = J(x_0, \tilde{u}^{\text{crit}}(x_0)), \tag{2.17}$$

where the sesquilinear form $P(\, , \,)$ is defined in Definition 76.

Proof. Define $\tilde{w} := \tilde{u} - \tilde{u}^{\text{crit}}(x_0) \in \ell^2(\mathbf{Z}_+; Y)$. Then quite easily

$$\begin{aligned} J(x_0, \tilde{u}) &= J(x_0, \tilde{u}^{\text{crit}}(x_0) + \tilde{w}) \\ &= J(x_0, u^{\text{crit}}(x_0)) + J(0, w) \\ &+ 2Re\big\langle \bar{\pi}_+ \mathcal{D}^* \, J \, \mathcal{C} x_0 + \bar{\pi}_+ \mathcal{D}^* J \mathcal{D} \tilde{u}^{\text{crit}}(x_0), \tilde{w} \big\rangle_{\ell^2(\mathbf{Z}_+; Y)} \end{aligned} \tag{2.18}$$

But now the last term in the left of (2.18) vanishes because the critical control sequence $\tilde{u}^{\text{crit}}(x_0)$ satisfies formula (2.3). This immediately proves (2.16). Equation (2.17) is immediate from the definition of $\mathcal{C}^{\text{crit}}$. □

We consider a special case for an I/O stable and J-coercive DLS Φ, under the additional assumption $\bar{\pi}_+ \mathcal{D}^* J \mathcal{D} \bar{\pi}_+ \geq 0$. Then, by J-coercivity, $\bar{\pi}_+ \mathcal{D}^* J \mathcal{D} \bar{\pi}_+ \geq \epsilon \bar{\pi}_+$ for some $\epsilon > 0$. It follows that

$$J(x_0, \tilde{u}) \geq P_0^{\text{crit}}(x_0, x_0)$$

for all $x_0 \in \operatorname{dom}(\mathcal{C})$ and $\tilde{u} \in \ell^2(\mathbf{Z}_+; U)$ because

$$J(0, \tilde{w}) = \langle \bar{\pi}_+ \mathcal{D}^* J \mathcal{D} \bar{\pi}_+ \tilde{w}, \tilde{w} \rangle_{\ell^2(\mathbf{Z}_+; U)} \geq \epsilon ||\tilde{w}||^2_{\ell^2(\mathbf{Z}_+; U)}$$

for all $\tilde{w} \in \ell^2(\mathbf{Z}_+; U)$. Furthermore, equality $J(x_0, \tilde{u}) = P_0^{\text{crit}}(x_0, x_0)$ implies that $\tilde{u} = u^{\text{crit}}(x_0)$. This reveals the connection of the sesquilinear form $P_0^{\text{crit}}(\, , \,)$ to the unique solution of the cost optimization problem, associated to the DLS Φ and the cost operator J.

2.3 Factorization of the I/O map and the Popov operator

Let $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ be an I/O stable DLS and $J \in \mathcal{L}(Y)$ a cost operator. In this section we consider certain factorizations of the I/O map $\mathcal{D} : \ell^2(\mathbf{Z}; U) \to \ell^2(\mathbf{Z}; Y)$ and the Popov operator $\bar{\pi}_+ \mathcal{D}^* J \mathcal{D} \bar{\pi}_+ : \ell^2(\mathbf{Z}_+; U) \to \ell^2(\mathbf{Z}_+; U)$. The present approach is similar to that given in [83] and [86]. We remark that all the I/O maps considered in this section are I/O stable. Such I/O maps can be regarded as bounded linear operator on the space $\ell^2(\mathbf{Z}; U)$, see the discussion presented at the end of Section 1.7. By Lemma 37, we could as well speak about abstract bounded, shift-invariant causal operators on $\ell^2(\mathbf{Z}; U)$.

Definition 79. *Let $J \in \mathcal{L}(Y)$ be self-adjoint, and let $S \in \mathcal{L}(U)$ self-adjoint and invertible. Let $\mathcal{D}$ and $\mathcal{N}$ be I/O maps of I/O stable DLSs whose input space is U and output space is Y. Let $\mathcal{X}$ be an I/O map of an I/O stable DLS whose input space and output space is U.*

(i) The operator $E \in \mathcal{L}(U)$ is S-unitary, if $E^{-1} \in \mathcal{L}(U)$ and $E^* S E = S$.

(ii) The I/O map $\mathcal{N}$ is (J, S)-inner, if $\mathcal{N}^* J \mathcal{N} = S$.

(iii) The I/O map $\mathcal{X}$ is outer, if $\overline{\operatorname{range}(\mathcal{X}\bar{\pi}_+)} = \ell^2(\mathbf{Z}_+; U)$. If, in addition, $\mathcal{X}$ is injective and $\operatorname{range}(\mathcal{X}\bar{\pi}_+) = \ell^2(\mathbf{Z}_+; U)$, we say that $\mathcal{X}$ is outer with a bounded inverse.

In fact, part (iii) of previous definition is a reiteration of Definition 45. Basic properties of outer I/O maps have been considered in Proposition 46. We proceed to define the spectral factors.

Definition 80. *Let $J \in \mathcal{L}(Y)$ and $S \in \mathcal{L}(U)$ be self-adjoint. Let $\mathcal{D}$ be an I/O map of an I/O stable DLS. The mapping $\mathcal{X}$ is a stable S-spectral factor of $\mathcal{D}^* J \mathcal{D}$, if*

(i) $\mathcal{X}$ is an I/O map of an I/O stable DLS, whose input space and output space is U,

(ii) $\mathcal{X} : Seq(U) \to Seq(U)$ has an inverse $\mathcal{X}^{-1} : Seq(U) \to Seq(U)$ which is an I/O map of a DLS, and

(iii) $\mathcal{D}^* J \mathcal{D} = \mathcal{X}^* S \mathcal{X}$.

If, in addition, $\mathcal{X}^{-1}$ is an I/O map of an I/O stable DLS, then $\mathcal{X}$ is a stable outer S-spectral factor of $\mathcal{D}^* J \mathcal{D}$.

In this book, we consider only stable spectral factors and we will not state that explicitly from now on. In Chapter 3, nonouter spectral factors of $\mathcal{D}^* J \mathcal{D}$ are investigated. We also need factorizations of the I/O stable I/O map $\mathcal{D}$.

Definition 81. *Let $J \in \mathcal{L}(Y)$ and $S \in \mathcal{L}(U)$ be self-adjoint. Let $\mathcal{D}$ be the I/O map of an I/O stable DLS whose input space is U and output space is Y. Then $\mathcal{D}$ has a a (J, S)-inner-outer factorization $\mathcal{D} = \mathcal{N}\mathcal{X}$, if*

(i) the operator $\mathcal{N}$ is a (J, S)-inner I/O map of an I/O stable DLS whose input space is U and output space is Y

(ii) the operator $\mathcal{X}$ is an outer I/O map of an I/O stable DLS whose input space and output space is U, and

(iii) the operator equation $\mathcal{D} = \mathcal{N}\mathcal{X}$ holds.

We say that $\mathcal{N}$ is the (J, S)-inner factor of $\mathcal{D}$, and $\mathcal{X}$ is the outer factor of $\mathcal{D}$.

The operator S is called the sensitivity operator of the factorization in [87]. By Proposition 46, $\mathcal{X}$ is outer with a bounded inverse if and only if the Toeplitz operator $\mathcal{X}\bar{\pi}_+$ has a bounded inverse on $\ell^2(\mathbf{Z}_+; U)$. Expectedly, there is a strong link between S-spectral factorizations of $\mathcal{D}^* J \mathcal{D}$ and (J, S)-inner-outer factorizations of $\mathcal{D}$.

Proposition 82. *Let $\mathcal{D}$ be the I/O map of an I/O stable DLS. Then the following are equivalent:*

(i) $\mathcal{D} = \mathcal{N}\mathcal{X}$ is a (J, S)-inner-outer factorization of $\mathcal{D}$, where $\mathcal{X}$ is outer with a bounded inverse.

(ii) $\mathcal{X}$ is a stable outer S-spectral factor of $\mathcal{D}^* J \mathcal{D}$ for some $S \in \mathcal{L}(U)$, and $\mathcal{N} = \mathcal{D}\mathcal{X}^{-1}$.

Proof. Let us first show that (i) implies (ii). Assume that $\mathcal{D} = \mathcal{N}\mathcal{X}$ is a (J, S)-inner-outer factorization. Then

$$\mathcal{D}^* J \mathcal{D} = \mathcal{X}^* \left(\mathcal{N}^* J \mathcal{N}\right) \mathcal{X} = \mathcal{X}^* S \mathcal{X}$$

because all the operators are bounded. Because $\mathcal{X}$ is outer with a bounded inverse, the bounded inverse operator $\mathcal{X}^{-1}$ on $\ell^2(\mathbf{Z}; U)$ exists as an I/O map of an I/O stable DLS, by claim (ii) Proposition 46. Now claim (ii) follows.

To show that (ii) implies (i), assume that we have the outer S-spectral factorization $\mathcal{D}^* J \mathcal{D} = \mathcal{X}^* S \mathcal{X}$. Define $\mathcal{N} := \mathcal{D}\mathcal{X}^{-1}$. Because both $\mathcal{D}$ and $\mathcal{X}^{-1}$ are I/O maps of DLS, so is their product $\mathcal{N}$ by claim (ii) of Proposition 17. It is trivial that $\mathcal{N}$ is I/O stable. We have

$$\mathcal{N}^* J \mathcal{N} = (\mathcal{X}^{-1})^* \left(\mathcal{D}^* J \mathcal{D}\right) (\mathcal{X}^{-1}) = (\mathcal{X}^{-1})^* \left(\mathcal{X}^* S \mathcal{X}\right) \mathcal{X}^{-1} = S,$$

which proves that $\mathcal{N}$ is (J,S)-inner. It is trivial consequence of causality that $(\mathcal{X}^{-1}\bar{\pi}_+)\mathcal{X}\bar{\pi}_+ = \mathcal{X}\bar{\pi}_+(\mathcal{X}^{-1}\bar{\pi}_+) = \bar{\pi}_+$. Thus $(\mathcal{X}\bar{\pi}_+)^{-1}$ exists on $\ell^2(\mathbf{Z}_+;U)$, and it equals the bounded operator $\mathcal{X}^{-1}\bar{\pi}_+$. Because $\mathcal{X}\bar{\pi}_+$ is a bounded bijection, $\mathcal{X}$ is outer with a bounded inverse. It follows that $\mathcal{D} = \mathcal{N}\mathcal{X}$ is a (J,S)-inner-outer factorization of $\mathcal{D}$ and $\mathcal{X}$ has a bounded inverse. The proof is complete. □

Not all operators of the form $\mathcal{D}^* J \mathcal{D}$ have S-spectral factorization for any S. Those that have the factorization are more interesting to us. If we know one (J,S)-inner-outer factorization of $\mathcal{D}$ for some S, then we know them all. This is because all the (J,S)-inner-outer factorization can be parameterized by the set of all S-unitary operators.

Proposition 83. *Let $J \in \mathcal{L}(Y)$ be self-adjoint and $\mathcal{D}$ be the I/O map of an I/O stable DLS. Let $\mathcal{D} = \mathcal{N}\mathcal{X}$ be a (J,S) -inner-outer factorization for some $S \in \mathcal{L}(U)$, such that the outer factor $\mathcal{X}$ has a bounded inverse. Then the set of all possible (J,S_E)-inner-outer factorizations $\mathcal{D} = \mathcal{N}_E\mathcal{X}_E$ (with the outer factor $\mathcal{X}_E$ having a bounded inverse) can be parameterized by*

(2.19) $$\mathcal{N}_E = \mathcal{N}E, \quad \mathcal{X}_E = E^{-1}\mathcal{X}, \quad S_E = E^* S E,$$

where E ranges over the set all boundedly invertible operators in $\mathcal{L}(U)$. In particular, if we in addition require that $S_E = S$, the E is allowed to range over the set of all S-unitary operators $E \in \mathcal{L}(U)$.

Proof. We first show that for each invertible E we have the factorization as claimed. So let $E \in \mathcal{L}(U)$ be boundedly invertible and $\mathcal{D} = \mathcal{N}\mathcal{X}$ be a (J,S)-inner-outer factorization for some $S \in \mathcal{L}(U)$. The operators $\mathcal{N}_E$, $\mathcal{X}_E$ and $\mathcal{X}_E^{-1}$ in equation (2.19) are I/O maps of I/O stable DLSs, and trivially $\mathcal{D} = \mathcal{N}\mathcal{X} = \mathcal{N}_E\mathcal{X}_E$. We have

(2.20) $$\mathcal{N}_E^* J \mathcal{N}_E = (\mathcal{N}E)^* J (\mathcal{N}E) = E^* \mathcal{N}^* J \mathcal{N} E = E^* S E = S_E,$$

i.e. $\mathcal{N}_E$ is (J,S_E)-inner. It is trivial that $\mathcal{X}_E$ is outer with a bounded inverse.

In order to prove the remaining part, we must show that if there is another (J,S')-inner-outer factorization $\mathcal{D} = \mathcal{N}'\mathcal{X}'$, then it is of the form $\mathcal{N}' = \mathcal{N}_E$ and $\mathcal{X}' = \mathcal{X}_E$ for some boundedly invertible $E \in \mathcal{L}(U)$. Because all the operator $\mathcal{X}$, $\mathcal{X}^{-1}$, $\mathcal{X}'$ and $(\mathcal{X}')^{-1}$ are assumed to be I/O maps of I/O stable DLSs, both the operators $\mathcal{U} := \mathcal{X}'\mathcal{X}^{-1}$ and $\mathcal{U}^{-1} := \mathcal{X}(\mathcal{X}')^{-1}$ are I/O maps of I/O stable DLSs, by claim (ii) of Proposition 17. By definition, the following identity on $\ell^2(\mathbf{Z};U)$ holds

(2.21) $$\mathcal{N} = \mathcal{N}'\mathcal{U}.$$

Now, because $\mathcal{N}$ is (J,S)-inner, and $\mathcal{N}'$ is (J,S')-inner, we have

$$S = \mathcal{N}^* J \mathcal{N} = (\mathcal{N}'\mathcal{U})^* J (\mathcal{N}'\mathcal{U}) = \mathcal{U}^*(\mathcal{N}'^* J \mathcal{N}')\mathcal{U} = \mathcal{U}^* S' \mathcal{U},$$

which implies

$$S\mathcal{U}^{-1} = \mathcal{U}^* S', \tag{2.22}$$

where the operators are considered as bounded shift-invariant operators on $\ell^2(\mathbf{Z};U)$. Both S and S' are (extended to) static operators. $\mathcal{U}^*$ is anticausal and $\mathcal{U}^{-1}$ causal. It is a triviality that the right side of equation (2.22) is causal and the left side is anticausal shift-invariant operator. It follows that the both sides of equation (2.22) are static in the sense of part (iv) of Definition 5. By Proposition 6, $\mathcal{U}^{-1}$ is equal to a componentwise multiplication by some $E \in \mathcal{L}(U)$. Because the same is true for $\mathcal{U}^*$, it follows that E has a bounded inverse. This together with equation (2.21) implies $\mathcal{N}' = \mathcal{N}E = \mathcal{N}_E$ and also by the definition of $\mathcal{U}$ we obtain $\mathcal{X}' = E^{-1}\mathcal{X} = \mathcal{X}_E$. Finally (2.22) gives $S' = E^* S E = S_E$. The statement about the S-unitary parameterizations is trivial, and the proof of the proposition is now completed. □

The existence of (J,S)-inner-outer factorization $\mathcal{D} = \mathcal{N}\mathcal{X}$ affects the properties of the Popov operator $\bar{\pi}_+\mathcal{D}^* J \mathcal{D}\bar{\pi}_+$. The following lemma is the main result of this section.

Lemma 84. *Let* $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ *be an I/O stable DLS. Let* $J \in \mathcal{L}(Y)$ *be self-adjoint and* $S \in \mathcal{L}(U)$ *self adjoint with bounded inverse. Assume that the I/O map* $\mathcal{D}$ *has a* (J,S)*-inner-outer factorization* $\mathcal{D} = \mathcal{N}\mathcal{X}$*, such that the outer factor* $\mathcal{X}$ *has a bounded inverse.*

(i) Φ *is* J*-coercive.*

(ii) The inverse of the Popov operator operator $\bar{\pi}_+\mathcal{D}^* J \mathcal{D}\bar{\pi}_+$ *satisfies*

$$(\bar{\pi}_+\mathcal{D}^* J \mathcal{D}\bar{\pi}_+)^{-1} = (\bar{\pi}_+\mathcal{X}^{-1}\bar{\pi}_+)\, S^{-1}(\bar{\pi}_+(\mathcal{X}^*)^{-1}\bar{\pi}_+).$$

(iii) The critical operators A^{crit}*,* $\mathcal{C}^{\mathrm{crit}}$ *and* $\mathcal{K}^{\mathrm{crit}}$ *can be written in forms*

$$\begin{aligned} A^{\mathrm{crit}} &= A - \mathcal{B}\mathcal{X}^{-1}\tau^* S^{-1}\bar{\pi}_+\mathcal{N}^* J \mathcal{C}, \\ \mathcal{C}^{\mathrm{crit}} &= \mathcal{C} - \mathcal{N} S^{-1}\bar{\pi}_+\mathcal{N}^* J \mathcal{C}, \\ \mathcal{K}^{\mathrm{crit}} &= -\mathcal{X}^{-1} S^{-1}\bar{\pi}_+\mathcal{N}^* J \mathcal{C}. \end{aligned}$$

Proof. We prove parts (i) and (ii) at the same time. Given an arbitrary $\bar{\pi}_+\tilde{w} \in \ell^2(\mathbf{Z}_+;U)$, we try to solve $\bar{\pi}_+\tilde{u}$ in equation

$$\bar{\pi}_+\mathcal{D}^* J \mathcal{D}\bar{\pi}_+\tilde{u} = \bar{\pi}_+\tilde{w}. \tag{2.23}$$

We first replace $\mathcal{D}$ by $\mathcal{N}\mathcal{X}$ and use the fact that $\mathcal{N}$ is (J,S)-inner to get

$$\bar{\pi}_+\tilde{w} = \bar{\pi}_+\mathcal{X}^* S\mathcal{X}\bar{\pi}_+\tilde{u} \tag{2.24}$$

Because the outer factor $\mathcal{X}$ is assumed to be outer with a bounded inverse, its inverse $\mathcal{X}^{-1}$ exists on $\ell^2(\mathbf{Z};U)$ as an I/O map of an I/O stable DLS, by claim (ii) of Proposition 46. By the theory of the bounded linear operator, the adjoint $(\mathcal{X}^{-1})^*$ exists and satisfies $(\mathcal{X}^{-1})^* = (\mathcal{X}^*)^{-1}$. Because the existence of $S^{-1} \in \mathcal{L}(U)$ is assumed, the operator $S^{-1}\bar{\pi}_+(\mathcal{X}^*)^{-1}\bar{\pi}_+ : \ell^2(\mathbf{Z}_+;U) \to \ell^2(\mathbf{Z}_+;U)$ is bounded, and we can multiply equation (2.24) from the left by it, to obtain

$$\begin{aligned}(S^{-1}\bar{\pi}_+(\mathcal{X}^*)^{-1})\bar{\pi}_+\tilde{w} &= S^{-1}(\bar{\pi}_+(\mathcal{X}^*)^{-1}\bar{\pi}_+ \cdot \bar{\pi}_+\mathcal{X}^*\bar{\pi}_+)S\mathcal{X}\bar{\pi}_+\tilde{u} \\ &= S^{-1}(\bar{\pi}_+(\mathcal{X}^*)^{-1}\mathcal{X}^*\bar{\pi}_+)S\mathcal{X}\bar{\pi}_+\tilde{u} = \mathcal{X}\bar{\pi}_+\tilde{u},\end{aligned} \tag{2.25}$$

where we have used $\bar{\pi}_+(\mathcal{X}^{-1})^*\pi_- = 0$, implied by the causality of the outer factor $\mathcal{X}$. Equation (2.25) is equivalent to

$$\bar{\pi}_+\tilde{u} = \mathcal{X}^{-1}S^{-1}\bar{\pi}_+(\mathcal{X}^*)^{-1}\bar{\pi}_+\tilde{w} \tag{2.26}$$

which is the equation of claim (ii). This $\bar{\pi}_+\tilde{u}$ is the only possible solution to equation (2.23). In particular, it follows that $\bar{\pi}_+\mathcal{D}^*J\mathcal{D}\bar{\pi}_+$ is injective on $\ell^2(\mathbf{Z}_+;U)$. To check that $\bar{\pi}_+\tilde{u}$ given by (2.26), indeed, is a solution, it suffices to compute

$$\begin{aligned}&(\bar{\pi}_+\mathcal{D}^*J\mathcal{D}\bar{\pi}_+)\,\mathcal{X}^{-1}S^{-1}\bar{\pi}_+(\mathcal{X}^*)^{-1}\bar{\pi}_+\tilde{w} = \bar{\pi}_+\mathcal{D}^*J(\mathcal{D}\mathcal{X}^{-1})S^{-1}\bar{\pi}_+(\mathcal{X}^*)^{-1}\bar{\pi}_+\tilde{w} \\ &= \bar{\pi}_+\mathcal{X}^*(\mathcal{N}J\mathcal{N}S^{-1})\bar{\pi}_+(\mathcal{X}^*)^{-1}\bar{\pi}_+\tilde{w} = (\bar{\pi}_+\mathcal{X}^*(\mathcal{X}^*)^{-1}\bar{\pi}_+)\,\bar{\pi}_+\tilde{w} = \bar{\pi}_+\tilde{w}.\end{aligned}$$

So there is a solution for each $\bar{\pi}_+\tilde{w} \in \ell^2(\mathbf{Z}_+;U)$, and it follows that $\bar{\pi}_+\mathcal{D}^*J\mathcal{D}\bar{\pi}_+$ is surjective.

We have shown that $\bar{\pi}_+\mathcal{D}^*J\mathcal{D}\bar{\pi}_+$ is a bounded bijection on $\ell^2(\mathbf{Z}_+;U)$, and it follows that $\bar{\pi}_+\mathcal{D}^*J\mathcal{D}\bar{\pi}_+$ has a bounded inverse; i.e. Φ is J-coercive. This proves the first two claims of the lemma. In order to prove the remaining claim (iii), is is sufficient to apply the formula of claim (ii) to the formulae of Definition 70. This completes the proof of the lemma. □

Corollary 85. *Assume that $\mathcal{D}$ is an I/O map of an I/O stable DLS Φ having a (J,S)-inner-outer factorization $\mathcal{D} = \mathcal{N}\mathcal{X}$. Then outer factor $\mathcal{X}$ is outer with a bounded inverse and $S^{-1} \in \mathcal{L}(U)$ if and only if Φ is J-coercive.*

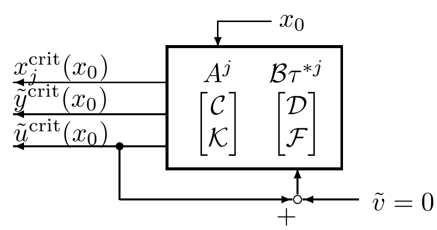
Proof. To prove the "if" part, assume that Φ is J-coercive. For a (J,S)-inner-outer factorization $\mathcal{D} = \mathcal{N}\mathcal{X}$ we have $\bar{\pi}_+\mathcal{D}^*J\mathcal{D}\bar{\pi}_+ = \bar{\pi}_+\mathcal{X}^*S\mathcal{X}\bar{\pi}_+$. The outer factor $\mathcal{X}$ is an I/O map of an I/O stable DLS, say Φ'. Trivially, Φ is J-coercive if and only if Φ' is S-coercive. Proposition 69 implies that the Toeplitz operator $\mathcal{X}\bar{\pi}_+$ is coercive. Thus $\mathrm{range}\,(\mathcal{X}\bar{\pi}_+)$ is closed and $\mathcal{X}\bar{\pi}_+$ is injective on $\ell^2(\mathbf{Z}_+;U)$. Because $\mathcal{X}$ is outer, $\mathrm{range}\,(\mathcal{X}\bar{\pi}_+) = \overline{\mathrm{range}\,(\mathcal{X}\bar{\pi}_+)} = \ell^2(\mathbf{Z}_+;U)$, and it follows that $\mathcal{X}$ is outer with a bounded inverse. Because $S = (\mathcal{X}^*)^{-1} (\mathcal{D}^* J \mathcal{D}) \mathcal{X}^{-1}$ as a static operator, and all the bounded operators $(\mathcal{X}^*)^{-1}$, $\mathcal{D}^* J \mathcal{D}$ and $\mathcal{X}^{-1}$ are boundedly invertible, so is S, as a static operator. It immediately follows from Proposition 6 that $S^{-1} \in \mathcal{L}(U)$. The "only if" part is given by claim (i) of Lemma 84. □

2.4 Critical control and state feedback

Let $J \in \mathcal{L}(Y)$ be a cost operator, an $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ be an I/O stable and J-coercive DLS. In this section, we give necessary and sufficient conditions for a class of critical control problems to be solvable by state feedback. This class is associated to Φ and J as in Section 2.2, but now we additionally require that we have a bounded critical one step feedback operator $K^{\mathrm{crit}} = \pi_0 \mathcal{K}^{\mathrm{crit}} : \mathrm{dom}\,(\mathcal{C}) \to H$, see part (ii) of Definition 70. Here $\mathrm{dom}\,(\mathcal{C})$ is given the norm of H. We remark that this additional well-posedness requirement is imposed on the common structure of Φ and J, and not on these objects separately. The formulations and proofs of the results are divided into two Lemmas 87 and 88, and then stated in Theorem 89. Analogous considerations for the continuous time well-posed linear systems can be found in [85].

Let Φ be an I/O stable and J-coercive DLS, such that well-posedness assumption $K^{\mathrm{crit}} \in \mathcal{L}(\mathrm{dom}\,(\mathcal{C}, H))$ holds. Because $\overline{\mathrm{dom}\,(\mathcal{C})} = H$ by our standing assumption, we may (and always will) assume that $K^{\mathrm{crit}} \in \mathcal{L}(H)$ because a unique extension exists, by the continuity and the completeness of H. Then, the critical closed loop semigroup generator A^{crit}, satisfying $A^{\mathrm{crit}} = A + BK^{\mathrm{crit}}$ on $\mathrm{dom}\,(\mathcal{C})$, can also be extended to a bounded operator on H, see Lemma 75. By Lemma 74, we see that the closed loop feedback map $\mathcal{K}^{\mathrm{crit}}$ is a valid observability map for any DLS whose semigroup generator is A^{crit}, provided that no trouble emerges with the right hand column of the DLS in question. So as to the closed loop critical observability map $\mathcal{C}^{\mathrm{crit}} = \mathcal{C} + \mathcal{D}\mathcal{K}^{\mathrm{crit}}$, the same holds.

After these preliminary considerations, we are led to ask the following question: Given a cost operator J and an I/O stable and J-coercive DLS Φ, is there an I/O stable and outer feedback pair $[\mathcal{K}, \mathcal{F}]$ for Φ such that the closed loop extended DLS $\Phi_{\diamond}^{\mathrm{ext}} := [\Phi, [\mathcal{K}, \mathcal{F}]]$ outputs the critical state trajectory $\{x_j^{\mathrm{crit}}(x_0)\}_{j \geq 0}$, critical output sequence $\tilde{y}^{\mathrm{crit}}(x_0)$ and critical control sequence $\tilde{u}^{\mathrm{crit}}(x_0)$? Here $x_0 \in \mathrm{dom}\,(\mathcal{C})$ is an arbitrary initial state of Φ, the sequences $\{x_j^{\mathrm{crit}}(x_0)\}_{j \geq 0}$, $\tilde{y}^{\mathrm{crit}}(x_0)$ and $\tilde{u}^{\mathrm{crit}}(x_0)$ are given by Lemma 71, and the external input sequence to the feedback loop is $\tilde{v} = 0$. The feedback connection and the signals are illustrated in the following feedback diagram.



Definition 86. *Let $J \in \mathcal{L}(Y)$ be a cost operator and $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ be an I/O stable J-coercive DLS.*

If the critical state trajectory $\{x_\diamond^{\mathrm{crit}}(x_0)\}$ *and the critical sequences* $\tilde{u}^{\mathrm{crit}}(x_0)$ *and* $\tilde{y}^{\mathrm{crit}}(x_0)$ *are given by the signals of the closed loop extended DLS*

$$\Phi_\diamond^{\mathrm{ext}} := [\Phi, [\mathcal{K}, \mathcal{F}]]_\diamond = \begin{bmatrix} A^j + \mathcal{B}\tau^{*j}(\mathcal{I} - \mathcal{F})^{-1}\mathcal{K} & \mathcal{B}\tau^{*j}(\mathcal{I} - \mathcal{F})^{-1} \\ \begin{bmatrix} \mathcal{C} + \mathcal{D}(\mathcal{I} - \mathcal{F})^{-1}\mathcal{K} \\ (\mathcal{I} - \mathcal{F})^{-1}\mathcal{K} \end{bmatrix} & \begin{bmatrix} \mathcal{D}(\mathcal{I} - \mathcal{F})^{-1} \\ (\mathcal{I} - \mathcal{F})^{-1} - \mathcal{I} \end{bmatrix} \end{bmatrix},$$

for some I/O stable and outer feedback pair $[\mathcal{K}, \mathcal{F}]$ *as explained above, we say that the critical control problem (associated to* Φ *and* J*) is solvable by state feedback. Any such feedback pair* $[\mathcal{K}, \mathcal{F}]$ *is called a critical feedback for* Φ *and* J*. Any such* $\Phi_\diamond^{\mathrm{ext}}$ *is called a critical closed loop DLS.*

We remark that even if a critical feedback pair exists, it (and consequently, a critical closed loop DLS) is not unique. In the following Lemma, we associate a critical feedback pair to each (J,S)-inner-outer factorization $\mathcal{D} = \mathcal{N}\mathcal{X}$ of the I/O map. Such factorizations are parameterized in Proposition 83.

Lemma 87. *Let* $J \in \mathcal{L}(Y)$ *be a cost operator. Let* $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ *be an I/O stable DLS. Assume that*

(i) $\mathcal{D}$ *has a* (J,S)*-inner-outer factorization* $\mathcal{D} = \mathcal{N}\mathcal{X}$ *for some boundedly invertible* $S \in \mathcal{L}(U)$*, such the outer factor* $\mathcal{X}$ *is outer with a bounded inverse, and*

(ii) $K^{\mathrm{crit}} := \pi_0 \mathcal{K}^{\mathrm{crit}} : \mathrm{dom}\,(\mathcal{C}) \to U$ *is bounded.*

Then the following holds:

(i) Φ *is* J*-coercive.*

(ii) There is a unique I/O stable and outer feedback pair $[\mathcal{K}, \mathcal{F}]$ *for* Φ*, such that the mappings* $\mathcal{K} : H \to Seq(U)$ *and* $\mathcal{F} : Seq(U) \to Seq(U)$ *satisfy*

$$\mathcal{K}x_0 = -S^{-1}\bar{\pi}_+\mathcal{N}^* J\mathcal{C}x_0 \quad \text{for all} \quad x_0 \in \mathrm{dom}\,(\mathcal{C}), \tag{2.27}$$

$$\mathcal{F}\tilde{u} = (\mathcal{I} - \mathcal{X})\tilde{u} \quad \text{for all} \quad \tilde{u} \in \ell^2(\mathbf{Z}_+; U). \tag{2.28}$$

(iii) The critical control problem, associated to Φ *and* J*, is solvable by state feedback. Furthermore,* $[\mathcal{K}, \mathcal{F}]$ *is a critical feedback pair.*

(iv) Assume, in addition, that Φ *is output stable. Then* $[\mathcal{K}, \mathcal{F}]$ *is a stable feedback pair for* Φ*; i.e.* $\mathcal{K} : H \to \ell^2(\mathbf{Z}_+; U)$ *is bounded. Furthermore, the requirement* $K^{\mathrm{crit}} \in \mathcal{L}(\mathrm{dom}\,(\mathcal{C}), U)$ *need not explicitly assumed.*

Proof. Claim (i) is an implication of Corollary 85. In order to prove claim (ii) we first show that equations (2.27) and (2.28) in fact uniquely define a feedback pair $[\mathcal{K}, \mathcal{F}]$ for Φ. This is nontrivial because the right hand side of equation (2.27) does not make sense for $x_0 \in H \setminus \mathrm{dom}(\mathcal{C})$ which is a nonempty set if Φ is not output stable. However, an observability map of any DLS is defined on all of the state space, not just on a dense subset (here $\mathrm{dom}(\mathcal{C})$) of it.

Define the linear operator $\mathcal{K}' := -S^{-1}\bar{\pi}_+\mathcal{N}^* J\mathcal{C}$ on $\mathrm{dom}(\mathcal{C})$. Define the linear operator $K := \pi_0\mathcal{K}' : \mathrm{dom}(\mathcal{C}) \to U$. By claim (i) of Proposition 46, the feed-through operator $X := \pi_0\mathcal{X}\pi_0 \in \mathcal{L}(U)$ has a bounded inverse $X^{-1} = \pi_0\mathcal{X}^{-1}\pi_0 \in \mathcal{L}(U)$. Now we obtain on $\mathrm{dom}(\mathcal{C})$ the identity

$$\begin{aligned} X^{-1}K &= \pi_0\mathcal{X}^{-1}\pi_0\mathcal{K}' = \pi_0\mathcal{X}^{-1}\mathcal{K}' \\ &= -\pi_0\mathcal{X}^{-1}S^{-1}\bar{\pi}_+\mathcal{N}^*J\mathcal{C} = \pi_0\mathcal{K}^{\mathrm{crit}} = K^{\mathrm{crit}}, \end{aligned}$$

where the fourth equality is by claim (iii) of Lemma 84. Because $K^{\mathrm{crit}} : \mathrm{dom}(\mathcal{C}) \to U$ is bounded by assumption, so is $K : \mathrm{dom}(\mathcal{C}) \to U$. Because of our standing assumption $\overline{\mathrm{dom}(\mathcal{C})} = H$, the operator K has a unique bounded extension to an element of $\mathcal{L}(H;U)$, also denoted by K.

On the other hand, we have on $\mathrm{dom}(\mathcal{C})$

$$\begin{aligned} \mathcal{K}'A &= -S^{-1}\bar{\pi}_+\mathcal{N}^*J\mathcal{C}A = -S^{-1}\bar{\pi}_+\mathcal{N}^*\bar{\pi}_+\tau^*J\mathcal{C} = -S^{-1}\bar{\pi}_+\mathcal{N}^*\tau^*J\mathcal{C} \\ &= -S^{-1}\bar{\pi}_+\tau^*\mathcal{N}^*J\mathcal{C} = -\bar{\pi}_+\tau^*(S^{-1}\bar{\pi}_+\mathcal{N}^*J\mathcal{C}) = \bar{\pi}_+\tau^*\mathcal{K}', \end{aligned}$$

where we have used the fact that $\mathcal{N}^*$ is anticausal and shift-invariant, and S is regarded as a static operator on $\ell^2(\mathbf{Z};U)$. It now follows for all $j \geq 0$ and $x_0 \in \mathrm{dom}(\mathcal{C})$ that

$$KA^jx_0 = \pi_0\mathcal{K}'A^jx_0 = \pi_0\tau^{*j}\mathcal{K}'x_0 = \tau^{*j}\pi_j\mathcal{K}'x_0,$$

and thus $\mathcal{K}'x_0 = \{KA^jx_0\}_{j\geq 0} =: \mathcal{K}x_0$. Here $\mathcal{K} := \mathcal{C}_\phi$ for any DLSs of the form $\phi = \left(\begin{smallmatrix} A & B \\ K & F \end{smallmatrix}\right)$, where $F \in \mathcal{L}(U)$ is arbitrary. In particular, $\mathcal{K}$ is a closed extension of $\mathcal{K}'$, and $\mathrm{dom}(\mathcal{C}) \subset \mathrm{dom}(\mathcal{K})$. Because $\mathcal{I} - \mathcal{X}$ is shift-invariant and causal on $\ell^2(\mathbf{Z};U) \cap Seq(U)$, it can be uniquely extended to a shift-invariant and causal linear mapping $\mathcal{F} : Seq(U) \to Seq(U)$, by using the causality. We now proceed to show that the quadruple of mappings $\left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{K} & \mathcal{F} \end{smallmatrix}\right]$ can be regarded as a DLS.

The output operator candidate for $\left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{K} & \mathcal{F} \end{smallmatrix}\right]$ is the familiar $K := \pi_0\mathcal{K}$, which belongs to $\mathcal{L}(H;U)$ as discussed above. Also $F := \pi_0\mathcal{F}\pi_0 = \pi_0(\mathcal{I} - \mathcal{X})\pi_0 = I - X \in \mathcal{L}(U)$, where $\mathrm{range}(\pi_0)$ and U have been identified, and I denotes the identity operator on U. By construction, $\mathcal{K}$ is an observability map of the DLS $\left(\begin{smallmatrix} A & B \\ K & F \end{smallmatrix}\right)$, which clearly equals $\left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{K} & \mathcal{F} \end{smallmatrix}\right]$ if the latter is a DLS at all. Finally, we have on $Seq_-(U) \subset \ell^2(\mathbf{Z}_-;U)$

$$\begin{aligned} \mathcal{K}\mathcal{B} &= -S^{-1}\bar{\pi}_+\mathcal{N}^*J\mathcal{C}\mathcal{B} = -S^{-1}\bar{\pi}_+\mathcal{N}^*(\bar{\pi}_+J\mathcal{D}\pi_-) \\ &= -S^{-1}\bar{\pi}_+\mathcal{N}^*J\mathcal{D}\pi_- = -S^{-1}\bar{\pi}_+(\mathcal{N}^*J\mathcal{N})\mathcal{X}\pi_- \\ &= -S^{-1}S\bar{\pi}_+\mathcal{X}\pi_- = -\bar{\pi}_+\mathcal{X}\pi_- = -\bar{\pi}_+(\mathcal{I} - \mathcal{F})\pi_- = \bar{\pi}_+\mathcal{F}\pi_-. \end{aligned}$$

We have now shown that $\left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{K} & \mathcal{F} \end{smallmatrix}\right]$ is a DLS. It is I/O stable, because $\mathcal{X}$ is an I/O map of an I/O stable DLS, by assumption. Because also $\operatorname{dom}(\mathcal{C}) \subset \operatorname{dom}(\mathcal{K})$, it follows that the pair $[\mathcal{K}, \mathcal{F}]$ is an I/O stable feedback pair for Φ. It is an outer feedback pair because $(\mathcal{I} - \mathcal{F})^{-1} = \mathcal{X}^{-1}$ is an I/O map of an I/O stable DLS. Now claim (ii) follows.

The proof of claim (iii) is now rather straightforward. The closed loop extended DLS $\Phi_\diamond^{\text{ext}} := \left[\Phi, [\mathcal{K}, \mathcal{F}]\right]_\diamond$ is I/O stable by claim (ii) of Theorem 48, and it is given by

$$\Phi_\diamond^{\text{ext}} = \begin{bmatrix} A_\diamond^j & \mathcal{B}_\diamond \tau^{*j} \\ \begin{bmatrix} \mathcal{C}_\diamond \\ \mathcal{K}_\diamond \end{bmatrix} & \begin{bmatrix} \mathcal{D}_\diamond \\ \mathcal{F}_\diamond \end{bmatrix} \end{bmatrix}$$
$$:= \begin{bmatrix} A^j + \mathcal{B}\tau^{*j}(\mathcal{I} - \mathcal{F})^{-1}\mathcal{K} & \mathcal{B}\tau^{*j}(\mathcal{I} - \mathcal{F})^{-1} \\ \begin{bmatrix} \mathcal{C} + \mathcal{D}(\mathcal{I} - \mathcal{F})^{-1}\mathcal{K} \\ (\mathcal{I} - \mathcal{F})^{-1}\mathcal{K} \end{bmatrix} & \begin{bmatrix} \mathcal{D}(\mathcal{I} - \mathcal{F})^{-1} \\ (\mathcal{I} - \mathcal{F})^{-1} - \mathcal{I} \end{bmatrix} \end{bmatrix}.$$

For all $x_0 \in \operatorname{dom}(\mathcal{C})$, it clearly follows that

$$\mathcal{K}_\diamond x_0 = (\mathcal{I} - \mathcal{F})^{-1}\mathcal{K}x_0 = -\mathcal{X}^{-1}S^{-1}\bar{\pi}_+\mathcal{N}^* J\mathcal{C}x_0 = \mathcal{K}^{\text{crit}}x_0 = \tilde{u}^{\text{crit}}(x_0),$$

by claim (iii) of Lemma 84 and Lemma 71. Then, also

$$A_\diamond{}^j x_0 = A^j x_0 + \mathcal{B}\tau^{*j}\mathcal{K}^{\text{crit}}x_0 = (A^{\text{crit}})^j x_0 = x_j^{\text{crit}}(x_0),$$

and

$$\mathcal{C}_\diamond x_0 = \mathcal{C}x_0 + \mathcal{D}\mathcal{K}^{\text{crit}}x_0 = \mathcal{C}^{\text{crit}}x_0 = \tilde{y}^{\text{crit}}(x_0),$$

by Lemmas 71 and 72. This verifies claim (iii). Claim (iv) is trivial, and the proof of this lemma is complete. $\square$

Let us now consider more carefully the case when the DLS Φ, in addition, is output stable. In this case $\operatorname{dom}(\mathcal{C}) = H$ and the linear operator $\mathcal{K}' := -S^{-1}\bar{\pi}_+\mathcal{N}^*J\mathcal{C} : \operatorname{dom}(\mathcal{C}) \to \ell^2(\mathbf{Z}_+; U)$ equals its (bounded) extension $\mathcal{K} : H \to \ell^2(\mathbf{Z}_+; U)$ that has been constructed in the previous proof. Now, the corresponding critical feedback pair is given by

$$[\mathcal{K}, \mathcal{F}] = \left[-S^{-1}\bar{\pi}_+\mathcal{N}^*J\mathcal{C},\ \mathcal{I} - \mathcal{X}\right]$$

where $\mathcal{X}$ is regarded as the linear extension (by causality) of $\mathcal{X}|\left(\ell^2(\mathbf{Z}; U) \cap Seq(U)\right)$ to all of $Seq(U)$. The critical closed loop DLS $\Phi_\diamond^{\text{ext}} = [\Phi, [\mathcal{K}, \mathcal{F}]]_\diamond$ is given by

$$\Phi_\diamond^{\text{ext}} = \begin{bmatrix} A^j - \mathcal{B}\mathcal{X}^{-1}\tau^{*j}S^{-1}\bar{\pi}_+\mathcal{N}^*J\mathcal{C} & \mathcal{B}\mathcal{X}^{-1}\tau^{*j} \\ \begin{bmatrix} \mathcal{C} - \mathcal{N}S^{-1}\bar{\pi}_+\mathcal{N}^*J\mathcal{C} \\ -\mathcal{X}^{-1}S^{-1}\bar{\pi}_+\mathcal{N}^*J\mathcal{C} \end{bmatrix} & \begin{bmatrix} \mathcal{N} \\ \mathcal{X}^{-1} - \mathcal{I} \end{bmatrix} \end{bmatrix} \tag{2.29}$$
$$= \begin{bmatrix} (A^{\text{crit}})^j & \mathcal{B}\mathcal{X}^{-1}\tau^{*j} \\ \begin{bmatrix} \mathcal{C}^{\text{crit}} \\ \mathcal{K}^{\text{crit}} \end{bmatrix} & \begin{bmatrix} \mathcal{N} \\ \mathcal{X}^{-1} - \mathcal{I} \end{bmatrix} \end{bmatrix}.$$

As we have noted earlier, Proposition 83 gives a parameterization for the critical feedback pairs. In fact, all the critical feedback pairs are parameterized this way because the previous lemma has a converse.

Lemma 88. *Let $J \in \mathcal{L}(Y)$ be a cost operator. Let $\Phi = \left[\begin{smallmatrix} \mathcal{A}^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ be an I/O stable DLS. Assume that*

(i) Φ is J-coercive, and

(ii) the critical control problem, associated to Φ and J, is solvable by state feedback.

Then the following holds:

(i) There exists a boundedly invertible $S \in \mathcal{L}(U)$ such that $\mathcal{D} = \mathcal{N}\mathcal{X}$ is a (J,S)-inner-outer factorization, where the outer factor $\mathcal{X}$ is outer with a bounded inverse. One such factorization is characterized by

$$\mathcal{X}\tilde{u} = (\mathcal{I} - \mathcal{F})\,\tilde{u} \quad \textit{for all} \quad \tilde{u} \in \ell^2(\mathbf{Z};U) \cap Seq(U), \tag{2.30}$$

$$\mathcal{N}\tilde{u} = \mathcal{D}\mathcal{X}^{-1}\tilde{u} \quad \textit{for all} \quad \tilde{u} \in \ell^2(\mathbf{Z};U) \cap Seq(U), \tag{2.31}$$

where $[\mathcal{K}, \mathcal{F}]$ is an I/O stable and outer, critical feedback pair for Φ.

(ii) $K^{\mathrm{crit}} := \pi_0\mathcal{K}^{\mathrm{crit}} : \mathrm{dom}\,(\mathcal{C}) \to U$ is bounded.

Proof. Much of the work in the proof of claim (i) lies in making various extensions and restrictions of linear operators. Let $[\mathcal{K}, \mathcal{F}]$ be an I/O stable and outer, critical feedback pair for the DLS Φ. Then equation (2.30) defines a densely defined, bounded linear operator $\mathcal{X} : \ell^2(\mathbf{Z};U) \cap Seq(U) \to \ell^2(\mathbf{Z};U) \cap Seq(U)$. By density, $\mathcal{X}$ has a bounded extension to all of $\ell^2(\mathbf{Z};U)$, denoted by $\bar{\mathcal{X}}$. The extended operator $\bar{\mathcal{X}}$ is shift-invariant and causal, by a simple continuity argument.

Because the feedback pair $[\mathcal{K}, \mathcal{F}]$ is assumed to be outer, $(\mathcal{I} - \mathcal{F})^{-1} : Seq(U) \to Seq(U)$ exists, and it is an I/O map of an I/O stable DLS. It follows that $\mathcal{G} := (\mathcal{I} - \mathcal{F})^{-1} : \ell^2(\mathbf{Z};U) \cap Seq(U) \to \ell^2(\mathbf{Z};U) \cap Seq(U)$ is a bounded, densely defined operator on $\ell^2(\mathbf{Z};U)$. By the definitions of $\bar{\mathcal{X}}$ and $\mathcal{G}$,

$$\mathcal{G}\bar{\mathcal{X}}\tilde{u} = \bar{\mathcal{X}}\mathcal{G}\tilde{u} = \tilde{u} \tag{2.32}$$

for all $\tilde{u} \in \ell^2(\mathbf{Z};U) \cap Seq(U)$. As before, $\mathcal{G}$ has a bounded extension to all of $\ell^2(\mathbf{Z};U)$, denoted by $\bar{\mathcal{G}}$. By continuity of $\bar{\mathcal{X}}$ and $\bar{\mathcal{G}}$, identity (2.32) holds for the extended $\bar{\mathcal{G}}$ and for all $\tilde{u} \in \ell^2(\mathbf{Z};U)$. Thus the bounded inverse $\bar{\mathcal{X}}^{-1} : \ell^2(\mathbf{Z};U) \to \ell^2(\mathbf{Z};U)$ exists and equals $\bar{\mathcal{G}}$. Because $\bar{\mathcal{X}}^{-1}$ coincides with the I/O map $(\mathcal{I} - \mathcal{F})^{-1}$ on the dense set $\ell^2(\mathbf{Z};U) \cap Seq(U)$, it follows that $\bar{\mathcal{X}}^{-1}$ is shift-invariant and causal, by continuity. We can now define $\mathcal{N} := \mathcal{D}\bar{\mathcal{X}}^{-1}$ as a bounded operator on $\ell^2(\mathbf{Z}; U)$. From now on, we write $\mathcal{X}$ in place of $\bar{\mathcal{X}}$, too. Clearly, such $\mathcal{N}$ and $\mathcal{X}$ satisfy equations (2.30) and (2.31).

It remains to be shown that $\mathcal{D} = \mathcal{N}\mathcal{X}$ is a (J, S)-inner-outer factorization for some $S \in \mathcal{L}(U)$. Because Φ is I/O stable, by Proposition 82 it is sufficient to show that $\mathcal{X}$ is an outer S-spectral factor of $\mathcal{D}^* J \mathcal{D}$ for some $S \in \mathcal{L}(U)$; i.e.

$$\mathcal{D}^* J \mathcal{D} = \mathcal{X}^* S \mathcal{X}. \tag{2.33}$$

Let the bounded shift-invariant operator $\mathcal{Z} : \ell^2(\mathbf{Z}; U) \to \ell^2(\mathbf{Z}; U)$ be defined through its adjoint

$$\mathcal{Z}^* := \mathcal{D}^* J \mathcal{D} \mathcal{X}^{-1} = \mathcal{D}^* J \mathcal{N}.$$

We next show that $\mathcal{Z}^*$ is anticausal; i.e. $\bar{\pi}_+ \mathcal{Z}^* \pi_- = 0$.

The critical closed loop DLS $\Phi_\diamond^{\text{ext}} = [\Phi, [\mathcal{K}, \mathcal{F}]]$, associated to the critical feedback pair $[\mathcal{K}, \mathcal{F}]$, is given by

$$\Phi_\diamond^{\text{ext}} = \begin{bmatrix} A^j + \mathcal{B}\tau^{*j}(\mathcal{I} - \mathcal{F})^{-1}\mathcal{K} & \mathcal{B}(\mathcal{I} - \mathcal{F})^{-1}\tau^{*j} \\ \begin{bmatrix} \mathcal{C} + \mathcal{D}(\mathcal{I} - \mathcal{F})^{-1}\mathcal{K} \\ (\mathcal{I} - \mathcal{F})^{-1}\mathcal{K} \end{bmatrix} & \begin{bmatrix} \mathcal{D}(\mathcal{I} - \mathcal{F})^{-1} \\ (\mathcal{I} - \mathcal{F})^{-1} - \mathcal{I} \end{bmatrix} \end{bmatrix}.$$

Because $\Phi_\diamond^{\text{ext}}$ is a critical closed loop DLS, we have a unique critical output sequence for all $x_0 \in \operatorname{dom}(\mathcal{C})$, given by

$$\tilde{y}^{\text{crit}}(x_0) = \left(\mathcal{C} + \mathcal{D}(\mathcal{I} - \mathcal{F})^{-1}\mathcal{K}\right) x_0. \tag{2.34}$$

Let $\tilde{u} \in \operatorname{dom}(\mathcal{B}) := Seq_-(U)$ be arbitrary. Then $\pi_-(\mathcal{I} - \mathcal{F})^{-1}\pi_-\tilde{u} \in \operatorname{dom}(\mathcal{B})$ by causality, and because always $\mathcal{B} = \mathcal{B}\pi_-$ there exists

$$x(\tilde{u}) := \mathcal{B}(\mathcal{I} - \mathcal{F})^{-1}\pi_-\tilde{u} = \mathcal{B}\pi_-(\mathcal{I} - \mathcal{F})^{-1}\pi_-\tilde{u} \in \operatorname{range}(\mathcal{B}).$$

We have $\operatorname{range}(\mathcal{B}) \subset \operatorname{dom}(\mathcal{C})$, by Lemma 35 and I/O stability of Φ. Thus $x(\tilde{u}) \in \operatorname{dom}(\mathcal{C})$ for any $\tilde{u} \in \operatorname{dom}(\mathcal{B})$, and we can set $x_0 = x(\tilde{u})$ in equation (2.34). This gives for the critical output sequence the expression

$$\begin{aligned} \tilde{y}^{\text{crit}}(x(\tilde{u})) &= \left(\mathcal{C} + \mathcal{D}(\mathcal{I} - \mathcal{F})^{-1}\mathcal{K}\right) \cdot \left(\mathcal{B}(\mathcal{I} - \mathcal{F})^{-1}\right) \pi_-\tilde{u} \\ &= \bar{\pi}_+\mathcal{D}(\mathcal{I} - \mathcal{F})^{-1}\pi_-\tilde{u} = \bar{\pi}_+\mathcal{D}\mathcal{X}^{-1}\pi_-\tilde{u} = \bar{\pi}_+\mathcal{N}\pi_-\tilde{u}. \end{aligned} \tag{2.35}$$

Here the third equality follows from claim (iv) of Lemma 12 because $\left(\mathcal{C} + \mathcal{D}(\mathcal{I} - \mathcal{F})^{-1}\mathcal{K}\right)$ is a component of the observability map and $\mathcal{B}(\mathcal{I} - \mathcal{F})^{-1}$ is the controllability map of the DLS $\Phi_\diamond^{\text{ext}}$. The third equality holds because $\pi_-\tilde{u} \in Seq_-(U) \subset \ell^2(\mathbf{Z}_-; U)$ and $\mathcal{X}^{-1}$ coincides with $(\mathcal{I} - \mathcal{F})^{-1}$ on $\ell^2(\mathbf{Z}; U) \cap Seq(U)$. Because $x(\tilde{u}) \in \operatorname{dom}(\mathcal{C})$, we have by formula (2.4) of Lemma 67

$$\bar{\pi}_+\mathcal{D}^* J \tilde{y}^{\text{crit}}(x(\tilde{u})) = 0. \tag{2.36}$$

Now the equations (2.35) and (2.36) together give for all $\tilde{u} \in \operatorname{dom}(\mathcal{B})$

$$\bar{\pi}_+ \mathcal{D}^* J \mathcal{N} \pi_- \tilde{u} = \bar{\pi}_+ \mathcal{Z}^* \pi_- \tilde{u} = 0,$$

by the anticausality of $\mathcal{D}^*$. Because $\operatorname{dom}(\mathcal{B})$ is dense in $\ell^2(\mathbf{Z}_-; U)$, it follows that the bounded shift-invariant operator $\mathcal{Z}^*$ is anticausal.

Because $\mathcal{D}^* J \mathcal{D}$ is self-adjoint, we have

$$\mathcal{D}^* J \mathcal{D} = \mathcal{Z}^* \mathcal{X} = \mathcal{X}^* \mathcal{Z},$$

or equivalently,

$$(\mathcal{X}^*)^{-1} \mathcal{D}^* J \mathcal{D} \mathcal{X}^{-1} = (\mathcal{Z}\mathcal{X}^{-1})^* = \mathcal{Z}\mathcal{X}^{-1}. \tag{2.37}$$

Because $(\mathcal{Z}\mathcal{X}^{-1})^*$ is anticausal and $\mathcal{Z}\mathcal{X}^{-1}$ is causal, it follows that $(\mathcal{X}^*)^{-1}\mathcal{D}^* J \mathcal{D}\mathcal{X}^{-1}$ is a static operator. By Proposition 6, it is a componentwise multiplication by a self-adjoint operator $S \in \mathcal{L}(U)$. Thus the spectral factorization (2.33) follows.

We have now shown that $\mathcal{X}$ is an outer stable S-spectral factor of $\mathcal{D}^* J \mathcal{D}$. Proposition 82 implies that $\mathcal{D} = \mathcal{N}\mathcal{X}$ is a (J, S)-inner-outer factorization, where the outer factor $\mathcal{X}$ is outer with a bounded inverse. The sensitivity operator S has a bounded inverse $S^{-1} \in \mathcal{L}(U)$, by the assumed J-coercivity of Φ and Corollary 85. This completes the proof of claim (i). The second claim (ii) holds because for all $x_0 \in \operatorname{dom}(\mathcal{C})$

$$K^{\text{crit}} x_0 = \pi_0 \mathcal{K}^{\text{crit}} x_0 = \pi_0 (\mathcal{I} - \mathcal{F})^{-1} \mathcal{K} x_0 = \pi_0 \mathcal{X}^{-1} \pi_0 \cdot \pi_0 \mathcal{K} x_0$$

where the feed-through part of the outer factor has a bounded inverse, by Proposition 46, and $\pi_0 \mathcal{K} : H \to U$ is bounded, by the definition of the feedback pair. This completes the proof. $\square$

Now we are ready to present the first main result of this chapter. Under certain conditions, the (J, S)-inner-outer factorization problem of the I/O map $\mathcal{D}$ of Φ is equivalent with the problem of solving the critical control problem, associated to Φ and J, by state feedback.

Theorem 89. *Let $J \in \mathcal{L}(Y)$ be a cost operator. Let $\Phi = \left[\begin{smallmatrix} \mathcal{A}^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ be an I/O stable DLS. Then the following conditions (i) and (ii) are equivalent:*

(i) *a)* *Φ is J-coercive, and*

b) *the critical control problem, associated to Φ and J, is solvable by state feedback. The critical feedback pair $[\mathcal{K}, \mathcal{F}]$ for Φ is I/O stable and outer.*

(ii) *a)* *There is a self-adjoint boundedly invertible operator* $S \in \mathcal{L}(U)$ *such that* $\mathcal{D}$ *has a* (J,S)*-inner-outer factorization* $\mathcal{D} = \mathcal{N}\mathcal{X}$*, where the outer factor* $\mathcal{X}$ *is outer with a bounded inverse, and*

b) $\pi_0 \mathcal{N}^* J\mathcal{C} \in \mathcal{L}(\mathrm{dom}\,(\mathcal{C})\,; U)$*, where* $\mathrm{dom}\,(\mathcal{C})$ *is given the norm of* H*.*

Furthermore, if the above conditions hold, then $K^{\mathrm{crit}} \in \mathcal{L}(\mathrm{dom}\,(\mathcal{C})\,; U)$*, and both the open loop extended DLS* $\Phi^{\mathrm{ext}} := \left[\Phi, [\mathcal{K}, \mathcal{F}]\right]$ *and the critical closed loop extended DLS* $\Phi^{\mathrm{ext}}_{\diamond}$ *are I/O stable. Assume, in addition, that* Φ *is output stable. Then the critical feedback pair* $[\mathcal{K}, \mathcal{F}]$ *is stable and outer, both* Φ^{ext} *and* $\Phi^{\mathrm{ext}}_{\diamond}$ *are output stable, and part (b)) of condition (ii) need not be stated explicitly.*

Proof. The implication (i) $\Rightarrow$ (ii) is given in Lemma 88. We prove the converse implication (ii) $\Rightarrow$ (i). By claim (iii) of Lemma 84 and causality of $\mathcal{X}^{-1}$ we have

$$K^{\mathrm{crit}} = \pi_0 \mathcal{K}^{\mathrm{crit}} = -\pi_0 \mathcal{X}^{-1} \pi_0 S^{-1} \cdot \pi_0 \mathcal{N}^* J\mathcal{C}$$

where $\pi_0 \mathcal{X}^{-1} \pi_0 \in \mathcal{L}(U)$ is bounded with bounded inverse, by claim (i) of Proposition 46. It follows that $K^{\mathrm{crit}} : \mathrm{dom}\,(\mathcal{C}) \to U$ is bounded if and only if $\pi_0 \mathcal{N}^* J\mathcal{C}$ is. Thus, condition (ii) implies condition (i) of Lemma 87. But now condition (i) follows, by Lemma 87. The I/O stability of Φ^{ext} and $\Phi^{\mathrm{ext}}_{\diamond}$ follows from Theorem 48. The additional claim involving the output stability of Φ is a trivial consequence of Theorem 48 and the fact that an observability map is bounded if and only if its domain is all of the state space. $\square$

Further stability results for the critical closed loop DLS $\Phi^{\mathrm{ext}}_{\diamond}$ and its semigroup generator A^{crit} are given by Theorems 50 and 51.

2.5 Weak algebraic Riccati equation

Let $J \in \mathcal{L}(Y)$ be a cost operator and $\Phi = \left[\begin{smallmatrix} \mathcal{A}^j & \mathcal{B}_{\mathcal{T}^{*j}} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ be an I/O stable and J-coercive DLS. In the remaining part of this chapter, a weak form of a discrete time algebraic Riccati equation is introduced. Such an equation is associated to the minimax control problem that has been introduced in Section 2.2. More precisely, we show that the critical cost sesquilinear form $P_0^{\mathrm{crit}}(\,,\,)$, as introduced in Definition 76, satisfies an algebraic Riccati equation, provided that $\mathcal{D}$ has a (J,S)-inner-outer factorization. A converse result to this is given in Section 2.6.

Definition 90. *Let $J \in \mathcal{L}(Y)$ be a cost operator, and $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ be an I/O stable DLS. Let $P(\,,\,) : H \times H \supset \mathrm{dom}\,(\mathcal{C}) \times \mathrm{dom}\,(\mathcal{C}) \to \mathbf{C}$ denote a conjugate symmetric sesquilinear form. Then the conjugate symmetric sesquilinear form $\Lambda_P(\,,\,)$ on $U \times U$ defined by*

$$\Lambda_P(u_0, w_0) := \langle Du_0, JDw_0 \rangle_Y + P(Bu_0, Bw_0)$$

is the indicator of the sesquilinear form $P(\,,\,)$, associated to DLS ϕ and cost operator J.

The indicator $\Lambda_P(\,,\,)$ is well-defined on the whole of $U \times U$. The possible problem would arise if we had to go outside the domain $\mathrm{dom}\,(\mathcal{C}) \times \mathrm{dom}\,(\mathcal{C})$ of $P(\,,\,)$ for some $u_0, w_0 \in U$. However, I/O stability of ϕ implies the inclusion $BU \subset \mathrm{dom}\,(\mathcal{C})$, see Lemma 35. Even more can be said about the critical sesquilinear form $P_0^{\mathrm{crit}}(\,,\,)$ of Definition 76.

Proposition 91. *Let $J \in \mathcal{L}(Y)$ be a cost operator. Let $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) = \left[\begin{smallmatrix} \mathcal{A}^j & \mathcal{B}_{\mathcal{T}^{*j}} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ be an I/O stable and J-coercive DLS. By $P_0^{\mathrm{crit}}(\,,\,)$ denote the critical sesquilinear form as defined in Definition 76. Then there exists a unique self-adjoint operator $\Lambda_{P_0^{\mathrm{crit}}} \in \mathcal{L}(U)$ such that the indicator sesquilinear form $\Lambda_{P_0^{\mathrm{crit}}}(\,,\,)$ can be represented by*

$$\Lambda_{P_0^{\mathrm{crit}}}(u_0, w_0) = \left\langle \Lambda_{P_0^{\mathrm{crit}}} u_0, w_0 \right\rangle_U,$$

where

$$\Lambda_{P_0^{\mathrm{crit}}} := D^* J D + (\mathcal{C}^{\mathrm{crit}} B)^* J (\mathcal{C}^{\mathrm{crit}} B).$$

Proof. The claim immediately follows, once we remember that the I/O stability of ϕ implies that $\mathcal{C}B \in \mathcal{L}(U, \ell^2(\mathbf{Z}_+; Y))$. Then

$$\mathcal{K}^{\mathrm{crit}} B = -(\bar{\pi}_+ \mathcal{D}^* J \mathcal{D} \bar{\pi}_+)^{-1} \bar{\pi}_+ \mathcal{D}^* J \mathcal{C} B,$$

is bounded, and so is $\mathcal{C}^{\mathrm{crit}} B = \left(\mathcal{C} + \mathcal{D}\mathcal{K}^{\mathrm{crit}}\right) B$, too. This makes it possible to speak about $(\mathcal{C}^{\mathrm{crit}} B)^*$ as an adjoint of a bounded operator. The self-adjointness and uniqueness of $\Lambda_{P_0^{\mathrm{crit}}}$ is clear. □

The operator $\Lambda_{P_0^{\rm crit}}$ is the critical indicator operator, associated to Φ and J. In the following lemma, we couple the critical sesquilinear form $P_0^{\rm crit}(\, ,\,)$, its indicator operator $\Lambda_{P_0^{\rm crit}}$ and the critical one step feedback operator $K^{\rm crit}$ together.

Lemma 92. *Let $J \in \mathcal{L}(Y)$ be a cost operator. Let $\Phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ be an I/O stable and J-coercive DLS, such that $K^{\rm crit} := \pi_0 \mathcal{K}^{\rm crit} : \mathrm{dom}\,(\mathcal{C}) \to U$ is bounded. Let the critical indicator operator $\Lambda_{P_0^{\rm crit}}$ be given by Proposition 91. Then the critical sesquilinear cost $P_0^{\rm crit}(\, ,\,)$ satisfies the equations*

$$P_0^{\rm crit}(A^{\rm crit}x_0, Bw_0) + \left\langle C^{\rm crit}x_0, JDw_0 \right\rangle_Y = 0, \tag{2.38}$$

$$P_0^{\rm crit}(Ax_0, Bw_0) + \left\langle (\Lambda_{P_0^{\rm crit}} K^{\rm crit} + D^*JC)x_0, w_0 \right\rangle_U = 0, \tag{2.39}$$

$$\Lambda_{P_0^{\rm crit}} K^{\rm crit} x_0 = -\left((\mathcal{C}^{\rm crit}B)^* J \mathcal{C}^{\rm crit} A - D^*JC \right) x_0, \tag{2.40}$$

for all $x_0 \in \mathrm{dom}\,(\mathcal{C})$ and $w_0 \in U$. Furthermore,

$$\begin{aligned} &P_0^{\rm crit}(Ax_0, Ax_1) - P_0^{\rm crit}(x_0, x_1) \\ &= \left\langle ((K^{\rm crit})^* \Lambda_{P_0^{\rm crit}} K^{\rm crit} - C^*JC)x_0, x_1 \right\rangle_H \end{aligned} \tag{2.41}$$

for all $x_0, x_1 \in \mathrm{dom}\,(\mathcal{C})$, where $A^{\rm crit} = A + BK^{\rm crit}$ and $C^{\rm crit} := C + DK^{\rm crit} = \pi_0 \mathcal{C}^{\rm crit}$.

Proof. In order to establish equation (2.38), we consider the following perturbed critical control sequences for arbitrary $x_0 \in \mathrm{dom}\,(\mathcal{C})$, $\tilde{w} = \{w_j\}_{j \ge 0} \in \ell^2(\mathbf{Z}_+; U)$ and $\epsilon \in \mathbf{R}$

$$\begin{aligned} &\mathcal{K}^{\rm crit}x_0 + \epsilon\left(\pi_0 \tilde{w} + \tau \mathcal{K}^{\rm crit} B w_0\right) \\ &= \pi_0\left(\mathcal{K}^{\rm crit}x_0 + \epsilon\tilde{w}\right) + \tau\left(\bar{\pi}_+ \tau^* \mathcal{K}^{\rm crit} x_0 + \mathcal{K}^{\rm crit} B \epsilon w_0\right) \\ &= \pi_0\left(\mathcal{K}^{\rm crit}x_0 + \epsilon\tilde{w}\right) + \tau \mathcal{K}^{\rm crit}\left(A^{\rm crit}x_0 + B\epsilon w_0\right) \\ &= \pi_0\left(\mathcal{K}^{\rm crit}x_0 + \epsilon\tilde{w}\right) + \tau \mathcal{K}^{\rm crit}\left(Ax_0 + B(K^{\rm crit}x_0 + \epsilon w_0)\right) \\ &= \pi_0\left(\tilde{u}^{\rm crit}(x_0) + \epsilon\tilde{w}\right) + \tau \tilde{u}^{\rm crit}\left(Ax_0 + B(u_0^{\rm crit}(x_0) + \epsilon w_0)\right) \end{aligned}$$

where the second equality is by Lemma 74 and the critical control sequence satisfies $\tilde{u}^{\rm crit}(x) = \{u_j^{\rm crit}(x)\}_{j \ge 0} = \mathcal{K}^{\rm crit}x$ for all $x \in \mathrm{dom}\,(\mathcal{C})$, by Lemma 71. This gives us the identity

$$\begin{aligned} &J\left(x_0, \mathcal{K}^{\rm crit}x_0 + \epsilon(\pi_0\tilde{w} + \tau\mathcal{K}^{\rm crit}Bw_0)\right) \\ &= \left\langle Cx_0 + D\left(u_0^{\rm crit}(x_0) + \epsilon w_0\right), J(-,,-)\right\rangle_Y \\ &+ J\left(Ax_0 + B\left(u_0^{\rm crit}(x_0) + \epsilon w_0\right), \tilde{u}^{\rm crit}\left(Ax_0 + B(u_0^{\rm crit}(x_0) + \epsilon w_0)\right)\right) \\ &= \left\langle C^{\rm crit}x_0 + \epsilon D w_0, J(-,,-)\right\rangle_Y \\ &+ P_0^{\rm crit}\left(Ax_0 + B(u_0^{\rm crit}(x_0) + \epsilon w_0), (-,,-)\right), \\ &= \left\langle C^{\rm crit}x_0 + \epsilon D w_0, J(-,,-)\right\rangle_Y \\ &+ P_0^{\rm crit}\left(A^{\rm crit}x_0 + \epsilon B w_0, (-,,-)\right), \end{aligned} \tag{2.42}$$

where the second equality follows because $C^{\rm crit}x_0 = Cx_0 + DK^{\rm crit}x_0 = Cx_0 + Du_0^{\rm crit}(x_0)$, and equation (2.17) of Proposition 78 holds.

Now we Frechet differentiate identity (2.42) with respect to ϵ at $\epsilon = 0$ where $\tilde{w} := \{w_j\}_{j\geq 0} \in \ell^2(\mathbf{Z}_+;U)$ is arbitrary. This Frechet derivative must equal zero for any $\tilde{w}$, by the definition of the critical control $\tilde{u}^{\rm crit}(x_0) = \mathcal{K}^{\rm crit}x_0$ appearing on the left hand side. We obtain the equality

$$Re\left(P_0^{\rm crit}(A^{\rm crit}x_0, Bw_0) + \left\langle C^{\rm crit}x_0, JDw_0\right\rangle_Y\right) = 0.$$

By replacing x_0 with ix_0, we see that this is true for the imaginary part, as well. Equation (2.38) now follows.

The proof of equation (2.39) is based upon equation (2.38). We have by a straightforward calculation, starting from the definition of the indicator operator $\Lambda_{P_0^{\rm crit}}$

$$\begin{aligned}
&\Lambda_{P_0^{\rm crit}}(K^{\rm crit}x_0, w_0) := P_0^{\rm crit}(BK^{\rm crit}x_0, Bw_0) + \left\langle DK^{\rm crit}x_0, JDw_0\right\rangle_Y\\
&= P_0^{\rm crit}(A^{\rm crit}x_0, Bw_0) + \left\langle C^{\rm crit}x_0, JDw_0\right\rangle_Y\\
&- P_0^{\rm crit}(Ax_0, Bw_0) - \langle D^*JCx_0, w_0\rangle_Y\,.
\end{aligned}$$

This proves that equation (2.39) is equivalent to equation (2.38).

Equation (2.40) follows immediately from equation (2.39) and the definition of $P_0^{\rm crit}(\,,\,)$. The proof of equation (2.41) is based on Lemma 74 and the first part of this lemma. Lemma 74 implies

$$\begin{aligned}
&P_0^{\rm crit}(A^{\rm crit}x_0, A^{\rm crit}x_1) = \left\langle \mathcal{C}^{\rm crit}A^{\rm crit}x_0, J\mathcal{C}^{\rm crit}A^{\rm crit}x_1\right\rangle_{\ell^2(\mathbf{Z}_+;Y)}\\
&= \left\langle \bar{\pi}_+\tau^*\mathcal{C}^{\rm crit}x_0, \bar{\pi}_+\tau^*J\mathcal{C}^{\rm crit}x_1\right\rangle_{\ell^2(\mathbf{Z}_+;Y)}\\
&= \left\langle \mathcal{C}^{\rm crit}x_0, J\mathcal{C}^{\rm crit}x_1\right\rangle_{\ell^2(\mathbf{Z}_+;Y)} - \left\langle C^{\rm crit}x_0, JC^{\rm crit}x_1\right\rangle_Y\\
&= P_0^{\rm crit}(x_0, x_1) - \left\langle C^{\rm crit}x_0, JC^{\rm crit}x_1\right\rangle_Y\,.
\end{aligned}$$

We conclude that that

$$P_0^{\rm crit}(A^{\rm crit}x_0, A^{\rm crit}x_1) - P_0^{\rm crit}(x_0, x_1) + \left\langle C^{\rm crit}x_0, JC^{\rm crit}x_1\right\rangle_Y = 0 \tag{2.43}$$

for all $x_0, x_1 \in \mathrm{dom}\,(\mathcal{C})$. Now, a straightforward calculation, based on the identities $A^{\rm crit} = A + BK^{\rm crit}$ and $C^{\rm crit} = C + DK^{\rm crit}$, gives

$$\begin{aligned}
&P_0^{\rm crit}(Ax_0, Ax_1) - P_0^{\rm crit}(x_0, x_1) + \langle C^*JCx_0, x_1\rangle_H\\
&= P_0^{\rm crit}(A^{\rm crit}x_0, A^{\rm crit}x_1) - P_0^{\rm crit}(x_0, x_1) + \left\langle C^{\rm crit}x_0, JC^{\rm crit}x_1\right\rangle_Y\\
&- P_0^{\rm crit}(BK^{\rm crit}x_0, A^{\rm crit}x_1) - P_0^{\rm crit}(Ax_0, BK^{\rm crit}x_1)\\
&- \left\langle DK^{\rm crit}x_0, JC^{\rm crit}x_1\right\rangle_Y - \left\langle Cx_0, JDK^{\rm crit}x_1\right\rangle_Y\\
&= -\left(P_0^{\rm crit}(BK^{\rm crit}x_0, A^{\rm crit}x_1) + \left\langle JDK^{\rm crit}x_0, C^{\rm crit}x_1\right\rangle_Y\right)\\
&- \left(P_0^{\rm crit}(Ax_0, BK^{\rm crit}x_1) + \left\langle D^*JCx_0, K^{\rm crit}x_1\right\rangle_Y\right)
\end{aligned}$$

where equation (2.43) has been used. By equation (2.38), the (conjugate of) first term on the right hand side vanished. The second term equals

$$\begin{aligned} &P_0^{\mathrm{crit}}(Ax_1, BK^{\mathrm{crit}}x_0) + \left\langle D^*JCx_1, K^{\mathrm{crit}}x_0\right\rangle_U \\ &= -\left\langle (K^{\mathrm{crit}})^*\Lambda_{P_0^{\mathrm{crit}}}K^{\mathrm{crit}}x_1, x_0\right\rangle_U \end{aligned}$$

by equation (2.39). This completes the proof. □

Under certain conditions, the indicator operator $\Lambda_{P_0^{\mathrm{crit}}}$ has a bounded inverse in $\mathcal{L}(U)$. At the same time we get a connection between the (J,S)-inner-outer factorization of $\mathcal{D}$ and the indicator operator. This is the contents of the following lemma.

Lemma 93. *Let $J \in \mathcal{L}(Y)$ be a cost operator, and $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right] = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ be an I/O stable DLS. Assume that the equivalent conditions of Theorem 89 are satisfied. By $P_0^{\mathrm{crit}}(\,,\,)$ denote the critical cost sesquilinear form. Then the indicator operator $\Lambda_{P_0^{\mathrm{crit}}} \in \mathcal{L}(U)$ has a bounded inverse.*

Proof. Because the equivalent conditions of Theorem 89 hold, it follows that Φ is J-coercive and the critical cost sesquilinear form $P_0^{\mathrm{crit}}(\,,\,)$ of Definition 76 is defined. By Theorem 89, there is an (J,S')-inner-outer factorization $\mathcal{D} = \mathcal{N}'\mathcal{X}'$, where the outer factor $\mathcal{X}'$ is outer with a bounded inverse and $S'^{-1} \in \mathcal{L}(U)$. By claim (i) of Proposition 46, the feed-forward part $X' = \pi_0\mathcal{X}'\pi_0$ has a bounded inverse. Choosing $E = X'$ in Proposition 83, we see that there is another (J,S)-inner-outer factorization $\mathcal{D} = \mathcal{N}\mathcal{X}$, where the outer factor $\mathcal{X}'$ is outer with a bounded inverse and its feed-forward part satisfies $\pi_0\mathcal{X}'\pi_0 = I$; here range$\,(\pi_0)$ and U are identified, and I denotes the identity operator in $\mathcal{L}(U)$. Finally, the sensitivity operator of the factorization is given by $S = X'^*S'X'$, and this is boundedly invertible because S' is.

Let $[\mathcal{K},\mathcal{F}]$ be the critical feedback pair as in the proof of Lemma 87, corresponding to the above constructed factorization $\mathcal{D} = \mathcal{N}\mathcal{X}$. By $\Phi_\diamond^{\mathrm{ext}} = [\Phi,[\mathcal{K},\mathcal{F}]]_\diamond$ denote the critical closed loop DLS. Let $\tilde{u} = \{u_j\}_{j\geq 0} \in \ell^2(\mathbf{Z}_+;U)$ be arbitrary. By the basic properties of the DLS, the impulse response of the I/O map of $\Phi_\diamond^{\mathrm{ext}}$ is given by

$$\begin{aligned} &\begin{pmatrix} \mathcal{D}(\mathcal{I}-\mathcal{F})^{-1} \\ (\mathcal{I}-\mathcal{F})^{-1}-\mathcal{I} \end{pmatrix}\pi_0\tilde{u} \\ &= \begin{pmatrix} D(I-F)^{-1} \\ (I-F)^{-1}-I \end{pmatrix}\pi_0\tilde{u} + \tau\begin{pmatrix} \mathcal{C}+\mathcal{D}(\mathcal{I}-\mathcal{F})^{-1}\mathcal{K} \\ (\mathcal{I}-\mathcal{F})^{-1}\mathcal{K} \end{pmatrix} B(I-F)^{-1}u_0 \end{aligned} \tag{2.44}$$

where $(K,F) = [\mathcal{K},\mathcal{F}]$ is the critical feedback pair in difference equation form, and we have used Lemma 26 to find the feed-through operator of the closed loop DLS $\Phi_\diamond^{\mathrm{ext}}$. In fact, the feed-through operator of the feedback pair satisfies

$F = 0$ because $F = I - X$ and we have normalized the feed-through operator $X = \pi_0 \mathcal{X} \pi_0$ of the outer factor to the identity operator on U.

By Lemma 35 $Bu_0 \in \operatorname{dom}(\mathcal{C})$, and trivially $\pi_0 \tilde{u} \in \ell^2(\mathbf{Z}; U) \cap Seq(U)$. Because $\Phi_\diamond^{\mathrm{ext}}$ is a critical closed loop DLS, it follows that

$$(\mathcal{I} - \mathcal{F})^{-1} \mathcal{K} x_0 = \mathcal{K}^{\mathrm{crit}} x_0, \quad \left(\mathcal{C} + \mathcal{D}(\mathcal{I} - \mathcal{F})^{-1}\mathcal{K}\right) x_0 = \mathcal{C}^{\mathrm{crit}} x_0,$$

for all $x_0 \in \operatorname{dom}(\mathcal{C})$, and in particular for $x_0 = Bu_0$. Similarly, $\mathcal{F}\tilde{w} = (\mathcal{I} - \mathcal{X})\tilde{w}$ and $(\mathcal{I} - \mathcal{F})^{-1}\tilde{w} = \mathcal{X}^{-1}\tilde{w}$ for $\tilde{w} \in \ell^2(\mathbf{Z}; U) \cap Seq(U)$, and in particular $\tilde{w} = \pi_0 \tilde{u}$. The impulse response formula (2.44) for $\Phi_\diamond^{\mathrm{ext}}$ takes now the form

$$\begin{pmatrix} \mathcal{N} \\ \mathcal{X}^{-1} - \mathcal{I} \end{pmatrix} \pi_0 \tilde{u} = \begin{pmatrix} D \\ 0 \end{pmatrix} \pi_0 \tilde{u} + \tau \begin{pmatrix} \mathcal{C}^{\mathrm{crit}} \\ \mathcal{K}^{\mathrm{crit}} \end{pmatrix} Bu_0. \tag{2.45}$$

We need only the lower row of equation (2.45) which implies

$$\mathcal{X}^{-1} \pi_0 \tilde{u} = \pi_0 \tilde{u} + \tau \mathcal{K}^{\mathrm{crit}} Bu_0 = \pi_0 \tilde{u} + \tau \tilde{u}^{\mathrm{crit}}(Bu_0) \tag{2.46}$$

for all $\tilde{u} = \{u_j\}_{j \geq 0} \in \ell^2(\mathbf{Z}_+; U)$.

By using the spectral factorization $\mathcal{D}^* J \mathcal{D} = \mathcal{X}^* S \mathcal{X}$, equation (2.46) implies

$$\begin{aligned} & J(0, \pi_0 \tilde{u} + \tau \tilde{u}^{\mathrm{crit}}(Bu_0)) \\ &= \left\langle \mathcal{D}^* J \mathcal{D}(\pi_0 \tilde{u} + \tau \tilde{u}^{\mathrm{crit}}(Bu_0)), (-, , -) \right\rangle_{\ell^2(\mathbf{Z}_+; U)} \\ &= \left\langle S\mathcal{X}(\pi_0 \tilde{u} + \tau \tilde{u}^{\mathrm{crit}}(Bu_0)), \mathcal{X}(-, , -) \right\rangle_{\ell^2(\mathbf{Z}_+; U)} \\ &= \left\langle S\mathcal{X}(\mathcal{X}^{-1}\pi_0 \tilde{u}), \mathcal{X}(\mathcal{X}^{-1}\pi_0 \tilde{u}) \right\rangle_{\ell^2(\mathbf{Z}_+; U)} = \langle Su_0, u_0 \rangle_U \end{aligned} \tag{2.47}$$

On the other hand, for all $\tilde{u} = \pi_0 \tilde{u}$ we have

$$\begin{aligned} & J(0, \pi_0 \tilde{u} + \tau \tilde{u}^{\mathrm{crit}}(Bu_0)) \\ &= \langle Du_0, JDu_0 \rangle_Y + P_0^{\mathrm{crit}}(Bu_0, Bu_0) =: \Lambda_{P_0^{\mathrm{crit}}}(u_0, u_0). \end{aligned} \tag{2.48}$$

Now the combination of equations (2.47) and (2.48) gives

$$\langle Su_0, u_0 \rangle_U = \Lambda_{P_0^{\mathrm{crit}}}(u_0, u_0) = \left\langle \Lambda_{P_0^{\mathrm{crit}}} u_0, u_0 \right\rangle_U$$

for all $u_0 \in U$, where $S \in \mathcal{L}(U)$ is self-adjoint with a bounded inverse. The last equality is by Proposition 91. By [79, Theorem 12.7], $\Lambda_{P_0^{\mathrm{crit}}} = S$, and is thus boundedly invertible. □

We proceed to discuss the critical feedback pair $[\mathcal{K}, \mathcal{F}] = (K, F)$, associated to the (J, S)-inner-outer factorization $\mathcal{D} = \mathcal{N}\mathcal{X}$, where the feed-through part of the outer factor $\mathcal{X}$ is normalized to identity, as has been done in the proof of Lemma 93. We have already indicated that the feed-forward operator of this feedback pair satisfies $F = 0$. By comparing the formula for $\mathcal{K}^{\mathrm{crit}}$ in claim (iii) of Lemma 84 and equation (2.27) of Lemma 87 for the observability map $\mathcal{K}$ of the critical feedback pair, we see that the output operator of the critical feedback pair satisfies $K = K^{\mathrm{crit}} = \pi_0\mathcal{K}^{\mathrm{crit}}$ because $\pi_0\mathcal{X}\pi_0 = I$. Thus the critical feedback pair, associated to this specially normalized factorization, satisfies $[\mathcal{K}, \mathcal{F}] = \left(K^{\mathrm{crit}},\ 0\right)$.

The proof of Lemma 93 reveals the system theoretic meaning of the indicator operator $\Lambda_{P_0^{\mathrm{crit}}}$, too. Let $\tilde{u} = \{u_j\}_{j\geq 0} \in \ell^2(\mathbf{Z}_+; U)$ be arbitrary. Cost of the (external perturbation) impulse $\pi_0\tilde{u}$ to the critical closed loop DLS $\Phi_\diamond^{\mathrm{crit}} = [\Phi, [\mathcal{K}, \mathcal{F}]]_\diamond$, initially resting at zero initial state $x_0 = 0$, is given by

$$J(0, \pi_0\tilde{u} + \tau\tilde{u}^{\mathrm{crit}}(Bu_0)) = \left\langle \Lambda_{P_0^{\mathrm{crit}}} u_0, u_0 \right\rangle_U$$

because $x_1 = Bu_0$. Of course, one can see this identity from the definition of $\Lambda_{P_0^{\mathrm{crit}}}$ and equation (2.17), too. Note that the indicator $\Lambda_{P_0^{\mathrm{crit}}}$ does not depend on the particular (J, S)-inner-outer factorization $\mathcal{D} = \mathcal{N}\mathcal{X}$ that has been used to form the critical feedback pair. The sensitivity operator S is generally different for different (J, S)-inner-outer factorizations. Always

$$\Lambda_{P_0^{\mathrm{crit}}} = X^* S X$$

where $X = \pi_0\mathcal{X}\pi_0$. If we normalize the outer factor by requiring $\pi_0\mathcal{X}\pi_0 = I$, we get the identity $\Lambda_{P_0^{\mathrm{crit}}} = S$.

Now we are ready to approach the main result of this section, namely Lemma 96. We show there that the critical sesquilinear form $P_0^{\mathrm{crit}}(\, ,\,)$ satisfies the weak discrete time Riccati equation, defined as follows.

Definition 94. *Let $J \in \mathcal{L}(Y)$ be a cost operator. Let $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ be a strongly H^2 stable DLS. We say that the conjugate symmetric sesquilinear form $P(\, ,\,) : \operatorname{dom}(\mathcal{C}) \times \operatorname{dom}(\mathcal{C}) \to \mathbf{C}$ is a solution of the weak discrete time Riccati equation (WDARE), associated to ϕ and J, if there exists $Q_P \in \mathcal{L}(H; U)$ and a boundedly invertible self-adjoint $\Lambda_P \in \mathcal{L}(U)$ such that*

$$\left\{ \begin{array}{l} P(Ax, Ax') - P(x, x') + \langle C^* J C x, x'\rangle_H = \left\langle Q_P^* \Lambda_P^{-1} Q_P x, x'\right\rangle_H \\ \langle \Lambda_P u, u'\rangle_U = \langle D^* J D u, u'\rangle_U + P(Bu, Bu') \\ \langle Q_P x'', u''\rangle_U = -\langle D^* J C x'', u''\rangle_U - P(Ax'', Bu'') \end{array} \right. \tag{2.49}$$

for all $u, u', u'' \in U$ and $x, x', x'' \in \operatorname{dom}(\mathcal{C})$. The operator Λ_P is the indicator of the solution $P(\, ,\,)$. The linear operator $K_P := \Lambda_P^{-1} Q_P \in \mathcal{L}(H; U)$ is the feedback operator of the solution $P(\, ,\,)$.

By Proposition 34, Lemma 35 and the strong H^2 stability requirement, each of the terms $P(Ax, Ax')$, $P(Bu, Bu')$ and $P(Ax'', Bu'')$ that appear in the

WDARE are well-defined complex numbers. Given $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$, J and $P(\, , \,)$, it is clear that at most one bounded operator can serve as the indicator operator Λ_P in equation (2.49). Because of our standing assumption $\overline{\mathrm{dom}\,(\mathcal{C})} = H$, the same holds also for the operator Q_P, too. We conclude that the indicator Λ_P and the feedback operator K_P of any solution $P(\, , \,)$ are uniquely defined.

In general, WDARE (2.49) has a plenty of solutions. We need to classify these solutions.

Definition 95. *Let $J \in \mathcal{L}(Y)$ be cost operator, and $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ be an I/O stable DLS. Let $P(\, , \,)$ be a solution of WDARE (2.49).*

(i) The solution $P(\, , \,)$ satisfies the ultra weak residual cost condition, if

$$P(A^j x_0, A^j x_0) \to 0 \quad \textit{as} \quad j \to \infty$$

for all $x_0 \in \mathrm{range}\,(\mathcal{B}_\phi)$.

(ii) The DLS

$$\phi_P := \begin{pmatrix} A & B \\ -K_P & I \end{pmatrix} \tag{2.50}$$

is the spectral DLS of the solution $P(\, , \,)$, associated to ϕ and J. Here K_P is the feedback operator of the solution $P(\, , \,)$, given in Definition 94.

(iii) The solution $P(\, , \,)$ critical, if it satisfies the ultra weak residual cost condition, the spectral DLS ϕ_P is I/O stable, and the I/O map $\mathcal{D}_{\phi_P}$ is outer with a bounded inverse.

Now the main result of this section:

Lemma 96. *Let $J \in \mathcal{L}(Y)$ be self-adjoint, and $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right] = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ be an I/O stable DLS. Assume that the equivalent conditions of Theorem 89 are satisfied, and by $P_0^{\mathrm{crit}}(\, , \,)$ denote the critical sesquilinear form of Definition 76.*

Then $P_0^{\mathrm{crit}}(\, , \,)$ is a critical solution of the weak discrete time algebraic Riccati equation of Definition 94.

Proof. Because the equivalent conditions of Theorem 89 hold, it follows that Φ is J-coercive and $P_0^{\mathrm{crit}}(\, , \,)$ is defined. By Definition 90, Proposition 91 and Lemma 93 there is an unique self-adjoint, boundedly invertible operator, namely the critical indicator $\Lambda_{P_0^{\mathrm{crit}}}$, such that

$$\left\langle \Lambda_{P_0^{\mathrm{crit}}} u, u' \right\rangle_U = \langle D^* J D u, u' \rangle_U + P_0^{\mathrm{crit}}(Bu, Bu')$$

is satisfied. Because the conditions of Theorem 89 hold, the one step feedback operator $K^{\rm crit} = \pi_0 \mathcal{K}^{\rm crit} : {\rm dom}\,(\mathcal{C}) \to U$ is bounded, and identifiable to its bounded extension $K^{\rm crit} \in \mathcal{L}(H;U)$. Define the operator $Q_{P_0^{\rm crit}} := \Lambda_{P_0^{\rm crit}} K^{\rm crit} \in \mathcal{L}(H;U)$. Equations (2.39) and (2.41) of Lemma 92 imply that $P_0^{\rm crit}(\,,\,)$ is a solution of WDARE (2.49).

It remains to be shown that $P_0^{\rm crit}(\,,\,)$ is a critical solution. By Lemma 35, ${\rm range}\,(\mathcal{B}) \subset {\rm dom}\,(\mathcal{C})$. By Proposition 77, it follows that $P_0^{\rm crit}(\,,\,)$ satisfies the ultra weak residual cost condition of part (i) of Definition 95. It remains to show that the spectral DLS

$$\phi_{P_0^{\rm crit}} = \begin{pmatrix} A & B \\ -K_{P_0^{\rm crit}} & I \end{pmatrix} = \begin{pmatrix} A & B \\ -K^{\rm crit} & I \end{pmatrix}$$

is I/O stable, and its I/O map is outer with bounded inverse.

Because the conditions of Theorem 89 hold, we have a (J,S')-inner-outer factorization $\mathcal{D} = \mathcal{N}'\mathcal{X}'$ for some $S' \in \mathcal{L}(U)$, where the outer factor $\mathcal{X}'$ is outer with a bounded inverse. As in the first part of the proof of Lemma 93, there is another (J,S)-inner-outer factorization $\mathcal{D} = \mathcal{N}\mathcal{X}$ such that the feed-through operator of the outer factor satisfies $\pi_0 \mathcal{X} \pi_0 = I$, the identity operator in $\mathcal{L}(U)$. As in Lemma 87, this factorization is associated to a critical feedback pair $[\mathcal{K},\mathcal{F}]$ that has been defined through equations (2.27) and (2.28). Because $[\mathcal{K},\mathcal{F}]$ is a critical feedback pair, it is I/O stable and outer, by Definition 86. This means that the DLS $\Phi^{\rm fb} = \left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{K} & \mathcal{F} \end{smallmatrix}\right]$ is I/O stable, and $(\mathcal{I}-\mathcal{F})^{-1}$ is an I/O map of an I/O stable DLS. Because $\bar{\pi}_+ = (\mathcal{I}-\mathcal{F})^{-1}\bar{\pi}_+ \cdot (\mathcal{I}-\mathcal{F})\bar{\pi}_+ = (\mathcal{I}-\mathcal{F})\bar{\pi}_+ \cdot (\mathcal{I}-\mathcal{F})^{-1}\bar{\pi}_+$ on $\ell^2(\mathbf{Z}_+;U)$, it follows that $(\mathcal{I}-\mathcal{F})\bar{\pi}_+ : \ell^2(\mathbf{Z}_+;U) \to \ell^2(\mathbf{Z}_+;U)$ is a bounded bijection. It follows that the DLS

$$\Phi' := \begin{bmatrix} A^j & \mathcal{B}\tau^{*j} \\ -\mathcal{K} & \mathcal{I}-\mathcal{F} \end{bmatrix}.$$

is I/O stable, and its I/O map is outer with a bounded inverse.

However, the critical feedback pair is given in the difference equation form by $[\mathcal{K},\mathcal{F}] = \big(K^{\rm crit},\, 0\big)$, where $K^{\rm crit} \in \mathcal{L}(H;U)$ is the extension of the bounded one step feedback operator $\pi_0\mathcal{K}^{\rm crit} : {\rm dom}\,(\mathcal{C}) \to U$; for details see the discussion following Lemma 93. It now follows from Theorem 15 that $\Phi' = \phi_{P_0^{\rm crit}}$, and thus the spectral DLS $\phi_{P_0^{\rm crit}}$ is I/O stable, and its I/O map is outer with a bounded inverse. By part (iii) of Definition 95, $P_0^{\rm crit}(\,,\,)$ is a critical solution. This completes the proof. $\square$

2.6 Solution of the weak algebraic Riccati equation

In this section we give a number of partial converses to Lemma 92. We show that if WDARE (2.49) of Definition 94 has a solution $P^{\mathrm{crit}}(\, , \,)$ of a special kind, then the equivalent conditions of Theorem 89 are satisfied under some extra conditions. The specialty of the solution $P^{\mathrm{crit}}(\, , \,)$ is that it must be a critical solution, in the sense of part (iii) of Definition 95. Such a critical solution of WDARE is associated to the existence of a $(J, \Lambda_{P^{\mathrm{crit}}})$-inner-outer factorization the I/O map. However, stronger assumptions are required to guarantee that the critical closed loop is well-posed, i.e. $K^{\mathrm{crit}} \in \mathcal{L}(\mathrm{dom}\,(\mathcal{C})\,; U)$. In fact, this well-posedness problem, present for non-output stable DLSs, in fact, is deeper than the factorization problem. We start with a preliminary proposition which almost solves the question of the spectral factorization.

Proposition 97. *Let $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix} \right)$ be a strongly H^2 stable DLS. Let $P(\, , \,)$ be a solution of WDARE (2.49). Let $\tilde{u} = \{u_j\}_{j \geq 0} \in \ell^2(\mathbf{Z}_+; U)$ and $x_0 \in \mathrm{dom}\,(\mathcal{C})$ be arbitrary. Denote the corresponding state trajectory by $x_k = x_k(x_0, \tilde{u}) = A^k x_0 + \mathcal{B}\tau^{*k}\tilde{u}$ for all $k \geq 0$.*

(i) We have

$$\begin{aligned} &P(x_k, x_k) - P(x_{k+1}, x_{k+1}) \\ &= \langle J(Cx_k + Du_k), (-, , -)\rangle_Y \\ &- \langle \Lambda_P(-K_P x_k + u_k), (-, , -)\rangle_U \end{aligned} \tag{2.51}$$

for all $k \geq 0$. For all $n \geq 1$, we have

$$\begin{aligned} &P(x_0, x_0) - P(x_{n+1}, x_{n+1}) \\ &= \sum_{k=0}^{n} \langle J(Cx_k + Du_k), (-, , -)\rangle_Y \\ &- \sum_{k=0}^{n} \langle \Lambda_P(-K_P x_k + u_k), (-, , -)\rangle_U . \end{aligned} \tag{2.52}$$

(ii) Assume, in addition, that ϕ is I/O stable, and $\lim_{k \to \infty} P(x_k, x_k) = 0$. Then

$$J(x_0, \tilde{u}) = P(x_0, x_0) + \sum_{k=0}^{\infty} \langle \Lambda_P(-K_P x_k + u_k), (-, , -)\rangle_U , \tag{2.53}$$

where the sum converges.

(iii) *Assume, in addition, that both* ϕ *and* ϕ_P *are I/O stable,* $x_0 \in \mathrm{dom}\,(\mathcal{C}) \cap \mathrm{dom}\,(\mathcal{C}_{\phi_P})$*, and* $\lim_{k\to\infty} P(x_k, x_k) = 0$*. Then*

$$J(x_0, \tilde{u}) = P(x_0, x_0) + \langle \Lambda_P(\mathcal{C}_{\phi_P} x_0 + \mathcal{D}_{\phi_P}\tilde{u}), (-,,-)\rangle_{\ell^2(\mathbf{Z}_+;U)}\,. \tag{2.54}$$

Proof. Claim (i) is proved by first calculating

$$\begin{aligned}
&P(x_k, x_k) - P(x_{k+1}, x_{k+1}) = P(x_k, x_k) - P(Ax_k + Bu_k, Ax_k + Bu_k)\\
&= P(x_k, x_k) - P(Ax_k, Ax_k) - P(Ax_k, Bu_k) - P(Bu_k, Ax_k) - P(Bu_k, Bu_k).
\end{aligned}$$

Because the sesquilinear form $P(\,,\,)$ satisfies WDARE (2.49), the previous equals

$$\begin{aligned}
&= \left\langle (C^*JC - Q_P^*\Lambda_P^{-1}Q_P)x_k, x_k\right\rangle_H\\
&+ \langle (Q_P + D^*JC)x_k, u_k\rangle_U + \langle u_k, (Q_P + D^*JC)x_k\rangle_U\\
&+ \langle (D^*JD - \Lambda_P)u_k, u_k\rangle_U\\
&= (\langle C^*JCx_k, x_k\rangle_H + \langle D^*JCx_k, u_k\rangle_U + \langle u_k, D^*JCx_k\rangle_U + \langle D^*JDu_k, u_k\rangle_U)\\
&+ \left(\left\langle -Q_P^*\Lambda_P^{-1}Q_Px_k, x_k\right\rangle_H + \langle Q_Px_k, u_k\rangle_U + \langle u_k, Q_Px_k\rangle_U - \langle \Lambda_P u_k, u_k\rangle_U\right)\\
&= \langle J(Cx_k + Du_k), (-,,-)\rangle_Y - \left\langle \Lambda_P^{-1}(-Q_Px_k + \Lambda_P u_k), (-,,-)\right\rangle_U
\end{aligned}$$

where the last equality is obtained simply by grouping terms. This proves equation (2.51), by replacing the feedback operator $K_P = \Lambda_P^{-1}Q_P$. Equation (2.52) is now an immediate consequence. Claim (ii) is proved by inspection of equation (2.52). We have for each $n \geq 1$

$$\begin{aligned}
&\sum_{k=0}^{n} \langle \Lambda_P(-K_Px_k + u_k), (-,,-)\rangle_U\\
&= -P(x_0, x_0) + P(x_{n+1}, x_{n+1}) + \sum_{k=0}^{n} \langle J(Cx_k + Du_k), (-,,-)\rangle_Y
\end{aligned}$$

Because ϕ is I/O stable, $x_0 \in \mathrm{dom}\,(\mathcal{C})$ and $\tilde{u} \in \ell^2(\mathbf{Z}_+; U)$, we have

$$\{Cx_k + Du_k\}_{k\geq 0} = \mathcal{C}x_0 + \mathcal{D}\bar{\pi}\tilde{u} \in \ell^2(\mathbf{Z}_+; Y).$$

On the other hand, we have for each $k \geq 0$

$$|\,\langle J(Cx_k + Du_k), (-,,-)\rangle_Y\,| \leq ||J||_{\mathcal{L}(Y)} \cdot ||Cx_k + Du_k||_Y^2.$$

It follows that the sum in the right hand side converges absolutely as $n \to \infty$. By assumption, also $P(x_{n+1}(x_0, \tilde{u}), x_{n+1}(x_0, \tilde{u})) \to 0$ as $n \to \infty$ for this particular $\tilde{u}$. It follows that $\lim_{n\to\infty} \sum_{k=0}^{n} \langle \Lambda_P(-K_Px_k + u_k), (-,,-)\rangle_U$ exists and satisfies (2.53).

In order to prove the final claim (iii), note that the I/O stability of ϕ_P implies for $x_0 \in \mathrm{dom}\,(\mathcal{C}_{\phi_P})$ and $\tilde{u} \in \ell^2(\mathbf{Z}_+; U)$

$$\{-K_Px_k + u_k\}_{k\geq 0} = \mathcal{C}_{\phi_P}x_0 + \mathcal{D}_{\phi_P}\bar{\pi}_+\tilde{u} \in \ell^2(\mathbf{Z}_+; U),$$

by the definition of the spectral DLS ϕ_P. Then the sum in (2.53) is majorizes by

$$|\langle \Lambda_P(-K_P x_k + I u_k), (-, , -)\rangle_U| \leq ||\Lambda_P||_{\mathcal{L}(U)} \cdot || - K_P x_k + u_k||_U^2,$$

and it thus converges absolutely. □

Note that the intersection $\operatorname{dom}(\mathcal{C}) \cap \operatorname{dom}(\mathcal{C}_{\phi_P})$ in claim (iii) of Proposition 97 is far from empty for I/O stable ϕ and ϕ_P. In particular, because $\mathcal{B} = \mathcal{B}_{\phi_P}$, and for I/O stable systems always $\operatorname{range}(\mathcal{B}) \subset \operatorname{dom}(\mathcal{C})$, it follows that $\operatorname{range}(\mathcal{B}) \subset \operatorname{dom}(\mathcal{C}) \cap \operatorname{dom}(\mathcal{C}_{\phi_P})$. The connection between a solution of Riccati equation system and a class of stable spectral factorizations of the Popov operator is given below.

Lemma 98. *Let $J \in \mathcal{L}(Y)$ be a cost operator, and $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ be an I/O stable DLS. Let $P(,)$ be the solution of the Riccati equation system (2.49) of Definition 94 such that the spectral DLS ϕ_P is I/O stable and the ultra weak residual cost condition of part (i) of Definition 95 holds; i.e.*

$$P(A^k x_0, A^k x_0) \to 0 \quad as \quad k \to \infty$$

for all $x_0 \in \operatorname{range}(\mathcal{B})$. Then the following stable spectral factorization identity holds on $\ell^2(\mathbf{Z}; U)$.

$$\mathcal{D}^* J \mathcal{D} = \mathcal{D}_{\phi_P}^* \Lambda_P \mathcal{D}_{\phi_P}. \tag{2.55}$$

Proof. The both sides of (2.55) are bounded, causal and shift invariant operators. Let $j, k \geq 0$ be arbitrary. For all $\tilde{u} \in \ell^2(\mathbf{Z}_+; U)$ we have by definition

$$J(0, \pi_{[0,j]}\tilde{u}) = \left\langle \mathcal{D}^* J \mathcal{D} \pi_{[0,j]}\tilde{u}, \pi_{[0,j]}\tilde{u} \right\rangle_{\ell^2(\mathbf{Z}_+;U)}. \tag{2.56}$$

By linearity of $P(x_0, x_1)$ in x_0 we get $P(0,0) = 0$. Because we use inputs of form $\pi_{[0,j]}\tilde{u}$ and the initial state $x_0 = 0$, we have

$$x_k(x_0, \pi_{[0,j]}\tilde{u}) = A^{k-j}\left(A^j x_0 + \mathcal{B}\tau^{*j}\pi_{[0,j]}\tilde{u}\right) = A^{k-j} \cdot \mathcal{B}\tau^{*j}\pi_{[0,j]}\tilde{u},$$

for all $k \geq j$. Because the solution $P(,)$ satisfies the ultra weak residual cost condition, $\lim_{k\to\infty} P(x_k(x_0, \pi_{[0,j]}\tilde{u}), x_k(x_0, \pi_{[0,j]}\tilde{u})) = 0$. By claim (iii) of Proposition 97 we have for all $\tilde{u} \in \ell^2(\mathbf{Z}_+; U)$

$$J(0, \pi_{[0,j]}\tilde{u}) = \left\langle \mathcal{D}_{\phi_P}^* \Lambda_P \mathcal{D}_{\phi_P} \pi_{[0,j]}\tilde{u}, \pi_{[0,j]}\tilde{u} \right\rangle_{\ell^2(\mathbf{Z}_+;U)}. \tag{2.57}$$

By combining equations (2.56) and (2.57),

$$\left\langle (\pi_{[0,j]}\mathcal{D}^* J \mathcal{D}\pi_{[0,j]} - \pi_{[0,j]}\mathcal{D}_{\phi_P}^* \Lambda_P \mathcal{D}_{\phi_P}\pi_{[0,j]})\tilde{u}, \tilde{u} \right\rangle_{\ell^2(\mathbf{Z}_+;U)} = 0,$$

because $\tilde{u} \in \ell^2(\mathbf{Z}_+;U)$ was arbitrary. It follows that the truncated Popov operators satisfy $\pi_{[0,j]}\mathcal{D}^* J \mathcal{D}\pi_{[0,j]} = \pi_{[0,j]}\mathcal{D}^*_{\phi_P}\Lambda_P\mathcal{D}_{\phi_P}\pi_{[0,j]}$ for all $j \geq 0$, by [79, Theorem 12.7].

Clearly $\mathcal{T} := \mathcal{D}^* J\mathcal{D} - \mathcal{D}^*_{\phi_P}\Lambda_P\mathcal{D}_{\phi_P}$ is a self-adjoint shift-invariant operator on $\ell^2(\mathbf{Z};U)$. For contradiction, assume that $\mathcal{T} \neq 0$. Then there is $\tilde{u} \in \ell^2(\mathbf{Z};U)$ such that $||\mathcal{T}\tilde{u}||_{\ell^2(\mathbf{Z};U)} \geq 4\nu > 0$. Because $\mathcal{T}$ is bounded, there is $j_1 > 0$ such that $||\mathcal{T}\pi_{[-j_1,j_1]}\tilde{u}||_{\ell^2(\mathbf{Z};U)} \geq 2\nu$. Similarly, there is a $j_2 > 0$ such that $||\pi_{[-j_2,j_2]}\mathcal{T}\pi_{[-j_1,j_1]}\tilde{u}||_{\ell^2(\mathbf{Z};U)} \geq \nu$. Denote note $j = \max(j_1, j_2)$. Then

$$||\pi_{[j-j_2,+j+j_2]}\mathcal{T}\pi_{[j-j_1,+j+j_1]}\tilde{u}||_{\ell^2(\mathbf{Z};U)} = ||\pi_{[-j_2,j_2]}\mathcal{T}\pi_{[-j_1,j_1]}\tilde{u}||_{\ell^2(\mathbf{Z};U)} \geq \nu.$$

But this is a contradiction against $\pi_{[0,j]}\mathcal{D}^* J\mathcal{D}\pi_{[0,j]} = \pi_{[0,j]}\mathcal{D}^*_{\phi_P}\Lambda_P\mathcal{D}_{\phi_P}\pi_{[0,j]}$, and the proof is complete. □

If the solution $P(\, , \,)$ is critical, then the factorization of Lemma 98 can be put in a more familiar form:

Corollary 99. *Let $J \in \mathcal{L}(Y)$ be a cost operator, and $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D}\end{smallmatrix}\right]$ be an I/O stable DLS. Let $P^{\mathrm{crit}}(\, , \,)$ be a critical solution of WDARE (2.49). Then $\mathcal{D} = \mathcal{N}\mathcal{X}$ is a $(J, \Lambda_{P^{\mathrm{crit}}})$-inner-outer factorization, where*

$$\mathcal{N} := \mathcal{D}\mathcal{D}^{-1}_{\phi_{P^{\mathrm{crit}}}}, \quad \mathcal{X} := \mathcal{D}_{\phi_{P^{\mathrm{crit}}}}.$$

The outer factor $\mathcal{X}$ is outer with a bounded inverse, and the critical sesquilinear form $P_0^{\mathrm{crit}}(\, , \,)$ exists.

Proof. The the existence of the factorization follows from equation (2.55) and Proposition 82. In particular, the outer factor $\mathcal{X}$ is outer with a bounded inverse, by condition (iii) of Definition 95. The DLS Φ is J-coercive, by Lemma 84 and the fact that $\Lambda_{P^{\mathrm{crit}}}$ is boundedly invertible. By Definition 76, the critical sesquilinear form $P_0^{\mathrm{crit}}(\, , \,)$ exists. □

Note that when the conditions of Corollary 99 are satisfied, the I/O map $\mathcal{D}_{\phi_P}$ equals the outer spectral factor $\mathcal{X}$ of $\mathcal{D}^* J\mathcal{D}$, such that the feed-through operator $\pi_0\mathcal{X}\pi_0 = I$. The previous results are collected in the following lemma, the main result of this section. It is the first partial converse of Lemma 92.

Lemma 100. *Let $J \in \mathcal{L}(Y)$ be a cost operator. Let $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D}\end{smallmatrix}\right]$ be an output stable and I/O stable DLS. Assume that the sesquilinear form $P^{\mathrm{crit}}(\, , \,) : \mathrm{dom}(\mathcal{C}) \times \mathrm{dom}(\mathcal{C}) \to \mathbf{C}$ is a critical solution of WDARE (2.49).*

Then the equivalent conditions of Theorem 89 hold. In particular, $\Lambda_{P^{\mathrm{crit}}} = \Lambda_{P_0^{\mathrm{crit}}}$, where $P_0^{\mathrm{crit}}(\, , \,)$ is the critical sesquilinear form of Definition 76.

Proof. By Corollary 99, we have the $(J, \Lambda_{P^{\mathrm{crit}}})$-inner-outer factorization $\mathcal{D} = \mathcal{N}\mathcal{X}$, where $\mathcal{X} = \mathcal{D}_{\phi_{P^{\mathrm{crit}}}}$ and $\mathcal{N} = \mathcal{D}\mathcal{D}_{\phi_{P^{\mathrm{crit}}}}^{-1}$. Also, the outer factor $\mathcal{X}$ is outer with a bounded inverse. Because Φ is output stable, $\mathrm{dom}\,(\mathcal{C}) = H$ and $\pi_0\mathcal{N}^* J\mathcal{C} : H \to U$ is bounded. It follows that condition (ii) of Theorem 89 holds. In particular, the critical sesquilinear form $P_0^{\mathrm{crit}}(\,,\,)$ exists and satisfies the WDARE (2.49), by Lemma 96. This gives us another $(J, \Lambda_{P^{\mathrm{crit}}})$-inner-outer factorization $\mathcal{D} = \mathcal{N}'\mathcal{X}'$, where the outer factor $\mathcal{X}$ is outer with a bounded inverse. Both the outer factors satisfy the normalization $\pi_0\mathcal{X}\pi_0 = \pi_0\mathcal{X}'\pi_0 = I$. By Proposition 83, $\mathcal{X} = \mathcal{X}'$ and $\Lambda_{P^{\mathrm{crit}}} = \Lambda_{P_0^{\mathrm{crit}}}$. The proof is complete. $\square$

In previous lemma, we have made a rather strong assumption of output stability of Φ to ensure that the well-posedness condition $\pi_0\mathcal{N}^* J\mathcal{C} \in \mathcal{L}(\mathrm{dom}\,(\mathcal{C})\,; U)$ holds. We now try to motivate why this has been necessary. We also give to variants of Lemma 100.

Assume that the conditions of Corollary 99 hold, and by $\mathcal{D} = \mathcal{N}\mathcal{X}$ denote the $(J, \Lambda_{P^{\mathrm{crit}}})$-inner-outer factorization. By writing the spectral DLS ϕ_P in I/O form

$$\phi_{P^{\mathrm{crit}}} = \begin{pmatrix} A & B \\ -K_{P^{\mathrm{crit}}} & I \end{pmatrix} = \begin{bmatrix} A^j & \mathcal{B}\tau^{*j} \\ -\mathcal{K} & \mathcal{I} - \mathcal{F} \end{bmatrix}, \tag{2.58}$$

we define the observability map $\mathcal{K} = -\mathcal{C}_{\phi_{P^{\mathrm{crit}}}}$ and the I/O map $\mathcal{F} = \mathcal{I} - \mathcal{D}_{\phi_{P^{\mathrm{crit}}}}$. Clearly, $[\mathcal{K}, \mathcal{F}]$ is a feedback pair for Φ. By definition, $(\mathcal{I} - \mathcal{F})\tilde{u} = \mathcal{X}\tilde{u}$ for all $\tilde{u} \in \ell^2(\mathbf{Z}; U) \cap Seq(U)$. Because ϕ_P is I/O stable and its I/O map $\mathcal{D}_{\phi_{P^{\mathrm{crit}}}}$ is outer with a bounded inverse, it follows that $[\mathcal{K}, \mathcal{F}]$ is an I/O stable and outer feedback pair for Φ, provided we have the inclusion $\mathrm{dom}\,(\mathcal{C}) \subset \mathrm{dom}\,\left(\mathcal{C}_{\phi_{P^{\mathrm{crit}}}}\right)$. We make this additional assumption explicitly in Lemma 102.

We proceed to consider the operator

$$\mathcal{K}' = -\Lambda_{P^{\mathrm{crit}}}^{-1}\bar{\pi}_+\mathcal{N}^* J\mathcal{C} : \mathrm{dom}\,(\mathcal{C}) \to \ell^2(\mathbf{Z}_+; U).$$

A similar calculation as in the proof of Lemma 87 implies that $\mathcal{K}'A = \bar{\pi}_+\tau^*\mathcal{K}'$ on $\mathrm{dom}\,(\mathcal{C})$, and $-\mathcal{K}'\mathcal{B} = \bar{\pi}_+\mathcal{X}\pi_- = \bar{\pi}_+(\mathcal{I} - \mathcal{F})\pi_-$ on $\mathrm{dom}\,(\mathcal{B})$. If we could verify that $-\Lambda_{P_0^{\mathrm{crit}}}\pi_0\mathcal{K}' = \pi_0\mathcal{N}^* J\mathcal{C} : \mathrm{dom}\,(\mathcal{C}) \to U$ is bounded, then the quadruple of linear mappings

$$\begin{bmatrix} A^j & \mathcal{B}\tau^{*j} \\ \overline{\mathcal{K}'} & \mathcal{I} - \mathcal{F} \end{bmatrix} \tag{2.59}$$

would be an I/O stable DLS, whose I/O map is outer with a bounded inverse. Here the observability map $\overline{\mathcal{K}'} : \mathrm{dom}\,\left(\overline{\mathcal{K}'}\right) \to \ell^2(\mathbf{Z}_+; U)$ is a closed extension of $\mathcal{K}' : \mathrm{dom}\,(\mathcal{C}) \to \ell^2(\mathbf{Z}_+; U)$, and $\mathrm{dom}\,(\mathcal{C}) \subset \mathrm{dom}\,\left(\overline{\mathcal{K}'}\right)$. From equation (2.58) and the fact $-\mathcal{K}'\mathcal{B} = \bar{\pi}_+(\mathcal{I} - \mathcal{F})\pi_-$ we conclude that

$$\begin{aligned} &- \Lambda_{P^{\mathrm{crit}}}\left(\Lambda_{P^{\mathrm{crit}}}^{-1} K_{P^{\mathrm{crit}}} + \pi_0\mathcal{N}^* J\mathcal{C}\right) x \\ &= (\pi_0\mathcal{K}' - \pi_0\mathcal{K})x = 0 \quad \text{for all} \quad x \in \mathrm{range}\,(\mathcal{B}) \subset \mathrm{dom}\,(\mathcal{C}) \end{aligned} \tag{2.60}$$

with the natural identification of range (π_0) and U. Suppose now that we have no *a priori* knowledge of the boundedness of $\pi_0\mathcal{N}^*J\mathcal{C} : \mathrm{dom}\,(\mathcal{C}) \to U$ but $\overline{\mathrm{range}\,(\mathcal{B})} = H$. Then $\Delta Q := \left(\Lambda^{-1}_{P^{\mathrm{crit}}} K_{P^{\mathrm{crit}}} + \pi_0\mathcal{N}^*J\mathcal{C}\right) : \mathrm{dom}\,(\mathcal{C}) \to U$ is a densely defined operator, whose null space contains the dense set range $(\mathcal{B})$. We cannot conclude that $\Delta Q = 0$ because there exists a densely defined linear operator on a Hilbert space, whose null space is dense. For example, define the vector space by

$$\mathrm{dom}\,(T) := \{\tilde{u} = \{u_j\}_{j\geq 0} \in \ell^2(\mathbf{Z}_+;\mathbf{C}) \mid \lim_{j\to\infty} ju_j \text{ exists}\}$$

and the linear functional $T : \mathrm{dom}\,(T) \mapsto \mathbf{C}$ by $T\{u_j\}_{j\geq 0} := \lim_{j\to\infty} ju_j$. Then T is densely defined, $T \neq 0$ but $T\tilde{u} = 0$ for all $\tilde{u}$ that have only finitely many nonzero components. Because such $\tilde{u}$ are dense in $\ell^2(\mathbf{Z}_+;\mathbf{C})$, it follows that $\ker\,(T)$ is dense. See also [79, Theorem 1.18].

Another variant of Lemma 100 is the following proposition.

Proposition 101. *Let $J \in \mathcal{L}(Y)$ be a cost operator. Let $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ be an I/O stable DLS, such that $\overline{\mathrm{range}\,(\mathcal{B})} = H$. Assume that the sesquilinear form $P^{\mathrm{crit}}(\,,\,) : \mathrm{dom}\,(\mathcal{C}) \times \mathrm{dom}\,(\mathcal{C}) \to \mathbf{C}$ is a critical solution of WDARE (2.49), and $\pi_0\mathcal{N}^*J\mathcal{C} : \mathrm{dom}\,(\mathcal{C}) \to U$ is closable, where $\mathcal{N} := \mathcal{D}\mathcal{D}^{-1}_{\phi_{P_0^{\mathrm{crit}}}}$.*

Then the equivalent conditions of Theorem 89 hold. In particular, $\pi_0\mathcal{N}^*J\mathcal{C} \in \mathcal{L}(\mathrm{dom}\,(\mathcal{C})\,; U)$.

Proof. As in the proof of Lemma 100, we have the $(J, \Lambda_{P_0^{\mathrm{crit}}})$-inner-outer factorization $\mathcal{D} = \mathcal{N}\mathcal{X}$, where $\mathcal{X} = \mathcal{D}_{\phi_{P^{\mathrm{crit}}}}$ and $\mathcal{N} = \mathcal{D}\mathcal{D}^{-1}_{\phi_{P^{\mathrm{crit}}}}$. The outer factor $\mathcal{X}$ is outer with a bounded inverse, too.

Suppose now that $\pi_0\mathcal{N}^*J\mathcal{C} : \mathrm{dom}\,(\mathcal{C}) \to U$ is closable. Let $\mathrm{dom}\,(\mathcal{C}) \ni x_j \to 0$ be such that $\Delta Q x_j \to u$, where $\Delta Q = \Lambda^{-1}_{P^{\mathrm{crit}}} K_{P^{\mathrm{crit}}} + \pi_0\mathcal{N}^*J\mathcal{C}$. Because $\Lambda^{-1}_{P^{\mathrm{crit}}} K_{P^{\mathrm{crit}}} \in \mathcal{L}(H;U)$ by Definition 94 of WDARE, it follows that

$$||\pi_0\mathcal{N}^*J\mathcal{C}x_j - u|| \leq ||\Delta Q x_j - u|| + ||\Lambda^{-1}_{P^{\mathrm{crit}}} K_{P^{\mathrm{crit}}} x_j|| \to 0 \quad \text{as} \quad j \to \infty.$$

Because $\pi_0\mathcal{N}^*J\mathcal{C}$ is closable, it follows that $u = 0$. Thus ΔQ is closable, and it has a minimal closed extension $\overline{\Delta Q}$. By the discussion following equation (2.60), ΔQ vanishes on the vector space range $(\mathcal{B})$. Now, range $(\mathcal{B}) \subset \ker\left(\overline{\Delta Q}\right)$ and then $\overline{\Delta Q} = 0$ because the null space of a closed operator is closed. We conclude that $\Delta Q = 0$ and $\pi_0\mathcal{N}^*J\mathcal{C} = -\Lambda^{-1}_{P^{\mathrm{crit}}} K_{P^{\mathrm{crit}}}$ on $\mathrm{dom}\,(\mathcal{C})$. It follows that $\pi_0\mathcal{N}^*J\mathcal{C} \in \mathcal{L}(\mathrm{dom}\,(\mathcal{C})\,; U)$. $\square$

It is a consequence of the compactness of the unit ball in finite dimensional spaces, that finite rank closed operators are bounded. Thus, assuming $\pi_0\mathcal{N}^*J\mathcal{C} : \mathrm{dom}\,(\mathcal{C}) \to U$ to be closable is equivalent to assuming it bounded, if $\dim U < \infty$.

If the approximate controllability assumption is strengthened, then we obtain another analogue of Lemma 100.

Lemma 102. *Let $J \in \mathcal{L}(Y)$ be a cost operator. Let $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ be an I/O stable DLS. Assume that $\overline{\mathrm{range}\,(\mathcal{B})} = E$, where the Hilbert space $E = \mathrm{dom}\,(\mathcal{C})$ is given in Definition 38, and the closure is taken in the norm of E. Assume that the sesquilinear form $P^{\mathrm{crit}}(\,,\,) : \mathrm{dom}\,(\mathcal{C}) \times \mathrm{dom}\,(\mathcal{C}) \to \mathbf{C}$ is a critical solution of WDARE (2.49), such that $\mathrm{dom}\,(\mathcal{C}) \subset \mathrm{dom}\,(\mathcal{C}_{\phi_{P^{\mathrm{crit}}}})$.*

Then the equivalent conditions of Theorem 89 hold. In particular, $\pi_0 \mathcal{N}^* J \mathcal{C} \in \mathcal{L}(\mathrm{dom}\,(\mathcal{C})\,; U)$.

Proof. As before, we have the $(J, \Lambda_{P_0^{\mathrm{crit}}})$-inner-outer factorization $\mathcal{D} = \mathcal{N}\mathcal{X}$, where the outer factor $\mathcal{X}$ is outer with a bounded inverse. Our task is to show that $\pi_0 \mathcal{N}^* J \mathcal{C} \in \mathcal{L}(\mathrm{dom}\,(\mathcal{C})\,; U)$.

Because Φ and the spectral DLS $\phi_{P^{\mathrm{crit}}}$ share the same semigroup generator A, we can form an extended DLS whose observability map is

$$\begin{bmatrix} \mathcal{C} \\ \mathcal{C}_{\phi_{P^{\mathrm{crit}}}} \end{bmatrix} : \mathrm{dom}\left(\begin{bmatrix} \mathcal{C} \\ \mathcal{C}_{\phi_{P^{\mathrm{crit}}}} \end{bmatrix}\right) \to \ell^2(\mathbf{Z}_+; Y \oplus U).$$

As in the proof of Theorem 48, $\mathrm{dom}\left(\left[\begin{smallmatrix} \mathcal{C} \\ \mathcal{C}_{\phi_{P^{\mathrm{crit}}}} \end{smallmatrix}\right]\right) = \mathrm{dom}\,(\mathcal{C})$, because $\mathrm{dom}\,(\mathcal{C}) \subset \mathrm{dom}\,(\mathcal{C}_{\phi_{P^{\mathrm{crit}}}})$ is assumed. By Definition 38, the vector space $\mathrm{dom}\,(\mathcal{C})$ gets two norms

$$\begin{aligned} ||x||_E^2 &= ||x||_H^2 + ||\mathcal{C}x||^2_{\ell^2(\mathbf{Z}_+;Y)}, \quad \text{and} \\ ||x||_E'^2 &= ||x||_H^2 + ||\begin{bmatrix} \mathcal{C} \\ \mathcal{C}_{\phi_{P^{\mathrm{crit}}}} \end{bmatrix} x||^2_{\ell^2(\mathbf{Z}_+;Y\oplus U)} \\ &= ||x||_H^2 + ||\mathcal{C}x||^2_{\ell^2(\mathbf{Z}_+;Y)} + ||\mathcal{C}_{\phi_{P^{\mathrm{crit}}}} x||^2_{\ell^2(\mathbf{Z}_+;U)}. \end{aligned}$$

By E and E' denote the vector space $\mathrm{dom}\,(\mathcal{C})$, equipped with the norms $||\cdot||_E$ and $||\cdot||_{E'}$, respectively. By Lemma 39, both E and E' are Hilbert spaces. Because $||x||_E \leq ||x||_{E'}$ for all $x \in \mathrm{dom}\,(\mathcal{C})$, it follows that the inclusion operator $Inc : E' \to E$ is a bounded operator. It is a bijection because $E = E'$, as vector spaces. Thus the inverse operator $Inc^{-1} : E \to E'$ is bounded. We now have for each $x \in E = \mathrm{dom}\,(\mathcal{C}) \subset \mathrm{dom}\,(\mathcal{C}_{P^{\mathrm{crit}}})$ the estimate

$$\begin{aligned} &||\mathcal{C}_{P^{\mathrm{crit}}} x||^2_{\ell^2(\mathbf{Z}_+;U)} \leq ||x||_H^2 + ||\mathcal{C}x||^2_{\ell^2(\mathbf{Z}_+;Y)} + ||\mathcal{C}_{\phi_{P^{\mathrm{crit}}}} x||^2_{\ell^2(\mathbf{Z}_+;U)} \\ &= ||x||^2_{E'} = ||Inc^{-1}x||^2_{E'} \leq ||Inc^{-1}||^2_{E \to E'} \cdot ||x||^2_E. \end{aligned}$$

We conclude that there is a constant $M < \infty$ such that $||\mathcal{C}_{\phi_{P^{\mathrm{crit}}}} x||_{\ell^2(\mathbf{Z}_+;U)} \leq M\,||x||_E$ for all $x \in E$.

It follows that $||K_{P^{\mathrm{crit}}} x|| \leq M\,||x||_E$ for all $x \in E$. Because $\mathcal{C} : E \to \ell^2(\mathbf{Z}_+; U)$ is bounded by Lemma 39, so is the operator $\pi_0 \mathcal{N}^* J \mathcal{C}$. We conclude that $\Delta Q =$

$\Lambda_{P^{\mathrm{crit}}}^{-1} K_{P^{\mathrm{crit}}} + \pi_0 \mathcal{N}^* J \mathcal{C} : E \to U$ is bounded, and it vanished on the vector subspace $\mathrm{range}\,(\mathcal{B}) \subset \mathrm{dom}\,(\mathcal{C}) = E$. By assumption, $\overline{\mathrm{range}\,(\mathcal{B})} = E$ in the norm of E. By continuity, $\Delta Q = 0$ on E, and thus $\pi_0 \mathcal{N}^* J \mathcal{C} = -\Lambda_{P^{\mathrm{crit}}}^{-1} K_{P^{\mathrm{crit}}}$ on $\mathrm{dom}\,(\mathcal{C})$ because $E = \mathrm{dom}\,(\mathcal{C})$ as vector spaces. Because $\Lambda_{P^{\mathrm{crit}}}^{-1} K_{P^{\mathrm{crit}}} \in \mathcal{L}(H; U)$, we conclude that $\pi_0 \mathcal{N}^* J \mathcal{C} \in \mathcal{L}(\mathrm{dom}\,(\mathcal{C})\,; U)$, where the original norm of H is used on $\mathrm{dom}\,(\mathcal{C})$. This completes the proof. $\square$

We remark that the inclusion $\mathrm{dom}\,(\mathcal{C}) \subset \mathrm{dom}\left(\mathcal{C}_{\phi_{P^{\mathrm{crit}}}}\right)$ is necessary to make $[\mathcal{K}, \mathcal{F}] = \left[-\mathcal{C}_{\phi_{P^{\mathrm{crit}}}}, \mathcal{I} - \mathcal{D}_{\phi_{P^{\mathrm{crit}}}}\right]$ an I/O stable and outer feedback pair for Φ, see Definition 44. As a consequence of the approximate controllability assumptions, it follows that the closed loop DLSs $(\phi, (K_{P^{\mathrm{crit}}}, 0))_{\diamond}$ are critical in Proposition 101 and Lemma 102. So as to Lemma 100, the same need not be true.

2.7 Equivalence theorem for the critical control

The main theorem of this chapter is a conclusion of Theorem 89 and Lemmas 92, 100. In Theorem 114 of Chapter 3 we give the same result in the particular case that the DLS Φ is output stable, and the solution of the Riccati equation are self-adjoint operators, rather than conjugate symmetric sesquilinear forms.

Theorem 103. *Let $J \in \mathcal{L}(Y)$ be a cost operator and $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ an I/O stable DLS. Enumerate the conditions (i), (ii) and (iii) as follows:*

(i) *a) Φ is J-coercive, and*
b) the critical control problem, associated to Φ and J, is solvable by state feedback. The critical feedback pair $[\mathcal{K}, \mathcal{F}]$ for Φ is I/O stable and outer.

(ii) *a) There is a self-adjoint, boundedly invertible operator $S \in \mathcal{L}(U)$ such that $\mathcal{D}$ has a (J, S)-inner-outer factorization $\mathcal{D} = \mathcal{N}\mathcal{X}$, where the outer factor $\mathcal{X}$ is outer with a bounded inverse, and*
b) $\pi_0 \mathcal{N}^* J \mathcal{C} \in \mathcal{L}(\mathrm{dom}\,(\mathcal{C})\,; U)$, where $\mathrm{dom}\,(\mathcal{C})$ is given the norm of H.

(iii) *There is a critical solution $P^{\mathrm{crit}}(\ ,\ )$ of the weak discrete time algebraic Riccati equation (2.49).*

Then (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iii). If, in addition, the DLS Φ is output stable, then (iii) $\Rightarrow$ (ii).

When the equivalent conditions (i) and (ii) hold, the critical sesquilinear form $P_0^{\mathrm{crit}}(\ ,\ )$ of Definition 76 exists, it is a critical solution of WDARE (2.49) and satisfies

$$P_0^{\mathrm{crit}}(x_j(x_0, \tilde{u}), x_j(x_0, \tilde{u})) \to 0 \quad \text{as} \quad j \to \infty$$

for all $x_0 \in \mathrm{dom}\,(\mathcal{C})$ and $\tilde{u} \in \ell^2(\mathbf{Z}_+; U)$. If, in addition, the feed-through operator of the outer factor $\mathcal{X}$ is normalized to identity, then $\Lambda_{P_0^{\mathrm{crit}}} = S$.

For other sufficient conditions for the implication (iii) $\Rightarrow$ (ii) to hold, see Proposition 101 and Lemma 102. For analogous results, see [45, Theorem 2.1] for equivalence of type (i) $\Leftrightarrow$ (ii), and [45, Theorem 4.1] for equivalence of type (ii) $\Leftrightarrow$ (iii). In continuous time, we refer to [64], [83], [86], and [103].

In the light of claim (iii) of Theorem 103, sufficient conditions for the I/O stability of ϕ_P in terms of the solution $P(\ ,\ )$ would be useful. We remind that for I/O stable and J-coercive DLS Φ, the critical sesquilinear form satisfies $P_0^{\mathrm{crit}}(x_k(x_0, \tilde{u}), x_k(x_0, \tilde{u})) \to 0$ for $x_0 \in \mathrm{dom}\,(\mathcal{C})$, $\tilde{u} \in \ell^2(\mathbf{Z}_+; U)$, by Proposition 77. Under a nonnegativity assumption of the indicator, an additional speed estimate appears to be the key observation.

Proposition 104. *Let $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ be an I/O stable DLS. Let $P(\ ,\ )$ be a solution of WDARE (2.49) such that its indicator $\Lambda_P > 0$ and*

$$P(x_k(0,\tilde{u}), x_k(0,\tilde{u})) \to 0 \quad \textit{for all} \quad \tilde{u} \in \ell^2(\mathbf{Z}_+;U),$$

where $x_k = x_k(0,\tilde{u}) = \mathcal{B}\tau^{*j}\tilde{u}$.

Then the spectral DLS ϕ_P is I/O stable if and only if

$$\sum_{k=0}^{\infty} |P(x_k, x_k) - P(x_{k+1}, x_{k+1})| < \infty \tag{2.61}$$

for all $\tilde{u} \in \ell^2(\mathbf{Z}_+;U)$.

Proof. For any nonnegative, boundedly invertible operator T on a Hilbert space, we can estimate

$$||T^{-1}||^{-1} \langle x, x\rangle \le \langle Tx, x\rangle \le ||T|| \langle x, x\rangle .$$

The latter equality is proved by $\langle Tx, x\rangle = ||Tx||\cdot||x|| = ||T||\cdot||x||^2 = ||T||\cdot\langle x, x\rangle$. The former inequality follows from the latter by choosing T^{-1} in place for T, and $T^{\frac{1}{2}}x$ in place for x. Because the nonnegative indicator of a solution $P(\ ,\ )$ is boundedly invertible, we can apply this with $T = \Lambda_P$ and obtain the equivalence:

$$\sum_{k=0}^{\infty} | \left\langle \Lambda_P(-K_P x_k + u_k), (-, , -)\right\rangle_U | < \infty \tag{2.62}$$

if and only if

$$\sum_{k=0}^{\infty} | \left\langle (-K_P x_k + u_k), (-, , -)\right\rangle_U | = ||\{-K_P x_k + u_k\}_{k\ge 0}||^2_{\ell^2(\mathbf{Z}_+;U)} < \infty. \tag{2.63}$$

We first show that inequality (2.63) is equivalent with the I/O stability of ϕ_P. By Lemma 31, the causal Toeplitz operator $\mathcal{D}_{\phi_P}\bar{\pi}_+ : \mathrm{dom}\,(\mathcal{D}_{\phi_P}\bar{\pi}_+) \to \ell^2(\mathbf{Z}_+;U)$ is closed when equipped with the domain

$$\mathrm{dom}\,(\mathcal{D}_{\phi_P}\bar{\pi}_+) := \{\tilde{u} \in \ell^2(\mathbf{Z}_+;U) \,|\, \mathcal{D}_{\phi_P}\bar{\pi}_+\tilde{u} \in \ell^2(\mathbf{Z}_+;U)\}.$$

By Definition 32, the spectral DLS ϕ_P is I/O stable if and only if $\mathrm{dom}\,(\mathcal{D}_{\phi_P}\bar{\pi}_+) = \ell^2(\mathbf{Z}_+;U)$.

Now, let $\tilde{u} \in \ell^2(\mathbf{Z}_+;U)$ be arbitrary and $x_k = x_k(0,\tilde{u}) = \mathcal{B}_{\phi_P}\tau^{*j}\tilde{u} = \mathcal{B}\tau^{*j}\tilde{u}$. Then $\{-K_P x_k + u_k\}_{k\ge 0} = \mathcal{D}_{\phi_P}\bar{\pi}_+\tilde{u} \in Seq_+(U)$, by the definition of the spectral DLS. It follows that (2.63) holds for all $\tilde{u} \in \ell^2(\mathbf{Z}_+;U)$ if and only if $\mathrm{dom}\,(\mathcal{D}_{\phi_P}\bar{\pi}_+) = \ell^2(\mathbf{Z}_+;U)$. We conclude that the inequality (2.62) holds if and only if ϕ_P is I/O stable.

So it remains to prove that the conditions of (2.61) and (2.62) are equivalent. From equation (2.51), we obtain the estimates

$$
\begin{aligned}
&|P(x_k, x_k) - P(x_{k+1}, x_{k+1})| \\
&\le ||J|| \cdot |\langle Cx_k + Du_k, (-,,-)\rangle_Y| + |\langle \Lambda_P(-K_P x_k + u_k), (-,,-)\rangle_U|, \\
&|\langle \Lambda_P(-K_P x_k + u_k), (-,,-)\rangle_U| \\
&\le ||J|| \cdot |\langle Cx_k + Du_k, (-,,-)\rangle_Y| + |P(x_k, x_k) - P(x_{k+1}, x_{k+1})|
\end{aligned}
$$

for all $k \ge 0$. The I/O stability of Φ implies that $\{Cx_k + Du_k\}_{k\ge0} = \mathcal{D}_\phi \bar{\pi}_+ \tilde{u} \in \ell^2(\mathbf{Z}_+; Y)$, and the sequence $\{\langle Cx_k + Du_k, (-,,-)\rangle_Y\}_{k\ge0}$ in equation (2.51) is absolutely summable for all $\tilde{u} \in \ell^2(\mathbf{Z}_+; U)$. But then the sequence $\{\langle \Lambda_P(-K_P x_k + u_k), (-,,-)\rangle_U\}_{k\ge0}$ is absolutely summable if and only if $\{|P(x_k, x_k) - P(x_{k+1}, x_{k+1})|\}_{k\ge0}$ is absolutely summable. This completes the proof. $\square$

So by Proposition 104, only the condition in claim (iii) of Theorem 103 that ϕ_P should be outer with bounded inverse remains less concrete. It is easy to see that for power stable systems this follows from the familiar requirement that $P(\,,\,)$ should be a (power) stabilizing solution of the Riccati equation: if both $\rho(A) < 1$ and $\rho(A + BK_P) < 1$ then ϕ_P is both I/O stable and outer (see [66], [67]). For infinite dimensional power stable result we refer to e.g. [44], [72].

2.8 Notes and references

Optimal control in discrete and continuous time

We make a short and superficial review on the literature and development of various cost optimization problems for linear systems. Because in this book we concentrate in the state feedback solutions, we consider only full information problems. This means that the whole state space is assumed to be visible for the optimal controller. Under favorable circumstances, such an optimal control problem can be shown to be equivalent to finding a solution of an algebraic Riccati equation. An example of an unfavorable situation is the failure of well-posedness of the closed loop that appeared in Section 2.6 for general non-output stable DLSs. For the continuous time WPLSs, it is generally not even possible to write down an algebraic Riccati equation in a standard way because a feed-through operator is needed for such an equation.

In the case when the proper algebraic Riccati equation can be written down, we remark that the information structure of the optimization problem affects the form of the algebraic Riccati equation. For the reason of notational simplicity, we now consider the case when the DARE is presented in the "strong" form, i.e.

$$
(2.64) \quad \begin{cases} A^*PA - P + C^*JC = (D^*JC + B^*PA)^* \, \Lambda_P^{-1} \, (D^*JC + B^*PA) \\ \Lambda_P = D^*JD + B^*PB, \end{cases}
$$

where the solutions $P \in \mathcal{L}(H)$ are required to be bounded self-adjoint operators. There is another variant of DARE

$$
(2.65) \quad \begin{cases} A^*PA - P + C^*JC = A^*PB \cdot \Lambda_P^{-1} \cdot B^*PA \\ \Lambda_P = D^*JD + B^*PB, \end{cases}
$$

characterized by the fact that the cross term $\mathbf{D}^*JC$ vanishes. The algebraic Riccati equation of type (2.65), together with its continuous time analogue, appears in linear quadratic control problems where $J \geq 0$, and a direct coercive cost is imposed on the input. For this reason, we call the cross term free equation (2.65) LQDARE, even if the cost operator J in indefinite. As discussed at the beginning of Section 2.2, each LQDARE (2.65) can be reduced to the form of DARE (2.64) by expanding the state space. The relation of these two equations is briefly considered in Section 4.9.

After these preparations, let us proceed to consider the literature. The continuous time linear quadratic control problem, corresponding to the dynamical system

$$
(2.66) \quad \begin{cases} x'(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t), \quad t \geq 0 \end{cases}
$$

with bounded operators on Hilbert spaces, is solved in [48, Section 3.5] (Kalman, Falb and Arbib, 1969). Both the time-variant finite and time-invariant infinite horizon problems are considered, and the latter is regarded as a limit case of the former. This leads to the use of both the Riccati differential equation and the algebraic Riccati equation. Both the finite and infinite time linear quadratic control problem is solved in [18, Chapter 6] (Curtain and Zwart, 1995) in the case when A generates a strongly continuous semigroup, but B, C and D in (2.66) are bounded operators. Also a nice survey of the history is presented there.

Early papers about spectral factorization techniques, feedback control and stabilizing solutions of matrix algebraic Riccati equations are [66] (Molinari, 1973) and [73] (Payne and Silverman, 1973) for discrete time, and [65] and [67] (Molinari, 1973, 1975) for continuous time. The classical infinite-dimensional discrete time operator Riccati equation reference is [45] (Helton, 1976) where LQDARE with a nondefinite cost operator is considered. References to this paper appear throughout this book, and we do not consider it here in length. For a nonlinear variant, see also [2] (Ball and Helton, 1991). In the monograph [44] (Halanay and Ionescu, 1995), the suboptimal (state space) disturbance attenuation problem is solved for exponentially stabilizable discrete time time-variant systems, and the solution is presented with the aid of stabilizing nonnegative solutions of two algebraic Riccati equations. The monograph [49] (Lancaster and Rodman, 1995) contains plenty of historical remarks and references. In particular, [49, Chapter 16] contains the solutions of both the continuous and discrete time linear quadratic control problem, with the algebraic Riccati equation of the general type (2.64) but an additional nondegeneracy condition is imposed, related to the lack of cross term. This additional condition is satisfied by all LQDAREs (2.65).

We proceed to consider the continuous time papers [74] and [75] (Pritchard and Salamon, 1985, 1987). In the latter of these papers, both the finite and the infinite horizon version of the linear quadratic control problems are covered for Pritchard–Salamon realizations. The Riccati integral and differential equations are derived for the finite horizon problem. The Riccati operator of the infinite cost problem, giving the optimal cost of an arbitrary initial state, is recovered by using a limit argument of finite horizon problems in the spirit of [48, Section 3.5]. It is shown that the Riccati operator is a minimal nonnegative solution of an algebraic Riccati equation. The Riccati operator corresponds to the critical sesquilinear form $P_0^{\mathrm{crit}}(\,,\,)$ of this paper. Under extra conditions, related to stabilizability and detectability of the system, it can be shown that the Riccati operator is the only nonnegative solution of the algebraic Riccati equation, and that the closed loop semigroup is exponentially stable. The paper [74] gives the solution of the linear quadratic control problem for retarded functional differential equations, using the tools of [75]. See also [43] (Grabowski, 1993), where examples of time delay systems are considered.

The infinite horizon optimal control problem can be solved in a conceptually different manner; not by an approximation with finite horizon problems but by solving a spectral factorization problem. The reader has surely noticed that the spectral factorization approach is the one that we have taken in this chapter for DLSs. Let us begin with the continuous time case when A generates a strongly continuous semigroup, the generating operators B, C and D are bounded, and the transfer function is in the matrix-valued Callier–Desoer algebra. The pioneering works are [11], [12] and [13] (Callier and Winkin, 1987, 1988, 1990) where the optimal state feedback operator for the linear quadratic control problem is connected to a nonnegative solution of an algebraic Riccati equation, and to an invertible spectral factor of the spectral (Popov) function. A necessary and sufficient coercivity condition for the existence of such a spectral factor (in the stable Callier–Desoer class) is given in terms of the determinant of the spectral function, in case the impulse response of the system does not contain delays. These ideas are extended in [14] (Callier and Winkin, 1992), and even an example involving the heat equation is worked out in detail. In their later work [15] Callier and Winkin extend the necessary and sufficient condition for the existence of a spectral factor to systems whose impulse response is allowed to have arbitrary delays.

So as to the WPLSs and the spectral factorization approach, the first works for WPLSs are [82] (Staffans, 1995), [83] (Staffans, 1997) for stable (regular) WLPSs and [103] (G. Weiss and M. Weiss, 1997) for stable (weakly regular) WPLSs. For stable WPLSs in the sense of [83, Definition 1], it is shown in [83] that the state feedback solution of the optimal control problem is equivalent to a spectral factorization problem, without ever appealing to the possibly unbounded input and output operators B and C. Furthermore, the Riccati operator, corresponding to the critical sesquilinear form $P_0^{\mathrm{crit}}(\, , \,)$ of this chapter, can be written down by an explicit formula. The trouble begins when an algebraic Riccati equation is to be written down; now the operators B and C are needed. Furthermore, the WPLS must be regular, together with its adjoint, so that the feed-through operator D is defined. However, even more is required. Also the (outer, stable) spectral factor $\mathcal{X}$ of $\mathcal{D}$ must be regular, together with its adjoint $\mathcal{X}^*$. Under these assumptions, an algebraic Riccati equation can be written down, and the Riccati operator solves it, but surprisingly, the feed-through operator X of the spectral factor appears in the equation. This was first reported in [82] and [103]. The converse result for WPLSs, concluding the existence of the spectral factor from the assumed existence of a critical solution for the algebraic Riccati equation, is given in [64] (Mikkola, 1997). Some results of [83] on the optimal control are extended to the corresponding critical control results in [85] (Staffans, 1998) when the unique saddle point of the cost functional is to be expressed by a static state feedback law. The unstable linear quadratic control problem is considered in [83] for jointly stabilizable and detectable WPLSs in the sense of [84] (Staffans, 1998). The information structure of the algebraic Riccati equation in [64], [83], [85] and [103] corresponds to that of DARE (2.64) with a nonvanishing cross term.

Let us compare the finite horizon approach and the spectral factorization approach. It appears in [75] that the unboundedness of the input and output operators B and C and the presence of a number of state spaces produce a lot of extra trouble in the solution of the linear quadratic control problem. This is true for the Pritchard–Salamon systems in particular, and a WPLSs in general. It is a clear advantage of the spectral factorization approach that the use of these unbounded operators can be avoided in the solution of the optimal control problem, since an algebraic Riccati equation need not be written down or solved. It is also an advantage that it gives an explicit formula for the Riccati operator rather than an existence result, as is the case with the limit process involved with the finite horizon approach. However, a natural analogy to the solution of the finite horizon problem is lost. A clear disadvantage is that the spectral factorization approach relies on the possibility of (numerical) computation of the spectral factor which is a somewhat elusive object. However, on the level of generality of WPLSs, even the (approximate numerical) solution of a proper algebraic Riccati equation would be a formidable task. We remark that there could exists practical ways of solving a spectral factorization problem without resorting to a difficult continuous time algebraic Riccati equation. In the scalar case, an outer (spectral) factor can be recovered from its boundary values by a well-known integral formula, see e.g. [78, Theorem 17.17]. In the operator-valued case, the Cayley transform of the transfer function to the unit disk is possible, and the outer factor can then be found by the discrete time methods, presented in this chapter.

Chapter 3

Spectral Factorization

3.1 Introduction

Let $\phi := \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ be an I/O stable DLS and $J \in \mathcal{L}(Y)$ a cost operator. By restricting the solution set of WDARE (2.49) of Definition 94 to sesquilinear forms that can be expressed with the aid of bounded self-adjoint operators, we obtain the corresponding discrete time algebraic Riccati equation (DARE),

$$\begin{cases} A^*PA - P + C^*JC = K_P^*\Lambda_P K_P, \\ \Lambda_P = D^*JD + B^*PB, \\ \Lambda_P K_P = -D^*JC - B^*PA, \end{cases} \tag{3.1}$$

which will be introduced in Section 3.2. Even though DARE $Ric(\phi, J)$ can be written for an arbitrary (even non-I/O stable) DLS ϕ, in the rest of this book we consider the special case when the DLS ϕ is output stable and I/O stable. Note that by the additional output stability, the critical (closed loop) observability map

$$\mathcal{C}_\phi^{\text{crit}} := \mathcal{C}_\phi - \bar{\pi}_+\mathcal{D}_\phi(\bar{\pi}_+\mathcal{D}_\phi^* J\mathcal{D}_\phi\bar{\pi}_+)^{-1}\bar{\pi}_+\mathcal{D}_\phi^* J\mathcal{C}_\phi : H \to \ell^2(\mathbf{Z}_+; Y)$$

is a bounded operator, provided that the equivalent conditions of Theorem 103 hold. Now the critical sesquilinear form

$$P_0^{\text{crit}}(x_1, x_2) := \left\langle J\mathcal{C}_\phi^{\text{crit}} x_1, \mathcal{C}_\phi^{\text{crit}} x_2 \right\rangle_{\ell^2(\mathbf{Z}_+;Y)}$$

can be represented by the bounded self-adjoint operator $P_0^{\text{crit}} := (\mathcal{C}_\phi^{\text{crit}})^* J\mathcal{C}_\phi^{\text{crit}} \in \mathcal{L}(H)$. Furthermore, the operator P_0^{crit} satisfies DARE (3.1). We conclude that replacing WDARE (2.49) by the less general DARE (3.1) is not entirely unsatisfactory because the theory of Chapter 2 can be written by using only DARE (3.1), under a standing hypothesis of output stability. Recall that a non-output stable DLS can be made output stable, by change of norm in the state space as discussed at the end of Section 2.8. In this chapter, we develop a more general spectral factorization theory for such output stable and I/O stable DLSs and their DAREs.

Let us give a technical outline of this chapter. Given a DLS $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ and a cost operator J, DARE (3.1) is denoted by $Ric(\phi, J)$. If $P \in \mathcal{L}(H)$ is a self-adjoint solution of $Ric(\phi, J)$, we write $P \in Ric(\phi, J)$. Thus the symbol $Ric(\phi, J)$ represents both the equation itself and its solution set. If, in addition, ϕ is output stable and I/O stable, we call equation (3.1) an H^∞DARE, and write $ric(\phi, J)$ instead of $Ric(\phi, J)$. In Definition 106, we associate to each $P \in Ric(\phi, J)$ an indicator operator Λ_P and two additional DLSs: the spectral DLS ϕ_P and the inner DLS ϕ^P, centered at $P \in Ric(\phi, J)$. These three objects are central in this book. In Section 4.2 they appear in the open and closed loop DLSs when certain state feedbacks, associated to solutions $P \in Ric(\phi, J)$, are applied to ϕ. The solutions of the H^∞DARE $ric(\phi, J)$ are classified in Definition 107 according to the stability properties of the spectral DLS ϕ_P, and in Definition 108 according to their residual cost behavior "at infinite time". The smallest subset of solutions for H^∞DARE is denoted by $ric_0(\phi, J)$ — the set of regular H^∞ solutions $P \in ric_0(\phi, J)$. Our strongest results are given in this subset. In Theorem 114 we specialize Theorem 103 for additionally output stable DLSs. In this process, WDARE (2.49) is replaced by DARE (3.1), and we get rid of all the well-posedness problems of the critical closed loop DLS that have been discussed at the end of Section 2.6. The three equivalent conditions of Theorem 114 hold if and only if $P_0^{\mathrm{crit}} := (\mathcal{C}_\phi^{\mathrm{crit}})^* J \mathcal{C}_\phi^{\mathrm{crit}} \in ric_0(\phi, J)$. The existence of such a solution P_0^{crit} is almost a standing hypothesis for the rest of this book. Well known sufficient conditions for the existence of $P_0^{\mathrm{crit}} \in ric_0(\phi, J)$, relying on the nonnegativity of J or $\mathcal{D}_\phi^* J \mathcal{D}_\phi$, are given in Proposition 117 and Corollary 118. In Section 3.4 we present some auxiliary results from the operator-valued function theory. A result of particular importance to us is Lemma 130, which allows us later to deal with an infinite-dimensional input space U, provided that the input operator $B \in \mathcal{L}(U; H)$ is restricted to be a compact Hilbert–Schmidt operator. This result has some application in Section 3.5.

Section 3.5 contains two spectral factorization results, namely Lemma 138 (the spectral factorization of truncated Toeplitz operators) and Proposition 139 (the spectral factorization of the Popov function $\mathcal{D}_\phi(e^{i\theta})^* J \mathcal{D}_\phi(e^{i\theta})$, constructed from the boundary trace of the H^2 transfer function $\mathcal{D}_\phi(e^{i\theta})$). Despite of this, our main interest lies in the characterization of the solution subset $ric_0(\phi, J) \subset Ric(\phi, J)$. So we must consider the output stability and I/O stability of the spectral DLS ϕ_P for various solutions $P \in Ric(\phi, J)$. The output stability of ϕ_P is easier, and it is treated in Proposition 136 by nonnegativity techniques. The I/O stability of ϕ_P is considered in Corollary 140 and the remarks following it.

In Section 3.6, a spectral factorization of the Popov operator

$$\mathcal{D}_\phi^* J \mathcal{D}_\phi = \mathcal{D}_{\phi_P}^* \Lambda_P \mathcal{D}_{\phi_P} \tag{3.2}$$

is associated to each solution of the Riccati equation $P \in ric_0(\phi, J)$ satisfying a certain residual cost condition. We say that such an operator $\mathcal{D}_{\phi_P}$ is a stable spectral factor of the Popov operator $\mathcal{D}_\phi^* J \mathcal{D}_\phi$. Also the converse it true: each such factorization induces a solution $P \in ric_0(\phi, J)$, if $\overline{\mathrm{range}\,(\mathcal{B}_\phi)} = H$. This is the content of Theorem 142. We remark that the factorization of the Popov operator does not require the cost operator J to be nonnegative, if we have an *a priori* knowledge that ϕ_P is output stable and I/O stable. For nonnegative J, this stability of ϕ_P follows as in the previous Section 3.5, under proper technical assumptions. If $P = P_0^{\mathrm{crit}} := (\mathcal{C}_\phi^{\mathrm{crit}})^* J \mathcal{C}_\phi^{\mathrm{crit}}$ is the regular critical solution in the sense of Theorem 114, then the factorization (3.2) is the $\Lambda_{P_0^{\mathrm{crit}}}$-spectral factorization $\mathcal{D}_\phi^* J \mathcal{D}_\phi = \mathcal{X}^* \Lambda_{P_0^{\mathrm{crit}}} \mathcal{X}$, where the spectral factor $\mathcal{X} := \mathcal{D}_{\phi_{P_0^{\mathrm{crit}}}}$ is I/O stable and outer with a bounded causal inverse. This leads to the $(J, \Lambda_{P_0^{\mathrm{crit}}})$-inner-outer factorization of the I/O map $\mathcal{D}_\phi = \mathcal{N}\mathcal{X}$, see Proposition 82. We remark that if $P \in ric_0(\phi, J)$ but $P \neq P_0^{\mathrm{crit}}$, then we do not always obtain an analogous factorization of $\mathcal{D}_\phi$, as a composition of two I/O stable I/O maps. The circumstances related to the partial ordering of $ric_0(\phi, J)$ when we get such stable factors, are considered in Chapter 4. Lemma 145 is an inertia result for the indicator operators Λ_P, $P \in ric_{uw}(\phi, J)$ in a possibly indefinite metric. The positive indicators are considered in Corollary 146.

In Proposition 147, the spectral factor $\mathcal{D}_{\phi_P}$ in equation (3.2) is $(\Lambda_P, \Lambda_{P_0^{\mathrm{crit}}})$-inner-outer factorized as $\mathcal{D}_{\phi_P} = \mathcal{N}_P \mathcal{X}$, under the assumption that the original DARE $ric(\phi, J)$ has a regular critical solution P_0^{crit}. Quite expectedly, the outer factor of $\mathcal{D}_{\phi_P}$ does not depend on the choice of the solution $P \in ric_0(\phi, J)$. Realizations for the factors are computed. Section 3.6 is concluded with Proposition 148, where a realization algebra is developed for the inner factors $\mathcal{N}_P$.

A preliminary version of the contents of this chapter is [61] (Malinen, 1999).

3.2 H^∞ algebraic Riccati equation

In this section we give basic definitions of the discrete time algebraic Riccati equation, associated to an output stable and I/O stable DLS Φ and a possibly indefinite cost operator $J \in \mathcal{L}(U)$. The solutions P of such equation are classified according to stability properties of an associated DLS ϕ_P, see Definitions 106 and 107. An additional classification is done according to the residual cost properties, as introduced in Definition 108. After that, inclusions of the various solution sets are considered.

Definition 105. *Let $J \in \mathcal{L}(Y)$ be self-adjoint, and $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right] = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ be a DLS. Then the following system of operator equations*

$$(3.3) \qquad \begin{cases} A^*PA - P + C^*JC = K_P^*\Lambda_P K_P \\ \Lambda_P = D^*JD + B^*PB \\ \Lambda_P K_P = -D^*JC - B^*PA \end{cases}$$

is called the discrete time algebraic Riccati equation (DARE) and denoted by $Ric(\Phi, J)$. The linear operators are required to satisfy $\Lambda_P, \Lambda_P^{-1} \in \mathcal{L}(U)$ and $K_P \in \mathcal{L}(H; U)$. Here P is a unknown self-adjoint operator to be solved. If $P \in \mathcal{L}(H)$ satisfies (3.3), we write $P \in Ric(\Phi, J)$.

We use the same symbol $Ric(\Phi, J)$ both for the solution set of a DARE, and the DARE itself. This should not cause confusion. Clearly the equations (3.3) can be put into form

$$(3.4) \qquad \begin{aligned} &A^*PA - P + C^*JC \\ &= (D^*JC + B^*PA)^* (D^*JD + B^*PB)^{-1} (D^*JC + B^*PA). \end{aligned}$$

This is the usual form of the DARE in the literature. Because Λ_P and K_P are quite fundamental objects in our treatment, the system (3.3) is used instead. For a given $P \in Ric(\Phi, J)$, the operator Λ_P is called the indicator of P, and the operator K_P is called the (state) feedback operator of solution P. The operators $A_P := A + BK_P$ and $C_P = C + DK_P$ are the closed loop semigroup generator and the closed loop output operator, respectively. Sometimes DARE (3.4) has a trivial solution; if we can write $(D^*JD)^{-1} = D^{-1}J^{-1}(D^{-1})^*$, then clearly $0 \in Ric(\Phi, J)$.

To each solution $P \in Ric(\Phi, J)$, two additional DLSs are associated:

Definition 106. *Let $J \in \mathcal{L}(Y)$ be self-adjoint, and $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right] = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ be a DLS. Let K_P, A_P and C_P be as above.*

(i) For $P \in Ric(\Phi, J)$, the DLS

$$\phi_P := \begin{pmatrix} A & B \\ -K_P & I \end{pmatrix}$$

is the spectral DLS, associated to the pair (Φ, J) *and centered at* P.

(ii) *For* $P \in Ric(\Phi, J)$, *the DLS*

$$\phi^P := \begin{pmatrix} A_P & B \\ C_P & D \end{pmatrix}$$

is called the inner DLS, associated to the pair (Φ, J) *and centered at* P.

In this work, we consider DAREs $Ric(\Phi, J)$, such that Φ is output stable and I/O stable. These are called H^∞DAREs, and defined as follows:

Definition 107. *Let the objects* Φ, J, $Ric(\Phi, J)$, $P \in Ric(\Phi, J)$, *and* ϕ_P *be as in Definitions 105 and 106. Assume that* Φ *is, in addition, I/O stable and output stable.*

(i) *We denote the DARE (3.3) by* $ric(\Phi, J)$ *instead of* $Ric(\Phi, J)$. *The DARE* $ric(\Phi, J)$ *is called* H^∞*DARE.*

(ii) *If* $P \in Ric(\Phi, J)$ *is such that the spectral DLS* ϕ_P *is I/O stable and output stable, then we say that* $P \in ric(\Phi, J)$. *We say that such* P *is an* H^∞ *solution of a* H^∞*DARE.*

When we write inclusions and equalities like $Ric(\Phi, J) \subset Ric(\Phi', J')$, $Ric(\Phi, J) = Ric(\Phi', J')$, then these symbols refer to the solution sets of the respective DAREs. We remark that a H^∞DARE $ric(\Phi, J)$ could have a non-H^∞ solution P. This this case we write $P \in Ric(\Phi, J)$ instead of $P \in ric(\Phi, J)$.

A number of residual cost conditions are required in our work.

Definition 108. *Let the objects* Φ, J, $Ric(\Phi, J)$, $P \in Ric(\Phi, J)$, *and* ϕ_P *be as in Definitions 105 and 106.*

(i) *If the residual cost operator*

$$L_{A,P} := \mathrm{s} - \lim_{j \to \infty} A^{*j} P A^j$$

exists as a bounded operator in $\mathcal{L}(H)$, *we write* $P \in Ric_{00}(\Phi, J)$.

(ii) *If* $L_{A,P} = 0$, *we write* $P \in Ric_0(\Phi, J)$. *Such* P *satisfies the strong residual cost condition.*

(iii) *If* $\left\langle PA^j x_0, A^j x_0 \right\rangle \to 0$ *for all* $x_0 \in H$, *we write* $P \in Ric_{000}(\Phi, J)$. *Such* P *satisfies the weak residual cost condition.*

(iv) *If* $\left\langle PA^j x_0, A^j x_0 \right\rangle \to 0$ *for all* $x_0 \in \mathrm{range}\,(\mathcal{B})$, *we write* $P \in Ric_{uw}(\Phi, J)$. *Such* P *satisfies the ultra weak residual cost condition.*

We also define the solution sets $ric_0(\Phi, J) := Ric_0(\Phi, J) \cap ric(\Phi, J)$, $ric_{00}(\Phi, J) := Ric_{00}(\Phi, J) \cap ric(\Phi, J)$, $ric_{000}(\Phi, J) := Ric_{000}(\Phi, J) \cap ric(\Phi, J)$ *and* $ric_{uw}(\Phi, J) := Ric_{uw}(\Phi, J) \cap ric(\Phi, J)$. *The elements of* $ric_0(\Phi, J)$ *are called regular* H^∞ *solutions.*

We remark that the residual cost conditions (i), (ii), and (iii) depend on the structure of the solution P in the whole state space H. The ultra weak residual cost condition (iii) imposes only requirements on P restricted to the (possibly nonclosed) controllable vector subspace $\mathrm{range}\,(\mathcal{B})$. Recall that $\mathrm{range}\,(\mathcal{B}) = \mathcal{B}(\mathrm{dom}\,(\mathcal{B}))$ where $\mathrm{dom}\,(\mathcal{B}) := Seq_-(U)$ consists of sequences in $\ell^2(\mathbf{Z}_-; U)$ with only finitely many nonzero components. Equivalently, $P \in Ric_{uw}(\Phi, J)$ if and only if $\lim_{j\to\infty} \langle P\mathcal{B}\tau^{*j}\tilde{u}, \mathcal{B}\tau^{*j}\tilde{u}\rangle = 0$ for all $\{u_j\}_{j\geq 0} = \tilde{u} \in \ell^2(\mathbf{Z}_+; U)$ having only finitely many nonzero components u_j. Solutions $P \in Ric_{uw}(\Phi, J)$ are of particular interest in the factorization theory of Theorem 114 and Theorem 142. The residual cost conditions (i) and (ii) of Definition 108 are convenient for the Liapunov equation techniques. The following inclusions are basic:

Proposition 109. *Let the objects* Φ, J, $Ric(\Phi, J)$, $P \in Ric(\Phi, J)$, *and* ϕ_P *be as in Definitions 105 and 108. Then the following holds*

(i) *If* A *is strongly stable, then* $Ric(\Phi, J) = Ric_0(\Phi, J)$.

(ii) $\{P \in Ric_{000}(\Phi, J) \mid P \geq 0\} \subset Ric_0(\Phi, J) \subset Ric_{000}(\Phi, J)$.

(iii) $Ric_0(\Phi, J) \cup Ric_{000}(\Phi, J) \subset Ric_{uw}(\Phi, J)$.

(iv) $Ric_{00}(\Phi, J) \cap Ric_{000}(\Phi, J) \subset Ric_0(\Phi, J)$. *If* $\overline{\mathrm{range}\,(\mathcal{B})} = H$, *then* $Ric_{00}(\Phi, J) \cap Ric_{uw}(\Phi, J) \subset Ric_0(\Phi, J)$.

(v) *If* $\overline{\mathrm{range}\,(\mathcal{B})} = H$ *and* A *is power bounded, then* $Ric_{uw}(\Phi, J) \subset Ric_{000}(\Phi, J)$ *and* $\{P \in Ric_{uw}(\Phi, J) \mid P \geq 0\} \subset Ric_0(\Phi, J)$.

(vi) *We have the inclusion:*

$$\{P \in Ric(\Phi, J) \mid \lim_{j\to\infty} \langle P\mathcal{B}\tau^{*j}\tilde{u}, \mathcal{B}\tau^{*j}\tilde{u}\rangle = 0 \text{ for all } \tilde{u} \in \ell^2(\mathbf{Z}_+; U)\}$$
$$\subset Ric_{uw}(\Phi, J).$$

If Φ *is, in addition, input stable, then the inclusion is equality.*

Proof. If A is strongly stable, then for all $x_0 \in H$ we have

$$||A^{*j}PA^j x_0|| \leq ||A^{*j}|| \cdot ||P|| \cdot ||A^j x_0|| \quad \text{for all} \quad j \geq 1.$$

By the strong stability of A, $||A^j x_0|| \to 0$ as $j \to \infty$. Furthermore, by Banach–Steinhaus Theorem, $\sup_{j\geq 1} ||A^j|| < \infty$ and thus also $\sup_{j\geq 1} ||A^{*j}|| < \infty$. Thus

$||A^{*j}PA^jx_0|| \to 0$ for all x_0 and $L_{A,P} := \mathrm{s} - \lim_{j\to\infty} A^{*j}PA^j = 0$. This verifies claim (i).

Assume that $P \in Ric_{000}(\Phi, J)$ is nonnegative. Then it follows that $\left\langle PA^jx_0, A^jx_0\right\rangle = ||P^{\frac{1}{2}}A^jx_0||^2 \to 0$ for all $x_0 \in H$. Again, by Banach–Steinhaus Theorem, $C := \sup_{j\geq 1} ||A^{*j}P^{\frac{1}{2}}|| < \infty$. It now follows that $||A^{*j}PA^jx_0|| \leq C \cdot ||P^{\frac{1}{2}}A^jx_0|| \to 0$, and thus $P \in Ric_0(\Phi, J)$. Now claim (ii) follows. Claim (iii) is trivial.

Let $P \in Ric_{00}(\Phi, J) \cap Ric_{000}(\Phi, J)$. Thus $L_{A,P}$ exists, and for all $x_0 \in H$ we have

$$0 = \lim_{j\to\infty} \left\langle PA^jx_0, A^jx_0\right\rangle = \lim_{j\to\infty} \left\langle A^{*j}PA^jx_0, x_0\right\rangle = \left\langle L_{A,P}x_0, x_0\right\rangle .$$

Now, [79, Theorem 12.7] implies that $L_{A,P} = 0$, and the first part of claim (iv) follows. Because $\overline{\mathrm{range}\,(\mathcal{B})} = H$ and $P \in Ric_{uw}(\Phi, J)$, it follows that $\lim_{j\to\infty} \left\langle PA^jx_0, A^jx_0\right\rangle = 0$ for all x_0 in a dense set. Thus $L_{A,P}x_0 = 0$ in a dense set, and vanishes, by continuity. Now claim (iv) follows.

To prove claim (v), assume that $\overline{\mathrm{range}\,(\mathcal{B})} = H$ and $\sup_{j\geq 0} ||A^j|| < \infty$. Because $P \in Ric_{uw}(\Phi, J)$, we have $\left\langle PA^jx, A^jx\right\rangle \to 0$ for all $x \in \mathrm{range}\,(\mathcal{B})$. Let $x_0 \in H$ be arbitrary, and let $\mathrm{range}\,(\mathcal{B}) \ni x_k \to x_0$ in the norm of H, as $k \to \infty$. Then

$$\begin{aligned}
&|\left\langle PA^jx_0, A^jx_0\right\rangle| \\
&\leq |\left\langle A^{*j}PA^jx_k, x_k\right\rangle| + |\left\langle A^{*j}PA^jx_k, (x_0 - x_k)\right\rangle| + |\left\langle A^{*j}PA^j(x_0 - x_k), x_0\right\rangle| \\
&\leq |\left\langle A^{*j}PA^jx_k, x_k\right\rangle| + \sup_{j\geq 0} ||A^{*j}PA^j|| \cdot ||x_0 - x_k|| \cdot (||x_k|| + ||x_0||)
\end{aligned}$$

Because $\{x_k\}$ is a convergent sequence, it is a bounded set. Because A is power bounded, $\sup_{j\geq 0} ||A^{*j}PA^j|| < \infty$. Then, by first increasing k sufficiently the latter term get arbitrarily small, and the former term gets small as j is increased. Now $\left\langle PA^jx_0, A^jx_0\right\rangle \to 0$ for all $x_0 \in H$, not just $x_0 \in \mathrm{range}\,(\mathcal{B})$; or $P \in Ric_{000}(\Phi, J)$. The additional claim for $P \geq 0$ follows from claim (ii) of this Proposition.

The inclusion part of claim (vi) is trivial. For the rest, let $\epsilon > 0$, $\tilde{u} \in \ell^2(\mathbf{Z}_+; U)$ and $P \in Ric_{uw}(\Phi, J)$ be arbitrary. Let $K \geq 0$ so large that $||\pi_{[k,\infty]}\tilde{u}|| \leq \epsilon/||\mathcal{B}||$ for all $k \geq K$, where the input stability is used. Then for $j > k \geq K$,

$$\begin{aligned}
(3.5)\quad &|\left\langle P\mathcal{B}\tau^{*j}\tilde{u}, \mathcal{B}\tau^{*j}\tilde{u}\right\rangle| \\
&\leq |\left\langle P\mathcal{B}\tau^{*j}\pi_{[0,k-1]}\tilde{u}, \mathcal{B}\tau^{*j}\pi_{[0,k-1]}\tilde{u}\right\rangle| + |\left\langle P\mathcal{B}\tau^{*j}\pi_{[0,k-1]}\tilde{u}, \mathcal{B}\tau^{*j}\pi_{[k,\infty]}\tilde{u}\right\rangle| \\
&+ |\left\langle P\mathcal{B}\tau^{*j}\pi_{[k,\infty]}\tilde{u}, \mathcal{B}\tau^{*j}\pi_{[0,k-1]}\tilde{u}\right\rangle| + |\left\langle P\mathcal{B}\tau^{*j}\pi_{[k,\infty]}\tilde{u}, \mathcal{B}\tau^{*j}\pi_{[k,\infty]}\tilde{u}\right\rangle| \\
&\leq 2\,||P|| \cdot ||\mathcal{B}|| \cdot ||\tilde{u}|| \cdot \epsilon + ||P|| \cdot \epsilon^2 + |\left\langle P\mathcal{B}\tau^{*j}\pi_{[0,k-1]}\tilde{u}, \mathcal{B}\tau^{*j}\pi_{[0,k-1]}\tilde{u}\right\rangle|.
\end{aligned}$$

Now we estimate the latter term. Because $j \geq k$, $\mathcal{B}\tau^{*j}\pi_{[0,k-1]}\tilde{u} = A^{j-k}x_0$, where $x_0 = \mathcal{B}\tau^{*k}\pi_{[0,k-1]}\tilde{u} \in \operatorname{range}(\mathcal{B})$. But because $P \in Ric_{uw}(\Phi, J)$ by assumption, $\left\langle PA^{j-k}x_0, A^{j-k}x_0\right\rangle \to 0$ as $j \to \infty$. So there is $J \geq K$ such that the latter term satisfies $|\left\langle P\mathcal{B}\tau^{*j}\pi_{[0,k-1]}\tilde{u}, \mathcal{B}\tau^{*j}\pi_{[0,k-1]}\tilde{u}\right\rangle| < \epsilon$ for all $j > J$. Now the claim follows from estimate (3.5). $\square$

We proceed to consider the H^∞ solutions. The symbols $ric(\phi, J)$ and $ric_{00}(\phi, J)$ can be used synonymously, as far as they refer to the solution sets.

Proposition 110. *Let Φ be an output stable and I/O stable DLS. Let J be a cost operator. Then $ric(\Phi, J) = ric_{00}(\Phi, J)$, and we have*

$$P - L_{A,P} = \mathcal{C}^* J \mathcal{C} - \mathcal{C}_{\phi_P}^* J \mathcal{C}_{\phi_P}.$$

Proof. By iterating on DARE (3.3), we obtain for all $j \geq 0$:

$$\begin{aligned} (3.6)\quad P - (A^*)^{j+1}PA^{j+1} &= \left(\pi_{[0,j]}\mathcal{C}\right)^* J \left(\pi_{[0,j]}\mathcal{C}\right) - \left(\pi_{[0,j]}\mathcal{C}_{\phi_P}\right)^* \Lambda_P \left(\pi_{[0,j]}\mathcal{C}_{\phi_P}\right) \\ &= \mathcal{C}^* J \cdot \pi_{[0,j]}\mathcal{C} - \mathcal{C}_{\phi_P}^* J \cdot \pi_{[0,j]}\mathcal{C}_{\phi_P}, \end{aligned}$$

where we have written the adjoints by the assumed output stabilities. Clearly $\mathcal{C}^*J\mathcal{C} = \mathcal{C}^*J\pi_{[0,j]}\mathcal{C} + \mathcal{C}^*J\pi_{[j+1,\infty]}\mathcal{C}$. Now $\mathrm{s}-\lim_{j\to\infty}\pi_{[j+1,\infty]}\mathcal{C} = 0$ because $\mathcal{C} : H \to \ell^2(\mathbf{Z}_+; Y)$. Because $\mathcal{C}^*J$ is bounded, $\mathrm{s}-\lim_{j\to\infty}\mathcal{C}^*J\pi_{[j+1,\infty]}\mathcal{C} = 0$ and thus $\mathrm{s}-\lim_{j\to\infty}\mathcal{C}^*J\pi_{[0,j]}\mathcal{C} = \mathcal{C}^*J\mathcal{C}$. Similarly $\mathrm{s}-\lim_{j\to\infty}\mathcal{C}_{\phi_P}^*\Lambda_P\pi_{[0,j]}\mathcal{C}_{\phi_P} = \mathcal{C}_{\phi_P}^*\Lambda_P\mathcal{C}_{\phi_P}$. Now we see from (3.6) that the strong limit $L_{A,P} := \mathrm{s}-\lim_{j\to\infty}(A^*)^{j+1}PA^{j+1}$ on the left hand side exists, and the claim follows. Also the identity immediately follows. $\square$

Note that the I/O stability of ϕ and ϕ_P played no part in the proof of previous proposition.

The question to what extent the operators Λ_P, K_P (or, equivalently the indicator Λ_P and the spectral DLS ϕ_P in case $\overline{\operatorname{range}(\mathcal{B}_\phi)} = H$) uniquely define a solution $P \in Ric(\phi, J)$, is discussed in the following.

Proposition 111. *Let $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ be an I/O stable output stable DLS. Let J be a self-adjoint operator. Let $P_1, P_2 \in Ric(\phi, J)$ be such that $\Lambda_{P_1} = \Lambda_{P_2}$ and $K_{P_1} = K_{P_2}$.*

(i) If either P_1 or $P_2 \in Ric_{00}(\phi, J)$, then they both are in $Ric_{00}(\phi, J)$. In this case, $P_1 - P_2 = L_{A,P_1} - L_{A,P_2}$.

(ii) If, in addition, $P_1, P_2 \in Ric_0(\phi, J)$, then $P_1 = P_2$. This is, in particular, always the case when A is strongly stable.

Proof. It follows from equation (3.3) that $A^*P_1A - P_1 = A^*P_2A - P_2$, and immediately $P_1 - P_2 = A^{*j}(P_1 - P_2)A^j = A^{*j}P_1A^j - A^{*j}P_2A^j$ for all $j \geq 1$. Now, if $A^{*j}P_2A^j \to L_{A,P_2} \in \mathcal{L}(H)$ in the strong operator topology, $A^{*j}P_1A^j$ converges in the strong operator topology, too. Now $L_{A,P_1} - L_{A,P_2} = P_1 - P_2$ and claim (i) follows. The other claim is trivial. $\square$

Proposition 112. *Let $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ be a DLS and $J \in \mathcal{L}(Y)$ a cost operator. Let $P \in Ric_{00}(\phi)$ be arbitrary. If $B^*L_{A,P}B = 0$ and $B^*L_{A,P}A = 0$ then $P' = P - L_{A,P} \in Ric(\phi, J)$, and $\Lambda_P = \Lambda_{P'}$, $K_P = K_{P'}$.*

Proof. The claim immediately follows, by noting that $A^*L_{A,P}A - L_{A,P} = 0$. $\square$

Under stronger assumptions, it in fact follows that $L_{A,P} = 0$ and then $P' = P$, see Lemma 144. In this case, the indicator Λ_P and the spectral DLS ϕ_P uniquely determine $P \in ric(\phi, J)$.

3.3 Critical solutions of H^∞DARE

Let $J \in \mathcal{L}(Y)$ be a cost operator and $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ an I/O stable DLS. As has been considered in Chapter 2 and concluded in Theorem 103, there are fundamental connections between the feedback solution of a certain critical control problem, the existence of a certain factorization of the I/O map $\mathcal{D}$, and the existence of a critical solution of WDARE (2.49) of Definition 94. In the special case, when Φ is, in addition, output stable, this connection is the equivalence of the following Theorem 114. In part (iii) of Definition 95, we have introduced the notion of the critical solution of WDARE (2.49). So as to DARE (3.3), the critical solutions are defined analogously.

Definition 113. *Let $J \in \mathcal{L}(Y)$ be a cost operator. Let $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ be an I/O stable and output stable DLS. The solution $P^{\mathrm{crit}} \in Ric_{uw}(\Phi, J)$ is critical, if such the spectral DLS $\phi_{P^{\mathrm{crit}}}$ is I/O stable, and its I/O map $\mathcal{D}_{\phi_{P^{\mathrm{crit}}}}$ is outer with a bounded inverse. If a critical solution P^{crit} lies in $ric_0(\Phi, J)$, we call it a regular critical solution.*

Theorem 114. *Let $J \in \mathcal{L}(Y)$ be a cost operator, and let $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ be an I/O stable and output stable DLS. Then the following conditions (i), (ii) and (iii) are equivalent:*

(i) a) Φ is J-coercive, and

b) the critical control problem, associated to Φ and J, is solvable by state feedback. The critical feedback pair $[\mathcal{K}, \mathcal{F}]$ for Φ is I/O stable, output stable and outer.

(ii) There is a self-adjoint, boundedly invertible operator $S \in \mathcal{L}(U)$ such that $\mathcal{D}$ has a (J,S)-inner-outer factorization $\mathcal{D} = \mathcal{N}\mathcal{X}$, where the outer part $\mathcal{X}$ has a bounded inverse.

(iii) There is a critical solution $P^{\mathrm{crit}} \in Ric_{uw}(\Phi, J)$ of DARE (3.3).

If, in addition, the feed-through operator of the outer factor $\mathcal{X}$ is normalized to identity, then $\Lambda_{P^{\mathrm{crit}}} = S$.

Proof. The equivalence of claims (i) and (ii) is a particular case of Theorem 89, applied to an additionally output stable DLS Φ. Note that the assumed output stability trivializes the condition $\pi_0 \mathcal{N}^* J \mathcal{C} \in \mathcal{L}(\mathrm{dom}\,(\mathcal{C})\,; U)$ of Theorem 89. To study condition (iii), assume that the equivalent conditions (i) and (ii) of this theorem hold. We first note that the critical (closed loop) observability map $\mathcal{C}^{\mathrm{crit}}$ of equation

$$\mathcal{C}^{\mathrm{crit}} := \left(\mathcal{I} - \bar{\pi}_+ \mathcal{D} (\bar{\pi}_+ \mathcal{D}^* J \mathcal{D} \bar{\pi}_+)^{-1} \bar{\pi}_+ \mathcal{D}^* J\right) \mathcal{C} \tag{3.7}$$

is bounded, because all its operators are bounded. In particular, the inverse of the Popov operator $(\bar{\pi}_+\mathcal{D}^*J\mathcal{D}\bar{\pi}_+)^{-1}$ is bounded because Φ is J-coercive, by condition (i). The observability map $\mathcal{C}$ is bounded because Φ is assumed to be output stable. By Lemma 96, the conjugate symmetric sesquilinear form

$$P_0^{\mathrm{crit}}(x_0, x_1) := \left\langle \mathcal{C}^{\mathrm{crit}} x_0, J\mathcal{C}^{\mathrm{crit}} x_1 \right\rangle$$

for all $(x_0, x_1) \in \mathrm{dom}\,(\mathcal{C}) \times \mathrm{dom}\,(\mathcal{C}) = H \times H$, satisfies the weak algebraic Riccati equation (WDARE) of Definition 94. Because $\mathcal{C}^{\mathrm{crit}}$ is bounded, the sesquilinear form $P_0^{\mathrm{crit}}(\,,\,)$ can be written $P_0^{\mathrm{crit}}(x_0, x_1) = \left\langle P_0^{\mathrm{crit}} x_0, x_1 \right\rangle$, where $P_0^{\mathrm{crit}} := (\mathcal{C}^{\mathrm{crit}})^* J \mathcal{C}^{\mathrm{crit}}$ is a bounded self-adjoint operator. Now P_0^{crit} satisfies the DARE of Definition 105 because $P_0^{\mathrm{crit}}(\,,\,)$ satisfies the WDARE of Definition 94.

The spectral DLS $\phi_{P_0^{\mathrm{crit}}}$ is I/O stable and outer with a bounded inverse, by Lemma 96. Because $\mathcal{C}^{\mathrm{crit}} = (\mathcal{I} - \bar{\pi}\mathcal{D}(\bar{\pi}_+\mathcal{D}^*J\mathcal{D}\bar{\pi}_+)^{-1}\bar{\pi}_+\mathcal{D}^*J)\mathcal{C} = \Pi\mathcal{C}$, $x_0 \in H$, we have

$$||A^{*j} P_0^{\mathrm{crit}} A^j x_0|| = ||A^{*j}\mathcal{C}^*\Pi^*\Pi\mathcal{C}x_0|| \leq ||\mathcal{C}^*|| \cdot ||\Pi^*\Pi|| \cdot ||\bar{\pi}_+\tau^*\mathcal{C}x_0|| \to 0.$$

It now follows that $L_{A,P_0^{\mathrm{crit}}} = 0$, and in particular, $P_0^{\mathrm{crit}} \in ric_0(\Phi, J) \subset Ric_{uw}(\Phi, J)$. Claim (iii) immediately follows.

For the converse direction, assume that (iii) holds. We indicate how condition (ii) follows. The solution $P^{\mathrm{crit}} \in Ric_{uw}(\phi, J)$ defines a conjugate symmetric sesquilinear form $P^{\mathrm{crit}}(\,,\,)$ as above. Lemma 98 and Corollary 99 imply that $\mathcal{D} = \mathcal{N}\mathcal{D}_{\phi_{P^{\mathrm{crit}}}}$, $\mathcal{N} := \mathcal{D}\mathcal{D}_{\phi_{P^{\mathrm{crit}}}}^{-1}$, is a $(J, \Lambda_{P^{\mathrm{crit}}})$-inner-outer factorization, where the outer factor has a bounded inverse. But this is condition (ii), thus completing the proof of the equivalence part. To see that $\Lambda_{P_0^{\mathrm{crit}}} = S$, note that each critical solution gives a $(J, \Lambda_{P^{\mathrm{crit}}})$-inner-outer factorization of $\mathcal{D}$, such that the feedthrough operator of the outer factor $\mathcal{D}_{\phi_{P^{\mathrm{crit}}}}$ is identity. The equivalence of the sensitivity operators S and $\Lambda_{P^{\mathrm{crit}}}$ follows from Proposition 83, as in the proof claim (iii) of Proposition 115. □

Note that Theorem 114 takes no position whether a critical solution P^{crit}, when it exists, is unique in the solution set $Ric_{uw}(\Phi, J)$. We also remark that the spectral DLS $\phi_{P^{\mathrm{crit}}}$ of a critical solution is not required to be output stable, and thus P^{crit} is not required to be a H^∞ solution. However, the proof of Theorem 114 indicates that if a critical $P^{\mathrm{crit}} \in Ric_{uw}(\Phi, J)$ exists, then also a regular critical solution exists, and one of those can be given by an explicit formula $P_0^{\mathrm{crit}} := (\mathcal{C}^{\mathrm{crit}})^* J \mathcal{C}^{\mathrm{crit}}$ where $\mathcal{C}^{\mathrm{crit}}$ is given by (3.7). It follows that a critical solution $P^{\mathrm{crit}} \in Ric_{uw}(\Phi, J)$ exists if and only if the regular critical solution $P_0^{\mathrm{crit}} \in ric_0(\Phi, J)$ exists. Note that the critical observability map $\mathcal{C}^{\mathrm{crit}}$ does not necessarily make sense as a bounded operator, if the conditions of Theorem 114 do not hold. Even if $\mathcal{C}^{\mathrm{crit}}$ is bounded and $P_0^{\mathrm{crit}} = (\mathcal{C}^{\mathrm{crit}})^* J \mathcal{C}^{\mathrm{crit}}$ is well-defined, we cannot conclude that it solves the DARE $Ric(\Phi, J)$ because its indicator $\Lambda_{P_0^{\rm crit}} := D^* jD + B^* P_0^{\rm crit} B$ could fail to have a bounded inverse. To exclude this possibility, we have required in Lemma 93 that the equivalent conditions of Theorem 89 hold. Under the present output stability assumption, it is equivalent to require that the equivalent conditions (i) and (ii) of Theorem 114 hold. We conclude that a successful construction of the operator $P_0^{\rm crit} = (\mathcal{C}^{\rm crit})^* J \mathcal{C}^{\rm crit}$ does not allow us to conclude that the equivalent condition of Theorem 114 hold. The special regular critical solution $P_0^{\rm crit}$ is considered in the following.

Proposition 115. *Let $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ be an I/O stable and output stable DLS, and $J \in \mathcal{L}(Y)$ be self-adjoint. Assume that a critical solution $P^{\rm crit} \in Ric_{uw}(\Phi, J)$ exists. Then*

(i) $P_0^{\rm crit} := (\mathcal{C}^{\rm crit})^* J \mathcal{C}^{\rm crit} \in ric_0(\Phi, J)$ is a critical solution,

(ii) the residual cost operator $L_{A^{\rm crit}, P_0^{\rm crit}}$ exists and vanishes, where $A^{\rm crit} := A + BK^{\rm crit} = A + BK_{P_0^{\rm crit}}$, and

(iii) the indicators of all critical solutions are equal to $\Lambda_{P_0^{\rm crit}}$.

Proof. Because a critical solution $P^{\rm crit} \in Ric_{uw}(\Phi, J)$ exists, the equivalent conditions of Theorem 114 hold. It has been shown in the proof of Theorem 114 that $P_0^{\rm crit} := (\mathcal{C}^{\rm crit})^* J \mathcal{C}^{\rm crit}$ is a critical solution of DARE $Ric(\Phi, J)$, and it satisfies the strong residual cost condition $L_{A, P_0^{\rm crit}} = 0$, too. We already know that the spectral DLS $\phi_{P_0^{\rm crit}}$ is I/O stable by Definition 113. To show that $P_0^{\rm crit}$ is an H^∞ solution, it remains to consider the output stability of the spectral DLS $\phi_{P_0^{\rm crit}}$.

Let $\mathcal{D} = \mathcal{N}\mathcal{X}$ be a $(J, \Lambda_{P_0^{\rm crit}})$-inner-outer factorization of Corollary 99, where the outer factor $\mathcal{X} := \mathcal{D}_{\phi_{P_0^{\rm crit}}}$ is outer with a bounded inverse. The critical (one step, state) feedback operator takes the form

$$K^{\rm crit} := \pi_0 \mathcal{K}^{\rm crit} = -\pi_0 \mathcal{X}^{-1} \Lambda_{P_0^{\rm crit}}^{-1} \bar{\pi}_+ \mathcal{N}^* J \mathcal{C} = -\Lambda_{P_0^{\rm crit}}^{-1} \pi_0 \mathcal{N}^* J \mathcal{C},$$

where we have used claim (iii) of Lemma 84 and the fact that the feed-through operator of $\mathcal{X}$ is identity. It follows that $K^{\rm crit} \in \mathcal{L}(H, U)$, because the DLS Φ is assumed to be output stable.

It follows from equation (2.40) of Lemma 92 that $\Lambda_{P_0^{\rm crit}} K^{\rm crit} = -D^* JC - B^* P_0^{\rm crit} A = Q_{P_0^{\rm crit}}$ in the whole of $\mathrm{dom}\,(\mathcal{C}) = H$. The invertibility of the indicator $\Lambda_{P_0^{\rm crit}}$ implies that $K^{\rm crit} = K_{P_0^{\rm crit}}$, by Definition 105 of DARE. The fact that we know this in the whole of H, and not only in $\overline{\mathrm{range}\,(\mathcal{B})}$, is a specialty of this particular critical solution $P_0^{\rm crit}$. We now conclude that the observability map

$\mathcal{C}_{\phi_{P_0^{\mathrm{crit}}}} = \{-K_{P_0^{\mathrm{crit}}} A^j\}_{j\geq 0} = \{-K^{\mathrm{crit}} A^j\}_{j\geq 0}$ equals $-\mathcal{K}$, where $\mathcal{K}$ is the observability map of the critical feedback pair $[\mathcal{K}, \mathcal{F}]$, see equations (2.27) and (2.28) of Lemma 87. Because $\mathcal{K} = -\Lambda_{P_0^{\mathrm{crit}}} \bar{\pi}_+ \mathcal{N}^* J \mathcal{C}$, where $\mathcal{N}$ is the $(J, \Lambda_{P_0^{\mathrm{crit}}})$-inner factor of $\mathcal{D}$, it follows that $\mathcal{K} : H \to \ell^2(\mathbf{Z}_+; U)$ is bounded. We conclude that $\phi_{P_0^{\mathrm{crit}}}$ is output stable, and $P_0^{\mathrm{crit}} \in ric_0(\Phi, J)$.

The proof of claim (ii) is analogous to the proof of $L_{A,P} = 0$. Because $\mathcal{C}^{\mathrm{crit}} A^{\mathrm{crit}} = \bar{\pi}_+ \tau^* \mathcal{C}^{\mathrm{crit}}$ by Lemma 74, we have

$$
\begin{aligned}
&||(A^{\mathrm{crit}*})^j P_0^{\mathrm{crit}} (A^{\mathrm{crit}})^j x_0|| = ||(A^{\mathrm{crit}*})^j (\mathcal{C}^{\mathrm{crit}})^* J \mathcal{C}^{\mathrm{crit}} (A^{\mathrm{crit}})^j x_0|| \\
&\leq ||(\mathcal{C}^{\mathrm{crit}})^* J|| \cdot ||\bar{\pi}_+ \tau^{*j} \mathcal{C}^{\mathrm{crit}} x_0|| \to 0
\end{aligned}
$$

as $j \to \infty$. It now follows that $L_{A, P_0^{\mathrm{crit}}} = 0$.

It remains to prove claim (iii). Assume that both P_1^{crit} and P_2^{crit} are critical solutions. Then, by Corollary 99, both the I/O maps $\mathcal{D}_{\phi_{P_1^{\mathrm{crit}}}}$ and $\mathcal{D}_{\phi_{P_2^{\mathrm{crit}}}}$ are outer factors in the $(J, \Lambda_{P_1^{\mathrm{crit}}})$, $(J, \Lambda_{P_2^{\mathrm{crit}}})$-inner-outer factorizations that they induce, respectively. With the aid of Proposition 83, we conclude that there is a boundedly invertible $E \in \mathcal{L}(U)$, such that

$$
\mathcal{D}_{\phi_{P_1^{\mathrm{crit}}}} = E^{-1} \mathcal{D}_{\phi_{P_2^{\mathrm{crit}}}}, \quad \text{and} \quad \Lambda_{P_1^{\mathrm{crit}}} = E^* \Lambda_{P_2^{\mathrm{crit}}} E.
$$

Because the feed-through operators of both $\mathcal{D}_{\phi_{P_1^{\mathrm{crit}}}}$ and $\mathcal{D}_{\phi_{P_2^{\mathrm{crit}}}}$ are identity operators, it follows that $E = I$, $\mathcal{D}_{\phi_{P_1^{\mathrm{crit}}}} = \mathcal{D}_{\phi_{P_2^{\mathrm{crit}}}}$ and $\Lambda_{P_1^{\mathrm{crit}}} = \Lambda_{P_2^{\mathrm{crit}}}$. This completes the proof. □

Without the approximate controllability assumption $\overline{\mathrm{range}\,(\mathcal{B})} = H$, we cannot conclude that a regular critical solution is unique in the set $ric_0(\Phi, J)$. However, the following uniqueness result is basic:

Corollary 116. *Let $J \in \mathcal{L}(Y)$ be a cost operator. Let $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ be an I/O stable and output stable DLS, such that $\overline{\mathrm{range}\,(\mathcal{B})} = H$. Assume that a critical solution $P^{\mathrm{crit}} \in Ric_{uw}(\Phi, J)$ exists.*

- *(i) Then P_0^{crit} is the unique critical solution in the set $Ric_{00}(\Phi, J)$. If A is strongly stable, then P_0^{crit} is the unique critical solution.*
- *(ii) Assume, in addition, that $P^{\mathrm{crit}} \geq 0$. If $P^{\mathrm{crit}} \notin Ric_0(\Phi, J)$, then $\sup_{j\geq 0} ||(P^{\mathrm{crit}})^{\frac{1}{2}} A^j|| = \infty$.*

Proof. Let P_0^{crit} be as in Proposition 115. By claim (iii) of Proposition 115, we have $\mathcal{D}_{\phi_{P_0^{\mathrm{crit}}}} = \mathcal{D}_{\phi_{P^{\mathrm{crit}}}}$ and $\Lambda_{P_0^{\mathrm{crit}}} = \Lambda_{P^{\mathrm{crit}}}$. Also the restrictions satisfy

$K_{P_0^{\rm crit}}|\text{range}\,(\mathcal{B}) = K_{P^{\rm crit}}|\text{range}\,(\mathcal{B})$ because the controllability maps both spectral DLSs $\phi_{P_0^{\rm crit}}$ and $\phi_{P^{\rm crit}}$ equal to $\mathcal{B}$. Because $\overline{\text{range}\,(\mathcal{B})} = H$, it follows $K_{P_0^{\rm crit}} = K_{P^{\rm crit}}$.

By the definition of a critical solution, $P^{\rm crit} \in Ric_{uw}(\Phi, J)$. Now, if $P^{\rm crit} \in Ric_{00}(\Phi, J)$, then $P^{\rm crit} \in Ric_0(\Phi, J)$, by claim (iv) of Proposition 109 and the approximate controllability assumption $\overline{\text{range}\,(\mathcal{B})} = H$. Proposition 111 implies that $P^{\rm crit} \in Ric_{00}(\Phi, J)$ satisfies

$$P^{\rm crit} = P_0^{\rm crit} + L_{A,P^{\rm crit}} - L_{A,P_0^{\rm crit}} = P_0^{\rm crit},$$

because $P_0^{\rm crit} \in ric_0(\Phi, J)$, by Proposition 115. Now claim (i) follows.

By the definition of a critical solution, $P^{\rm crit} \in Ric_{uw}(\Phi, J)$. Because $P^{\rm crit} \geq 0$, it follows that $||(P^{\rm crit})^{\frac{1}{2}} A^j x|| \to 0$ for all $x \in \text{range}\,(\mathcal{B})$. Assume that $\sup_{j\geq 0} ||(P^{\rm crit})^{\frac{1}{2}} A^j|| < \infty$. Let $\text{range}\,(\mathcal{B}) \ni x_k \to x \in H \setminus \text{range}\,(\mathcal{B})$. Then,

$$||(P^{\rm crit})^{\frac{1}{2}} A^j x|| \leq \sup_{j\geq 0} ||(P^{\rm crit})^{\frac{1}{2}} A^j|| \cdot ||x - x_k|| + ||(P^{\rm crit})^{\frac{1}{2}} A^j x_k||.$$

The first term on the right can be made small by increasing k, and the latter by increasing j. It follows that $\lim_{j\to\infty} ||(P^{\rm crit})^{\frac{1}{2}} A^j x|| = 0$ and then $P^{\rm crit} \in Ric_0(\Phi, J)$, by the Banach–Steinhaus theorem. This completes the proof. □

The rest of this section is devoted to the study of sufficient conditions that guarantee that (one and hence all of) the equivalent conditions of Theorem 114 hold. We remark that this is practically a standing hypothesis in this work.

Proposition 117. *Let $J \in \mathcal{L}(Y)$ be a cost operator. Let $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ be an I/O stable DLS whose input space U is separable. If $\bar\pi_+ \mathcal{D}^* J \mathcal{D} \bar\pi_+ \geq \epsilon \bar\pi_+ > 0$ for some $\epsilon > 0$, then the equivalent conditions of 114 hold. In particular, this is true if Φ is J-coercive and $J \geq 0$, or Φ is J-coercive and there is $P \in ric_{uw}(\Phi, J)$ such that $\Lambda_P > 0$.*

Proof. Assume that $\bar\pi_+ \mathcal{D}^* J \mathcal{D} \bar\pi_+$ is a nonnegative self-adjoint Toeplitz operator on $\ell^2(\mathbf{Z}_+; U)$ with a bounded inverse. By [77, Theorem 3.7], there is an I/O stable I/O map $\mathcal{G} \in \mathcal{L}(\ell^2(\mathbf{Z}; U))$ such that $\bar\pi_+ \mathcal{D}^* J \mathcal{D} \bar\pi_+ = \bar\pi_+ \mathcal{G}^* \mathcal{G} \bar\pi_+$. By this trick we get rid of the output space Y. By [77, Theorem 3.4], $\bar\pi_+ \mathcal{D}^* J \mathcal{D} \bar\pi_+ = \bar\pi_+ \mathcal{G}^* \mathcal{G} \bar\pi_+ = \bar\pi_+ \mathcal{H}^* \mathcal{H} \bar\pi_+$, where $\mathcal{H}$ is outer in the sense of [77, Definition 1.6]. Now, two problems are present. Firstly, $\text{range}\,(\mathcal{H}\bar\pi_+)$ for the outer operator $\mathcal{H}$ of [77, Definition 1.6] need not be even dense in $\ell^2(\mathbf{Z}_+; U)$. It is required that the closure of $\text{range}\,(\mathcal{H}\bar\pi_+)$ reduces the unilateral shift and is consequently of the form $\ell^2(\mathbf{Z}_+; U')$ for some Hilbert *subspace* $U' \subset U$. Secondly, even if $U' = U$, we must have $\text{range}\,(\mathcal{H}\bar\pi_+)$ closed, so that $\mathcal{H}$ is outer with a bounded inverse in the sense of Definition 79. The latter of these problems is easy to resolve. By

Proposition 69, the coercivity $\bar{\pi}_+\mathcal{H}^*\mathcal{H}\bar{\pi}_+ \geq \epsilon\bar{\pi}_+ > 0$ implies that the Toeplitz operator $\mathcal{H}\bar{\pi}_+$ has a closed range.

To attack the former problem, note that $U' \subset U$ implies $\dim U' \leq \dim U$. We proceed to prove that also $\dim U' \geq \dim U$ holds. By $H := \pi_0\mathcal{H}\pi_0 : U \to U'$ denote the feed-through operator of $\mathcal{H}$. We show that $\ker(H) = \{0\}$. For contradiction, assume that $Hu_0 = 0$ for some nonzero $u_0 \in U$. Denote $\tilde{u} := \{u_j\}_{j\geq 0}$ where $u_j = 0$ for all $j \geq 1$. Then

$$\tilde{w} := \mathcal{H}\tilde{u} = \pi_+\mathcal{H}\tilde{u} \in \ell^2(\mathbf{Z}_+; U') \tag{3.8}$$

and also $\tau^*\tilde{w} \in \ell^2(\mathbf{Z}_+; U')$. Because range$(\mathcal{H}\bar{\pi}_+) = \ell^2(\mathbf{Z}_+; U')$, there is $\tilde{u}' \in \ell^2(\mathbf{Z}_+; U)$ such that $\tau^*\tilde{w} = \mathcal{H}\bar{\pi}_+\tilde{u}'$ and

$$\tilde{w} = \tau\mathcal{H}\bar{\pi}_+\tilde{u}' = \mathcal{H}\tau\bar{\pi}_+\tilde{u}' = \mathcal{H}\pi_+\tau\tilde{u}'. \tag{3.9}$$

From equations (3.8) and (3.9) we conclude that $\mathcal{H}\bar{\pi}_+(\tilde{u} - \pi_+\tau\tilde{u}') = 0$. Because $\bar{\pi}_+\mathcal{H}^*\mathcal{H}\bar{\pi}_+$ is coercive, it follows that $\bar{\pi}_+(\tilde{u} - \pi_+\tau\tilde{u}') = 0$ and thus $\pi_0\tilde{u} - \pi_0\pi_+\tau\tilde{u}' = \pi_0\tilde{u} = 0$. But then $u_0 = 0$, and this is a contradiction. We conclude that $H : U \to U'$ is an injection, and thus $\dim U' \geq \dim U$.

Because $\dim U' = \dim U$, there is a unitary $E \in \mathcal{L}(U'; U)$ such that $E^*E = I$. Define $\mathcal{X} := E\mathcal{H}$. This is a stable I-spectral factor of $\mathcal{D}^*J\mathcal{D}$, see Definition 80. Because $\mathcal{H}\bar{\pi}_+$, together with $\mathcal{X}\bar{\pi}_+$, is coercive on $\ell^2(\mathbf{Z}_+; U)$, it follows that $\mathcal{X}$ is an I/O stable I/O map that is outer with a bounded inverse, see Definition 79. By claim (i) of Proposition 46, $\mathcal{X}^{-1}$ is an I/O stable I/O map. By Definition 80, $\mathcal{X}$ is a stable outer I-spectral factor of $\mathcal{D}^*J\mathcal{D}$, and by Proposition 82, $\mathcal{D} = \mathcal{N}\mathcal{X}$ is a (J, I)-inner-outer factorization, where $\mathcal{N} := \mathcal{D}\mathcal{X}^{-1}$ and the outer factor $\mathcal{X}$ is outer with a bounded inverse. Now condition (ii) of Theorem 114 holds.

If there is a solution in $P \in ric_{uw}(\Phi, J)$ such that $\Lambda_P > 0$, then we obtain the factorization of the Popov operator $\bar{\pi}_+\mathcal{D}^*J\mathcal{D}\bar{\pi}_+ = \bar{\pi}_+\mathcal{D}^*_{\phi_P}J\mathcal{D}_{\phi_P}\bar{\pi}_+$, by Lemma 98 or claim (i) of Theorem 142. We can now proceed as above, with $\mathcal{D}$ replaced by $\mathcal{D}_{\phi_P}$. □

For a further comment on the nonnegativity of the indicators Λ_P, see Lemma 145 and Corollary 146. The following equivalence is now an immediate corollary:

Corollary 118. *Let $J \in \mathcal{L}(Y)$ be a cost operator. Let $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ be an I/O stable and output stable DLS, such that the input space U is separable. Then the following are equivalent:*

(i) $\bar{\pi}_+\mathcal{D}^*J\mathcal{D}\bar{\pi}_+ \geq \epsilon\bar{\pi}_+$ *for some* $\epsilon > 0$.

(ii) The Popov operator $\bar{\pi}_+\mathcal{D}^*J\mathcal{D}\bar{\pi}_+$ *is nonnegative, and the equivalent conditions of Theorem 114 hold.*

Proof. The implication (i) $\Rightarrow$ (ii) is in Proposition 117. The converse direction is claim (i) of Theorem 114. $\square$

The case the nonnegative Popov operator $\bar{\pi}_+\mathcal{D}^*J\mathcal{D}\bar{\pi}_+$ occurs in applications, e.g. in the study of linear quadratic optimal control problems and in the factorization versions of Bounded and Positive Real Lemmas, see [86, Section 8]. In the latter two applications, the cost operator J is not nonnegative. We remark that it is practically a standing hypothesis of this work that the equivalence of Theorem 114 holds.

3.4 Function theoretic definitions and tools

In this section, we present some relevant results from the operator-valued function theory. Let $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ be a DLS. We work in terms of the boundary trace $\mathcal{D}_\phi(e^{i\theta})$ of the transfer function $\mathcal{D}_\phi(z)$ that must now be of bounded type $\mathcal{D}_\phi(z) \in N(\mathbf{D}; \mathcal{L}(U))$. This requires the separability of the Hilbert spaces U and Y. To obtain the full results, we must make a compactness assumption of an input operator B of ϕ, as we shall later see.

The adjoints of transfer functions are considered in Proposition 119. Several types of inner transfer functions are given in Definition 120. In Proposition 121, inner from the left transfer functions are characterized via the associated I/O maps and their Toeplitz operators. In Proposition 122, the inner functions are characterized in the set $H^2(\mathbf{D}; \mathcal{L}(U;Y))$, rather than in $H^\infty(\mathbf{D}; \mathcal{L}(U;Y))$. In Proposition 123, we remind that an inner from the left analytic function on a finite dimensional space is inner from both sides. In Lemma 125, we show, under a compactness assumption, that an analytic function $\Theta(z) \in H^2(\mathbf{D}; \mathcal{L}(U))$, whose boundary trace $\Theta(e^{i\theta})$ is injective almost everywhere, has the property that $\Theta(e^{i\theta})$ is boundedly invertible almost everywhere. In Corollary 126, we obtain an infinite dimensional generalization of Proposition 123. This allows us to conclude that certain inner from the left operator-valued transfer functions are, in fact, inner from both sides. Transfer functions and boundary traces of outer I/O maps are considered in Proposition 127.

The Hilbert–Schmidt class of compact operators is introduced in Definition 128. In Lemma 130 and Corollary 131, we use the Hilbert–Schmidt property of the input operator B to conclude that the transfer function of an output stable DLS $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ is in $H^2(\mathbf{D}; \mathcal{L}(U;Y))$. In Lemma 134, we show that the nontangential limit $\mathcal{D}_\phi(e^{i\theta}) \in \mathcal{L}(U;Y)$ of an output stable, I/O stable and J-coercive DLS $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ is invertible almost everywhere, provided that the input operator B is a Hilbert–Schmidt operator and the feed-through operator D is boundedly invertible. Under the same assumptions, the invertibility properties of the (extended topological) I/O map $\mathcal{D}_\phi : \ell^2(\mathbf{Z}; U) \to \ell^2(\mathbf{Z}; Y)$ and its Toeplitz operator $\mathcal{D}_\phi \bar{\pi}_+ : \ell^2(\mathbf{Z}_+; U) \to \ell^2(\mathbf{Z}_+; Y)$ are considered in Proposition 135.

We start by introducing some function theoretic notions. Let $\Theta(z)$ be an analytic $\mathcal{L}(U;Y)$-valued function in $\mathbf{D}$. The adjoint function $\tilde{\Theta}(z)$ is defined by

$$\tilde{\Theta}(z) := \Theta(\bar{z})^* \quad \text{for all} \quad z \in \mathbf{D}.$$

If $\Theta(z) = \sum_{j \geq 0} c_j z^j$ for $\{c_j\}_{j \geq 0} \subset \mathcal{L}(U,Y)$, then $\tilde{\Theta}(z) = \sum_{j \geq 0} c_j^* z^j$. It is trivial that for all $1 \leq p \leq \infty$, $\tilde{\Theta}(z) \in H^p(\mathbf{D}; \mathcal{L}(Y;U))$ if and only if $\Theta(z) \in H^p(\mathbf{D}; \mathcal{L}(U;Y))$. The nontangential boundary limits of adjoint bounded analytic functions behave expectedly.

Proposition 119. *Let $1 \leq p \leq \infty$ and $\Theta(z) \in H^p(\mathbf{D};\mathcal{L}(U;Y))$ be arbitrary, and let $\tilde{\Theta}(z)$ denote the adjoint function. Then $\tilde{\Theta}(e^{i\theta}) = \Theta(e^{-i\theta})^*$ for almost all $e^{i\theta} \in \mathbf{T}$. Furthermore, there exists a set $E \subset T$ of measure zero such that*

$$\mathrm{s} - \lim_{z_j \to e^{i\theta}} \Theta(z_j) = \Theta(e^{i\theta}) \quad \textit{and} \quad \mathrm{s} - \lim_{z_j \to e^{i\theta}} \Theta(z_j)^* = \Theta(e^{i\theta})^*$$

for all $e^{i\theta} \in \mathbf{T} \setminus E$ and any sequence $\{z_j\}_{j\geq 0}$ approaching nontangentially $e^{i\theta}$.

Proof. Note first that the nontangential limit function $\tilde{\Theta}(e^{i\theta})$ exists as a strong operator limit a.e. $e^{i\theta} \in \mathbf{T}$ because $\tilde{\Theta}(z) \in H^p(\mathbf{D};\mathcal{L}(Y;U))$. Denote the exceptional sets of measure zero for $\Theta(z)$ and $\tilde{\Theta}(z)$ by E_1 and E_2, such that the nontangential limits $\Theta(e^{i\theta})$ and $\tilde{\Theta}(e^{i\theta})$ exists in sets $\mathbf{T} \setminus E_1$ and $\mathbf{T} \setminus E_2$, respectively. Define the exceptional set $E := E_1 \cup \bar{E_2}$, where bar denotes the complex conjugation. It can be shown that the set $\bar{E_2}$ is measurable (in the Lebesgue completed σ-algebra of the Borel σ-algebra of the unit circle $\mathbf{T}$) and of measure zero. It follows that the set E is of measure zero.

Let $e^{i\theta} \in \mathbf{T} \setminus E$ be arbitrary. Let $z_j \to e^{i\theta}$ be an arbitrary nontangentially approaching sequence. Trivially, the sequence of conjugates $\bar{z}_j \to e^{-i\theta}$ nontangentially, too. By the definition of E, $\mathrm{s} - \lim_{j\to\infty} \Theta(z_j) = \Theta(e^{i\theta})$ and

$$\mathrm{s} - \lim_{j\to\infty} \Theta(z_j)^* = \mathrm{s} - \lim_{j\to\infty} \tilde{\Theta}(\bar{z}_j) = \tilde{\Theta}(e^{-i\theta}). \tag{3.10}$$

But for a general sequence of bounded operator $\{T_j\}_{j\geq 0} \subset \mathcal{L}(U;Y)$, such that both the strong limits $T := \mathrm{s} - \lim T_j$ and $S := \mathrm{s} - \lim T_j^*$ exist as bounded operators, we have

$$\begin{aligned}
\langle u, (T^* - S)y \rangle &= \langle Tu, y \rangle - \langle u, Sy \rangle = \lim_{j\to\infty} \langle T_j u, y \rangle - \lim_{j\to\infty} \langle u, T_j^* y \rangle \\
&= \lim_{j\to\infty} \left(\langle T_j u, y \rangle - \langle u, T_j^* y \rangle \right) = \lim_{j\to\infty} \left(\langle T_j u, y \rangle - \langle T_j u, y \rangle \right) = \lim_{j\to\infty} 0 = 0,
\end{aligned}$$

where $u \in U$ and $y \in Y$ are arbitrary. It follows that $S = T^*$. From equation (3.10) we conclude that $\tilde{\Theta}(e^{-i\theta}) = \Theta(e^{i\theta})^*$ a.e. $e^{i\theta} \in \mathbf{T}$. This completes the proof. □

We proceed to introduce the inner functions with the aid of the boundary traces. In Proposition 121 we state that this definition is in harmony with Definition 79 of inner I/O maps.

Definition 120. *Let $\Theta(z) \in H^\infty(\mathbf{D};\mathcal{L}(U;Y))$, where U and Y are separable. Then*

(i) $\Theta(z)$ is inner from the left if $\Theta(e^{i\theta}) \in \mathcal{L}(U,Y)$ is an isometry for almost all $e^{i\theta} \in \mathbf{T}$,

(ii) $\Theta(z)$ *is inner from the right if the adjoint function* $\tilde{\Theta}(z)$ *is inner from the left,*

(iii) $\Theta(z)$ *is inner if* $\Theta(e^{it})$ *is unitary a.e.* $e^{i\theta} \in \mathbf{T}$.

Clearly $\Theta(z)$ is inner from the left if and only if $\tilde{\Theta}(z)$ is inner from the right. The nontangential limit $\Theta(e^{i\theta})$ of a function inner from the right is co-isometric for almost all $e^{i\theta} \in \mathbf{T}$. Also $\Theta(z)$ is inner if and only if it is inner from the left and right. In this case we can say, for clarity, that $\Theta(z)$ is inner from both sides or two-sided inner. In [27, p. 234 and 242], $\Theta(z)$ is inner ($*$-inner) if $\Theta(e^{i\theta})$ is isometric (co-isometric, respectively) for almost all $e^{i\theta} \in \mathbf{T}$. The same notation is used in [90, p. 190]. In [77], inner function is an element of $H^\infty(\mathbf{D}; \mathcal{L}(U))$ such that the values of the boundary trace are partial isometries almost everywhere, and their initial spaces are constant, see [77, Theorem 5.3A and Example 1 on p. 106]. The transfer functions of the isometric and unitary Toeplitz operators of I/O maps $\mathcal{N}\bar{\pi}_+ : \ell^2(\mathbf{Z}_+; U) \to \ell^2(\mathbf{Z}_+; Y)$ are of particular interest.

Proposition 121. *Let* $\mathcal{N} : \ell^2(\mathbf{Z}; U) \to \ell^2(\mathbf{Z}; Y)$ *be an (extended topological) I/O map of an I/O stable DLS, with* U *and* Y *separable. Then the following are equivalent:*

(i) $\mathcal{N}$ *is* (I, I)*-inner in the sense of Definition 79,*

(ii) $\mathcal{N}\bar{\pi}_+ : \ell^2(\mathbf{Z}_+; U) \to \ell^2(\mathbf{Z}_+; Y)$ *is an isometry, and*

(iii) the transfer function $\mathcal{N}(z)$ *is inner from the left.*

Furthermore, $\mathcal{N}\bar{\pi}_+$ *is unitary if and only if* $\mathcal{N}(z)$ *is a unitary constant function.*

Proof. This is [27, part (c) of Theorem 1.1 and Corollary 1.2, Chapter IX]. □

In Definition 120, we have required that the inner function $\Theta(z)$ is *a priori* in $H^\infty(\mathbf{D}; \mathcal{L}(U; Y))$. This makes it possible to speak about nontangential limits, defined a.e. on $\mathbf{T}$. Actually, it would have been sufficient to require that $\Theta(z)$ lies in $H^2(\mathbf{D}; \mathcal{L}(U; Y))$ or even in $N_+(\mathbf{D}; \mathcal{L}(U; Y))$.

Proposition 122. *Let* $T(z) \in H^2(\mathbf{D}; \mathcal{L}(U; Y))$*, with* U *and* Y *separable. Assume that the boundary trace satisfies* $\operatorname{ess\,sup}_{e^{i\theta} \in \mathbf{T}} ||T(e^{i\theta})|| < \infty$. *Then* $T(z) \in H^\infty(\mathbf{D}; \mathcal{L}(U; Y))$. *In particular, if* $T(z) \in H^2(\mathbf{D}; \mathcal{L}(U; Y))$ *has the isometry-valued boundary trace* $T(e^{i\theta})$ *for almost all* $e^{i\theta} \in \mathbf{T}$*, then* $T(z)$ *is inner from the left.*

Proof. By the same comment that is present in the proof of Proposition 55, we need to consider only the case $Y = U$. In this case, [77, Theorem 4.7A] proves the claim because $H^2(\mathbf{D}; \mathcal{L}(U; Y)) \subset N_+(\mathbf{D}; \mathcal{L}(U; Y))$. □

On several occasions, it will be necessary to conclude that an inner from the left function is in fact inner. If the Hilbert spaces U and Y are finite dimensional with the same dimension, it is easy to show that inner from the left implies inner from the both sides. This is because all isometries in a finite dimensional space are unitary, by a basic dimension counting argument.

Proposition 123. *Assume that $\Theta(z) \in H^\infty(\mathbf{D};\mathcal{L}(U))$ is inner from the left, where $\dim U < \infty$. Then $\Theta(z)$ is inner from both sides.*

If the involved Hilbert spaces are infinite dimensional, much less it true. However, a sufficient generalization of Proposition 123 holds, see Corollary 126. The following preliminary result allows us to conclude from the strong operator convergence the convergence in the operator norm, under a compactness assumption.

Proposition 124. *Let $K \in \mathcal{LC}(U)$ and $\kappa_j \in \mathcal{L}(U)$ for all $j \geq 0$. Assume that $\kappa_j u \to \kappa u$ for all $u \in U$. Then $||\kappa K - \kappa_j K||_{\mathcal{L}(U)} \to 0$ as $j \to \infty$.*

Proof. For contradiction, assume that $||\kappa K - \kappa_j K||_{\mathcal{L}(U)}$ does not converge to zero as $j \to \infty$. Then there is a sequence $\{u_i\}_{i\geq 0} \subset U$, $||u_i||_U = 1$, and a subsequence $\{j(i)\}_{j\geq 0}$ such that for all $i \geq 0$ we have

$$||(\kappa - \kappa_{j(i)})K u_i||_U \geq \nu \tag{3.11}$$

for some constant $\nu > 0$. Because K is compact, there exists a subsequence $\{u_{i(h)}\}_{h\geq 0}$ such that $K u_{i(h)}$ converges to a limit, say $u \in U$. We now estimate for all $h \geq 0$

$$\begin{aligned} &||(\kappa - \kappa_{j(i(h))})K u_{i(h)}||_U \leq ||(\kappa - \kappa_{j(i(h))})u||_U + ||(\kappa - \kappa_{j(i(h))})(K u_{i(h)} - u)||_U \\ &\leq ||(\kappa - \kappa_{j(i(h))})u||_U + \left(||\kappa||_{\mathcal{L}(U)} + \sup_{j\geq 0} ||\kappa_j||_{\mathcal{L}(U)} \right) \cdot ||K u_{i(h)} - u||_U. \end{aligned}$$

The first term on the right hand side converges to zero because $u \in U$ and $\kappa_j \to \kappa$ in the strong operator topology. Because $\kappa_j \to \kappa$ in the strong operator topology, it follows from the Banach–Steinhaus Theorem that $\sup_{j\geq 0} ||\kappa_j||_{\mathcal{L}(U)} < \infty$. We conclude that the latter term converges to zero, by the choice of the convergent subsequence $\{K u_{i(h)}\}_{h\geq 0}$. Thus $||(\kappa - \kappa_{j(i(h))})K u_{i(h)}||_U \to 0$ as $h \to \infty$, but this is a contradiction against the existence of a nonnegative lower bound ν in equation (3.11). This completes the proof. □

Lemma 125. *Let $K \in \mathcal{LC}(U)$ and $\kappa(z) \in H^2(\mathbf{D};\mathcal{L}(U))$ be arbitrary. Define*

$$\Theta(z) := I + \kappa(z)K, \quad z \in \mathbf{D}.$$

(i) *Then* $\Theta(z) \in H^2(\mathbf{D}; \mathcal{L}(U))$. *Furthermore, there is an exceptional set* $E \subset \mathbf{T}$ *of measure zero, such that the nontangential limits*

$$\Theta(e^{i\theta}) = \lim_{z_j \to e^{i\theta}} \Theta(z_j), \quad \Theta(e^{i\theta})^* = \lim_{z_j \to e^{i\theta}} \Theta(z_j)^* \tag{3.12}$$

exist and converge in the operator norm of $\mathcal{L}(U)$, *for all* $e^{i\theta} \in \mathbf{T} \setminus E$ *and for any sequence* $\{z_j\}_{j \geq 0} \subset \mathbf{D}$ *that converges to* $e^{i\theta}$ *nontangentially. The boundary trace satisfies*

$$\Theta(e^{i\theta}) = I + \kappa(e^{i\theta})K \tag{3.13}$$

for all $e^{i\theta} \in \mathbf{T} \setminus E$.

(ii) *Assume, in addition, that the boundary trace* $\Theta(e^{i\theta})$ *is injective for almost all* $e^{i\theta} \in \mathbf{T}$. *Then* $\Theta(e^{i\theta})$ *is boundedly invertible for almost all* $e^{i\theta} \in \mathbf{T}$.

Proof. It is a triviality that $\Theta(z) \in H^2(\mathbf{D}; \mathcal{L}(U))$. Because $\kappa(z) \in H^2(\mathbf{D}; \mathcal{L}(U))$, there exists an exceptional set $E \subset \mathbf{T}$ of measure zero, such that for all $e^{i\theta} \in \mathbf{T} \setminus E$, the strong limit

$$\kappa(e^{i\theta}) = \mathrm{s} - \lim_{z_j \to e^{i\theta}} \kappa(z_j)$$

converges, where $\{z_j\}_{j \geq 0} \subset \mathbf{D}$ is an arbitrary sequence that converges nontangentially to $e^{i\theta}$. Furthermore, the strong nontangential limit

$$\Theta(e^{i\theta}) := \mathrm{s} - \lim_{z_j \to e^{i\theta}} \Theta(z_j)$$

exists and satisfies equation (3.13) for all $e^{i\theta} \in \mathbf{T} \setminus E$. For the rest of the proof, fix an arbitrary $e^{i\theta} \in \mathbf{T} \setminus E$ and a nontangential sequence $\{z_j\}_{j \geq 0} \subset \mathbf{D}$, converging to $e^{i\theta}$.

Define $\kappa_j := \kappa(z_j)$ and $\kappa := \kappa(e^{i\theta})$. Then $\kappa_j \to \kappa$ in the strong operator topology, and $\kappa_j K \to \kappa K$ in the norm of $\mathcal{L}(U)$, by Proposition 124. But now we have

$$\begin{aligned} &||\Theta(z_j) - \Theta(e^{i\theta})||_{\mathcal{L}(U)} = ||\,(I - \kappa(z_j)K) - \big(I - \kappa(e^{i\theta})K\big)\,||_{\mathcal{L}(U)} \\ &= ||\kappa K - \kappa_j K||_{\mathcal{L}(U)} \to 0. \end{aligned}$$

We conclude that $\Theta(z_j) \to \Theta(e^{i\theta})$ in the norm of $\mathcal{L}(U)$, and immediately $\Theta(z_j)^* \to \Theta(e^{i\theta})^*$ in the norm of $\mathcal{L}(U)$, too. This completes the proof of claim (i).

We consider now claim (ii). By assumption, there is an exceptional set $E' \subset \mathbf{T}$ of measure zero, such that the evaluation of the nontangential limit $\Theta(e^{i\theta}) \in \mathcal{L}(U)$ is injective for $e^{i\theta} \in \mathbf{T} \setminus E'$. For the rest of this proof, fix an arbitrary

$e^{i\theta} \in \mathbf{T} \setminus (E \cup E')$, where E is as in claim (i), and $E \cup E'$ is a set of measure zero. The operator $\kappa(e^{i\theta})K \in \mathcal{L}(U)$ is compact because $\kappa(e^{i\theta}) \in \mathcal{L}(U)$ and $K \in \mathcal{LC}(U)$. Because the spectrum of a compact operator consists of eigenvalues and possibly the point zero, then either $-1 \in \sigma_p(\kappa(e^{i\theta})K)$ or $\Theta(e^{i\theta}) = I + \kappa(e^{i\theta})K$ is boundedly invertible. But if $-1 \in \sigma_p(\kappa(e^{i\theta})K)$, then $I + \kappa(e^{i\theta})K$ is not even injective, which is against the choice of $e^{i\theta}$. We conclude that $\Theta(e^{i\theta}) = I+\kappa(e^{i\theta})K$ is boundedly invertible, and because $e^{i\theta} \in \mathbf{T}\setminus(E\cup E')$ was arbitrary, the proof is complete. □

Corollary 126. *Let $\kappa(z) \in H^2(\mathbf{D}; \mathcal{L}(U))$ and $K \in \mathcal{LC}(U)$ be arbitrary. Assume that the function*

$$\Theta(z) = \Theta_0 + z\kappa(z)K, \quad z \in \mathbf{D}.$$

is inner from the left, and $\Theta_0 \in \mathcal{L}(U)$ is boundedly invertible. Then $\Theta(z)$ is inner from both sides.

Proof. Write $\Theta_0^{-1}\Theta(z) = I + z\Theta_0^{-1}\kappa(z)K$, and apply Lemma 125. □

We remark that if $\dim U = \infty$, the class of inner functions, considered in the previous Corollary 126, is rather restricted. The values of the boundary trace $\Theta(e^{i\theta})$ are unitary operator of form $\Theta_0 + zK(e^{i\theta})$, where $K(z)$ is compact a.e. $e^{i\theta} \in \mathbf{T}$. In this book, this restriction holds in all the instances where we must conclude that an inner from the left function is inner from both sides. However, even if we could deal with the more general inner functions, our results would not be more general because we are compelled to make a compactness assumption for other reasons (see Lemma 130), leading to this restricted type of inner factors.

Now that we have dealt with the matters concerning the boundary behavior of the inner functions, we proceed to study the outer functions and general transfer functions of certain DLSs. The basic properties of the outer I/O maps have been considered in Proposition 46. Now we consider the corresponding transfer functions and boundary traces.

Proposition 127. *Let $\mathcal{X} : \ell^2(\mathbf{Z}; U) \to \ell^2(\mathbf{Z}; U)$ be an I/O map of an I/O stable DLS, which is outer with a bounded inverse. Then the following holds.*

(i) *$\mathcal{X}^{-1} : \ell^2(\mathbf{Z}; U) \to \ell^2(\mathbf{Z}; U)$ exists boundedly, and it is an I/O map of an I/O stable DLS.*

(ii) *$\mathcal{X}(z)^{-1} \in \mathcal{L}(U)$ exists for all $z \in \mathbf{D}$, and $\mathcal{X}(z)^{-1} = \mathcal{X}^{-1}(z)$. Furthermore, $\sup_{z\in\mathbf{D}} ||\mathcal{X}(z)^{-1}||_{\mathcal{L}(U)} < \infty$ and thus $\mathcal{X}(z)^{-1} \in H^\infty(\mathbf{D}; \mathcal{L}(U))$.*

(iii) *If, in addition, U is separable, then the nontangential boundary limit $\mathcal{X}(e^{i\theta})$ exists and is boundedly invertible for almost all $e^{i\theta} \in \mathbf{T}$. We have $\mathcal{X}(e^{i\theta})^{-1} = \mathcal{X}^{-1}(e^{i\theta})$ for almost all $e^{i\theta} \in \mathbf{T}$. In particular, $\mathcal{X}(e^{i\theta})^{-1} \in H^\infty(\mathbf{T}; \mathcal{L}(U))$.*

Proof. Claim (i) is shown in claim (i) of Proposition 46. To prove claim (ii), we show that $\mathcal{X}(z)^{-1} = \mathcal{X}^{-1}(z)$ for all $z \in \mathbf{D}$. Let ϕ' be a realization such that $\mathcal{X} = \mathcal{D}_{\phi'}$. Then $\mathcal{X}^{-1} = \mathcal{D}_{\phi'}^{-1} = \mathcal{D}_{(\phi')^{-1}}$, by Proposition 17, and $\mathcal{I} = \mathcal{D}_{(\phi')^{-1}}\mathcal{D}_{\phi'}$. By Corollary 54, $I = \mathcal{D}_{(\phi')^{-1}}(z)\mathcal{D}_{\phi'}(z)$ and $I = \mathcal{D}_{\phi'}(z)\mathcal{D}_{(\phi')^{-1}}(z)$ for all $z \in \mathbf{D}$. It follows that $\mathcal{D}_{\phi'}(z) = \mathcal{X}(z) : U \to U$ is a bounded bijection and has a bounded inverse $\mathcal{X}(z)^{-1}$, for all $z \in \mathbf{D}$. Also $\mathcal{X}(z)^{-1} = \mathcal{D}_{\phi'}^{-1}(z) = \mathcal{X}^{-1}(z) \in H^\infty(\mathbf{D}; \mathcal{L}(U))$, by claim (i). The last claim (iii) follows now from the theory of boundary traces of H^∞-functions, see the discussion following Definition 58 or [77, p. 88]. $\square$

As we have stated earlier, functions in the Nevanlinna class $N(\mathbf{D}; X)$ can be adequately described by their nontangential boundary limit functions for $X = U$, $X = \mathcal{L}(U)$ or $X = \mathcal{L}(U; Y)$, when U and Y are separable Hilbert spaces. Unfortunately, a general $\mathrm{sH}^2(\mathbf{D}; \mathcal{L}(U; Y))$ function need not be in $N(\mathbf{D}; \mathcal{L}(U; Y))$ if $\dim U = \infty$. It is even more unfortunate that the strong H^2 stability of the transfer function is an important notion because it is implied by the output stability of any of its realizations. From the state space representation of a transfer function, output stability of the realization is often best we can achieve by Liapunov type methods.

In order to work with the boundary traces $\mathcal{D}_\phi(e^{i\theta})$ of an output stable DLS $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix} \right)$, we have to make an extra assumption. The question is about a compactness assumption of the input operator B which, in a sense, forbids the DLS ϕ to be "too" infinite-dimensional. With this restriction, we can conclude that $\mathcal{D}_\phi(z) \in H^2(\mathbf{D}; \mathcal{L}(U; Y)) \subset N_+(\mathbf{D}; \mathcal{L}(U; Y))$, by Lemma 130.

Definition 128. *Let H_1, H_2 be separable Hilbert spaces, and $T \in \mathcal{L}(H_1, H_2)$. Let $\{e_j\}_{j\geq 0}$ be an orthonormal basis for H_1. We say that T is a Hilbert–Schmidt operator if*

$$||T||^2_{HS(H_1;H_2)} := \sum_{j\geq 0} ||Te_j||^2_{H_2(H_1;H_2)}$$

is finite. In this case we write $T \in HS(H_1; H_2)$. The number $||T||_{HS(H_1;H_2)}$ is the Hilbert–Schmidt norm of T.

Basic references about the Hilbert–Schmidt operators are [24, Chapter XI.6] and [41]. Also [109, Chapter 1] is quite useful. It is customary to consider the Hilbert–Schmidt operators on a single Hilbert space. Because Hilbert spaces of same cardinality can be unitarily identified, this is only a technical problem. It can be shown that the class $HS(H_1; H_2)$ is well defined, and the norm $|| \cdot ||_{HS(H_1;H_2)}$ is independent of the choice of the basis $\{e_j\}_{j\geq 0}$. All Hilbert–Schmidt operators are compact, and each finite rank operator is trivially Hilbert–Schmidt. The adjoint of a Hilbert–Schmidt operator is Hilbert–Schmidt. The set $HS(H_1; H_2)$ is a vector space, and the norm $|| \cdot ||_{HS(H_1;H_2)}$

makes it a Banach space. In the matrix case, the $HS(H_1;H_2)$-norm is the familiar Frobenius matrix norm. The vector space $HS(H_1,H_2)$ is a Hilbert space under the inner product

$$[T_1,T_2]_{HS(H_1,H_2)} := \sum_{j\geq 0} \langle T_1 e_j, T_2^* e_j\rangle.$$

If $H_1 = H_2$, then $HS(H_1;H_2)$ is a Banach algebra where the involution $T \mapsto T^*$ satisfies $||T||_{HS(H_1)} = ||T^*||_{HS(H_1)}$. The Hilbert–Schmidt operators are exactly those compact operators T whose singular values satisfy $\sum_{j\geq 0}\sigma_j(T)^2 < \infty$, where the singular values are defined as the eigenvalues of the radial part $|T| := (T^*T)^{\frac{1}{2}}$. In fact, $||T||^2_{HS(H_1;H_2)} = \sum_{j\geq 0}\sigma_j(T)^2$. The singular values are know as s-numbers in [41], and characteristic numbers in [24]. There is a number of equivalent characterizations for the singular values of a compact operator. It is a matter of taste, which one is chosen to be the definition. The following result is important enough to stated formally, and can be found in [24, Corollary 5 in Chapter XI, Section 6].

Proposition 129. *Let $T \in HS(H_1;H_2)$ and $S \in \mathcal{L}(H_2;H_3)$. Then $ST \in HS(H_1;H_2)$ and $||ST||_{HS(H_2;H_3)} \leq ||S||_{\mathcal{L}(H_2;H_3)}\,||T||_{HS(H_1;H_2)}$.*

Our first application of the Hilbert–Schmidt operators is the following lemma.

Lemma 130. *Let $\Theta(z) \in \mathrm{sH}^2(\mathbf{D};\mathcal{L}(U;Y))$, with U and Y separable. Assume that the linear mapping*

$$U \ni u \mapsto \Theta(z)u \in H^2(\mathbf{D};Y) \tag{3.14}$$

is a Hilbert–Schmidt operator. Then $\Theta(z) \in H^2(\mathbf{D};\mathcal{L}(U;Y))$.

Proof. Let $\{e_j\}_{j\geq 0}$ be an countable orthonormal basis for the separable U. Define the analytic functions $\Theta_j(z) := \Theta(z)e_j$. Each $\Theta_j(z)$ belongs to $H^2(\mathbf{D};Y)$ because $\Theta(z) \in \mathrm{sH}^2(\mathbf{D};\mathcal{L}(U;Y))$. The Hilbert–Schmidt assumption means that

$$\sum_{j\geq 0} ||\Theta_j(z)||^2_{H^2(\mathbf{D};Y)} < \infty, \tag{3.15}$$

where

$$||\Theta_j(z)||^2_{H^2(\mathbf{D};Y)} := \sup_{0<r<1} \frac{1}{2\pi}\int_0^{2\pi} ||\Theta_j(re^{i\theta})||^2_Y \, d\theta.$$

For all $z \in D$, $\Theta(z) \in \mathcal{L}(U;Y)$. Let $u = \sum_{j\geq 0} c_j e_j \in U$ be arbitrary, such that only a finite number of c_j's are nonzero. Then for all $z \in \mathbf{D}$ we have

$$||\Theta(z)u||^2_Y = ||\sum_{j\geq 0} c_j\,\Theta_j(z)||^2_Y \leq \sum_{j\geq 0}|c_j|^2 \sum_{j\geq 0}||\Theta_j(z)||^2_Y = ||u||^2_U \cdot \sum_{j\geq 0}||\Theta_j(z)||^2_Y$$

Because above the set of u's is dense in U, it follows

$$||\Theta(z)||^2_{\mathcal{L}(U;Y)} \leq \sum_{j\geq 0} ||\Theta_j(z)||^2_Y \tag{3.16}$$

for all $z \in \mathbf{D}$.

Now, let $0 < r < 1$ be arbitrary. Then each function $e^{i\theta} \mapsto ||\Theta_j(re^{i\theta})||^2_Y$ is a smooth (and thus a measurable) function, by the analyticity of $\Theta_j(z)$ in $\mathbf{D}$. The function $e^{i\theta} \mapsto \sum_{j\geq 0} ||\Theta_j(re^{i\theta})||^2_Y$ is measurable because the partial sums are increasing, and the supremum of a countable collection of measurable functions is measurable, by [78, Theorem 1.14]. Similarly, because $\Theta(z)$ is analytic inside $\mathbf{D}$, the function $e^{i\theta} \mapsto ||\Theta(re^{i\theta})||^2_Y$ is measurable, too. Now equation (3.16) gives for all $0 < r < 1$

$$\begin{aligned} \frac{1}{2\pi}\int_0^{2\pi} ||\Theta(re^{i\theta})||^2_{\mathcal{L}(U;Y)}\, d\theta &\leq \frac{1}{2\pi}\int_0^{2\pi} \left(\sum_{j\geq 0} ||\Theta_j(re^{i\theta})||^2_Y\right) d\theta \\ &= \sum_{j\geq 0}\left(\frac{1}{2\pi}\int_0^{2\pi} ||\Theta_j(re^{i\theta})||^2_Y\, d\theta\right), \end{aligned} \tag{3.17}$$

where the latter equality is by the Lebesgues Monotone Convergence theorem [78, Theorem 1.26] implies (or its immediate corollary [78, Theorem 1.27]), because the partial sums are nondecreasing. Taking supremum over r, gives

$$\begin{aligned} &||\Theta(z)||^2_{H^2(\mathbf{D};\mathcal{L}(U;Y))} \leq \sum_{j\geq 0}\left(\sup_{0<r<1}\frac{1}{2\pi}\int_0^{2\pi} ||\Theta_j(re^{i\theta})||^2_Y\, d\theta\right) \\ &= \sum_{j\geq 0} ||\Theta_j(z)||^2_{H^2(\mathbf{D};Y)}. \end{aligned}$$

Using the Hilbert–Schmidt assumption in the form of equation (3.15) shows that $\Theta(z) \in H^2(\mathbf{D};\mathcal{L}(U;Y))$. The proof is now complete. □

Corollary 131. *Let $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ be an output stable DLS, such that the spaces U and Y are separable. Assume that the input operator $B \in \mathcal{L}(U;H)$ is Hilbert–Schmidt. Then $\mathcal{D}_\phi(z) \in H^2(\mathbf{D};\mathcal{L}(U;Y))$.*

Proof. Because ϕ is output stable, $\mathcal{D}_\phi(z) - D \in \mathrm{sH}^2(\mathbf{D};\mathcal{L}(U;Y))$, by Proposition 57. We also have $(\mathcal{D}_\phi(z) - D)u_0 = \sum_{j\geq 1} CA^{j-1}Bu_0 z^j = z\cdot(\mathcal{F}_z\mathcal{C}_\phi Bu_0)(z)$, where $\mathcal{F}_z$ denotes the unitary z-transform from $\ell^2(\mathbf{Z}_+;Y)$ onto $H^2(\mathbf{D};Y)$. By output stability, the composition $\mathcal{F}_z\mathcal{C}_\phi : H \to H^2(\mathbf{D};Y)$ is well defined and bounded. It follows from Proposition 129 that the mapping

$$U \ni u_0 \mapsto (\mathcal{F}_z\mathcal{C}_\phi Bu_0)(z) \in H^2(Y)$$

is Hilbert–Schmidt because the input operator B is. Because the multiplication of the variable z in $H^2(\mathbf{D};Y)$ is isometric, the mapping

$$U \ni u_0 \mapsto (\mathcal{D}_\phi(z) - D)u_0 \in H^2(Y)$$

is Hilbert–Schmidt. Lemma 130 implies now that $\mathcal{D}_\phi(z) - D \in H^2(\mathbf{D};\mathcal{L}(U;Y))$. This completes the proof. $\square$

The same conclusion can be made, if A^jB is Hilbert–Schmidt, for some $j \geq 0$.

We can always "steal" from the convergence of a singular values enough, to be able to factorize any Hilbert–Schmidt operator as follows.

Proposition 132. *Let U be a separable Hilbert space, and $T \in HS(U)$. Then $T = T_1T_2$, where $T_1 \in HS(U)$ and $T_2 \in \mathcal{LC}(U)$.*

Proof. We use the canonical decomposition (the Schmidt expansion) of a compact operator, see [41, Chapter II, Section 2] and [109, Theorem 1.3.11 and the associated remarks]. Let $T = V|T|$ be the polar decomposition, where $|T| := (T^*T)^{\frac{1}{2}}$ is a nonnegative self-adjoint operator. By $\{e_k\}_{k\geq 1}$ denote an orthonormal system of eigenvectors of $|T|$ whose linear span is dense in range $(|T|)$, see [1, Sections 55 and 61]. Then by the canonical decomposition

$$|T|u = \sum_{k\geq 1} \sigma_k(T)\, \langle u, e_k\rangle_U\, e_k$$

for all $u \in U$, where the series converge in the norm of U. The sequence $\{\sigma_k(T)\}_{k\geq 1}$ is the sequence of the singular values of T, which are, by definition, the eigenvalues of $|T|$. Because T is a Hilbert–Schmidt operator, the singular values satisfy $\sum_{k\geq 1}\sigma_k(T)^2 = ||T||^2_{HS(U)} < \infty$, see [41, Chapter III, Section 9] and [109, Theorem 1.4.2].

We need now an auxiliary result. Let $\{a_k\}_{k\geq 1}$ be a sequence of nonnegative real numbers, such that $\sum_{k\geq 1} a_k < \infty$. For each $l \geq 1$, choose a positive integer m_l such that

$$\sum_{k\geq m_l} a_k \leq \frac{1}{l^3}.$$

Clearly, such a sequence $\{m_l\}_{l\geq 1}$ exists, and we may assume that it is nonincreasing. Define the sequence $\{b_k\}_{k\geq 1}$ by setting

$$b_k := la_k \quad \text{for} \quad m_l \leq k < m_{l+1}.$$

Then

$$\sum_{k\geq 1} b_k = \sum_{l\geq 1}\sum_{k=m_l}^{m_l-1} b_k = \sum_{l\geq 1}\left(l\cdot \sum_{l=m_l}^{m_l-1} a_k\right) \leq \sum_{l\geq 1}\left(l\cdot \sum_{k\geq m_l} a_k\right) \leq \sum_{l\geq 1}\frac{1}{l^2} < \infty.$$

We conclude that for any square summable sequence $\{\sigma_k(T)\}_{k\geq 1}$ (of singular values), there exists a nonincreasing sequence $\{\tau_k\}_{k\geq 1}$ such that $\tau_k \to 0$ as $k \to \infty$, but $\{\sigma_k(T)\tau_k^{-1}\}_{k\geq 1}$ is still square summable. Define for all $u \in U$

$$T_1' u := \sum_{k\geq 1} \left(\sigma_k(T)\tau_k^{-1}\right) \langle u, e_k\rangle_U \, e_k$$

and

$$T_2 u := \sum_{k\geq 1} \tau_k \langle u, e_k\rangle_U \, e_k.$$

Then the mappings $u \mapsto T_1' u$ and $u \mapsto T_2 u$ are compact operators in $\mathcal{LC}(U)$, by [109, Theorem 1.4.2], and T_1' is, in addition, a Hilbert–Schmidt operator because its singular values $\{\sigma_k(T)\tau_k^{-1}\}_{k\geq 0}$ are square summable. We prove now that $|T| = T_1' T_2$. Let $u \in U$ be an arbitrary finite linear combination of the basis vectors $\{e_k\}_{k\geq 1}$. Then

$$\begin{aligned} T_1' T_2 u &= \sum_{k\geq 1} \tau_k \langle u, e_k\rangle_U \, T_1' e_k = \sum_{k\geq 1} \tau_k \langle u, e_k\rangle_U \left(\sigma_k(T)\tau_k^{-1}\right) e_k \\ &= \sum_{k\geq 1} \sigma_k(T) \langle u, e_k\rangle_U \, e_k = |T| u \end{aligned}$$

where all the sums are finite, and we have used the immediate fact $T_1' e_k = \left(\sigma_k(T)\tau_k^{-1}\right) e_k$ for all $k \geq 1$. Because the set of such u's is dense, and all the operators $|T|$, T_1' and T_2 are bounded, the equality $|T| = T_1' T_2$ follows. Define $T_1 := V T_1'$. Then $T = T_1 T_2$ and T_1 is Hilbert–Schmidt, by Proposition 129. This completes the proof. □

Proposition 133. *Let $J \in \mathcal{L}(Y)$ be a cost operator, and $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ an I/O stable and J-coercive DLS, whose input space U is separable. Then the values of the Popov function $\mathcal{D}(e^{i\theta})^* J \mathcal{D}(e^{i\theta}) \in \mathcal{L}(U)$ are boundedly invertible operators, and the nontangential limit $\mathcal{D}(e^{i\theta}) \in \mathcal{L}(U;Y)$ is coercive for almost all $e^{i\theta} \in \mathbf{T}$.*

Proof. The J-coercivity of Φ means that the Popov operator $\bar{\pi}_+ \mathcal{D}^* J \mathcal{D} \bar{\pi}_+$ has a bounded inverse on $\ell^2(\mathbf{Z}_+;U)$, see Definition 68. By the shift-invariance, we conclude that the bounded, shift-invariant operator $\mathcal{D}^* J \mathcal{D}$ is coercive on $\ell^2(\mathbf{Z};U)$, where $\mathcal{D} : \ell^2(\mathbf{Z};U) \to \ell^2(\mathbf{Z};Y)$ denotes the (extended topological) I/O map of Φ. Because $\mathcal{D}^* J \mathcal{D}$ is self-adjoint, it follows that it is boundedly invertible on $\ell^2(\mathbf{Z};U)$. Because $\mathcal{D}^* J \mathcal{D}$ is shift-invariant, a trivial argument shows that $(\mathcal{D}^* J \mathcal{D})^{-1}$ is shift-invariant, too. Define

$$\mathcal{P} := \mathcal{F}_U^* \mathcal{D}^* J \mathcal{D} \mathcal{F}_U, \quad \mathcal{Q} := \mathcal{F}_U^* \left(\mathcal{D}^* J \mathcal{D}\right)^{-1} \mathcal{F}_U,$$

where $\mathcal{F}_U : L^2(\mathbf{T};U) \to \ell^2(\mathbf{Z};U)$ is the unitary Fourier transform, as introduced in Section 1.10. By Proposition 62, the operators $\mathcal{P}$ and $\mathcal{Q}$ on $L^2(\mathbf{T};U)$ commute with the multiplication M_ξ operator by the function $\xi(e^{i\theta}) := e^{i\theta}$ on $\mathbf{T}$. By [27, Theorem 1.1(a) in Chapter IX], we conclude that $\mathcal{P} = M_{\mathcal{P}(e^{i\theta})}$, which is the multiplication operator by a function $\mathcal{P}(e^{i\theta}) \in L^\infty(\mathbf{T};\mathcal{L}(U))$. Similarly, $\mathcal{Q} = M_{\mathcal{Q}(e^{i\theta})}$. Because both $\mathcal{Q}(e^{i\theta}), \mathcal{P}(e^{i\theta}) \in L^\infty(\mathbf{T};\mathcal{L}(U))$, the evaluations satisfy $\mathcal{Q}(e^{i\theta}), \mathcal{P}(e^{i\theta}) \in \mathcal{L}(U)$ for all $e^{i\theta} \in \mathbf{T} \setminus E'$, where the exceptional set E' is of measure zero.

Because $(\mathcal{D}^* J \mathcal{D})^{-1}$ is the inverse of $\mathcal{D}^* J \mathcal{D}$ and $\mathcal{F}_U$ is unitary, we must have $\mathcal{P}\mathcal{Q} = \mathcal{Q}\mathcal{P} = \mathcal{I}$, the identity operator on $L^2(\mathbf{T};U)$. Let $\{u_j\}_{j\geq 0} \subset U$ be a dense countable subset which exists by the separability of the Hilbert space U. By applying the identity $\mathcal{P}\mathcal{Q} = \mathcal{I}$ to each u_j, regarded as constant a function in $L^2(\mathbf{T};U)$, we conclude that there exists a sequence $\{E_j\}_{j\geq 0}$ of exceptional sets of measure zero, such that for all $j \geq 0$

$$\mathcal{Q}(e^{i\theta})\mathcal{P}(e^{i\theta})u_j = u_j \quad \text{for all} \quad e^{i\theta} \in \mathbf{T} \setminus (E_j \cup E').$$

Define $E := (\cup_{j\geq 0} E_j) \cup E'$. Then, for all $e^{i\theta} \in \mathbf{T} \setminus E$,

$$\mathcal{Q}(e^{i\theta})\mathcal{P}(e^{i\theta}) = I,$$

the identity operator in $\mathcal{L}(U)$, because $\mathcal{Q}(e^{i\theta})\mathcal{P}(e^{i\theta})$ is a bounded operator, and $\{u_j\}_{j\geq 0}$ is a dense subset. Similarly, $\mathcal{P}(e^{i\theta})\mathcal{Q}(e^{i\theta}) = I$ a.e. $e^{i\theta} \in \mathbf{T}$, and it follows that $\mathcal{P}(e^{i\theta})$ is boundedly invertible for almost all $e^{i\theta} \in \mathbf{T}$.

It remains to recognize the function $\mathcal{P}(e^{i\theta})$. By the identification of the I/O stable I/O maps with $H^\infty(\mathbf{T};\mathcal{L}(U;Y))$ functions, and the Parseval identity, we have for all $\tilde{u}, \tilde{w} \in \ell^2(\mathbf{Z};U)$

$$\langle \mathcal{D}^* J \mathcal{D}\tilde{u}, \tilde{w}\rangle_{\ell^2(\mathbf{Z};U)} = \frac{1}{2\pi}\int_{\mathbf{T}} \left\langle \mathcal{D}(e^{i\theta})^* J \mathcal{D}(e^{i\theta})\tilde{u}(e^{i\theta}), \tilde{w}(e^{i\theta})\right\rangle_U \, d\theta \tag{3.18}$$

where $\tilde{u}(e^{i\theta}), \tilde{w}(e^{i\theta}) \in L^2(\mathbf{T};U)$ are the Fourier transforms of $\tilde{u}$ and $\tilde{w}$, and the Popov function $e^{i\theta} \mapsto \mathcal{D}(e^{i\theta})^* J \mathcal{D}(e^{i\theta})$ is the Popov function, belonging to $L^\infty(\mathbf{T};\mathcal{L}(U))$. We conclude that $\mathcal{P}(e^{i\theta}) = \mathcal{D}(e^{i\theta})^* J \mathcal{D}(e^{i\theta})$ a.e. $e^{i\theta} \in \mathbf{T}$. Thus the values of the Popov function are bounded, boundedly invertible operators almost everywhere on $\mathbf{T}$.

It remains to conclude the coercivity of $\mathcal{D}(e^{i\theta})$. Let $e^{i\theta} \in \mathbf{T}$ be such that $\mathcal{D}(e^{i\theta})^* J \mathcal{D}(e^{i\theta})$ is bounded and boundedly invertible. Assume for contradiction that there is a sequence $\{u_j\} \subset U$, $||u_j||_U = 1$, such that $\mathcal{D}(e^{i\theta})u_j \to 0$ as $j \to 0$. Because $\mathcal{D}(e^{i\theta})$ is bounded, so is $\mathcal{D}(e^{i\theta})^* J$. But then $\mathcal{D}(e^{i\theta})^* J \mathcal{D}(e^{i\theta})u_j \to 0$ as $j \to 0$. This is a contradiction against the choice of $e^{i\theta}$ from a set of full measure. The proof is complete. □

Lemma 134. *Let $J \in \mathcal{L}(Y)$ be a cost operator. Let $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ be an output stable, I/O stable and J-coercive DLS. Assume that the input operator $B \in \mathcal{L}(U;H)$ is Hilbert–Schmidt, and the feed-through operator $D \in \mathcal{L}(U;Y)$ is boundedly invertible.*

(i) Then the values of the boundary trace $\mathcal{D}_\phi(e^{i\theta})$ are bounded, boundedly invertible operators for almost all $e^{i\theta} \in \mathbf{T}$.

(ii) Assume, in addition, that J is nonnegative and boundedly invertible, and $\mathcal{D}_\phi = \mathcal{N}\mathcal{X}$ is a (J,S)-inner-outer factorization. Then the outer factor $\mathcal{X}$ is outer with a bounded inverse, the sensitivity operator S is boundedly invertible, and the normalized inner factor $J^{\frac{1}{2}}\mathcal{N}S^{-\frac{1}{2}}$ is inner from both sides.

Proof. Because $\dim \overline{\operatorname{range}(B)} \leq \dim U$, there is a partial isometry $V : H \to U$ whose initial space is $\overline{\operatorname{range}(B)}$. Because the input operator B is Hilbert–Schmidt, so is the operator $VB \in \mathcal{L}(U)$, see Proposition 129. By Proposition 132, we have the factorization $VB = B_1' K$ such that $B_1 \in HS(U)$ and $K \in \mathcal{LC}(U)$. But then $B = B_1 K$, where $B_1 := V^* B_1' \in \mathcal{L}(U;H)$ is a Hilbert–Schmidt operator.

Define the DLS $\phi' := \left(\begin{smallmatrix} A & B_1 \\ D^{-1}C & 0 \end{smallmatrix}\right)$. Because ϕ is output stable, and $\mathcal{C}_{\phi'} = D^{-1}\mathcal{C}_\phi$, the DLS ϕ' is output stable, too. Because the input operator B_1 is Hilbert–Schmidt, Corollary 131 implies that $\mathcal{D}_{\phi'}(z) \in H^2(\mathbf{D};\mathcal{L}(U))$. Now for all $z \in \mathbf{D}$,

$$\mathcal{D}_\phi(z) = D\left(I + zD^{-1}C(I - zA)^{-1}B_1 K\right) = D\left(I + \mathcal{D}_{\phi'}(z)K\right)$$

and

$$\Theta(z) := D^{-1}\mathcal{D}_\phi(z) = I + \kappa(z)K, \tag{3.19}$$

where $\kappa(z) := \mathcal{D}_{\phi'}(z)$ and $K \in \mathcal{LC}(U)$. By Proposition 133 and the J-coercivity assumption, the nontangential limit $\mathcal{D}_\phi(e^{i\theta}) \in \mathcal{L}(U;Y)$ is coercive a.e. $e^{i\theta} \in \mathbf{T}$. In particular, $\ker\left(\Theta(e^{i\theta})\right) = \ker\left(\mathcal{D}_\phi(e^{i\theta})\right) = \{0\}$ a.e. $e^{i\theta} \in \mathbf{T}$. By claim (ii) of Lemma 125 and equation (3.19), the operator $\Theta(e^{i\theta}) \in \mathcal{L}(U)$ is boundedly invertible a.e. $e^{i\theta} \in \mathbf{T}$. This completes the proof of claim (i).

We proceed to prove claim (ii). Because $\mathcal{D}_\phi = \mathcal{N}\mathcal{X}$ is a (J,S)-inner-outer factorization and Φ is assumed to be J-coercive, it follows that $\mathcal{X}$ is outer with a bounded inverse and the sensitivity operator S is boundedly invertible, by Corollary 85. Because all the I/O maps $\mathcal{D}_\phi$, $\mathcal{N}$ and $\mathcal{X}$ are I/O stable, we obtain the factorization of the boundary traces $\mathcal{D}_\phi(e^{i\theta}) = \mathcal{N}(e^{i\theta})\mathcal{X}(e^{i\theta})$ a.e. $e^{i\theta} \in T$. By claim (iii) of Proposition 127, $\mathcal{X}(e^{i\theta})$ is boundedly invertible a.e. $e^{i\theta} \in \mathbf{T}$. From claim (i) we conclude that $\mathcal{D}_\phi(e^{i\theta})$ is boundedly invertible a.e. $e^{i\theta} \in \mathbf{T}$. We conclude that $\mathcal{N}(e^{i\theta}) = \mathcal{D}_\phi(e^{i\theta})\mathcal{X}(e^{i\theta})^{-1}$ is a boundedly invertible operator a.e. $e^{i\theta} \in \mathbf{T}$.

From the (J,S)-inner-outer factorization $\mathcal{D}_\phi = \mathcal{N}\mathcal{X}$ we conclude the spectral factorization $\mathcal{D}_\phi^* J \mathcal{D}_\phi = \mathcal{X}^* S \mathcal{X}$. Because $\mathcal{X}$ has a bounded causal inverse $\mathcal{X}^{-1}$, we have $S = (\mathcal{X}^{-1})^* \mathcal{D}_\phi^* J \mathcal{D}_\phi \mathcal{X}^{-1}$ where S is regarded as a static operator on $\ell^2(\mathbf{Z};U)$. Because J is nonnegative, it follows that $\mathcal{D}_\phi^* J \mathcal{D}_\phi \geq 0$ and $S \geq 0$ as a static operator on $\ell^2(\mathbf{Z};U)$. But then, $S \geq 0$ also as a self-adjoint element of $\mathcal{L}(U)$. Thus the square root $S^{-\frac{1}{2}}$ is uniquely defined as a nonnegative operator. We conclude that the normalized I/O map $J^{\frac{1}{2}}\mathcal{N}S^{-\frac{1}{2}}$ is (I,I)-inner, and thus its transfer function is inner from the left. Because both $J^{\frac{1}{2}}$ and $S^{-\frac{1}{2}}$ are boundedly invertible, and $\mathcal{N}(e^{i\theta})$ is boundedly invertible a.e. $e^{i\theta} \in \mathbf{T}$, we conclude that $J^{\frac{1}{2}}\mathcal{N}(e^{i\theta})S^{-\frac{1}{2}}$ is a boundedly invertible isometry, i.e. unitary a.e. $e^{i\theta} \in \mathbf{T}$. Thus $J^{\frac{1}{2}}\mathcal{N}S^{-\frac{1}{2}}$ is inner from both sides. This completes the proof. $\square$

An an important application, we consider the noncausal shift-invariant inverse of the I/O map. This result is used in Lemma 145.

Proposition 135. *Let $J \in \mathcal{L}(Y)$ be cost operator. Let $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ be an output stable, I/O stable and J-coercive DLS, with input space U and output space Y. Then*

(i) both the (extended topological) I/O map $\mathcal{D}_\phi : \ell^2(\mathbf{Z};U) \to \ell^2(\mathbf{Z};Y)$ and the Toeplitz operator $\mathcal{D}_\phi\bar{\pi}_+ : \ell^2(\mathbf{Z}_+;U) \to \ell^2(\mathbf{Z}_+;Y)$ are coercive.

(ii) Assume, in addition, that U and Y are separable, the input operator $B \in \mathcal{L}(U;H)$ is Hilbert–Schmidt, and the feed-through operator $D \in \mathcal{L}(U;Y)$ is boundedly invertible. Then $\overline{\mathrm{range}\,(\mathcal{D}_\phi)} = \mathrm{range}\,(\mathcal{D}_\phi) = \ell^2(\mathbf{Z};Y)$. In this case, the inverse operator $\mathcal{D}_\phi^{-1} : \ell^2(\mathbf{Z};Y) \to \ell^2(\mathbf{Z};U)$ exists, and it is bounded and shift-invariant. ($\mathcal{D}_\phi^{-1}$ is not causal, unless $\mathcal{D}_\phi$ is outer with a bounded inverse.)

Proof. The claim about the Toeplitz operator $\mathcal{D}\bar{\pi}_+$ is Proposition 69. It follows by a density argument from the shift-invariance, causality and boundedness of the (extended topological) I/O map $\mathcal{D} : \ell^2(\mathbf{Z};U) \to \ell^2(\mathbf{Z};Y)$ that $\mathcal{D}_\phi\bar{\pi}_+$ and $\mathcal{D}$ are simultaneously coercive in the indicated spaces.

Consider now claim (ii). Because of the separability of the spaces U and Y, we can study the problem in terms of multiplication operators by the boundary traces. Because $\mathcal{D}_\phi\bar{\pi}_+$ is coercive, it follows that the Popov operator $\bar{\pi}_+\mathcal{D}_\phi^*\mathcal{D}_\phi\bar{\pi}_+ \geq \epsilon\bar{\pi}_+$ for some $\epsilon > 0$. Now Corollary 118 implies that we have the factorization $\mathcal{D}_\phi = \mathcal{N}'\mathcal{X}'$, where $\mathcal{N}'$ is (I,S)-inner, $\mathcal{X}'$ is outer with a bounded inverse and $S \in \mathcal{L}(U)$ has a bounded inverse. By normalizing the outer factor properly, we may assume that $S = I$, see Proposition 83. In terms of the boundary traces, this means

$$\mathcal{D}_\phi(e^{i\theta}) = \mathcal{N}'(e^{i\theta})\mathcal{X}'(e^{i\theta}) \tag{3.20}$$

a.e. $e^{i\theta} \in \mathbf{T}$. By Definition 120 and Proposition 121, the boundary trace of the inner (from the left) factor $\mathcal{N}'(e^{i\theta})$ is $\mathcal{L}(U;Y)$-valued isometry a.e. $e^{i\theta} \in \mathbf{T}$. By Proposition 127, the boundary trace of the outer factor $\mathcal{X}'(e^{i\theta})$ has a bounded inverse a.e. $e^{i\theta} \in \mathbf{T}$, and $\mathcal{X}'(e^{i\theta})^{-1} \in H^\infty(\mathbf{T};\mathcal{L}(U))$. By claim (i) of Lemma 134, the boundary trace $\mathcal{D}_\phi(e^{i\theta})$ is bounded and boundedly invertible a.e. $e^{i\theta} \in \mathbf{T}$. Because $\mathcal{N}'(e^{i\theta}) = \mathcal{D}_\phi(e^{i\theta})\mathcal{X}'(e^{i\theta})^{-1}$, we conclude that $\mathcal{N}'(e^{i\theta})$ is a boundedly invertible isometry, and thus an unitary operator in $\mathcal{L}(U;Y)$ for almost all $e^{i\theta} \in \mathbf{T}$. It now follows that $\mathcal{N}(e^{i\theta})$ is inner from both sides. This means that $\mathcal{N}'(e^{i\theta})\mathcal{N}'(e^{i\theta})^* = I$ a.e. $e^{i\theta} \in \mathbf{T}$. Also, $\mathcal{N}'(e^{i\theta})^* \in L^\infty(\mathbf{T};\mathcal{L}(Y;U))$ because it is trivially weakly measurable.

Now we can attack the claim about the density of range$(\mathcal{D}_\phi)$. Let $\tilde{y}(e^{i\theta}) \in L^2(\mathbf{T};Y)$ be arbitrary. Define $\tilde{w}(e^{i\theta}) := \mathcal{N}'(e^{i\theta})^*\tilde{y}(e^{i\theta})$ away from a set of measure zero. Because $\mathcal{N}'(e^{i\theta})^* \in L^\infty(\mathbf{T};\mathcal{L}(Y;U))$ and $\tilde{y}(e^{i\theta}) \in L^2(\mathbf{T};Y)$, [27, part (a) of Theorem 1.1, Chapter IX] implies that $\tilde{w}(e^{i\theta}) \in L^2(\mathbf{T};U)$. Similarly, $\tilde{u}(e^{i\theta}) := \mathcal{X}'(e^{i\theta})^{-1}\tilde{w}(e^{i\theta}) \in L^2(\mathbf{T};U)$. But now,

$$\mathcal{D}_\phi(e^{i\theta})\tilde{u}(e^{i\theta}) = \mathcal{N}'(e^{i\theta})\mathcal{X}'(e^{i\theta})\mathcal{X}'(e^{i\theta})^{-1}\mathcal{N}'(e^{i\theta})^*\tilde{y}(e^{i\theta}) = \tilde{y}(e^{i\theta})$$

almost everywhere on $\mathbf{T}$. Because $\tilde{y}(e^{i\theta})$ is arbitrary, this means in the time domain that range$(\mathcal{D}_\phi) = \ell^2(\mathbf{Z};Y)$ because the Fourier transform is an isometric isomorphism. We conclude that $\mathcal{D}_\phi : \ell^2(\mathbf{Z};U) \to \ell^2(\mathbf{Z};Y)$ is a bounded coercive operator with a full range, i.e. a bounded bijection. But then the bounded inverse operator $\mathcal{D}_\phi^{-1}$ exists. It is a triviality that such an inverse is shift-invariant. □

3.5 Factorization of the truncated Popov operator

Our main interest is in the H^∞DARE, associated to an output stable and I/O stable DLS Φ. As we have seen, this stability requirement makes some solution of DARE more interesting than others. In Section 3.2 we have sorted out the more interesting solutions from the less interesting.

In this section, we consider additional conditions that make the spectral DLS ϕ_P is either output stable, or I/O stable, or both, for a particular $P \in Ric(\Phi, J))$. More specifically, we introduce additional assumptions that allow us to conclude

$$P \in Ric(\Phi, J) \Rightarrow P \in ric(\Phi, J),$$

when Φ is known to be output stable and I/O stable. The basic tool to obtain the most general of these results is the factorization of the truncated Popov operator, as given in Lemma 138.

Let us first discuss the trivial cases. If Φ itself is power stable, then so are ϕ_P for all $P \in Ric(\Phi, J)$ because they have a common semigroup generator A. More generally, if the Wiener class type condition $\sum ||A^j B|| < \infty$ holds, then $\mathcal{D}_{\phi_P}$ is I/O stable for all $P \in Ric(\Phi, J)$. Now the common input structure (i.e. the common operators A and B) determine the I/O stability of both the systems Φ and ϕ_P. In the case when Φ is output stable and I/O stable, it is easy to see that ϕ_P is I/O stable (output stable) if and only if $\phi' = \left(\begin{smallmatrix} A & B \\ B^* P & 0 \end{smallmatrix} \right)$ is I/O stable (output stable, respectively) but this is just a restatement that is impossible to use in practice.

More general results are obtained by Liapunov type methods that require some type of nonnegativity, either in the cost operator J, the Popov operator $\mathcal{D}^* J \mathcal{D}$, or indicator Λ_P of the solution P. We start with discussing the case of output stability.

Proposition 136. *Let $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix} \right]$ be an output stable DLS and $J \in \mathcal{L}(Y)$ be a self-adjoint operator. Let $P \in Ric(\Phi, J)$ such that $\Lambda_P > 0$. Then*

(i) ϕ_P is output stable if and only if the strong limit $L_{A,P} := \mathrm{s} - \lim_{j \to \infty} A^{*j} P A^j$ exists as a bounded operator. When this equivalence holds, we have

$$L_{A,P} - P = \mathcal{C}_{\phi_P}^* \Lambda_P \mathcal{C}_{\phi_P} - \mathcal{C}^* J \mathcal{C}. \tag{3.21}$$

(ii) In particular, if A is strongly stable, then ϕ_P is output stable.

(iii) If $P \geq 0$ and $L_{A,P} = 0$, we have

$$\mathcal{C}^* J\mathcal{C} \geq \mathcal{C}^* J\mathcal{C} - P = \mathcal{C}^*_{\phi_P} \Lambda_P \mathcal{C}_{\phi_P}$$

Proof. We prove one direction of claim (i). Assume that $\Lambda_P > 0$ and $L_{A,P} = \mathrm{s} - \lim_{j\to\infty} A^{*j}PA^j$ exists. We can iterate on the Riccati equation (3.3) and obtain for all $j \geq 0$

$$A^{*(j+1)}PA^{j+1} - A^{*j}PA^j = A^{*j}K_P^*\Lambda_P K_P A^j - A^{*j}C^*JCA^j.$$

Telescope summing this up to $n \geq 0$ gives for all $x_0 \in H$

$$\begin{aligned} &\langle x_0, (A^{*n}PA^n - P)x_0\rangle \\ &= \left\langle x_0, \sum_{j=0}^{n-1} A^{*j}K_P^*\Lambda_P K_P A^j x_0 \right\rangle - \left\langle x_0, \sum_{j=0}^{n-1} A^{*j}C^*JCA^j x_0 \right\rangle \end{aligned} \tag{3.22}$$

By assumption, the left hand side of the previous equation converges to a finite limit $\langle x_0, (L_{A,P} - P)x_0\rangle$. On the right hand side, we have

$$\begin{aligned} &\left\langle x_0, \sum_{j=0}^{n-1} A^{*j}C^*JCA^j x_0 \right\rangle = \sum_{j=0}^{n-1} \left\langle CA^j x_0, JCA^j x_0 \right\rangle \\ &= \left\langle \pi_{[0,n-1]}\mathcal{C}x_0, J\pi_{[0,n-1]}\mathcal{C}x_0 \right\rangle_{\ell^2(\mathbf{Z}_+;Y)} \end{aligned}$$

which converges absolutely to a bounded limit $\langle x_0, \mathcal{C}^*J\mathcal{C}x_0\rangle$ as $n \to \infty$, by the assumed output stability of Φ.

Because everything else in (3.22) converges to a finite limit and $\Lambda_P > 0$, it follows that remaining term

$$\begin{aligned} &\left\langle x_0, \sum_{j=0}^{n-1} A^{*j}K_P^*\Lambda_P K_P A^j x_0 \right\rangle = \sum_{j=0}^{n-1} \left\langle K_P A^j x_0, \Lambda_P K_P A^j x_0 \right\rangle \\ &= ||\Lambda_P^{\frac{1}{2}}\pi_{[0,n-1]}\mathcal{C}_{\phi_P}x_0\}||^2_{\ell^2(\mathbf{Z}_+;U)} \end{aligned}$$

converges (increases) to a finite limit, equaling $||\{\Lambda_P^{\frac{1}{2}}K_P A^j x_0\}_{j\geq 0}||^2_{\ell^2(\mathbf{Z}_+;U)}$, as $n \to \infty$. Because Λ_P^{-1} is bounded and $x_0 \in H$ arbitrary, this is equivalent to the output stability of ϕ_P. This completes the proof of the first direction. The converse part in contained in the proof of Proposition 110 where also equation (3.21) is given. Claim (ii) follows trivially from the fact that strongly stable A implies that the strong limit operator $L_{A,P}$ always exists and equals 0. Claim (iii) is a trivial consequence of equation (3.21). ☐

Corollary 137. *Let $J \in \mathcal{L}(Y)$ be self-adjoint. Assume that ϕ is a I/O stable and output stable DLS, such that $\overline{\mathrm{range}\,(\mathcal{B})} = H$. Then $ric_{uw}(\phi, J) = ric_0(\phi, J)$.*

Proof. Trivially $ric_0(\phi, J) \subset ric_{uw}(\phi, J)$, and the converse inclusion is shown below. Because $P \in ric_{uw}(\phi, J)$, both ϕ and ϕ_P are output stable. We have for all $j \geq 1$

$$A^{*j} P A^j - P = \mathcal{C}^*_{\phi_P} \Lambda_P \pi_{[0,j-1]} \mathcal{C}_{\phi_P} - \mathcal{C}^* J \pi_{[0,j-1]} \mathcal{C},$$

as in equation (3.6) of Proposition 110. By the output stabilities, both $\pi_{[0,j-1]}\mathcal{C} \to \mathcal{C}$ and $\pi_{[0,j-1]}\mathcal{C}_{\phi_P} \to \mathcal{C}_{\phi_P}$ strongly. It follows that $L_{A,P}$ exists and $P \in Ric_{00}(\phi, J)$. Now claim (iv) of Proposition 109, together with the assumed approximate controllability, shows that $P \in Ric_0(\phi, J)$. $\square$

We proceed to study the I/O stability of the spectral DLS ϕ_P. For solutions such that $\lim_{j\to\infty} \left\langle P\mathcal{B}\tau^{*j}\tilde{u}, \mathcal{B}\tau^{*j}\tilde{u}\right\rangle = 0$ for all $\tilde{u} \in \ell^2(\mathbf{Z}_+; U)$, a necessary and sufficient condition for ϕ_P to be I/O stable is the following speed estimate

$$\sum_{j\geq 0} | \langle x_j, P x_j \rangle - \langle x_{j+1}, P x_{j+1} \rangle | < \infty$$

for all trajectories $x_j = \mathcal{B}\tau^{*j}\tilde{u}$ where $\tilde{u} \in \ell^2(\mathbf{Z}_+; U)$ is arbitrary, see Proposition 104. Unfortunately, this condition is not practical for our purposes.

We continue by giving an unsuccessful attempt that, however, reveals something about the nature of the problem. Assume that Φ is input stable and I/O stable, and $J \geq 0$. Suppose we already know ϕ_P to be output stable. Claim (iii) of Proposition 136 implies that

$$\infty > ||\pi_- \mathcal{D}^* J \mathcal{D} \pi_-|| \geq \mathcal{B}^* \mathcal{C}^* J \mathcal{C} \mathcal{B} \geq \mathcal{B}^*_{\phi_P} \mathcal{C}^*_{\phi_P} \Lambda_P \mathcal{C}_{\phi_P} \mathcal{B}_{\phi_P},$$

if $P \geq 0$ and $L_{A,P} = 0$, because $\mathcal{B}_{\phi_P} = \mathcal{B}$. So the Hankel operator $\mathcal{C}_{\phi_P}\mathcal{B}_{\phi_P} = \bar{\pi}_+ \mathcal{D}_{\phi_P} \pi_-$ is bounded in $\ell^2(\mathbf{Z}; U)$, but this does not allow us directly conclude the I/O stability of $\mathcal{D}_{\phi_P}$.

We are not far from having ϕ_P I/O stable, provided that we have the *a priori* knowledge that $\mathcal{D}_{\phi_P}(z) \in N(\mathbf{D}; \mathcal{L}(U))$ so that the nontangential limit function $\mathcal{D}_{\phi_P}(e^{i\theta})$ makes sense. More precisely, denote by Γ the bounded Hankel operator $\mathcal{C}_{\phi_P}\mathcal{B}_{\phi_P}$, and assume, for simplicity that everything is complex-valued, i.e. $U = Y = \mathbf{C}$. By [27, Theorem 3.3, Chapter IX], $\Gamma = \Gamma(Q)$, where $Q(e^{i\theta}) \in L^\infty(\mathbf{T}; d\theta)$ is a *bounded* symbol for Γ (we have omitted one unitary flip operator in the definition of the Hankel operator but this is immaterial). Write $Q(e^{i\theta})$ as the Fourier series $Q(e^{i\theta}) \sim \sum_{j\in\mathbf{Z}} q_j\, e^{ij\theta}$. Now $q_j = -K_P A^{j-1} B$ for $j \geq 1$ because $\mathcal{D}_{\phi_P}(e^{i\theta})$ is also a (possibly unbounded) symbol for Γ. It is well known that $L^\infty(\mathbf{T}; d\theta) \subset L^p(\mathbf{T}; d\theta)$ for all $1 < p < \infty$, and that the Szegö projection $\Pi : L^p \to H^p$ (zeroing the negatively indexed Fourier coefficients) is bounded for $1 < p < \infty$. But now $\mathcal{D}_{\phi_P}(e^{i\theta}) = \Pi Q(e^{i\theta}) \in \cap_{1<p<\infty} H^p(\mathbf{T}; \mathbf{C})$. Unfortunately, the inclusion $H^\infty(\mathbf{T}; \mathbf{C}) \subset \cap_{1<p<\infty} H^p(\mathbf{T}; \mathbf{C})$ is strict, and we cannot conclude $\mathcal{D}_{\phi_P}(e^{i\theta}) \in H^\infty(\mathbf{T}; \mathbf{C})$.

After one impractical and another unsuccessful attempt, we approach the I/O stability problem of ϕ_P from a third direction. We begin with factorization lemma of the truncated Popov operator for strongly H^2 stable DLSs. Recall that impulse response operator $\mathcal{D}\pi_0 : U \to \ell^2(\mathbf{Z}_+; Y)$ of a strongly H^2 stable DLS is bounded, by definition. It then immediately follows, by the shift invariance, that the truncated Toeplitz operators $\mathcal{D}\pi_{[0,m]}$ are bounded, for all $m \geq 0$.

Lemma 138. *Let $J \in \mathcal{L}(Y)$ be a self-adjoint cost operator, and $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ strongly H^2 stable. Let $P \in Ric_{uw}(\Phi, J)$; i.e.*

$$\left\langle PA^j x_0, A^j x_0 \right\rangle \to 0 \quad \textit{for all} \quad x_0 \in \operatorname{range}(\mathcal{B}) \tag{3.23}$$

as $j \to \infty$. Assume also that the spectral DLS ϕ_P is strongly H^2 stable.

Then $\mathcal{D}\pi_{[0,m]} : \ell^2(\mathbf{Z}; U) \to \ell^2(\mathbf{Z}; Y)$ and $\mathcal{D}_{\phi_P}\pi_{[0,m]} : \ell^2(\mathbf{Z}; U) \to \ell^2(\mathbf{Z}; U)$ are bounded, and the truncated Popov operator has the factorization

$$(\mathcal{D}\pi_{[0,m]})^* J \mathcal{D}\pi_{[0,m]} = (\mathcal{D}_{\phi_P}\pi_{[0,m]})^* \Lambda_P \mathcal{D}_{\phi_P}\pi_{[0,m]} \tag{3.24}$$

for all $m \geq 0$.

Proof. Let $x_0 \in H$ and $\{u_j\}_{j\geq 0} = \tilde{u} \in \ell^2(\mathbf{Z}_+; U)$ be arbitrary. Denote $x_j = x_j(x_0, \tilde{u}) = A^j x_0 + \mathcal{B}\tau^{*j}\tilde{u}$ the trajectory of Φ with this given initial state and input. We have in claim (i) of Proposition 97 for all $n > 0$

(3.25)

$$\begin{aligned}&\langle Px_0, x_0\rangle - \langle Px_n, x_n\rangle \\ &= \sum_{j=0}^{n-1} \langle J(Cx_j + Du_j), Cx_j + Du_j\rangle - \sum_{j=0}^{n-1} \langle \Lambda_P(-K_P x_j + u_j), -K_P x_j + u_j\rangle\rangle.\end{aligned}$$

Consider now the special case when the input is otherwise arbitrary, but of form $\tilde{u} = \pi_{[0,m]}\tilde{u}$, for $m \geq 0$. Then, for $n > m$,

$$\begin{aligned}x_n &= x_n(x_0, \pi_{[0,m]}\tilde{u}) = A^{n-m-1} \cdot x_{m+1}(x_0, \pi_{[0,m]}\tilde{u}), \\ x_{m+1}(x_0, \pi_{[0,m]}\tilde{u}) &= A^{m+1}x_0 + \mathcal{B}\tau^{*(m+1)}\pi_{[0,m]}\tilde{u}.\end{aligned}$$

Let $x_0 = 0$. Because now $x_{m+1}(0, \pi_{[0,m]}\tilde{u}) \in \operatorname{range}(\mathcal{B})$, it follows from the residual cost condition (3.23) that $\langle Px_n, x_n\rangle \to 0$ as $n \to \infty$. It follows that the left hand side of (3.25) vanishes as $n \to \infty$.

We must now consider the right hand side of (3.25). Because both the operators $\mathcal{D}\pi_{[0,m]}$ and $\mathcal{D}_{\phi_P}\pi_{[0,m]}$ are bounded, by the H^2 stability assumption of ϕ_P, it is not difficult to see that the limit of the left hand side of (3.25) is actually

$$\left\langle J\mathcal{D}\pi_{[0,m]}\tilde{u}, \mathcal{D}\pi_{[0,m]}\tilde{u}\right\rangle_{\ell^2(\mathbf{Z}_+;Y)} - \left\langle \Lambda_P \mathcal{D}_{\phi_P}\pi_{[0,m]}\tilde{u}, \mathcal{D}_{\phi_P}\pi_{[0,m]}\tilde{u}\right\rangle_{\ell^2(\mathbf{Z}_+;Y)},$$

as $n \to \infty$. Adjoining this gives

$$\left\langle \tilde{u}, \left((\mathcal{D}\pi_{[0,m]})^* J \mathcal{D}\pi_{[0,m]} - (\mathcal{D}_{\phi_P}\pi_{[0,m]})^* \Lambda_P \mathcal{D}_{\phi_P}\pi_{[0,m]} \right) \tilde{u} \right\rangle_{\ell^2(\mathbf{Z}_+;Y)} = 0$$

for all $\tilde{u} \in \ell^2(\mathbf{Z}; U)$. Now an application of [79, Theorem 12.7] completes the proof. $\square$

The result of the previous lemma can be translated to the frequency plane by Corollary 131, provided that the input operator is Hilbert–Schmidt. With this additional structure, further conclusions can be drawn.

Proposition 139. *Let J be a self-adjoint cost operator. Let $\Phi = \left[\begin{smallmatrix} \mathcal{A}^j & \mathcal{B}_{\mathcal{T}}^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix} \right]$ output stable, such that the input operator B is Hilbert–Schmidt and the input space U is separable. Let $P \in Ric_{uw}(\Phi, J)$ be such that ϕ_P is output stable.*

Then the adjoints of the boundary traces $\mathcal{D}(e^{i\theta})^*$ and $\mathcal{D}_{\phi_P}(e^{i\theta})^*$ exists a.e. $e^{i\theta} \in \mathbf{T}$, and belong to $L^2(\mathbf{T}; \mathcal{L}(U;Y))$, $L^2(\mathbf{T}; \mathcal{L}(U))$, respectively. Both the self-adjoint operator-valued functions

$$\begin{aligned} &\mathbf{T} \ni e^{i\theta} \mapsto \mathcal{D}(e^{i\theta})^* J \mathcal{D}(e^{i\theta}) \in \mathcal{L}(U), \quad \textit{and} \\ &\mathbf{T} \ni e^{i\theta} \mapsto \mathcal{D}_{\phi_P}(e^{i\theta})^* \Lambda_P \mathcal{D}_{\phi_P}(e^{i\theta}) \in \mathcal{L}(U) \end{aligned}$$

are in $L^1(\mathbf{T}; \mathcal{L}(U))$. We have the factorization

$$\mathcal{D}(e^{i\theta})^* J \mathcal{D}(e^{i\theta}) = \mathcal{D}_{\phi_P}(e^{i\theta})^* \Lambda_P \mathcal{D}_{\phi_P}(e^{i\theta}) \quad \textit{a.e.} \quad e^{i\theta} \in \mathbf{T}.$$

Proof. Recall that the output stability implies strong H^2 stability. So we can apply Lemma 138. Equation (3.24) implies for all $\tilde{u}_1, \tilde{u}_2 \in \ell^2(\mathbf{Z}_+; U)$

$$\left\langle \mathcal{D}\pi_{[0,m]}\tilde{u}_1, J\mathcal{D}\pi_{[0,m]}\tilde{u}_2 \right\rangle_{\ell^2(\mathbf{Z}_+;Y)} = \left\langle \mathcal{D}_{\phi_P}\pi_{[0,m]}\tilde{u}_1, \Lambda_P \mathcal{D}_{\phi_P}\pi_{[0,m]}\tilde{u}_2 \right\rangle_{\ell^2(\mathbf{Z}_+;Y)}.$$

Because both Φ and ϕ_P are output stable, the transfer functions $\mathcal{D}(z)$ and $\mathcal{D}_{\phi_P}(z)$ are analytic in the whole of $\mathbf{D}$, by Proposition 57. We have also $\mathcal{D}(z)\tilde{p}(z) \in H^2(\mathbf{D}; Y)$, $\mathcal{D}_{\phi_P}(z)\tilde{p}(z) \in H^2(\mathbf{D}; U)$ for all U-valued trigonometric polynomials $p(z) \in H^\infty(\mathbf{D}; U)$. Now we can put the factorization in form

$$\langle \mathcal{D}(z)p_1(z), J\mathcal{D}(z)p_2(z) \rangle_{H^2(\mathbf{D};Y)} = \langle \mathcal{D}_{\phi_P}(z)p_1(z), \Lambda_P \mathcal{D}_{\phi_P}(z)p_2(z) \rangle_{H^2(\mathbf{D};U)}$$

where $p_1(z), p_2(z)$ are polynomials as above. This is as far as we get without assuming that B is Hilbert–Schmidt.

Because B is Hilbert–Schmidt, we can state the factorization in terms of the boundary traces $\mathcal{D}(e^{i\theta}) \in H^2(\mathbf{T}; \mathcal{L}(U;Y))$ and $\mathcal{D}_{\phi_P}(e^{i\theta}) \in H^2(\mathbf{T}; \mathcal{L}(U))$, by

Corollary 131. By choosing the trigonometric polynomials $p_1(e^{i\theta}) = e^{ip_1\theta}u_1$ and $p_2(e^{i\theta}) = e^{ip_2\theta}u_2$, $p_1, p_2 \in \mathbf{Z}$, $u_1, u_2 \in U$, we obtain

$$\begin{aligned}
&\frac{1}{2\pi}\int_0^{2\pi} \left\langle u_1, \mathcal{D}(e^{i\theta})^* J\mathcal{D}(e^{i\theta})e^{ip\theta}u_2\right\rangle_U \, d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} \left\langle \mathcal{D}(e^{i\theta})e^{ip_1\theta}u_1, J\mathcal{D}(e^{i\theta})e^{ip_2\theta}u_2\right\rangle_Y \, d\theta \\
&= \left\langle \mathcal{D}(e^{i\theta})e^{ip_1\theta}u_1, J\mathcal{D}(e^{i\theta})e^{ip_2\theta}u_2\right\rangle_{H^2(\mathbf{T};Y)} \\
&= \left\langle \mathcal{D}_{\phi_P}(e^{i\theta})e^{ip_1\theta}u_1, \Lambda_P\mathcal{D}_{\phi_P}(e^{i\theta})e^{ip_2\theta}u_2\right\rangle_{H^2(\mathbf{T};U)} \\
&= \frac{1}{2\pi}\int_0^{2\pi} \left\langle \mathcal{D}(e^{i\theta})e^{ip_1\theta}u_1, J\mathcal{D}(e^{i\theta})e^{ip_2\theta}u_2\right\rangle_Y \, d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} \left\langle u_1, \mathcal{D}_{\phi_P}(e^{i\theta})^*\Lambda_P\mathcal{D}_{\phi_P}(e^{i\theta})e^{ip\theta}u_2\right\rangle_U \, d\theta,
\end{aligned}$$

where $p = p_2 - p_1$. Let us stop for a moment to see that previous is true integration theoretically. The functions $\mathbf{T} \ni e^{i\theta} \mapsto \mathcal{D}(e^{i\theta})^* \in \mathcal{L}(Y;U)$, $\mathbf{T} \ni e^{i\theta} \mapsto \mathcal{D}_{\phi_P}(e^{i\theta})^* \in \mathcal{L}(U)$ are weakly measurable and also in the respective L^2-spaces, by a trivial argument involving adjoining. Now the products $\mathcal{D}(e^{i\theta})^* J\mathcal{D}(e^{i\theta})$ and $\mathcal{D}_{\phi_P}(e^{i\theta})^*\Lambda_P\mathcal{D}_{\phi_P}(e^{i\theta})$ are weakly measurable, and they both are in $L^1(\mathbf{T};U)$, by the Hölder inequality; some of this detail and further references have been discussed immediately after Definition 59.

We can now calculate the weak Fourier coefficients of the difference of these two functions (which lies in $L^1(\mathbf{T};\mathcal{L}(U))$) as follows:

$$\begin{aligned}
&\left\langle u_1, \left(\int_0^{2\pi} \left[\mathcal{D}(e^{i\theta})^* J\mathcal{D}(e^{i\theta}) - \mathcal{D}_{\phi_P}(e^{i\theta})^*\Lambda_P\mathcal{D}_{\phi_P}(e^{i\theta})\right] e^{ip\theta}\, d\theta\right) u_2\right\rangle_U \\
&= \int_0^{2\pi} \left\langle u_1, \left[\mathcal{D}(e^{i\theta})^* J\mathcal{D}(e^{i\theta}) - \mathcal{D}_{\phi_P}(e^{i\theta})^*\Lambda_P\mathcal{D}_{\phi_P}(e^{i\theta})\right] e^{ip\theta}u_2\right\rangle_U \, d\theta = 0
\end{aligned}$$

for all $u_1, u_2 \in U$ and $p \in \mathbf{Z}$. Proposition 63 implies that

$$\left[\mathcal{D}(e^{i\theta})^* J\mathcal{D}(e^{i\theta}) - \mathcal{D}_{\phi_P}(e^{i\theta})^*\Lambda_P\mathcal{D}_{\phi_P}(e^{i\theta})\right] u = 0,$$

for all $u \in U$ and $e^{i\theta} \in \mathbf{T} \setminus E_u$, where $mE_u = 0$. Choose a countable dense subsequence $\{u_j\} \in U$, and define the exceptional set $E := \cup_j E_{u_j}$ of measure zero. Because $\mathcal{D}(e^{i\theta})^* J\mathcal{D}(e^{i\theta}) - \mathcal{D}_{\phi_P}(e^{i\theta})^*\Lambda_P\mathcal{D}_{\phi_P}(e^{i\theta}) \in \mathcal{L}(U)$ for all $e^{i\theta} \in \mathbf{T} \setminus E'$, $mE' = 0$, we conclude now that

$$\mathcal{D}(e^{i\theta})^* J\mathcal{D}(e^{i\theta}) - \mathcal{D}_{\phi_P}(e^{i\theta})^*\Lambda_P\mathcal{D}_{\phi_P}(e^{i\theta}) = 0$$

$e^{i\theta} \in \mathbf{T} \setminus (E' \cup E)$, by the density of the sequence $\{u_j\}$. This completes the proof. $\square$

Corollary 140. *Let J be a self-adjoint cost operator. Let $\Phi = \left[\begin{smallmatrix} \mathcal{A}^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ be an output stable and I/O stable DLS. Furthermore, assume that the input operator B is Hilbert–Schmidt and the input space U is separable. Let $P \in Ric_{uw}(\Phi, J)$ be such that ϕ_P is output stable.*

If $\Lambda_P > 0$ then ϕ_P is I/O stable, and we can write $P \in ric(\Phi, J)$. Furthermore, we have the inclusion

$$\{P \in Ric_0(\Phi, J) \mid \Lambda_P > 0\} \subset ric_0(\Phi, J) \tag{3.26}$$

Proof. By Proposition 139, $\mathcal{D}(e^{i\theta})^* J \mathcal{D}(e^{i\theta}) = \mathcal{D}_{\phi_P}(e^{i\theta})^* \Lambda_P \mathcal{D}_{\phi_P}(e^{i\theta})$ a.e. $e^{i\theta} \in \mathbf{T}$. By the assumed I/O stability of Φ, $\operatorname{ess\,sup}_{e^{i\theta} \in T} ||\mathcal{D}(e^{i\theta})|| < \infty$. We conclude that $\operatorname{ess\,sup}_{e^{i\theta} \in \mathbf{T}} ||\Lambda_P^{\frac{1}{2}} \mathcal{D}_{\phi_P}(e^{i\theta})|| < \infty$. The output stability of ϕ_P and the Hilbert–Schmidt compactness of B imply that $\Lambda_P^{\frac{1}{2}} \mathcal{D}_{\phi_P}(e^{i\theta}) \in H^2(\mathbf{T}; \mathcal{L}(U))$, by Corollary 131. Now [77, Theorem 4.7A], as used in Lemma 122, implies that $\Lambda_P^{\frac{1}{2}} \mathcal{D}_{\phi_P}(e^{i\theta}) \in H^\infty(\mathbf{T}; \mathcal{L}(U))$. Because Λ_P has a bounded inverse, $\mathcal{D}_{\phi_P}(e^{i\theta}) \in H^\infty(\mathbf{T}; \mathcal{L}(U))$.

To verify inclusion (3.26), note that Proposition 136 implies that ϕ_P is output stable. Because $L_{A,P} = 0$, then $P \in Ric_{uw}(\Phi, J)$. Now the first part of this Corollary implies that ϕ_P is I/O stable, and so $P \in ric(\Phi, J)$. The proof is now complete. $\square$

A slight modification of the proof verifies also

$$\{P \in Ric_{00}(\Phi, J) \cap Ric_{uw}(\Phi, J) \mid \Lambda_P > 0\} \subset ric_{00}(\Phi, J) \cap ric_{uw}(\Phi, J) \tag{3.27}$$

under the assumptions of the previous corollary. If $\overline{\operatorname{range}(\mathcal{B})} = H$, then this reduces to inclusion (3.26), by claim (iv) of Proposition 109. We also have:

Corollary 141. *Let $J \geq 0$ be a cost operator, and $\Phi = \left[\begin{smallmatrix} \mathcal{A}^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ be an output stable and I/O stable DLS. Furthermore, assume that the input operator $B \in \mathcal{L}(U; H)$ is Hilbert–Schmidt, and the input space U is separable.*

(i) The set $ric_0(\Phi, J)$ of regular H^∞ solutions is downward complete in the sense that if $\tilde{P} \in Ric_0(\Phi, J)$, $\tilde{P} \geq 0$, then

$$\{P \in Ric(\Phi, J) \mid 0 \leq P \leq \tilde{P}\} \subset ric_0(\Phi, J).$$

(ii) In particular, if a regular critical solution $P_0^{\mathrm{crit}} \in ric_0(\phi, J)$ exists, then

$$\{P \in Ric(\Phi, J) \mid 0 \leq P \leq P_0^{\mathrm{crit}}\} \subset ric_0(\Phi, J). \tag{3.28}$$

Proof. To prove claim (i), let $\tilde{P} \in Ric_0(\Phi, J)$, $P \geq 0$ be arbitrary. But then for any $P \in Ric(\Phi, J)$ such that $0 \leq P \leq \tilde{P}$ and $x_0 \in H$ we have

$$||P^{\frac{1}{2}} A^j x_0||_H^2 = \left\langle PA^j x_0, A^j x_0 \right\rangle \leq \left\langle A^{*j} \tilde{P} A^j x_0, x_0 \right\rangle \leq ||A^{*j} \tilde{P} A^j x_0||_H \cdot ||x_0||_H,$$

which approaches zero as $j \to \infty$, because $L_{A,\tilde{P}} = 0$ by assumption. Thus $L_{A,P}$ exists and vanishes. Because $J \geq 0$, it follows that $\Lambda_P > 0$ for all nonnegative $P \in Ric(\Phi, J)$. An application of Corollary 140 proves now claim (i). The other claim (ii) is just a particular case. □

In Theorem 188, we consider the converse inclusion of formula (3.28). This gives us a full order-theoretic characterization of nonnegative regular H^∞ solutions, under the indicated technical assumptions. Another result in this direction is Lemma 191, showing that the set $ric_0(\phi, J)$ is, in a sense, an order-convex subset of $Ric(\phi, J)$.

3.6 Factorization of the Popov operator

Let Φ be an output stable and I/O stable DLS, and J a self-adjoint cost operator. In this section we show that there is a one-to-one correspondence between certain factorizations of the Popov operator $\mathcal{D}^*J\mathcal{D}$ and certain solutions of the H^∞DARE $ric(\Phi, J)$. It is worth noting that these factorizations do not depend on the nonnegativity of the cost operator J.

The factorizations of the Popov operator have a number of useful consequences. In Lemma 145 and its Corollary 146, we show that sometimes all interesting solutions of DARE have a positive indicator. Proposition 147 gives results of the $(\Lambda_P, \Lambda_{P^{\mathrm{crit}}})$-inner-outer factorization for the I/O map of the spectral DLS ϕ_P.

In Definition 68, the Popov operator was defined to be the Toeplitz operator $\bar\pi_+\mathcal{D}^*J\mathcal{D}\bar\pi_+$. We call the bounded shift-invariant (but noncausal) operator $\mathcal{D}^*J\mathcal{D}$ (the *symbol* of the Toeplitz operator $\bar\pi_+\mathcal{D}^*J\mathcal{D}\bar\pi_+$) Popov operator, too.

Theorem 142. *Let $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D}\end{smallmatrix}\right] = \left(\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right)$ be an I/O stable and output stable DLS. Let $J \in \mathcal{L}(Y)$ be a self-adjoint operator*

(i) To each solution $P \in ric_{uw}(\Phi, J)$, we can associate the following factorization of the Popov operator

$$\mathcal{D}^*J\mathcal{D} = \mathcal{D}^*_{\phi_P}\Lambda_P\mathcal{D}_{\phi_P}, \tag{3.29}$$

where ϕ_P is the spectral DLS (of Φ and J), centered at P.

(ii) Assume, in addition that $\overline{\mathrm{range}\,(\mathcal{B})} = H$. Assume that the Popov operator has a factorization of form

$$\mathcal{D}^*J\mathcal{D} = \mathcal{D}^*_{\phi'}\Lambda\mathcal{D}_{\phi'}, \tag{3.30}$$

where

$$\phi' := \begin{pmatrix} A & B \\ -K & I\end{pmatrix}, \quad K \in \mathcal{L}(H, U), \quad \Lambda = \Lambda^*, \Lambda^{-1} \in \mathcal{L}(U),$$

is an I/O stable and output stable DLS. Then $\phi' = \phi_P$ and $\Lambda = \Lambda_P$ for a $P \in ric_0(\Phi, J)$.

Proof. We prove claim (i). Let $P \in ric_{uw}(\Phi, J)$. By Lemma 138, we have for all $m \geq 0$

$$\pi_{[0,m]}\mathcal{D}^*J\mathcal{D}\pi_{[0,m]} = \pi_{[0,m]}\mathcal{D}^*_{\phi_P}\Lambda_P\mathcal{D}_{\phi_P}\pi_{[0,m]}, \tag{3.31}$$

where using the adjoints is legal because both $\mathcal{D}$ and $\mathcal{D}_{\phi_P}$ are assumed to be bounded. Let $\tilde{u} \in \ell^2(\mathbf{Z}_+;U)$ be arbitrary. Then

$$\begin{aligned}
&||\pi_{[0,m]}\mathcal{D}^* J\mathcal{D}\pi_{[0,m]}\tilde{u} - \bar{\pi}_+\mathcal{D}^* J\mathcal{D}\bar{\pi}_+\tilde{u}|| \\
&\le ||\pi_{[0,m]}\mathcal{D}^* J\mathcal{D}(\pi_{[0,m]}\tilde{u} - \bar{\pi}_+\tilde{u})|| + ||(\pi_{[0,m]} - \bar{\pi}_+)\mathcal{D}^* J\mathcal{D}\pi_{[0,m]}\bar{\pi}_+\tilde{u}|| \\
&\le ||\pi_{[0,m]}\mathcal{D}^* J\mathcal{D}|| \cdot ||\pi_{[m+1,\infty]}\tilde{u}|| + ||\pi_{[m+1,\infty]} \cdot \bar{\pi}_+\mathcal{D}^* J\mathcal{D}\bar{\pi}_+\tilde{u}||
\end{aligned}$$

Because both $\tilde{u}$ and $\bar{\pi}_+\mathcal{D}^* J\mathcal{D}\bar{\pi}_+\tilde{u}$ are in $\ell^2(\mathbf{Z}_+;U)$, it follows that $\mathrm{s}-\lim_{m\to\infty}\pi_{[0,m]}\mathcal{D}^* J\mathcal{D}\pi_{[0,m]} = \bar{\pi}_+\mathcal{D}^* J\mathcal{D}\bar{\pi}_+$. Similarly we obtain the limit $\mathrm{s}-\lim_{m\to\infty}\pi_{[0,m]}\mathcal{D}^*_{\phi_P} J\mathcal{D}_{\phi_P}\pi_{[0,m]} = \bar{\pi}_+\mathcal{D}^*_{\phi_P} J\mathcal{D}_{\phi_P}\bar{\pi}_+$. The uniqueness of the strong limit, together with equation (3.31), gives now factorization (3.29).

To prove the other claim (ii), we show that there is a conjugate symmetric sesquilinear form $P(\ ,\ )$ such that for all $\tilde{u} \in \ell^2(\mathbf{Z}_+;U)$, $x_0 \in H$

$$J(x_0,\tilde{u}) = P(x_0,x_0) + \langle \Lambda(\mathcal{C}_{\phi'}x_0 + \mathcal{D}_{\phi'}\bar{\pi}_+\tilde{u}), (-,,-)\rangle, \tag{3.32}$$

assuming that the factorization (3.30) exists. Here $J(x_0,\tilde{u}) := \langle J(\mathcal{C}x_0 + \mathcal{D}\bar{\pi}_+\tilde{u}), (-,,-)\rangle$ is a cost functional, see Section 2.2. Suppose that such a sesquilinear form $P(\ ,\ )$ exists and try to find an expression for it. By expanding (3.32) we obtain

$$\begin{aligned}
&\langle \mathcal{C}^* J\mathcal{C}x_0, x_0\rangle + \overbrace{2Re\,\langle \bar{\pi}_+\mathcal{D}^* J\mathcal{C}x_0, \tilde{u}\rangle}^{(i)} + \overbrace{\langle \bar{\pi}_+\mathcal{D}^* J\mathcal{D}\bar{\pi}_+\tilde{u}, \tilde{u}\rangle}^{(ii)} \\
&= P(x_0,x_0) + \left\langle \mathcal{C}^*_{\phi'}\Lambda\mathcal{C}_{\phi'}x_0, x_0\right\rangle + \\
&\overbrace{2Re\left\langle \bar{\pi}_+\mathcal{D}^*_{\phi'}\Lambda\mathcal{C}_{\phi'}x_0, \tilde{u}\right\rangle}^{(iii)} + \overbrace{\left\langle \bar{\pi}_+\mathcal{D}^*_{\phi'}\Lambda\mathcal{D}_{\phi'}\bar{\pi}_+\tilde{u}, \tilde{u}\right\rangle}^{(iv)}
\end{aligned} \tag{3.33}$$

for all $\tilde{u} \in \ell^2(\mathbf{Z}_+;U)$ and $x_0 \in H$ because both Φ and ϕ' are I/O stable and output stable. By equation (3.30), parts (ii) and (iv) are equal. To compare parts (i) and (iii), note that for $x := \mathcal{B}\tilde{w}$, $\tilde{w} \in \mathrm{dom}\,(\mathcal{B})$, we have, because $\mathcal{B} = \mathcal{B}_{\phi'}$

$$\begin{aligned}
&\bar{\pi}_+\mathcal{D}^* J\mathcal{C}x - \bar{\pi}_+\mathcal{D}^*_{\phi'}\Lambda\mathcal{C}_{\phi'}x = \bar{\pi}_+\mathcal{D}^* J\bar{\pi}_+\mathcal{D}\pi_-\tilde{w} - \bar{\pi}_+\mathcal{D}^*_{\phi'}\Lambda\bar{\pi}_+\mathcal{D}_{\phi'}\pi_-\tilde{w} \\
&= \bar{\pi}_+(\mathcal{D}^* J\mathcal{D} - \mathcal{D}^*_{\phi'}\Lambda\mathcal{D}_{\phi'})\pi_-\tilde{w} = 0
\end{aligned} \tag{3.34}$$

by (3.30), and the anticausality of $\mathcal{D}^*$ and $\mathcal{D}^*_{\phi'}$. Because $\overline{\mathrm{range}\,(\mathcal{B})} = H$ it follows that $\bar{\pi}_+\mathcal{D}^* J\mathcal{C}x - \bar{\pi}_+\mathcal{D}^*_{\phi'}\Lambda\mathcal{C}_{\phi'}x = 0$, for all $x \in H$, by I/O stability and output stability of Φ and ϕ'.

So the parts (i), (ii), (iii) and (iv) cancel each other out in equation (3.33). What remains allows us to conclude that the sesquilinear form of equation (3.32) exists and equals

$$P(x_0,x_0) = \left\langle \left(\mathcal{C}^* J\mathcal{C} - \mathcal{C}^*_{\phi'}\Lambda\mathcal{C}_{\phi'}\right)x_0, x_0\right\rangle =: \langle Px_0, x_0\rangle,$$

which gives us a unique self-adjoint operator $P \in \mathcal{L}(H)$. We note that for all $x_0 \in H$

$$\begin{aligned}\left\langle A^{*j} P A^j x_0, x_0 \right\rangle &= \left\langle J \mathcal{C} A^j x_0, \mathcal{C} A^j x_0 \right\rangle - \left\langle \Lambda \mathcal{C}_{\phi'} A^j x_0, \mathcal{C}_{\phi'} A^j x_0 \right\rangle \\ &= \left\langle J \pi_{[j,\infty]} \mathcal{C} x_0, \pi_{[j,\infty]} \mathcal{C} x_0 \right\rangle - \left\langle \Lambda \pi_{[j,\infty]} \mathcal{C}_{\phi'} x_0, \pi_{[j,\infty]} \mathcal{C}_{\phi'} x_0 \right\rangle .\end{aligned}$$

By the output stabilities of Φ and ϕ', both $\pi_{[j,\infty]}\mathcal{C}x_0 \to 0$ and $\pi_{[j,\infty]}\mathcal{C}_{\phi'}x_0 \to 0$ in $\ell^2(\mathbf{Z}_+; Y)$, $\ell^2(\mathbf{Z}_+; U)$, respectively. Thus $\left\langle P A^j x_0, A^j x_0 \right\rangle \to 0$ for all $x_0 \in H$, by the boundedness of Λ^{-1}.

We complete the proof by showing that $P \in Ric(\Phi, J)$, and that $K = K_P$, $\Lambda = \Lambda_P$. We have for Λ_P

$$\begin{aligned}\Lambda_P &= D^* J D + B^* P B \\ &= (D^* J D + (\mathcal{C}B)^* J (\mathcal{C}B)) - (I^* \Lambda I + (\mathcal{C}_{\phi'} B)^* \Lambda (\mathcal{C}_{\phi'} B)) + \Lambda \\ &= (D\pi_0 + \tau \mathcal{C} B)^* J (D\pi_0 + \tau \mathcal{C} B) - (\pi_0 + \tau \mathcal{C}_{\phi'} B)^* \Lambda (\pi_0 + \tau \mathcal{C}_{\phi'} B) + \Lambda \\ &= \bar{\pi}_+ \mathcal{D}^* J \mathcal{D} \pi_0 - \bar{\pi}_+ \mathcal{D}_{\phi'}^* \Lambda \mathcal{D}_{\phi'} \pi_0 + \Lambda = \Lambda,\end{aligned}$$

where the second to the last equality has been written with the identification of spaces U and range(π_0), allowing us to write $\mathcal{D}\pi_0 = D\pi_0 + \tau \mathcal{C} B$. The last identity follows directly from the factorization (3.30), and so $\Lambda_P = \Lambda$.

For $K_P = \Lambda_P^{-1}(-D^* J C - B^* P A)$ we calculate similarly

$$\begin{aligned}&- D^* J C - B^* P A \\ &= - (D^* J C + (\mathcal{C}B)^* J \mathcal{C} A) + (-I^* \Lambda K + (\mathcal{C}_{\phi'} B)^* \Lambda \mathcal{C}_{\phi'} A) + \Lambda K\end{aligned} \tag{3.35}$$

Now $D^* J C + (\mathcal{C}B)^* J \mathcal{C} A = (D\pi_0 + \tau \mathcal{C} B)^* J \mathcal{C} = (\mathcal{D}\pi_0)^* J \mathcal{C} = \pi_0 \mathcal{D}^* J \mathcal{C}$. Quite similarly $-\Lambda K + (\mathcal{C}_{\phi'} B)^* \Lambda \bar{\pi}_+ \tau^* \mathcal{C}_{\phi'} = (\mathcal{D}_{\phi'} \pi_0)^* \Lambda \mathcal{C}_{\phi'} = \pi_0 \mathcal{D}_{\phi'}^* \Lambda \mathcal{C}_{\phi'}$. Then we obtain from (3.35)

$$-D^* J C - B^* P A = -\pi_0 (\mathcal{D}^* J \mathcal{C} - \mathcal{D}_{\phi'}^* \Lambda \mathcal{C}_{\phi'}) + \Lambda K, \tag{3.36}$$

with the identification of spaces U and range(π_0).

For all $x = \mathcal{B}\tilde{w} = \mathcal{B}_{\phi'}\tilde{w}$, $\tilde{w} \in \mathrm{dom}(\mathcal{B}) = \mathrm{dom}(\mathcal{B}_{\phi'})$, we have

$$\pi_0 (\mathcal{D}^* J \mathcal{C} - \mathcal{D}_{\phi'}^* \Lambda \mathcal{C}_{\phi'}) x = \pi_0 (\mathcal{D}^* J \mathcal{D} - \mathcal{D}_{\phi'}^* \Lambda \mathcal{D}_{\phi'}) \pi_- \tilde{w} = 0,$$

by the factorization (3.30). Because $\overline{\mathrm{range}(\mathcal{B})} = H$, and $\pi_0 (\mathcal{D}^* J \mathcal{C} - \mathcal{D}_{\phi'}^* \Lambda \mathcal{C}_{\phi'})$ is continuous in H, it follows that vanishes in the whole of H. From (3.36) it now follows that $K = \Lambda^{-1}(-D^* J C - B^* P A) = \Lambda_P^{-1}(-D^* J C - B^* P A) = K_P$ because $\Lambda = \Lambda_P$ has been shown earlier.

It is now straightforward to show that $P \in Ric(\Phi, J)$:

$$\begin{aligned}&P(Ax_0, Ax_0) - P(x_0, x_0) \\ &= \langle \pi_+ \mathcal{C} x_0, J \pi_+ \mathcal{C} \rangle - \langle \pi_+ \mathcal{C}_{\phi'} x_0, \Lambda \pi_+ \mathcal{C}_{\phi'} \rangle - \langle \mathcal{C} x_0, J \mathcal{C} \rangle + \langle \mathcal{C}_{\phi'} x_0, \Lambda \mathcal{C}_{\phi'} \rangle \\ &= \langle -K x_0, -\Lambda\ K x_0 \rangle - \langle C x_0, J C x_0 \rangle = \langle K_P^* \Lambda_P K_P x_0, x_0 \rangle - \langle C^* J C x_0, x_0 \rangle .\end{aligned}$$

Because ϕ' is output stable and I/O stable, by assumption, and $\phi_P = \phi'$, it follows that P is a H^∞ solution: $P \in ric(\Phi, J)$.

It remains to prove the final claim about the residual cost operator. Because Φ and ϕ' are output stable by assumption, we have

$$\begin{aligned}
A^{*j} P A^j &= A^{*j} \mathcal{C}^* J \mathcal{C} A^j - A^{*j} \mathcal{C}_{\phi'}^* \Lambda \mathcal{C}_{\phi'} A^j \\
&= (\bar{\pi}_+ \tau^{*j} \mathcal{C})^* J (\bar{\pi}_+ \tau^{*j} \mathcal{C}) - (\bar{\pi}_+ \tau^{*j} \mathcal{C}_{\phi'})^* \Lambda (\bar{\pi}_+ \tau^{*j} \mathcal{C}_{\phi'}) \\
&= \mathcal{C}^* J \pi_{[j,\infty]} \mathcal{C} - \mathcal{C}_{\phi'}^* \Lambda \pi_{[j,\infty]} \mathcal{C}_{\phi'}.
\end{aligned}$$

Now $\mathrm{s}-\lim_{j\to\infty} \pi_{[j,\infty]}\mathcal{C} = \mathrm{s}-\lim_{j\to\infty} \pi_{[j,\infty]}\mathcal{C}_{\phi'} = 0$, and immediately $L_{A,P} = \mathrm{s}-\lim_{j\to\infty} A^{*j} P A^j = 0$. This completes the proof. □

For analogous spectral factorization results, see [49, Chapter 19], [45, Theorem 4.6] and [36] together with its references. In claim (ii) of Theorem 142, a requirement has been imposed on the spectral factor $\mathcal{D}_{\phi'}$ of the Popov operator: it must be realizable by using the same input structure as the original DLS Φ and all the spectral DLSs ϕ_P. It is necessary to make such an apriori requirement explicitly. To see this, consider the trivial case when $\mathcal{D} = \mathcal{I}$, the identity operator of $\ell^2(\mathbf{Z}; U)$. Then the Popov operator satisfies $\mathcal{D}^* J \mathcal{D} = \mathcal{I}$, if $J = I$, the identity operator of U. Each inner from the left operator $\mathcal{N}'$ is, by definition, a spectral factor of the Popov operator $\mathcal{I}$. There is a multitude of such inner operators; if $U = \mathbf{C}$, then these are parameterized by sequences in $\mathbf{D}$ satisfying the Blaschke condition and the singular positive measures on $\mathbf{T}$. However, the DLS $\Phi = \phi$ can be very trivial, say $\phi = \left(\begin{smallmatrix} 0 & 0 \\ 0 & I \end{smallmatrix}\right)$. The DARE $Ric(\phi, I)$ is trivially $I = I$, and all (self-adjoint) operators $P \in \mathcal{L}(H)$ are its solution. However, each of the spectral DLSs equal $\phi_P = \left(\begin{smallmatrix} 0 & 0 \\ 0 & I \end{smallmatrix}\right)$, and only one spectral factor of the Popov operator is covered by a solution of the DARE.

In the proof of Theorem 142, we never wrote down a state space realization for the Popov function $\mathcal{D}(e^{i\theta})^* J \mathcal{D}(e^{i\theta})$. Suppose $\mathcal{D}(z) \in H^\infty(\mathbf{D}; \mathcal{L}(U))$ would be analytic in an open set $\Omega \subset \mathbf{C}$, such that $\mathbf{D} \subset \Omega$ and $\mathbf{T} \setminus (\mathbf{T} \cap \Omega)$ is, say, a finite set of points. Then the Popov function $\mathcal{D}(e^{i\theta})^* J \mathcal{D}(e^{i\theta})$ would have an analytic continuation to a neighborhood of each $e^{i\theta_0} \in \mathbf{T} \cap \Omega$. This analytic continuation is given by $\widetilde{\mathcal{D}}(z^{-1}) J \mathcal{D}(z)$, and its realization ϕ^{Popov} can be formed by using the formula for the product realization. Now, the connection between the DARE and the spectral factorization of the Popov function can be studied by using ϕ^{Popov}, even for certain classes of unstable transfer functions $\mathcal{D}(z)$. However, a general $\mathcal{D}(z) \in H^\infty(\mathbf{D}; \mathcal{L}(U))$ does not allow this approach; there is a function in the complex-valued disk algebra $f(z) \in A(\mathbf{D})$ that does not allow analytic continuation to any set larger than $\mathbf{D}$, and in fact the boundary trace $f(e^{i\theta})$ can be smooth. Such a function is constructed in [78, Example 16.7]. Then $f(z)$ and $\widetilde{f}(z^{-1})$ are bounded analytic functions in open sets $\mathbf{D}$ and $(\overline{\mathbf{D}})^c$, with an empty intersection.

In a later result, Lemma 193, we shall need a different spectral factorization result, associated to solutions $P \in ric(\Phi, J)$ that need not satisfy the strong residual cost condition. The nonvanishing residual cost is included in the Popov operator. To achieve this, we must first define analogues (in I/O form) to the residual cost operator $L_{A,P} := \mathrm{s} - \lim_{j\to\infty} A^{*j} P A^j$.

Definition 143. *Let $J \in \mathcal{L}(Y)$ be a cost operator. Let $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right] = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ be a DLS, and $P \in Ric(\Phi, J)$. Let $n, m \geq 0$ be arbitrary. Define the linear operators in $\ell^2(\mathbf{Z}_+; U)$*

$$\mathcal{L}_{\Phi,P}^{(m,n)} := \left(\mathcal{B}\tau^{*n}\pi_{[0,m]}\right)^* P \left(\mathcal{B}\tau^{*n}\pi_{[0,m]}\right),$$

and

$$\mathcal{L}_{\Phi,P}^{(m)} := \mathrm{s} - \lim_{n\to\infty} \mathcal{L}_{\Phi,P}^{(m,n)}, \quad \mathcal{L}_{\Phi,P} := \mathrm{s} - \lim_{m\to\infty} \mathcal{L}_{\Phi,P}^{(m)},$$

provided that the strong limits exists. The operator $\mathcal{L}_{\Phi,P}$ is the residual cost operator (in I/O form), and the operator $\mathcal{L}_{\Phi,P}^{(n)}$ is the truncated residual cost operator (in I/O form).

The operator $\mathcal{B}\tau^{*n}\pi_{[0,m]} : \ell^2(\mathbf{Z}_+; U) \to H$ is a finite sum of products of the bounded operators A, B, the orthogonal projections π_j, and the unitary shift τ^* in $\ell^2(\mathbf{Z}_+; U)$. Thus it is bounded for all $m, n \geq 0$, and it follows that $\mathcal{L}_{\Phi,P}^{(m,n)}$ always exists as a bounded operator.

Lemma 144. *Let $J \in \mathcal{L}(Y)$ be a cost operator. Let $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right] = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ be an output stable and I/O stable DLS, and $P \in ric(\Phi, J)$. Then*

(i) Both the residual cost operators $L_{A,P} \in \mathcal{L}(H)$ and $\mathcal{L}_{\Phi,P} \in \mathcal{L}(\ell^2(\mathbf{Z}_+; U))$ exist.

(ii) We have the spectral factorization identity

$$\mathcal{L}_{\phi,P} + \bar{\pi}_+ \mathcal{D}^* J \mathcal{D} \bar{\pi}_+ = \bar{\pi}_+ \mathcal{D}_{\phi_P}^* \Lambda_P \mathcal{D}_{\phi_P} \bar{\pi}_+.$$

The residual cost operator $\mathcal{L}_{\Phi,P}$ is a self-adjoint Toeplitz operator.

(iii) Assume, in addition, that $\overline{\mathrm{range}\,(\mathcal{B})} = H$. Then both $B^* L_{A,P} A = 0$ and $B^* L_{A,P} B = 0$ if and only if $\mathcal{L}_{\Phi,P} = 0$ if and only if $L_{A,P} = 0$.

Proof. Because $P \in ric(\Phi, J)$, the residual cost operator $L_{A,P}$ exists by Proposition 110. We prove the rest of claim (i) and claim (ii) simultaneously. Let $x_0 \in H$ and $\{u_j\}_{j\geq 0} = \tilde{u} \in \ell^2(\mathbf{Z}_+; U)$ be arbitrary. Denote $x_j = x_j(x_0, \tilde{u}) =$

$A^j x_0 + \mathcal{B}\tau^{*j}\tilde{u}$ the trajectory of the DLS Φ with this given initial state and input. We have in claim (i) of Proposition i for all $n > 0$

$$\begin{aligned}\langle Px_0, x_0\rangle - \langle Px_n, x_n\rangle \\ &= \sum_{j=0}^{n-1} \langle J(Cx_j + Du_j), Cx_j + Du_j\rangle \\ &- \sum_{j=0}^{n-1} \langle \Lambda_P(-K_P x_j + u_j), -K_P x_j + u_j)\rangle.\end{aligned} \tag{3.37}$$

We now set $x_0 = 0$ and assume that the inputs are of form $\pi_{[0,m]}\tilde{u}$ for some fixed $m \geq 0$ and arbitrary $\tilde{u} \in \ell^2(\mathbf{Z}_+; U)$. In this case, $\langle Px_0, x_0\rangle = 0$ and equation (3.37) takes now the form

$$\begin{aligned}&\left\langle P\mathcal{B}\tau^{*n}\pi_{[0,m]}\tilde{u}, \mathcal{B}\tau^{*n}\pi_{[0,m]}\tilde{u}\right\rangle + \left\langle J\mathcal{D}\pi_{[0,m]}\tilde{u}, \pi_{[0,n-1]}\mathcal{D}\pi_{[0,m]}\tilde{u}\right\rangle_{\ell^2(\mathbf{Z}_+;Y)} \\ &= \left\langle \Lambda_P \mathcal{D}_{\phi_P}\pi_{[0,m]}\tilde{u}, \pi_{[0,n-1]}\mathcal{D}_{\phi_P}\pi_{[0,m]}\tilde{u}\right\rangle_{\ell^2(\mathbf{Z}_+;Y)},\end{aligned}$$

because $x_n = \mathcal{B}\tau^{*n}\pi_{[0,m]}\tilde{u}$.

Both the operators $\mathcal{D}\pi_{[0,m]}$ and $\mathcal{D}_{\phi_P}\pi_{[0,m]}$ are bounded, because Φ and ϕ_P are I/O stable DLSs by assumptions. Also the operators $\mathcal{B}\tau^{*n}\pi_{[0,m]}$ are bounded, as has been discussed after Definition 143. So the adjoints $\left(\mathcal{B}\tau^{*n}\pi_{[0,m]}\right)^*$, $\mathcal{D}^*$ and $\mathcal{D}^*_{\phi_P}$ make sense, and we can write

$$\begin{aligned}&\left\langle \mathcal{L}^{(m,n)}_{\Phi,P}\tilde{u}, \tilde{u}\right\rangle + \left\langle \pi_{[0,m]}\mathcal{D}^* J\pi_{[0,n-1]}\mathcal{D}\pi_{[0,m]}\tilde{u}, \tilde{u}\right\rangle_{\ell^2(\mathbf{Z}_+;Y)} \\ &= \left\langle \pi_{[0,m]}\mathcal{D}^*_{\phi_P}\Lambda_P\pi_{[0,n-1]}\mathcal{D}_{\phi_P}\pi_{[0,m]}\tilde{u}, \tilde{u}\right\rangle_{\ell^2(\mathbf{Z}_+;Y)},\end{aligned}$$

by Definition 143. Because $\tilde{u}$ is arbitrary, and all the operators $\mathcal{L}^{(m,n)}_{\Phi,P}$, $\mathcal{D}$ and $\mathcal{D}_{\phi_P}$ are bounded, [79, Theorem 12.7] implies that

$$\mathcal{L}^{(m,n)}_{\Phi,P} = -\pi_{[0,m]}\mathcal{D}^* J \cdot \pi_{[0,n-1]}\mathcal{D}\pi_{[0,m]} + \pi_{[0,m]}\mathcal{D}^*_{\phi_P}\Lambda_P \cdot \pi_{[0,n]}\mathcal{D}_{\phi_P}\pi_{[0,m]} \tag{3.38}$$

for all $m, n \geq 0$. Because $\mathcal{D}$ is bounded, $\mathrm{s}-\lim_{n\to\infty}\pi_{[0,n-1]}\mathcal{D}\pi_{[0,m]} = \mathcal{D}\pi_{[0,m]}$ and $\mathrm{s}-\lim_{n\to\infty}\pi_{[0,n-1]}\mathcal{D}_{\phi_P}\pi_{[0,m]} = \mathcal{D}_{\phi_P}\pi_{[0,m]}$. But then, the strong limit in the right hand side of (3.38) exists, and we conclude that the residual cost operator $\mathcal{L}^{(m)}_{\Phi,P} \in \mathcal{L}(\ell^2(\mathbf{Z}_+; U))$ exists as a bounded operator. We obtain

$$\mathcal{L}^{(m)}_{\Phi,P} = -\pi_{[0,m]}\mathcal{D}^* J\mathcal{D}\pi_{[0,m]} + \pi_{[0,m]}\mathcal{D}^*_{\phi_P}\Lambda_P\mathcal{D}_{\phi_P}\pi_{[0,m]} \tag{3.39}$$

for all $m \geq 0$. We proceed to show that $\mathrm{s}-\lim_{m\to\infty}\pi_{[0,m]}\mathcal{D}^* J\mathcal{D}\pi_{[0,m]}$ exists and equals the Popov operator $\bar{\pi}_+\mathcal{D}^* J\mathcal{D}\bar{\pi}_+$. For all $m \geq 0$ and $\tilde{u} \in \ell^2(\mathbf{Z}_+; U)$, we have

$$\begin{aligned}&||\pi_{[0,m]}\mathcal{D}^* J\mathcal{D}\pi_{[0,m]}\tilde{u} - \bar{\pi}_+\mathcal{D}^* J\mathcal{D}\bar{\pi}_+\tilde{u}||_{\ell^2(\mathbf{Z}_+;U)} \\ &\leq ||\pi_{[0,m]}\mathcal{D}^* J\mathcal{D}\pi_{[m+1,\infty]}\tilde{u}||_{\ell^2(\mathbf{Z}_+;U)} + ||\pi_{[m+1,\infty]}\mathcal{D}^* J\mathcal{D}\bar{\pi}_+\tilde{u}||_{\ell^2(\mathbf{Z}_+;U)} \\ &\leq ||J||_{\mathcal{L}(Y)} \cdot ||\mathcal{D}||_{\ell^2(\mathbf{Z};U)\to\ell^2(\mathbf{Z};Y)} \cdot ||\pi_{[m+1,\infty]}\tilde{u}||_{\ell^2(\mathbf{Z}_+;U)} \\ &+ ||\pi_{[m+1,\infty]} \cdot \bar{\pi}_+\mathcal{D}^* J\mathcal{D}\bar{\pi}_+\tilde{u}||_{\ell^2(\mathbf{Z}_+;U)}.\end{aligned}$$

Because both $\tilde{u}$ and $\bar{\pi}_+\mathcal{D}^*J\mathcal{D}\bar{\pi}_+\tilde{u}$ belong to $\ell^2(\mathbf{Z}_+;U)$, the right hand side of the previous equation converges to zero as $m \to \infty$. It follows that $\mathrm{s}-\lim_{m\to\infty}\pi_{[0,m]}\mathcal{D}^*J\mathcal{D}\pi_{[0,m]} = \bar{\pi}_+\mathcal{D}^*J\mathcal{D}\bar{\pi}_+$ and similarly $\mathrm{s}-\lim_{m\to\infty}\pi_{[0,m]}\mathcal{D}^*_{\phi_P}J\mathcal{D}_{\phi_P}\pi_{[0,m]} = \bar{\pi}_+\mathcal{D}^*_{\phi_P}J\mathcal{D}_{\phi_P}\bar{\pi}_+$. Because the right hand side of equation (3.39) converges strongly as $m\to\infty$, we obtain the spectral factorization

$$\mathcal{L}_{\Phi,P} = -\bar{\pi}_+\mathcal{D}^*J\mathcal{D}\bar{\pi}_+ + \bar{\pi}_+\mathcal{D}^*_{\phi_P}\Lambda_P\mathcal{D}_{\phi_P}\bar{\pi}_+ \tag{3.40}$$

where $\mathcal{L}_{\Phi,P}$ is the residual cost operator in I/O form, as introduced in Definition 143. Clearly $\mathcal{L}_{\Phi,P}$ is a self-adjoint Toeplitz operator, because the right hand side of equation (3.40) is such an operator. This proves claims (i) and (ii).

We proceed to prove claim (iii). We first calculate the block matrix elements $(\mathcal{L}_{\Phi,P})_{j_1,j_2} := \pi_{j_2}\mathcal{L}_{\Phi,P}\pi_{j_1}$ of $\mathcal{L}_{\Phi,P}$ for $j_1,j_2 \geq 0$. Let $\tilde{u},\tilde{w} \in \ell^2(\mathbf{Z}_+;U)$ be arbitrary. Then

$$\begin{aligned}
&\left\langle (\mathcal{L}_{\Phi,P})_{j_1,j_2}\tilde{u},\tilde{w}\right\rangle_{\ell^2(\mathbf{Z}_+;U)} = \left\langle \left(\mathrm{s}-\lim_{m\to\infty}\mathcal{L}^{(m)}_{\Phi,P}\right)\cdot\pi_{j_1}\tilde{u},\pi_{j_2}\tilde{w}\right\rangle_{\ell^2(\mathbf{Z}_+;U)}\\
&= \left\langle \lim_{m\to\infty}\left(\mathcal{L}^{(m)}_{\Phi,P}\pi_{j_1}\tilde{u}\right),\pi_{j_2}\tilde{w}\right\rangle_{\ell^2(\mathbf{Z}_+;U)} = \lim_{m\to\infty}\left\langle \mathcal{L}^{(m)}_{\Phi,P}\pi_{j_1}\tilde{u},\pi_{j_2}\tilde{w}\right\rangle_{\ell^2(\mathbf{Z}_+;U)}.
\end{aligned}$$

But if $m \geq j_1$, then $\mathcal{L}^{(m)}_{\Phi,P}\pi_{j_1}\tilde{u} = \mathcal{L}^{(j_1)}_{\Phi,P}\pi_{j_1}\tilde{u}$. It follows that the sequence in the right hand side of the previous equation stabilizes, and for $m \geq \max(j_1,j_2)$ we get

$$\begin{aligned}
&\left\langle (\mathcal{L}_{\Phi,P})_{j_1,j_2}\tilde{u},\tilde{w}\right\rangle_{\ell^2(\mathbf{Z}_+;U)}\\
&= \left\langle \mathcal{L}^{(m)}_{\Phi,P}\pi_{j_1}\tilde{u},\pi_{j_2}\tilde{w}\right\rangle_{\ell^2(\mathbf{Z}_+;U)} = \left\langle \left(\mathrm{s}-\lim_{n\to\infty}\mathcal{L}^{(m,n)}_{\Phi,P}\right)\cdot\pi_{j_1}\tilde{u},\pi_{j_2}\tilde{w}\right\rangle_{\ell^2(\mathbf{Z}_+;U)}\\
&= \left\langle \lim_{n\to\infty}\left(\mathcal{L}^{(m,n)}_{\Phi,P}\pi_{j_1}\tilde{u}\right),\pi_{j_2}\tilde{w}\right\rangle_{\ell^2(\mathbf{Z}_+;U)} = \lim_{n\to\infty}\left\langle \mathcal{L}^{(m,n)}_{\Phi,P}\pi_{j_1}\tilde{u},\pi_{j_2}\tilde{w}\right\rangle_{\ell^2(\mathbf{Z}_+;U)}\\
&= \lim_{n\to\infty}\left\langle P\mathcal{B}\tau^{*(n-j_1-1)}\pi_{-1}\tau^{*(j_1+1)}\tilde{u},\mathcal{B}\tau^{*(n-j_2-1)}\pi_{-1}\tau^{*(j_2+1)}\tilde{w}\right\rangle_H.
\end{aligned}$$

But now $\mathcal{B}\tau^{*(n-j-1)}\pi_{-1} = \mathcal{B}\tau^{*(n-j-1)}\pi_-\cdot\pi_{-1} = A^{n-j-1}\mathcal{B}\pi_{-1} = A^{n-j-1}B\pi_{-1}$, where be have used $\mathcal{B}\pi_{-1} = B\pi_{-1}$. Now, if $j := \max(j_1,j_2)$, then

$$\begin{aligned}
&\left\langle (\mathcal{L}_{\Phi,P})_{j_1,j_2}\tilde{u},\tilde{w}\right\rangle_{\ell^2(\mathbf{Z}_+;U)}\\
&= \lim_{n\to\infty}\left\langle A^{*(n-j-1)}PA^{n-j-1}\cdot A^{j-j_1}B\pi_{-1}\tau^{*(j_1+1)}\tilde{u},A^{j-j_2}B\pi_{-1}\tau^{*(j_2+1)}\tilde{w}\right\rangle_H\\
&= \left\langle \left(\mathrm{s}-\lim_{n\to\infty}A^{*(n-j-1)}PA^{n-j-1}\right)\cdot A^{j-j_1}B\pi_{-1}\tau^{*(j_1+1)}\tilde{u},A^{j-j_2}B\pi_{-1}\tau^{*(j_2+1)}\tilde{w}\right\rangle_H\\
&= \left\langle L_{A,P}\cdot A^{j-j_1}B\pi_{-1}\tau^{*(j_1+1)}\tilde{u},A^{j-j_2}B\pi_{-1}\tau^{*(j_2+1)}\tilde{w}\right\rangle_H.
\end{aligned}$$

This gives for the block matrix elements of $\mathcal{L}_{\Phi,P}$ the expression

$$\left\langle \left(\mathcal{L}_{\Phi,P}\right)_{j_1,j_2} \tilde{u}, \tilde{w} \right\rangle_{\ell^2(\mathbf{Z}_+;U)} \tag{3.41}$$

$$= \left\langle \pi_{j_2} B^* A^{*(j-j_2)} L_{A,P} A^{j-j_1} B \pi_{j_1} \cdot \tilde{u}, \tilde{w} \right\rangle_{\ell^2(\mathbf{Z}_+;U)}, \tag{3.42}$$

where $j = \max(j_1, j_2)$ and $\tilde{u}, \tilde{w} \in \ell^2(\mathbf{Z}_+; U)$ are arbitrary.

If both $B^* L_{A,P} A = 0$ and $B^* L_{A,P} B = 0$, then all the block matrix elements $\left(\mathcal{L}_{\Phi,P}\right)_{j_1,j_2}$ vanish, by equation (3.41). By a straightforward density argument, the bounded operator $\mathcal{L}_{\Phi,P}$ is seen to vanish.

Assume that $\mathcal{L}_{\Phi,P} = 0$. Then all the block matrix elements $\left(\mathcal{L}_{\Phi,P}\right)_{j_1,j_2}$ for $j_1, j_2 \geq 0$ vanish by their definition, and equation (3.40) implies that $B^* L_{A,P} A^k B = 0$ for all $k \geq 0$. It follows that $B^* L_{A,P} \mathcal{B} \tilde{u} = 0$ for all $\tilde{u} \in \operatorname{dom}(\mathcal{B})$, and thus $B^* L_{A,P} x = 0$ for all $x \in \operatorname{range}(\mathcal{B})$. Because B and $L_{A,P}$ are bounded, and $\overline{\operatorname{range}(\mathcal{B})} = H$, it follows that $B^* L_{A,P} = 0$, and also $L_{A,P} B = 0$ because $L_{A,P}$ is self-adjoint.

It is easy to see that $A^{*j} L_{A,P} A^j = L_{A,P}$ for all $j \geq 0$. Thus $A^{*j} L_{A,P} A^j B = L_{A,P} B = 0$ and immediately $B^* A^{*k} L_{A,P} A^j B = B^* A^{*(k-j)} \cdot A^{*j} L_{A,P} A^j B = 0$ for all $k \geq j$. By adjoining, we see that $B^* A^{*k} L_{A,P} A^j B = 0$ for arbitrary $j, k \geq 0$. But this implies that $\langle L_{A,B} \mathcal{B} \tilde{u}, \mathcal{B} \tilde{u} \rangle_H = 0$, for all $\tilde{u} \in \operatorname{dom}(\mathcal{B})$. By the assumed approximate controllability $\overline{\operatorname{range}(\mathcal{B})} = H$, boundedness of $L_{A,B}$, and [79, Theorem 12.7], it follows that $L_{A,B} = 0$.

Trivially, if $L_{A,B} = 0$ then both $B^* L_{A,P} A = 0$ and $B^* L_{A,P} B = 0$. This completes the proof. □

Recall that in Propositions 111 and 112 we asked whether the indicator Λ_P and the DLS ϕ_P uniquely determine the solution $P \in Ric(\phi, J)$. Under the indicated additional assumptions, claim (iii) of Lemma 144 provides an answer to this. Under the approximate controllability $\overline{\operatorname{range}(\mathcal{B})} = H$, it is exactly the solutions $P \in ric_0(\phi, J)$ (in the set $ric(\phi, J)$) that give us a spectral factorization of the Popov operator $\bar{\pi}_+ \mathcal{D}^* J \mathcal{D} \bar{\pi}_+$.

We proceed to consider the inertia of the indicator operator.

Lemma 145. *Let J be a self-adjoint cost operator. Let $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ be an I/O stable, output stable and J-coercive DLS, such that the input operator $B \in \mathcal{L}(U; H)$ is Hilbert–Schmidt, and the input space U is separable.*

Then there is a decomposition of U as an orthogonal direct sum $U = U_+ \oplus U_-$, such that for each $P \in ric_{uw}(\Phi, J)$, there is a boundedly invertible operator

$V_P \in \mathcal{L}(U)$ satisfying

$$\Lambda_P = V_P^* \begin{bmatrix} I_+ & 0 \\ 0 & -I_- \end{bmatrix} V_P,$$

where I_+, (I_-) is the identity of U_+, (U_-, respectively). In particular, if $\Lambda_{P_0} > 0$ for some $P_0 \in ric_{uw}(\Phi, J)$, then $\Lambda_P > 0$ for all $P \in ric_{uw}(\Phi, J)$.

Proof. Let $P_0 \in ric_{uw}(\Phi, J)$ be fixed, and $P \in ric_{uw}(\Phi, J)$ be arbitrary. Now, ϕ_{P_0} is output stable and I/O stable, because $P_0 \in ric_{uw}(\Phi, J)$, by assumption. The input operator of ϕ_{P_0} equals the Hilbert–Schmidt operator B. Because the feed-through operator of any spectral DLS is identity, it is definitely boundedly invertible. We proceed to conclude that the spectral DLS ϕ_{P_0} is Λ_{P_0}-coercive. By Proposition 139, we have the factorization of the Popov function

$$\mathcal{D}(e^{i\theta})^* J \mathcal{D}(e^{i\theta}) = \mathcal{D}_{\phi_{P_0}}(e^{i\theta})^* \Lambda_P \mathcal{D}_{\phi_{P_0}}(e^{i\theta}) \quad \text{a.e.} \quad e^{i\theta} \in \mathbf{T}.$$

Because both Φ and ϕ_{P_0} are I/O stable, we conclude by using the unitary Fourier transform that

$$\langle \bar{\pi}_+ \mathcal{D}^* J \mathcal{D} \bar{\pi}_+ \tilde{u}, \tilde{u} \rangle_{\ell^2(\mathbf{Z}_+;U)} = \left\langle \bar{\pi}_+ \mathcal{D}^*_{\phi_{P_0}} \Lambda_{P_0} \mathcal{D}_{\phi_{P_0}} \bar{\pi}_+ \tilde{u}, \tilde{u} \right\rangle_{\ell^2(\mathbf{Z}_+;U)}$$

for any $\tilde{u} \in \ell^2(\mathbf{Z}_+; U)$. Thus there is equality of the bounded self-adjoint operators $\bar{\pi}_+ \mathcal{D}^* J \mathcal{D} \bar{\pi}_+ = \bar{\pi}_+ \mathcal{D}^*_{\phi_{P_0}} \Lambda_{P_0} \mathcal{D}_{\phi_{P_0}} \bar{\pi}_+$, and the assumed J-coercivity of Φ is equivalent to the Λ_{P_0}-coercivity of ϕ_{P_0}. By claim (i) of Lemma 134, we conclude that $\mathcal{D}_{\phi_{P_0}}(e^{i\theta})^{-1}$ exists a.e. $e^{i\theta} \in \mathbf{T}$, and in fact the boundary trace function satisfies $\mathcal{D}_{\phi_{P_0}}(e^{i\theta})^{-1} \in L^\infty(\mathbf{T}; \mathcal{L}(U))$. Similarly, $\mathcal{D}_{\phi_P}(e^{i\theta})$ has a bounded inverse for almost all $e^{i\theta} \in \mathbf{T}$, too.

Because Λ_{P_0} is self-adjoint and boundedly invertible, we can work with the spectral projections of Λ_{P_0} on the disjoint spectral sets in negative and positive real axes. This gives $\Lambda_{P_0} = \Lambda_+ - \Lambda_-$, where $\Lambda_+ \in \mathcal{L}(U_+)$, $\Lambda_- \in \mathcal{L}(U_-)$, and both are positive invertible operators in their respective spectral subspaces that are reducing. Now

$$\Lambda_{P_0} = V^* \begin{bmatrix} I_+ & 0 \\ 0 & -I_- \end{bmatrix} V,$$

where $V^* := \left[\Lambda_+^{\frac{1}{2}} \; \Lambda_-^{\frac{1}{2}} \right] : U_+ \oplus U_- \to U$ has a bounded inverse. By Proposition 139, we can choose $e^{i\theta_0} \in \mathbf{T}$ from a set of full Lebesgue measure, such that

$$\mathcal{D}_{\phi_{P_0}}(e^{i\theta_0})^* \Lambda_{P_0} \mathcal{D}_{\phi_{P_0}}(e^{i\theta_0}) = \mathcal{D}_{\phi_P}(e^{i\theta_0})^* \Lambda_P \mathcal{D}_{\phi_P}(e^{i\theta_0}).$$

As discussed above, $e^{i\theta_0} \in \mathbf{T}$ can be chosen from a set of full Lebesgue measure so that both $\mathcal{D}_{\phi_{P_0}}(e^{i\theta_0})$ and $\mathcal{D}_{\phi_P}(e^{i\theta_0})$ are boundedly invertible. We now have

$$\left(V \mathcal{D}_{\phi_{P_0}}(e^{i\theta_0}) \mathcal{D}_{\phi_P}(e^{i\theta_0})^{-1} \right)^* \begin{bmatrix} I_+ & 0 \\ 0 & -I_- \end{bmatrix} \left(V \mathcal{D}_{\phi_{P_0}}(e^{i\theta_0}) \mathcal{D}_{\phi_P}(e^{i\theta_0})^{-1} \right) = \Lambda_P.$$

This proves the claim because $V_P := V\mathcal{D}_{\phi_{P_0}}(e^{i\theta_0})(\mathcal{D}_{\phi_P}(e^{i\theta_0})^{-1}$ is boundedly invertible. The claim involving the positivity of the indicators is now a triviality. $\square$

By dimension counting, we immediately see that if either of the spaces U_+, U_- is finite dimensional, then the dimension will be an invariant of all the solutions $P \in ric_{uw}(\Phi, J)$. For an analogous matrix result, see [49, Corollary 12.2.4].

Corollary 146. *Let $J \in \mathcal{L}(Y)$ be a self-adjoint operator. Let $\Phi = \left[\begin{smallmatrix} \mathcal{A}^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ be an output stable, I/O stable and J-coercive DLS. Assume that the input operator $B \in \mathcal{L}(U;H)$ of Φ is Hilbert–Schmidt, and the input space U is separable. Then the following are equivalent*

(i) $\bar{\pi}_+\mathcal{D}^*J\mathcal{D}\bar{\pi}_+ \geq \epsilon\bar{\pi}_+$ *for some* $\epsilon > 0$,

(ii) the regular critical solution $P_0^{\mathrm{crit}} := \left(\mathcal{C}^{\mathrm{crit}}\right)^* J\mathcal{C}^{\mathrm{crit}} \in ric_0(\Phi, J)$ *exists, and its indicator* $\Lambda_{P_0^{\mathrm{crit}}}$ *is positive, and*

(iii) the solution set $ric_{uw}(\Phi, J)$ *is not empty, and all* $P \in ric_{uw}(\Phi, J)$ *satisfy* $\Lambda_P > 0$.

Proof. We first show that (i) implies (ii). Assume (i). Corollary 118 implies that the equivalent conditions of Theorem 114 hold, and in particular a critical $P^{\mathrm{crit}} \in Ric_{uw}(\Phi, J)$ exists. Proposition 115 implies that we have the regular critical solution $P_0^{\mathrm{crit}} := \left(\mathcal{C}^{\mathrm{crit}}\right)^* J\mathcal{C}^{\mathrm{crit}} \in ric_0(\Phi, J)$. Thus the solution set $ric_{uw}(\Phi, J)$ is not empty. By Theorem 142, $\bar{\pi}_+\mathcal{X}^*\Lambda_{P_0^{\mathrm{crit}}}\mathcal{X}\bar{\pi}_+ = \bar{\pi}_+\mathcal{D}^*J\mathcal{D}\bar{\pi}_+ \geq \epsilon\bar{\pi}_+$ where $\mathcal{X} := \mathcal{D}_{\phi_{P_0^{\mathrm{crit}}}}$ is outer with a bounded inverse. By the boundedness and shift-invariance, also $\mathcal{X}^*\Lambda_{P_0^{\mathrm{crit}}}\mathcal{X} \geq \epsilon\mathcal{I}$, and then $\Lambda_{P_0^{\mathrm{crit}}} \geq \epsilon\mathcal{X}^{-*}\mathcal{X}^{-1} = \epsilon\left(\mathcal{X}\mathcal{X}^*\right)^{-1} > 0$, where $\Lambda_{P_0^{\mathrm{crit}}}$ is regarded as a static multiplication operator on $\ell^2(\mathbf{Z}; U)$. Immediately, $\Lambda_{P_0^{\mathrm{crit}}} > 0$ as an element of $\mathcal{L}(U)$, too. Thus claim (ii) follows.

Assume claim (ii). Then $P_0^{\mathrm{crit}} \in ric_0(\Phi, J) \subset ric_{uw}(\Phi, J)$ exists, and the latter set is not empty. Because the input operator B is Hilbert–Schmidt, an application of Lemma 145 proves now claim (iii). Assume claim (iii). Then there is a $P_0 \in ric_{uw}(\Phi, J)$ with a positive indicator. By Theorem 142, $\bar{\pi}_+\mathcal{D}^*J\mathcal{D}\bar{\pi}_+ = \bar{\pi}_+\mathcal{D}^*_{\phi_{P_0}}\Lambda_{P_0}\mathcal{D}_{\phi_{P_0}}\bar{\pi}_+ \geq 0$. But by the assumed J-coercivity of Φ, claim (i) follows. $\square$

We remark that claims (i) and (ii) of Corollary 146 are equivalent even if the input operator B is not Hilbert–Schmidt and U is not separable.

There is a one-to-one correspondence between (J, S)-inner-outer factorizations of $\mathcal{D} = \mathcal{N}\mathcal{X}$ (with the outer part having a bounded inverse $\mathcal{X}^{-1}$) and S-spectral factorizations of the Popov operator $\mathcal{D}^* J \mathcal{D}$, see Proposition 82. Applying this to the spectral DLSs gives the proposition:

Proposition 147. *Let $\Phi = \left[\begin{smallmatrix} \mathcal{A}^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ be an I/O stable and output stable DLS. Let J be a self-adjoint operator. Assume that the equivalent conditions of Theorem 114 hold, and by $P_0^{\text{crit}} := (\mathcal{C}^{\text{crit}})^* J \mathcal{C}^{\text{crit}} \in ric_0(\Phi, J)$ denote the regular critical solution. Let $P \in ric_{uw}(\Phi, J)$ be arbitrary. Then*

(i) $\mathcal{D}_{\phi_P}$ has an $(\Lambda_P, \Lambda_{P_0^{\text{crit}}})$-inner-outer factorization given by

$$\mathcal{D}_{\phi_P} = \mathcal{N}_P \mathcal{X},$$

where $\mathcal{X} = \mathcal{D}_{\phi_{P_0^{\text{crit}}}}$ is I/O stable, and $\mathcal{N}_P := \mathcal{D}_{\phi_P} \mathcal{D}^{-1}_{\phi_{P_0^{\text{crit}}}}$. The equivalent conditions of Theorem 114 hold for the DLS ϕ_P and the cost operator Λ_P. The outer factor of $\mathcal{D}_{\phi_P}$ does not depend upon the solution P. Both $\operatorname{range}(\mathcal{D}_{\phi_P} \bar{\pi}_+)$ and $\operatorname{range}(\mathcal{D}_{\phi_P})$ are closed.

(ii) If, in addition, the input operator B is Hilbert–Schmidt and the space U is separable, then $\operatorname{range}(\mathcal{D}_{\phi_P}) = \ell^2(\mathbf{Z}; U)$, and $\mathcal{D}_{\phi_P}$ has a bounded shift-invariant inverse on $\ell^2(\mathbf{Z}; U)$. If $J \geq 0$, then normalized inner factor $\Lambda_P^{\frac{1}{2}} \mathcal{N}_P \Lambda_{P_0^{\text{crit}}}^{-\frac{1}{2}}$ is inner from both sides.

(iii) $\mathcal{X}$ ($\mathcal{X}^{-1}$) is the I/O map of the spectral DLS $\phi_{P_0^{\text{crit}}}$ ($\phi^{-1}_{P_0^{\text{crit}}}$, respectively), with the realizations

$$\phi_{P_0^{\text{crit}}} = \begin{pmatrix} A & B \\ -K_{P_0^{\text{crit}}} & I \end{pmatrix}, \quad \phi^{-1}_{P_0^{\text{crit}}} = \begin{pmatrix} A_{P_0^{\text{crit}}} & B \\ K_{P_0^{\text{crit}}} & I \end{pmatrix},$$

and $\mathcal{N}_P$ is the I/O map of the DLS

$$\phi_P \phi^{-1}_{P_0^{\text{crit}}} = \begin{pmatrix} A_{P_0^{\text{crit}}} & B \\ K_{P_0^{\text{crit}}} - K_P & I \end{pmatrix},$$

where $A_{P_0^{\text{crit}}} := A + BK_{P_0^{\text{crit}}}$.

Proof. To prove claim (i), we note that we have the factorization of the Popov operator, for all $P \in ric_{uw}(\Phi, J)$

$$\mathcal{D}^* J \mathcal{D} = \mathcal{D}^*_{\phi_P} \Lambda_P \mathcal{D}_{\phi_P} = \mathcal{D}^*_{\phi_{P_0^{\text{crit}}}} \Lambda_{P_0^{\text{crit}}} \mathcal{D}_{\phi_{P_0^{\text{crit}}}},$$

by claim (i) of Theorem 142. But then, $\mathcal{X} := \mathcal{D}_{\phi_{P_0^{\text{crit}}}}$ is a $\Lambda_{P_0^{\text{crit}}}$-spectral factor of $\mathcal{D}^*_{\phi_P} \Lambda_P \mathcal{D}_{\phi_P}$, and then, by Proposition 82, $\mathcal{D}_{\phi_P} = \mathcal{N}_P \mathcal{X}$ where $\mathcal{N}_P := \mathcal{D}_{\phi_P} \mathcal{X}^{-1}$ is a $(\Lambda_P, \Lambda_{P_0^{\text{crit}}})$-inner-outer factorization, and the outer part has a bounded inverse. Both $\operatorname{range}(\mathcal{D}_{\phi_P}) \bar{\pi}_+$ and $\operatorname{range}(\mathcal{D}_{\phi_P})$ are closed because ϕ_P is Λ_P-coercive, by Proposition 69.

In order to prove claim (ii), note that claim (ii) of Proposition 135 implies that range $(\mathcal{D}_{\phi_P}) = \ell^2(\mathbf{Z}; U)$, because ϕ_P is Λ_P-coercive, the feed-through operator $\mathcal{D}_{\phi_P}(0) = I$ has a bounded inverse and the input operator B of ϕ_P is Hilbert–Schmidt. If $J \geq 0$, then both the indicators Λ_P and $\Lambda_{P_0^{\text{crit}}}$ are positive, by Corollary 146. It follows that $\Lambda_P^{\frac{1}{2}} \mathcal{N}_P \Lambda_{P_0^{\text{crit}}}^{-\frac{1}{2}}$ is inner from both sides, by claim (ii) of Lemma 134.

To prove claim (iii), Proposition 17 is used. Only the claim concerning $\mathcal{N}_P$ is somewhat nontrivial, and the outlines are given below. For a more complete presentation using the same technique, see the proof of (ii) of Proposition 148. First, the product DLS $\phi_P \phi_{P_0^{\text{crit}}}^{-1}$ is written

$$\phi_P \phi_{P_0^{\text{crit}}}^{-1} = \left(\begin{array}{cc} \begin{bmatrix} A & BK_{P_0^{\text{crit}}} \\ 0 & A_{P_0^{\text{crit}}} \end{bmatrix} & \begin{bmatrix} B \\ B \end{bmatrix} \\ \begin{bmatrix} -K_P & K_{P_0^{\text{crit}}} \end{bmatrix} & I \end{array} \right).$$

Its the semigroup generator is seen to satisfy

$$\begin{bmatrix} A & BK_{P_0^{\text{crit}}} \\ 0 & A_{P_0^{\text{crit}}} \end{bmatrix}^j = \begin{bmatrix} A^j & A^j_{P_0^{\text{crit}}} - A^j \\ 0 & A^j_{P_0^{\text{crit}}} \end{bmatrix}.$$

Finally, looking at the Taylor coefficients of the I/O map, we see

$$\begin{bmatrix} -K_P & K_{P_0^{\text{crit}}} \end{bmatrix} \begin{bmatrix} A^j & A^j_{P_0^{\text{crit}}} - A^j \\ 0 & A^j_{P_0^{\text{crit}}} \end{bmatrix} \begin{bmatrix} B \\ B \end{bmatrix} = (K_{P_0^{\text{crit}}} - K_P) A^j_{P_0^{\text{crit}}} B.$$

We consider this claim to be proved. $\square$

Let $P \in ric_{uw}(\Phi, J)$ be arbitrary. To the spectral DLS ϕ_P, we can associate a critical control problem with the cost operator Λ_P, see Section 2.2. It follows from Proposition 147 and Theorem 114 that if one of these problems is solvable (in the sense of Theorem 114), then they all are, together with the original critical control problem associated to Φ and J. This is true just because all the I/O maps have the same outer factor $\mathcal{X}$, if they have such factorization at all.

In Proposition 147, a particular fixed regular critical solution $P_0^{\text{crit}} \in ric_0(\Phi, J)$ was picked and the proposition was formulated relative to this solution. One should ask whether we would have obtained another factorization $\mathcal{D}_{\phi_P} = \mathcal{N}_P' \mathcal{X}'$ for another critical solution, say $P_2^{\text{crit}} \in Ric_{uw}(\phi, J)$. The answer in negative. In the proof of Corollary 116, we have seen that the indicators of the critical solutions are all the same: $\Lambda_{P^{\text{crit}}} = \Lambda_{P_2^{\text{crit}}}$. Then we might have two different $(\Lambda_P, \Lambda_{P^{\text{crit}}})$ -inner-outer factorizations $\mathcal{D}_{\phi_P} = \mathcal{N}_P \mathcal{X} = \mathcal{N}_P' \mathcal{X}'$. However, the feed-through parts of both $\mathcal{X}$ and $\mathcal{X}'$ are normalized to identity operator I, and this implies by Proposition 83 that $\mathcal{X} = \mathcal{X}'$ as I/O maps. It now follows that the factor $\mathcal{N}_P$ does not depend on the choice of the critical solution. However, the realizations $\phi_{P^{\mathrm{crit}}}$, $\phi_{P_2^{\mathrm{crit}}}$ for $\mathcal{X}$, $\mathcal{X}'$ might be different, because the feedback operators $K_{P_0^{\mathrm{crit}}}$, $K_{P_2^{\mathrm{crit}}}$ might differ. However, this can happen only in the orthogonal complement of $\overline{\mathrm{range}\,(\mathcal{B})}$. So, if $\overline{\mathrm{range}\,(\mathcal{B})} = H$, then $K_{P_0^{\mathrm{crit}}} = K_{P_2^{\mathrm{crit}}}$ as in the proof of Proposition 116, and the possible nonuniqueness of the realizations disappears.

The following proposition gives us realizations for chains of certain I/O maps. It is instructive to compare the DLS ϕ_{P_1,P_2} to the realization of $\mathcal{N}_P$, given in claim (iii) of Proposition 147. We remark that the following tedious calculations depend on the properties of the Riccati equation only in a very implicit manner, if at all.

Proposition 148. *Let Φ be an DLS, and J self-adjoint. Let $P_1, P_2, P_3 \in ric(\Phi, J)$ be arbitrary. Define the DLS*

$$\phi_{P_1,P_2} = \begin{pmatrix} A_{P_2} & B \\ K_{P_2} - K_{P_1} & I \end{pmatrix}$$

and we denote $\mathcal{N}_{P_1,P_2} := \mathcal{D}_{\phi_{P_1,P_2}}$. Then

(i) $\mathcal{N}_{P_1,P_2}^{-1} = \mathcal{N}_{P_2,P_1}$,

(ii) $\mathcal{N}_{P_1,P_2}\mathcal{N}_{P_2,P_3} = \mathcal{N}_{P_1,P_3}$,

(iii) Assume, in addition, that the conditions of Theorem 114 hold. Then $\mathcal{N}_{P_1,P_0^{\mathrm{crit}}} = \mathcal{N}_{P_1}$ is the $(\Lambda_{P_1}, \Lambda_{P_0^{\mathrm{crit}}})$-inner factor of $\mathcal{D}_{\phi_P}$. Also $\mathcal{N}_{P_1}\mathcal{N}_{P_2}^{-1} = \mathcal{N}_{P_1,P_2}$.

Proof. To prove claim (i), use claim (i) of Proposition 17. A direct calculation gives

$$\phi_{P_1,P_2}^{-1} = \begin{pmatrix} A_{P_2} - B(K_{P_2} - K_{P_1}) & B \\ -(K_{P_2} - K_{P_1}) & I \end{pmatrix} = \begin{pmatrix} A_{P_1} & B \\ K_{P_1} - K_{P_2} & I \end{pmatrix} = \phi_{P_2,P_1},$$

proving claim (i). To verify claim (ii), claim (ii) of Proposition 17 is now used. We obtain

$$\begin{aligned} \phi_{P_1,P_2}\phi_{P_2,P_3} &= \begin{pmatrix} A_{P_2} & B \\ K_{P_2} - K_{P_1} & I \end{pmatrix} \begin{pmatrix} A_{P_3} & B \\ K_{P_3} - K_{P_2} & I \end{pmatrix} \\ &= \begin{pmatrix} \begin{bmatrix} A_{P_2} & B(K_{P_3} - K_{P_2}) \\ 0 & A_{P_3} \end{bmatrix} & \begin{bmatrix} B \\ B \end{bmatrix} \\ \begin{bmatrix} (K_{P_2} - K_{P_1}) & (K_{P_3} - K_{P_2}) \end{bmatrix} & I \end{pmatrix}. \end{aligned} \tag{3.43}$$

Now we have to consider the I/O map of the product DLS $\phi_{P_1,P_2}\phi_{P_2,P_3}$. We first see that its feed-through operator I is that of $\mathcal{D}_{\phi_{P_1,P_3}}$. The rest is studied by applying the Taylor series formula (1.7) for the I/O map of a DLS on the right hand side of (3.43). The whole trick lies in noting that the semigroup generator satisfies $\left[\begin{smallmatrix} A_{P_2} & B(K_{P_3}-K_{P_2}) \\ 0 & A_{P_3} \end{smallmatrix}\right] = \left[\begin{smallmatrix} A_{P_2} & A_{P_3}-A_{P_2} \\ 0 & A_{P_3} \end{smallmatrix}\right]$, and we have for the block matrices of this kind

$$A^j_{(\phi_{P_1,P_2}\phi_{P_2,P_3})} = \begin{bmatrix} A_{P_2} & A_{P_3} - A_{P_2} \\ 0 & A_{P_3} \end{bmatrix}^j = \begin{bmatrix} A^j_{P_2} & A^j_{P_3} - A^j_{P_2} \\ 0 & A^j_{P_3} \end{bmatrix}$$

for all $j \geq 0$, as can easily be shown by induction. We now obtain for all $j \geq 0$

$$\begin{aligned}
&C_{(\phi_{P_1,P_2}\phi_{P_2,P_3})}A^j_{(\phi_{P_1,P_2}\phi_{P_2,P_3})}B_{(\phi_{P_1,P_2}\phi_{P_2,P_3})} \\
&= \begin{bmatrix}(K_{P_2} - K_{P_1}) & (K_{P_3} - K_{P_2})\end{bmatrix} \begin{bmatrix} A^j_{P_2} & A^j_{P_3} - A^j_{P_2} \\ 0 & A^j_{P_3} \end{bmatrix} \begin{bmatrix} B \\ B \end{bmatrix} \\
&= (K_{P_2} - K_{P_1})A^j_{P_2}B + (K_{P_2} - K_{P_1})(A^j_{P_3} - A^j_{P_2})B \\
&+ (K_{P_3} - K_{P_2})A^j_{P_3}B \\
&= (K_{P_2} - K_{P_1})A^j_{P_3}B + (K_{P_3} - K_{P_2})A^j_{P_3}B \\
&= (K_{P_3} - K_{P_1})A^j_{P_3}B.
\end{aligned}$$

But these equal the corresponding coefficients of ϕ_{P_1,P_3}, and claim (ii) is proved. Claim (iii) follows immediately from claim (iii) of Proposition 147. The last claim follows from the previous claims: $\mathcal{N}_{P_1}\mathcal{N}_{P_2}^{-1} = \mathcal{N}_{P_1,P_0^{\text{crit}}}\mathcal{N}_{P_2,P_0^{\text{crit}}}^{-1} = \mathcal{N}_{P_1,P_0^{\text{crit}}}\mathcal{N}_{P_0^{\text{crit}},P2} = \mathcal{N}_{P_1,P2}$. □

The I/O maps of the DLSs ϕ_{P_1,P_2} will play a crucial role in Chapter 5.

3.7 Notes and references

Spectral factorization and DARE

Let $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ be an output stable and I/O stable DLS, and $J \in \mathcal{L}(Y)$ a cost operator. In the present chapter, we have developed a spectral factorization theory for the Popov operator $\mathcal{D}_\phi^* J \mathcal{D}_\phi$ of DLS ϕ on infinite dimensional separable Hilbert spaces. We have seen that a subset $ric_0(\phi, J) \subset Ric(\phi, J)$ of the regular H^∞ solutions of the H^∞DARE

$$\text{(3.44)} \qquad \begin{cases} A^*PA - P + C^*JC = K_P^* \Lambda_P K_P, \\ \Lambda_P = D^*JD + B^*PB, \\ \Lambda_P K_P = -D^*JC - B^*PA, \end{cases}$$

can be used to parameterize (at least a nontrivial subset of) the stable spectral factors. Under nonnegativity assumptions, we have given characterizations of the subset $ric_0(\phi, J)$ in the full solution set $Ric(\phi, J)$.

We now briefly discuss the literature relating to spectral factorization and algebraic Riccati equations. The general idea of using the (matrix) Riccati equations for the canonical and spectral factorization of rational transfer functions is quite old. Both the continuous and discrete time finite dimensional case is considered in [49, Chapters 10 and 19] (Lancaster and Rodman, 1995) and the references therein. At the end of both chapters, a short account for the history of such factorizations is given. For the classical existence results and applications of canonical factorizations of matrix-valued meromorphic functions, see [16] (Clancey and Gohberg, 1981). Some computational aspects of the spectral factorization are considered in [101] (M. Weiss, 1994). The discrete time infinite dimensional result [45, Theorem 4.6] (Helton, 1976) is closely related to our Theorem 142 on the spectral factorization, but the information structure of the system and DARE is that of a LQDARE

$$\text{(3.45)} \qquad \begin{cases} A^*PA - P + C^*JC = A^*PB \cdot \Lambda_P^{-1} \cdot B^*PA, \\ \Lambda_P = D^*JD + B^*PB, \end{cases}$$

where the input is penalized by direct cost. The reasons why we discuss the more general DARE (3.44) instead of LQDARE (3.45) will be discussed in Section 4.9. In [45, Theorem 4.6], a “nonvanishing residual cost” has been included in the Popov function, whose spectral factor is to be calculated. A similar modification can be done to Theorem 142, see Definition 143 and Lemma 144.

The related results in [36] (Fuhrmann, 1995) and [39] (Fuhrmann and Hoffman, 1997) seem to be most complete. A reference to an earlier spectral factorization paper [26] (Finesso and Picci, 1982) is also given there. Spectral factorization of continuous time infinite-dimensional transfer functions are considered in [36] by using an algebraic Riccati equation. The paper [39] deals with the discrete time state space state space factorization of rational inner functions. There is a considerable overlap between our results in [61], [62] and those given in [36], [39]. We learned about this at MTNS98 conference (Padova, July 1998) in a discussion with Fuhrmann, after the papers [61] and [62] (Malinen, 1998) were completed in their original form and the related conference article [59] (Malinen, 1998) had been presented. In technical style and basic assumptions these works are quite different from ours, which makes is a hard but a rewarding task to compare the continuous time results of [36] to our discrete time results. It appears that the results are in harmony to each other in a beautiful way.

Fuhrmann approaches the general structure from the minimal spectral factorization point of view, rather that from the Riccati equation point of view that we have adopted. In [36], unstable systems and spectral factors are parameterized by solutions of an algebraic Riccati equation of a quite special kind. We can roughly say that our work is more complete with respect to the Riccati equations and classes of stable systems, whereas more general spectral factors and unstable systems are considered in [36]. The work [36] is written under the standing hypothesis of strict noncyclicity of the spectral function, known as the Popov function in our work. This implies that the spectral function has a meromorphic extension from the imaginary axis to the rest of the complex plane. In [36, Theorem 2.1], this assumption is associated to the existence of Douglas–Shapiro–Shields factorization of the spectral function, see [27], [35] and [22] (Douglas, Shapiro and Shields, 1970). We remark that many results such as [36, Theorem 6.1] are genuinely two-directional where our analogous results Lemma 138, Proposition 139 and Theorem 142 are not. For example, in Theorem 142 we do not prove that all spectral factors of $\mathcal{D}_\phi^* J \mathcal{D}_\phi$ can be associated to a solution of DARE. Only those spectral factors are parameterized by the solutions in $ric_0(\phi, J)$ that can be realized in a particular way, with the original semigroup generator A and the input operator B of DLS $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix} \right)$. The full parameterization of the minimal spectral factors in [36] comes from the additional minimality assumption of the used realization, and the use of a state space isomorphism result that does not hold in the full generality in our setting. The lack of a general state space isomorphism is quite disappointing, and it makes the state space idea somewhat "too good to be true" for general infinite-dimensional systems, see the discussion in [35, Chapter 3].

Further application of algebraic Riccati equations

A traditional application of the algebraic Riccati equation, associated to an unstable system, is to find a (nonnegative) solution, such that the (semigroup of the) closed loop system is (at least partially) (exponentially) stabilized; see e.g. [9] (Callier, Dumortier and Winkin, 1995), [23] (Dumortier, 1998), and [107] (Wimmer, 1996), to mention a few possible references.

In this book, we mainly consider DAREs associated to an output stable and I/O stable DLS ϕ. The impetus to look into DAREs associated to stable systems came to us from the works [83], [85] and [103] (Staffans and G. Weiss). The feedback stabilization of (the semigroup or the I/O map of) an unstable DLS is seen as a separate problem, to be discussed elsewhere. We assume that the DLS ϕ "already output and I/O stabilized" by some means — not necessarily by a static state feedback law, induced by some (nonnegative, stabilizing, maximal nonnegative) solution of the DARE. In this case, the stabilized system could possess a nontrivial outer factor, and we conclude that a DARE theory dealing with only inner I/O maps is not sufficient. We also remark that there exists genuinely I/O stable (discrete time) processes that need not be stabilized; consider, for example, a (discrete time) Lax–Phillips scattering where the scattering process is usually described by (a DLS that has an inner) H^∞ transfer function. Similar examples can be produced from realizations of the characteristic functions of C_{00} contractions. Because of the standing I/O stability assumption of the DLS ϕ, the connections to the operator-valued function theory become very important, as will be seen in Chapter 4. We conclude that it would be quite desirable to have a sufficiently general DARE theory of stable systems to deal with these situations.

The application of and references to the algebraic Riccati equations in linear quadratic control problems has already been discussed in Section 2.8. The algebraic Riccati equation appears (in an adjoint form) in the theory of the Kalman filter for the stochastic state estimation, see [48, Section 2.6] (Kalman, Falb and Arbib, 1969), [5, Chapter 10] (Bitmead and Gevers, 1991) and the references therein. The latter reference contains a nice overview of the various types and applications of the (matrix) algebraic, difference and differential Riccati equations, both in continuous and discrete time.

Chapter 4

Inner-Outer Factorization

4.1 Introduction

Let $\phi := \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ be an output stable and I/O stable DLS and $J \in \mathcal{L}(Y)$ a cost operator. In Chapter 3 we defined an algebraic Riccati equation, called H^∞DARE and denoted by $ric(\phi, J)$. It appeared that some solutions of an H^∞DARE are more interesting than others; these are the H^∞ solutions $P \in ric(\phi, J) \subset Ric(\phi, J)$ and the regular H^∞ solutions $P \in ric_0(\phi, J) \subset ric(\phi, J)$. Then the solutions $P \in ric_0(\phi, J)$ are associated to the stable spectral factorizations of the Popov operator $\mathcal{D}_\phi^* J \mathcal{D}_\phi$. The main theme of this chapter is to connect a subset

$$\{P \in ric_0(\phi, J) \mid P \geq 0\} \subset ric_0(\phi, J)$$

to the factorizations of the I/O map $\mathcal{D}_\phi$ into causal, shift-invariant and I/O stable factors. As a result, we obtain a theory of the regular H^∞ solutions of a H^∞DARE and simultaneously, an inner-outer type state space factorization theory for operator-valued bounded analytic functions.

In Chapter 3, many of the results did not require that the self-adjoint cost operator $J \in \mathcal{L}(Y)$ is nonnegative. However, in this chapter it is almost a standing hypothesis that J is nonnegative, and the Popov operator $\bar{\pi}_+ \mathcal{D}_\phi^* J \mathcal{D}_\phi \bar{\pi}_+$ is boundedly invertible on $\ell^2(\mathbf{Z}_+; U)$. This is a sufficient condition for the existence of the regular critical solution $P_0^{\mathrm{crit}} := \left(\mathcal{C}_\phi^{\mathrm{crit}}\right)^* J \mathcal{C}_\phi^{\mathrm{crit}} \in ric_0(\phi, J)$, by Proposition 117. As in Chapter 3, the input space U can be allowed to be infinite-dimensional, provided that the input operator B of ϕ is Hilbert–Schmidt.

We give a short outline of the contents of this chapter. To each solution $P \in Ric(\phi, J)$, two families of algebraic Riccati equations are introduced in Section

4.2. These are associated to the spectral DLS ϕ_P and the inner DLS ϕ^P, centered at the solution $P \in Ric(\phi, J)$. For the definitions of ϕ_P and ϕ^P, see Definition 95. The spectral DARE $Ric(\phi_P, \Lambda_P)$ is the DARE associated to the ordered pair (ϕ_P, Λ_P), where the cost operator $\Lambda_P := D^*JD + B^*PB$ is the indicator of the solution P. Analogously, the inner $Ric(\phi^P, J)$ is associated to the ordered pair (ϕ^P, J). The solution sets of spectral and inner DAREs have natural relations to the solution set $P \in Ric(\phi, J)$ of the original DARE, see Lemmas 156 and 157. The transitions from the original DLS ϕ to the inner DLS ϕ^P and the spectral DLS ϕ_P are basic operations that we use in Section 4.6 to obtain order-theoretic descriptions of the solution (sub)set $ric_0(\phi, J) \subset Ric(\phi, J)$. The results of Section 4.2 are proved by algebraic manipulations, and do not require DARE $Ric(\phi, J)$ to be a H^∞DARE.

We remark that if the spectral DLS ϕ_P, (the inner DLS ϕ^P) is I/O stable and output stable, then the DARE $Ric(\phi_P, J)$, $(Ric(\phi^P, J))$ is a H^∞DARE, and it is associated to the critical control problem of DLS ϕ_P with cost operator Λ_P, (DLS ϕ^P with cost operator J, respectively). Recall that for $P \in Ric(\phi, J)$, ϕ_P is I/O stable and output stable if and only if P is a H^∞ solution, by Definition 107. For this reason it is important that, under technical assumptions, all "reasonable" solutions $P \in Ric(\phi, J)$ are shown to be (even regular) H^∞ solutions, see Corollary 140 and Equation 3.27. We conclude that the question whether the spectral DARE $Ric(\phi_P, \Lambda_P)$ is an H^∞DARE has already been settled in Chapter 3. It requires further study to give analogous conditions for the inner DARE $Ric(\phi^P, J)$ to be a H^∞DARE, and this study is carried out in the present chapter. When this is done, we will have shown that the general class of H^∞DAREs is closed under the transitions to spectral and inner DAREs at regular H^∞ solutions $P \in ric_0(\phi, J)$.

A fair amount of stability theory for DLSs is needed for the further results. This is provided by the scratch of an infinite-dimensional Liapunov equation theory of Section 4.3. An essential part of the Liapunov theory is based on monotonicity techniques, requiring the nonnegativity of the cost operator J, or some closely related assumption. By Corollary 167, we conclude that ϕ^P is output stable if $P \in Ric(\Phi, J)$ is nonnegative and the cost operator $J > 0$ has a bounded inverse, under quite general assumptions. It requires more work and stronger assumptions to make the inner DLS ϕ^P I/O stable and $Ric(\phi^P, J)$ an H^∞DARE.

The first main results of this paper are given in Section 4.4. We conclude that each nonnegative $P \in ric_0(\phi, J)$ gives a factorization of the I/O map

$$J^{\frac{1}{2}}\mathcal{D}_\phi = J^{\frac{1}{2}}\mathcal{D}_{\phi^P} \cdot \mathcal{D}_{\phi_P}. \tag{4.1}$$

The causal, shift-invariant factor $J^{\frac{1}{2}}\mathcal{D}_{\phi^P} : \ell^2(\mathbf{Z}; U) \to \ell^2(\mathbf{Z}; Y)$ is densely defined, not necessarily I/O stable, but always strongly H^2 stable. This means that the I/O map $J^{\frac{1}{2}}\mathcal{D}_{\phi^P}$ has a bounded impulse response, and the mapping

$J^{\frac{1}{2}}\mathcal{D}_{\phi^P} : \ell^1(\mathbf{Z};U) \to \ell^2(\mathbf{Z};Y)$ is bounded. If the input operator B of the DLS $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ is a compact Hilbert–Schmidt operator, then this factorization becomes a partial inner-outer factorization where all factors are I/O stable, see Lemma 171 and Theorem 173. In particular, the (properly normalized) inner DARE $Ric(J^{\frac{1}{2}}\phi^P, I)$ (which is equivalent to the inner DARE $Ric(\phi^P, J)$) becomes now a H^∞DARE, provided $P \in ric_0(\phi, J)$. A generalized H^2 factorization is considered in Lemma 174. Furthermore, finite increasing chains of solution in $ric_0(\phi, J)$ give factorizations of the I/O map of Blaschke–Potapov product type, as stated in Theorem 175. However, neither the zeroes nor the singular inner factor of the transfer function $\mathcal{D}_\phi(z)$ (whatever these would mean in the present generality) play any explicit role in this construction.

In Section 4.5, we consider converse results to those given in the previous Section 4.4. In Lemma 181 we show that for $P \in ric_0(\phi, J)$, the I/O stability of $J^{\frac{1}{2}}\phi^P$ implies that $P \geq 0$. Here, an approximate controllability assumption $\overline{\mathrm{range}\,(\mathcal{B}_\phi)} = H$ is made. Theorem 182 is a combination of results given in Sections 4.4 and 4.5. It states, under restrictive technical assumptions, that among the state feedbacks associated to solutions $P \in ric_0(\phi, J)$, it is exactly the nonnegative solutions which output stabilize and I/O stabilize the (normalized closed loop) inner DLS $J^{\frac{1}{2}}\phi^P$. In other words, among the H^∞ solutions of the DARE $ric(\phi, J)$, it is exactly the nonnegative $P \in ric_0(\phi, J)$ which give the factorization (4.1) of the I/O map $\mathcal{D}_\phi$ so that all the factors are I/O stable.

In Section 4.6, we study the partial ordering of the elements of $ric_0(\phi, J)$, as self-adjoint operators. The maximal nonnegative solution in the set $ric_0(\phi, J)$ is considered in Corollary 186, and seen to be the unique regular critical solution $P_0^{\mathrm{crit}} := (\mathcal{C}_\phi^{\mathrm{crit}})^* J \mathcal{C}_\phi^{\mathrm{crit}}$, if the approximate controllability $\overline{\mathrm{range}\,(\mathcal{B}_\phi)} = H$ is assumed. An order-preserving correspondence between the set $ric_0(\phi, J)$ and a set of certain closed shift-invariant subspaces of $\ell^2(\mathbf{Z}_+;U)$ is given in Theorem 187, in the spirit of the classical Beurling–Lax–Halmos Theorem. An order-theoretic characterization of the nonnegative elements of $ric_0(\phi, J)$ is given in Theorem 188.

In Section 4.7 we consider the conditions when the spectral DARE $Ric(\phi_P, \Lambda_P)$ and the inner DARE $Ric(\phi^P, J)$ are H^∞DAREs. Furthermore, the regular H^∞ solutions and the regular critical solutions of both the spectral and inner DAREs are described. Our technical assumptions include approximate controllability $\overline{\mathrm{range}\,(\mathcal{B}_\phi)} = H$ and the Hilbert–Schmidt compactness of the input operator B of the DLS ϕ which is common to all inner and spectral DLSs ϕ^P and ϕ_P. The case of the spectral DARE is dealt in Lemma 189 and Corollary 190. As a byproduct, we see that the set $ric_0(\phi, J)$ is an order-convex subset of $Ric(\phi, J)$ in the following sense: if $P_1, P_2 \in ric_0(\phi, J)$ with $P_2 \leq P_1$, then all $P \in Ric(\phi, J)$ such that $P_2 \leq P \leq P_1$ satisfy $P \in ric_0(\phi, J)$. In Lemma 192 it is shown that the inner DARE $Ric(\phi^P, J)$ is an H^∞DARE if $P \in ric_0(\phi, J)$ is nonnegative and the cost operator $J > 0$ has a bounded inverse — in this case the same P is also the regular critical solution of DARE

$ric(\phi^P, J)$. The full description of the regular H^∞ solutions $ric_0(\phi^P, J)$ of the inner DARE is given in Lemma 193.

In the final section, it is shown that the structure of the H^∞DARE $ric(\phi, J)$ and its inner DARE $ric(\phi^{P_0^{\text{crit}}}, J)$ is similar, where $P_0^{\text{crit}} := (\mathcal{C}_\phi^{\text{crit}})^* J \mathcal{C}_\phi^{\text{crit}} \in ric_0(\phi, J)$ is the regular critical solution. This means that the outer factor of the I/O map $\mathcal{D}_\phi$ is nonessential, from the H^∞DARE point of view. The treatment is similar to that given in Lemmas 192 and 193 for general nonnegative $P \in ric_0(\phi, J)$ but now the cost operator $J \geq 0$ is not required to be boundedly invertible. This result has an application in Section 5.7.

A preliminary version of the contents of this chapter is [62] (Malinen, 1999). The conference article [59], containing many of the results of Chapter 3 and this chapter, has been presented in MTNS98 conference (Padova, July, 1998).

4.2 Chains of DAREs

In this section, we write down a number of algebraic properties associated to iterated transitions to inner and spectral minimax nodes, DLSs and DLSs. The algebraic Riccati equation, together with the spectral DLS ϕ_P and the inner DLS ϕ^P, has already been introduced in Section 3.2. The spectral DLS ϕ_P has been extensively used in Chapter 3 because its I/O map gives spectral factors for the Popov operator $\bar{\pi}_+\mathcal{D}_\phi^* J\mathcal{D}_\phi\bar{\pi}_+$. For the inner DLS ϕ^P we have not had much application until now. The results of this section are proved by purely algebraic manipulations, and do not require input, output or I/O stability of any of the DLSs considered. The definiteness of the cost operator J does not play any role, either. Later, in Sections 4.7 and 4.8, the analogous structure of the H^∞DARE is considered, for $J \geq 0$.

We associate two chains of DAREs to a given DARE $Ric(\phi, J)$. The elements of these chains are called the spectral and inner DAREs. Both the chains are indexed by the solutions $P \in Ric(\phi, J)$. These new DAREs make it easy to "move" in the solution set $Ric(\phi, J)$ of the original DARE, provided we can solve these Riccati equations. The presented structure (in some form) are well known to specialists in Riccati equations, but they are hard to locate in the literature. For us, the presented chains of DAREs are invaluable tools in sections 4.4 and 4.6.

Because DARE $Ric(\phi, J)$ does not solely depend on the DLS but also on the cost operator J, it is not sufficient to consider the DLS ϕ alone in this section. Instead, we have to consider the pairs (ϕ, J) that we call minimax nodes. Each minimax node defines a cost optimization problem, as defined in Chapter 2 for I/O stable DLSs. To this cost optimization problem, a Riccati equation is associated in a natural way. We first define two operations on the minimax nodes, and give their basic properties. The DARE is introduced in the familiar form in Definition 153.

Definition 149. *Let $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ be a DLS with input space U, the state space H and output space Y. Let $J = J^* \in \mathcal{L}(Y)$ be a cost operator. Let $P = P^* \in \mathcal{L}(H)$ be arbitrary, such that the operator $\Lambda_P := D^*JD + B^*PB$ has a bounded inverse.*

(i) The ordered pair (ϕ, J) is called the minimax node, associated to the DLS ϕ and cost operator J.

(ii) The spectral minimax node of (ϕ, J) at P is defined by

$$(\phi, J)_P := \left(\begin{pmatrix} A & B \\ -K_P & I \end{pmatrix}, \Lambda_P \right),$$

where $\Lambda_P := D^*JD + B^*PB$ and $\Lambda_P K_P := -D^*JC - B^*PA$. The operator Λ_P is called the indicator of P, and K_P is called the feedback operator of P.

(iii) The inner minimax node of (ϕ, J) *at* P *is defined by*

$$(\phi, J)^P := \left(\begin{pmatrix} A_P & B \\ C_P & D \end{pmatrix}, J \right),$$

where $A_P := A + BK_P$*,* $C_P = C + DK_P$*, and* K_P *is as above. The operator* A_P *is called the (closed loop) semigroup generator of* P*, and* C_P *is called the (closed loop) output operator of* P*.*

We call two DLSs equal, if their defining ordered operator quadruples (in difference equation form) are equal. Two minimax nodes are equal, if their DLSs are equal, and the cost operators are equal. In this case we write $(\phi_1, J_1) \equiv (\phi_2, J_2)$.

To each self-adjoint operator $P \in \mathcal{L}(H)$, two additional DLSs are associated:

Definition 150. *Let* (ϕ, J)*,* K_P*,* A_P *and* C_P *be as in Definition 149. Let* $P = P^* \in \mathcal{L}(H)$ *be arbitrary, such that* $D^*JD + B^*PB$ *has a bounded inverse.*

(i) The DLS

$$\phi_P := \begin{pmatrix} A & B \\ -K_P & I \end{pmatrix}$$

is the spectral DLS, associated to the minimax node (ϕ, J)*, and centered at* P*.*

(ii) The DLS

$$\phi^P := \begin{pmatrix} A_P & B \\ C_P & D \end{pmatrix}$$

is called the inner DLS, associated to the minimax node (ϕ, J)*, and centered at* P*.*

So, we can write (by definitions)

$$(\phi, J)_P = (\phi_P, \Lambda_P), \quad (\phi, J)^P = (\phi^P, J),$$

instead of formulae appearing in parts (ii) and (iii) of Definition 149. The iterated transitions to inner and spectral minimax nodes behave as follows.

Proposition 151. *Let* (ϕ, J) *be a minimax node. Then the following holds for*

$P_1 = P_1^* \in \mathcal{L}(H)$, $P_2 = P_2^* \in \mathcal{L}(H)$ and $\Delta P := P_2 - P_1$.

$$\left((\phi, J)^{P_1}\right)_{P_2} \equiv \left(\phi^{P_1}, J\right)_{P_2} \equiv \left(\begin{pmatrix} A_{P_1} & B \\ K_{P_1} - K_{P_2} & I \end{pmatrix}, \Lambda_{P_2}\right), \tag{4.2}$$

$$\left((\phi, J)^{P_1}\right)^{P_2} \equiv \left(\phi^{P_1}, J\right)^{P_2} \equiv \left(\phi^{P_2}, J\right), \tag{4.3}$$

$$\left((\phi, J)_{P_1}\right)_{\Delta P} \equiv \left(\phi_{P_1}, \Lambda_{P_1}\right)_{\Delta P} \equiv \left(\phi_{P_2}, \Lambda_{P_2}\right), \tag{4.4}$$

$$\left((\phi, J)_{P_1}\right)^{\Delta P} \equiv \left(\phi_{P_1}, \Lambda_{P_1}\right)^{\Delta P} \equiv \left(\begin{pmatrix} A_{P_2} & B \\ K_{P_2} - K_{P_1} & I \end{pmatrix}, \Lambda_{P_2}\right). \tag{4.5}$$

Proof. As before, denote by Λ_P, K_P the indicator and feedback operator, associated to the minimax node (ϕ, J) and $P \in \mathcal{L}(H)$. We start with proving equation (4.2). By $\tilde{\Lambda}_{P_2}$ and $\tilde{K}_{P_2}$ denote the indicator and feedback operator, associated to the minimax node (ϕ^{P_1}, J) and $P_2 \in \mathcal{L}(H)$. It is easy to see that $\tilde{\Lambda}_{P_2} = \Lambda_{P_2}$. The feedback operator of the inner DLS ϕ^{P_1} at P_2 satisfies $\tilde{K}_{P_2} = K_{P_2} - K_{P_1}$ because

$$\begin{aligned} \tilde{K}_{P_2} &= \Lambda_{P_2}^{-1}\left(-D^* J C_{P_1} - B^* P_2 A_{P_1}\right) \\ &= \Lambda_{P_2}^{-1}\left((-D^* J C - B^* P_2 A) - (D^* J D + B^* P_2 B) K_{P_1}\right) \\ &= \Lambda_{P_2}^{-1}\left(\Lambda_{P_2} K_{P_2} - \Lambda_{P_2} K_{P_1}\right) = K_{P_2} - K_{P_1}, \end{aligned} \tag{4.6}$$

where $A_{P_1} = A + BK_{P_1}$ and $C_{P_1} = C + DK_{P_1}$, by part (ii) of Definition 149. Now (4.2) follows.

We proceed to prove equality (4.3). By part (iii) of Definition 149, we have

$$\left(\phi^{P_1}, J\right)^{P_2} \equiv \left(\begin{pmatrix} \tilde{A}_{P_2} & B \\ \tilde{C}_{P_2} & I \end{pmatrix}, J\right),$$

where the semigroup generator satisfies

$$\tilde{A}_{P_2} = A_{P_1} + B\tilde{K}_{P_2} = (A + BK_{P_1}) + B(K_{P_2} - K_{P_1}) = A + BK_{P_2} = A_{P_2},$$

and for the output operator we have

$$\tilde{C}_{P_2} = C_{P_1} + D\tilde{K}_{P_2} = (C + DK_{P_1}) + D(K_{P_2} - K_{P_1}) = C + DK_{P_2} = C_{P_2}$$

because $\tilde{K}_{P_2} = K_{P_2} - K_{P_1}$, as already shown in the proof of claim (4.2). This proves claim (4.3).

From now on, let $\tilde{\Lambda}_{\Delta P}$ and $\tilde{K}_{\Delta P}$ denote the indicator and feedback operator, associated to the spectral minimax node (ϕ_{P_1}, J). Denote also $\Delta P := P_2 - P_1$. Then

$$\begin{aligned} \tilde{\Lambda}_{\Delta P} &= I^* \cdot \Lambda_{P_1} \cdot I + B^* \Delta P B \\ &= D^* J D + B^* P_1 B + B^*(P_2 - P_1)B = \Lambda_{P_2}, \end{aligned} \tag{4.7}$$

and

$$\Lambda_{P_2}\tilde{K}_{\Delta P} = \tilde{\Lambda}_{\Delta P}\tilde{K}_{\Delta P} = -I^* \cdot \Lambda_{P_1} \cdot (-K_{P_1}) - B^*\Delta PA \\ = -D^*JC - B^*P_1A - B^*(P_2 - P_1)A = \Lambda_{P_2}K_{P_2}, \tag{4.8}$$

or $\tilde{K}_{\Delta P} = K_{P_2}$. But this gives for the spectral minimax node

$$(\phi_{P_1}, \Lambda_{P_1})_{\Delta P} \equiv \left(\begin{pmatrix} A & B \\ -\tilde{K}_{\Delta P} & I \end{pmatrix}, \tilde{\Lambda}_{\Delta P} \right) \equiv \left(\begin{pmatrix} A & B \\ -K_{P_2} & I \end{pmatrix}, \Lambda_{P_2} \right),$$

and equality (4.4) follows. It remains to consider the minimax node $(\phi_{P_1}, \Lambda_{P_1})^{\Delta P}$. By part (iii) of Definition 149, we have

$$(\phi_{P_1}, J)^{P_2} \equiv \left(\begin{pmatrix} \tilde{A}_{\Delta P} & B \\ \tilde{C}_{\Delta P} & I \end{pmatrix}, \tilde{\Lambda}_{\Delta P} \right)$$

where $\tilde{\Lambda}_{\Delta P} = \Lambda_{P_2}$ as above,

$$\tilde{A}_{\Delta P} = A + B\tilde{K}_{\Delta P} = A + BK_{P_2} = A + BK_{P_2} = A_{P_2},$$

and

$$\tilde{C}_{\Delta P} = -K_{P_1} + \tilde{K}_{\Delta P} = -K_{P_1} + K_{P_2}.$$

This proves the final claim (4.5). □

The following "commutation" result will be important in applications:

Corollary 152. *Let (ϕ, J) be a minimax node, and $P_1, P_2 \in \mathcal{L}(H)$ self-adjoint. Then*

$$\left((\phi_{P_1})^{P_2 - P_1}, \Lambda_{P_1} \right) \equiv \left((\phi^{P_2})_{P_1}, \Lambda_{P_1} \right).$$

Proof. This is an immediate consequence of formulae (4.2) and (4.5) of Proposition 151. □

Now we have introduced the notion of a minimax node, and defined two algebraic operations on such nodes: transition to inner and spectral minimax nodes. In the following definition, a discrete time algebraic Riccati equation (DARE) is associated to each minimax node in the familiar form, see Definition 105.

Definition 153. *Let $(\phi, J) \equiv \left(\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix} \right), J \right)$ be a minimax node. Then the following system of operator equations*

$$\begin{cases} A^*PA - P + C^*JC = K_P^*\Lambda_P K_P \\ \Lambda_P = D^*JD + B^*PB \\ \Lambda_P K_P = -D^*JC - B^*PA \end{cases} \tag{4.9}$$

is called the discrete time algebraic Riccati equation (DARE) and denoted by $Ric(\phi, J)$. The linear operators are required to satisfy $\Lambda_P, \Lambda_P^{-1} \in \mathcal{L}(U)$ and $K_P \in \mathcal{L}(H; U)$. Here P is a unknown self-adjoint operator to be solved. If $P \in \mathcal{L}(H)$ satisfies (4.9), we write $P \in Ric(\phi, J)$.

As before, we use the same symbol $Ric(\phi, J)$ both for the solution set of a DARE, and the DARE itself. This should not cause confusion. When we write expressions such as

$$P \in Ric(\phi, J), \quad Ric(\phi, J) = Ric(\phi, J), \quad Ric(\phi, J) \subset Ric(\phi, J),$$

the symbol $Ric(\phi, J)$ denotes the solution set. Clearly, different minimax nodes can give the same DARE because the DARE depends on the operators C^*JC, D^*JC, and D^*JD, but not directly on C, D, or J. When two DAREs $Ric(\phi_1, J_1)$ and $Ric(\phi_2, J_2)$ equal in this way, we write $Ric(\phi_1, J_1) \doteq Ric(\phi_2, J_2)$. We have

$$(\phi_1, J_1) \equiv (\phi_2, J_2) \Rightarrow Ric(\phi_1, J_1) \doteq Ric(\phi_2, J_2) \Rightarrow Ric(\phi_1, J_1) = Ric(\phi_2, J_2),$$

and none of the implications is an equivalence. In particular, the equality $Ric(\phi, J) = Ric(\phi, J)$ does not imply that the two Riccati equations were same, and even less that the two minimax nodes were the same. If $(\phi_1, J_1) \equiv (\phi_2, J_2)$, then we write $Ric(\phi_1, J_1) \equiv Ric(\phi_2, J_2)$.

The inner and spectral minimax nodes of an original minimax node (ϕ, J) give rise to new DAREs: namely the inner and spectral DAREs, centered at the self-adjoint operator $P \in \mathcal{L}(U)$. In order to obtain something interesting, we must now require that in fact $P \in Ric(\phi, J)$.

Definition 154. *Let $(\phi, J) \equiv (\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right), J)$ be a minimax node. Let $P \in Ric(\phi, J)$ be arbitrary. Let ϕ_P and ϕ^P as given in Definition 150, and by Λ_P, K_P denote the indicator and feedback operators of P, respectively.*

(i) The DARE $Ric(\phi, J)_P :\equiv Ric(\phi_P, \Lambda_P)$

$$\left\{ \begin{array}{l} A^* \tilde{P} A - \tilde{P} + K_P^* \Lambda_P K_P = \tilde{K}_{\tilde{P}}^* \tilde{\Lambda}_{\tilde{P}} \tilde{K}_{\tilde{P}} \\ \tilde{\Lambda}_{\tilde{P}} = \Lambda_P + B^* \tilde{P} B \\ \tilde{\Lambda}_{\tilde{P}} \tilde{K}_{\tilde{P}} = \Lambda_P K_P - B^* \tilde{P} A \end{array} \right. \tag{4.10}$$

is the spectral (ϕ, J)-DARE, centered at $P \in Ric(\phi, J)$. Here $\tilde{P}$ is an unknown self-adjoint operator to be solved.

(ii) The DARE $Ric(\phi, J)^P :\equiv Ric(\phi^P, J)$

$$\left\{ \begin{array}{l} A_P^* \tilde{P} A_P - \tilde{P} + C_P^* J C_P = \tilde{K}_{\tilde{P}}^* \Lambda_{\tilde{P}} \tilde{K}_{\tilde{P}} \\ \Lambda_{\tilde{P}} = D^* J D + B^* \tilde{P} B \\ \Lambda_{\tilde{P}} \tilde{K}_{\tilde{P}} = -D^* J C_P - B^* \tilde{P} A_P, \end{array} \right. \tag{4.11}$$

is the inner (ϕ, J)-DARE, centered at $P \in Ric(\phi, J)$. Here $\tilde{P}$ is an unknown self-adjoint operator to be solved, and $A_P := A + BK_P$, $C_P := C + DK_P$.

We start with discussing the spectral Riccati equation $Ric(\phi, J)_P$. The following proposition is basic, and serves as a prerequisite for Lemma 156.

Proposition 155. *Let (ϕ, J) be a minimax node. Let $P \in Ric(\phi, J)$. Then $Ric(\phi, J)_P$ can be written in the equivalent form*

$$\begin{cases} A^*\tilde{P}A - \tilde{P} + K_P^*\Lambda_P K_P = K_{P+\tilde{P}}^*\Lambda_{P+P}K_{P+\tilde{P}} \\ \Lambda_{P+\tilde{P}} = D^*JD + B^*(P+\tilde{P})B \\ \Lambda_{P+\tilde{P}}K_{P+\tilde{P}} = -D^*JC - B^*(P+\tilde{P})A. \end{cases}$$

Proof. By equation (4.7), $\tilde{\Lambda}_{\tilde{P}} = \Lambda_{P+\tilde{P}}$, and by equation (4.8), $\tilde{K}_{\tilde{P}} = K_{P+\tilde{P}}$. □

Lemma 156. *Let (ϕ, J) be a minimax node. Let $P \in Ric(\phi, J)$ and $\tilde{P}$ be a bounded self-adjoint operator. Then the following are equivalent*

(i) $P + \tilde{P} \in Ric(\phi, J)$,

(ii) $\tilde{P} \in Ric(\phi, J)_P$.

Proof. Assume claim (i). Because both $P, (P + \tilde{P}) \in Ric(\phi, J)$, we have by Proposition 155

$$\begin{aligned} &A^*(P+\tilde{P})A - (P+\tilde{P}) + C^*JC = K_{P+\tilde{P}}^*\Lambda_{P+\tilde{P}}K_{P+\tilde{P}}, \\ &A^*PA - P + C^*JC = K_P^*\Lambda_P K_P. \end{aligned}$$

Here Λ_Q and K_Q denote the indicator and the feedback operator of the self-adjoint operator Q, relative to the original minimax node (ϕ, J). Subtracting these two Riccati equations we obtain

$$A^*\tilde{P}A - \tilde{P} + K_P^*\Lambda_P K_P = K_{P+\tilde{P}}^*\Lambda_{P+\tilde{P}}K_{P+\tilde{P}}.$$

But now, by Proposition 155, $\tilde{P} \in Ric(\phi, J)_P$, and claim (ii) follows.

For the converse direction, assume claim (ii). Let $P \in Ric(\phi, J)$, $P \in Ric(\phi_P, \Lambda_P)$ $= Ric(\phi, J)_P$ be arbitrary. By adding the DAREs $Ric(\phi, J)$ and $Ric(\phi, J)_P$ we obtain

$$A^*(P+\tilde{P})A - (P+\tilde{P}) + C^*JC = K_{P+\tilde{P}}^*\Lambda_{P+P}K_{P+\tilde{P}}$$

where Proposition 155 has been used again. Thus claim (i) immediately follows. □

The remaining part of this section is devoted to the study of the inner Riccati equation $Ric(\phi, J)^P$. Given any $P \in Ric(\phi, J)$, the relation between the solution sets of $Ric(\phi, J)^P$ and $Ric(\phi, J)$ appears to be very simple.

Lemma 157. *Let (ϕ, J) be a minimax node. Let $P \in Ric(\phi, J)$ be arbitrary. Then the following are equivalent:*

(i) $\tilde{P} \in Ric(\phi, J)^P$,

(ii) $\tilde{P} \in Ric(\phi, J)$.

Proof. We prove the direction (i) $\Rightarrow$ (ii); the proof of the other direction is obtained by reading this proof in the reverse direction. Let $\tilde{P} \in Ric(\phi, J)^P$. Then the left hand side of the first equation in (4.11) takes the form

$$\begin{aligned} (4.12) \qquad & A_P^* \tilde{P} A_P - \tilde{P} + C_P^* J C_P \\ & = A^* \tilde{P} A - \tilde{P} + C^* J C - K_P^* \Lambda_{\tilde{P}} K_{\tilde{P}} - K_{\tilde{P}}^* \Lambda_{\tilde{P}} K_P + K_P^* \Lambda_{\tilde{P}} K_P. \end{aligned}$$

Here Λ_Q and K_Q denote the indicator and the feedback operator of the self-adjoint operator Q, relative to the original minimax node (ϕ, J). By equation (4.6), $\tilde{K}_{\tilde{P}} = K_{\tilde{P}} - K_P$ and the right hand side of the first equation in (4.11) becomes

$$\tilde{K}_{\tilde{P}}^* \Lambda_{\tilde{P}} \tilde{K}_{\tilde{P}} = K_{\tilde{P}}^* \Lambda_{\tilde{P}} K_{\tilde{P}} - K_P^* \Lambda_{\tilde{P}} K_{\tilde{P}} - K_{\tilde{P}}^* \Lambda_{\tilde{P}} K_P + K_P^* \Lambda_{\tilde{P}} K_P.$$

This, together with equation (4.12) gives

$$A^* \tilde{P} A - \tilde{P} + C^* J C = K_{\tilde{P}}^* \Lambda_{\tilde{P}} K_{\tilde{P}}.$$

Thus $\tilde{P} \in Ric(\phi, J)$. This completes the proof. ☐

As an immediate corollary, we can put $Ric(\phi, J)^P$ in a different form

Proposition 158. *Let (ϕ, J) be a minimax node. Let $P \in Ric(\phi, J)$. Then $Ric(\phi, J)^P$ can be written in the equivalent form*

$$\begin{cases} A_P^* \tilde{P} A_P - \tilde{P} + C_P^* J C_P = (K_{\tilde{P}} - K_P)^* \Lambda_{\tilde{P}} (K_{\tilde{P}} - K_P) \\ \Lambda_{\tilde{P}} = D^* J D + B^* \tilde{P} B \\ \Lambda_{\tilde{P}} K_{\tilde{P}} = -D^* J C - B^* \tilde{P} A, \quad \Lambda_P K_P = -D^* J C - B^* P A. \end{cases}$$

Proof. This is because $\tilde{K}_{\tilde{P}} = K_{\tilde{P}} - K_P$, by equation (4.6). ☐

The results of Lemmas 156 and 157 can be given in a short form

$$\begin{aligned} Ric(\phi, J) &= P + Ric(\phi, J)_P = P + Ric(\phi_P, \Lambda_P), \\ Ric(\phi, J) &= Ric(\phi, J)^P = Ric(\phi^P, J) \end{aligned} \tag{4.13}$$

for all $P \in Ric(\phi, J)$. It now follows that the iterated transitions to inner and spectral DAREs satisfy the following rules of calculation.

Corollary 159. *Let* $(\phi, J) \equiv ((\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}), J)$ *be a minimax node. Let* $P_1, P_2 \in Ric(\phi, J)$*, and* $\Delta P := P_2 - P_1 \in Ric(\phi, J)_{P_1}$*. Then*

$$Ric(\phi^{P_1}, J)_{P_2} \equiv Ric(\begin{pmatrix} A_{P_1} & B \\ K_{P_1} - K_{P_2} & I \end{pmatrix}, \Lambda_{P_2}) = Ric(\phi, J) - P_2, \tag{4.14}$$

$$Ric(\phi^{P_1}, J)^{P_2} = Ric(\phi, J), \tag{4.15}$$

$$Ric(\phi_{P_1}, \Lambda_{P_1})_{\Delta P} = Ric(\phi, J) - P_2, \tag{4.16}$$

$$Ric(\phi_{P_1}, \Lambda_{P_1})^{\Delta P} \equiv Ric(\begin{pmatrix} A & B \\ K_{P_2} - K_{P_1} & I \end{pmatrix}, \Lambda_{P_2}) = Ric(\phi, J) - P_1. \tag{4.17}$$

We remark that the DLS $\phi_{P_2,P_1} := \left(\begin{smallmatrix} A_{P_1} & B \\ K_{P_1} - K_{P_2} & I \end{smallmatrix} \right)$ is familiar from Proposition 148.

4.3 Liapunov equation theory

The operator equation

$$A^*PA - P + C^*JC = 0, \tag{4.18}$$

is called the discrete time Liapunov equation or the (symmetric) Stein equation. As with the Riccati equation, the operators are as follows: the operator $A \in \mathcal{L}(H)$ is the semigroup generator, $C \in \mathcal{L}(H,Y)$ is the output operator, and the self-adjoint operator $J \in \mathcal{L}(Y)$ is the cost operator. The solution P is required to be self-adjoint. It is clear that the observability and controllability Gramians $\mathcal{C}^*\mathcal{C}$ and $\mathcal{B}\mathcal{B}^*$ of a DLS are solutions of Liapunov equations, see e.g. [108, p. 71].

A fairly complete Liapunov equation theory is given e.g. in [49] and [108] for the case when A, C and J are matrices, and $J > 0$. It is well known that the matrix Liapunov equation has a unique solution for any self-adjoint matrix C^*JC if and only if $\sigma(A) \cap \left(\overline{\sigma(A)}\right)^{-1} = \emptyset$, see [49, Theorem 5.2.3]. When this spectral separation holds, the solution P can be expressed as a Cauchy integral, see [49, Theorem 5.2.4]. When we do not have the spectral separation, the Cauchy integral cannot be defined because an integration contour cannot be drawn such that $\sigma(A)$ and $\left(\overline{\sigma(A)}\right)^{-1}$ lie on the "opposite sides" of the contour. The Cauchy integral solution makes perfect sense even for some operator Liapunov equations, provided that the required spectral separation exists. Even if we produced the dimension free variants of these results, the spectral separation would be too restrictive a condition to be useful for non-power stable but nevertheless strongly stable semigroup generators A. If $\sigma(A) \subset \overline{\mathbf{D}}$, then the spectral separation forces $\sigma(A) \subset \mathbf{D}$, and so A is power stable.

In the present work, our main interest is not in finding solutions for Liapunov equations. Quite conversely, we are given a nonnegative solution P of the Liapunov equation (4.18), with $J \geq 0$. Our task is to show that the output stability of an associated observability map $\mathcal{C}_{\phi'} := \{J^{\frac{1}{2}}CA^j\}_{j\geq 0}$ follows, see Lemma 166. Then, an expression can be found for the minimal nonnegative solution P_0 of (4.18), and the other solutions are parameterized by their residual cost operators $L_{A,P} := \mathrm{s}-\lim_{j\to\infty} A^{*j}PA^j$, see Corollary 163. Recall that the residual cost operator is defined as a strong limit $L_{A,P} := \mathrm{s}-\lim_{j\to\infty} A^{*j}PA^j$, see Definition 108.

We now briefly discuss the connection of the Liapunov equation to stability questions. The Liapunov equation is connected to the Liapunov stability theory of DLSs, see [51] for an exposition of the matrix case. For another view into this, suppose $Q \geq 0$ and $P > 0$ satisfies $A^*PA - P + Q = 0$. Then by writing for $x \neq 0$,

$$(4.19) \quad ||Ax||_P^2 - ||x||_P^2 := \left\langle P^{\frac{1}{2}}Ax, P^{\frac{1}{2}}Ax \right\rangle - \left\langle P^{\frac{1}{2}}x, P^{\frac{1}{2}}x \right\rangle = -\langle Qx, x \rangle \leq 0,$$

we see that such solution P defines an inner product topology such that the operator A becomes a contraction. Because P is bounded, we have $||x||_P \leq ||P|| \cdot ||x||$, which implies that the $||.||_P$-topology is generally weaker that the original. Clearly the topologies coincide if P has a bounded inverse. This gives some functional analytic meaning for the Liapunov stability theory of linear systems.

Another instance where a Liapunov equation arises is connected to DARE and given in the following proposition. Its proof is a straightforward calculation, and clearly connected to the inner Riccati equation $Ric(\phi, J)^P$ of Definition 154 and Lemma 157.

Proposition 160. *Let $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix} \right)$ be a DLS, and $J \in \mathcal{L}(Y)$ a self-adjoint cost operator. Then $P \in Ric(\phi, J)$ if and only if*

$$(4.20) \qquad A_P^* P A_P - P + C_P^* J C_P = 0,$$

where $A_P := A + BK_P$ and $C_P := C + DK_P$. Furthermore, $D^* J C_P + B^* P A_P = 0$.

By solving the Liapunov equation (4.20), the operator $P \in Ric(\phi, J)$ can be recovered from the operators Λ_P and K_P, provided that the solution of the Liapunov equation is unique or we know the residual cost operator $L_{A_P,P}$ apriori. Unfortunately, it is difficult to check (for uniqueness of P) the spectral separation $\sigma(A_P) \cap \left(\overline{\sigma(A_P)}\right)^{-1} = \emptyset$ for solutions $P \in Ric(\phi, J)$ of interest. By iteration, the following algebraic triviality is shown.

Proposition 161. *Assume that $A \in \mathcal{L}(H)$, $C \in \mathcal{L}(H, Y)$ and $J \in \mathcal{L}(Y)$. Assume that a possibly unbounded linear map $P : H \supset \mathrm{dom}\,(P) \to H$, $A\,\mathrm{dom}\,(P) \subset \mathrm{dom}\,(P)$, satisfies the Liapunov equation $A^* P A - P + C^* J C = 0$. Then*

$$Px = \sum_{j=0}^{n-1} A^{*j} C^* J C A^j x + A^{*n} P A^n x, \quad \textit{for all} \quad x \in \mathrm{dom}\,(P), n \geq 1.$$

We start to study solutions P of the Liapunov equation (4.18) for which the residual cost operator $L_{A,P}$ exists. The fact that the mapping $P \mapsto A^* P A - P$ is bounded and linear, gives the background for the following proposition:

Proposition 162. *Assume that the linear mappings $A \in \mathcal{L}(H)$, $C \in \mathcal{L}(H, Y)$ and $J \in \mathcal{L}(Y)$ self-adjoint. Then the following are equivalent:*

(i) *There is a solution P_0 of the Liapunov equation such that the residual cost operator vanishes: $L_{A,P_0} = 0$.*

(ii) *There is at least one solution $\tilde{P}$ of the Liapunov equation such that the residual cost operator $L_{A,\tilde{P}} \in \mathcal{L}(H)$ exists.*

(iii) *The Liapunov equation has at least one solution, and for all solutions P, the residual cost operator $L_{A,P} \in \mathcal{L}(H)$ exists.*

If, in addition, $J \geq 0$, then we have a third equivalent condition

(iv) *The DLS $\phi' := \left(\begin{smallmatrix} A & * \\ J^{\frac{1}{2}}C & * \end{smallmatrix}\right)$ is output stable.*

Proof. The implication (i) $\Rightarrow$ (ii) is trivial. To prove the implication (ii) $\Rightarrow$ (iii), note that by Proposition 161 $\sum_{j=0}^{n-1} A^{*j}C^*JCA^jx = \tilde{P}x - A^{*n}\tilde{P}A^nx$ for all $x \in H$. Thus $\mathrm{s}-\lim_{n\to\infty}\sum_{j=0}^{n-1} A^{*j}C^*JCA^j = \tilde{P} - L_{A,\tilde{P}}$ exists if (ii) holds. Now, for all solutions P of the Liapunov equation the strong limit $L_{A,P} = \mathrm{s}-\lim_{n\to\infty} A^{*n}PA^n$ exists, because the limit on the right hand side for the following equation exists

$$A^{*n}PA^nx = Px - \sum_{j=0}^{n-1} A^{*j}C^*JCA^jx$$

for all $x \in H$.

To prove the implication (iii) $\Rightarrow$ (i), assume $\tilde{P}$ is a solution such that $L_{A,\tilde{P}} \in \mathcal{L}(H)$ exists. It follows that the strong limit operator

$$P_0 := \mathrm{s}-\lim_{n\to\infty}\sum_{j=0}^{n-1} A^{*j}C^*JCA^j$$

exists and equals $\tilde{P} - L_{A,\tilde{P}} \in \mathcal{L}(H)$. We show that P_0 is a solution of the Liapunov equation such that $L_{A,P_0} = 0$. Let $x_1, x_2 \in H$ be arbitrary. Then

$$\begin{aligned}(4.21)\qquad \langle x_1, (A^*P_0A - P_0)x_2\rangle_H &= \left\langle Ax_1, (\mathrm{s}-\lim_{n\to\infty}\sum_{j=0}^{n-1} A^{*j}C^*JCA^j)Ax_2\right\rangle_H \\ &- \left\langle x_1, (\mathrm{s}-\lim_{n\to\infty}\sum_{j=0}^{n-1} A^{*j}C^*JCA^j)x_2\right\rangle_H .\end{aligned}$$

Now the latter part on the right hand side of equation (4.21) takes the form

$$\left\langle x_1, (\mathrm{s}-\lim_{n\to\infty}\sum_{j=0}^{n-1} A^{*j}C^*JCA^j)x_2\right\rangle_H = \left\langle x_1, \lim_{n\to\infty}(\sum_{j=0}^{n-1} A^{*j}C^*JCA^j x_2)\right\rangle_H$$

$$= \lim_{n\to\infty}\left\langle x_1, (\sum_{j=0}^{n-1} A^{*j}C^*JCA^j x_2)\right\rangle_H = \lim_{n\to\infty}\sum_{j=0}^{n-1}\left\langle x_1, A^{*j}C^*JCA^j x_2\right\rangle_H$$

$$= \sum_{j=0}^{\infty}\left\langle x_1, A^{*j}C^*JCA^j x_2\right\rangle_H,$$

where the second equality holds because $\langle x_1, \cdot\rangle_H$ is a continuous linear functional for each $x_1 \in H$. Similarly,

$$\left\langle Ax_1, (\mathrm{s}-\lim_{n\to\infty}\sum_{j=0}^{n-1} A^{*j}C^*JCA^j)Ax_2\right\rangle_H = \left\langle x_1, \sum_{j=0}^{\infty} A^{*(j+1)}C^*JCA^{(j+1)}x_2\right\rangle_H.$$

Subtracting these two limits, together with equation (4.21), gives

$$\langle x_1, (A^*P_0A - P_0)x_2\rangle_H = -\langle x_1, C^*JCx_2\rangle_H.$$

Because x_1 and x_2 are arbitrary, P_0 solves the Liapunov equation. To show that $L_{A,P_0} = \mathrm{s}-\lim_{n\to\infty} A^{*n}P_0A^n = 0$, we note that for each $x_1 \in H$, $n \in \mathbf{N}$

$$||A^{*n}P_0A^nx_1|| = ||P_0x_1 - \sum_{j=0}^{n-1} A^{*j}C^*JCA^jx_1||$$

$$= ||\lim_{m\to\infty}\sum_{j=0}^{m} A^{*j}C^*JCA^jx_1 - \sum_{j=0}^{n-1} A^{*j}C^*JCA^jx_1||$$

$$= ||\sum_{j=n}^{\infty} A^{*j}C^*JCA^jx_2|| \to 0,$$

as a tail of a convergent series. We complete the proof by studying the additional part (iv). Assume that both (ii) and (iii) hold, P is a solution of the Liapunov equation such that $L_{A,P}$ exists, and $J \geq 0$. Then both the bounded operators $J^{\frac{1}{2}}$ and $\mathrm{s}-\lim_{n\to\infty}\sum_{j=0}^{n-1} A^{*j}C^*JCA^j = P - L_{A,P}$ exist.

We calculate for any $x \in H$

$$
\begin{aligned}
&||P - L_P|| \cdot ||x||^2 \geq |\langle x, (P - L_P)x \rangle_H| \\
&= \left| \left\langle x, \left(\mathrm{s} - \lim_{n\to\infty} \sum_{j=0}^{n-1} A^{*j} C^* J C A^j \right) x \right\rangle_H \right| = \left| \left\langle x, lim_{n\to\infty} \sum_{j=0}^{n-1} A^{*j} C^* J C A^j x \right\rangle_H \right| \\
&= \left| \lim_{n\to\infty} \left\langle x, \sum_{j=0}^{n-1} (A^{*j} C^* J C A^j x) \right\rangle_H \right| = \lim_{n\to\infty} \sum_{j=0}^{n-1} \left\langle J^{\frac{1}{2}} C A^j x, J^{\frac{1}{2}} C A^j x \right\rangle_H \\
&= ||\{J^{\frac{1}{2}} C A^j x\}_{j\geq 0}||^2_{\ell^2(\mathbf{Z}_+;Y)} = ||\mathcal{C}_{\phi'} x||^2_{\ell^2(\mathbf{Z}_+;Y)},
\end{aligned}
$$

where the third equality holds because $\langle x, \cdot \rangle_H$ is a continuous linear functional for each $x \in H$.

It follows that the observability map $\mathcal{C}_{\phi'}$ of the DLS ϕ' maps all of a (complete) Hilbert space H into $\ell^2(\mathbf{Z}_+;Y)$. However, the observability map of a DLS is a closed operator by Lemma 31, and now the domain $\mathrm{dom}\,(\mathcal{C}_{\phi'}) = H$ is complete. The Closed Graph Theorem implies the boundedness of $\mathcal{C}_{\phi'}$; i.e. the output stability of ϕ'. So claim *(iv)* follows. The implication *(iv)* $\Rightarrow$ (i) follows because the output stability of ϕ' implies the strong convergence of the sum $\mathrm{s} - \lim_{n\to\infty} \sum_{j=0}^{n-1} A^{*j} C^* J C A^j$, thus defining the solution P_0 of the Liapunov equation. This completes the proof. $\square$

Compare the above proof to the proof of Proposition 136. An immediate consequence is the following:

Proposition 163. *If there is a solution P of the Liapunov equation (4.18) such that the residual cost operator $L_{A,P} \in \mathcal{L}(H)$ exists, then there is a solution P_0 such that $L_{A,P_0} = 0$. Such P_0 is unique, and given by $P_0 x_0 = \sum_{j=0}^{\infty} (A^{*j} C^* J C A^j x_0)$ for all $x_0 \in H$. All other bounded solutions P of the Liapunov equation satisfy*

$$P = P_0 + L_{A,P}, \quad L_{A,P} = \mathrm{s} - \lim_{j\to\infty} A^{*j} P A^j.$$

If A is strongly stable, then P_0 is the unique solution of the Liapunov equation.

Proof. The existence of P_0 is the matter of the implication (ii) $\Rightarrow$ (i) of Proposition 162. The formula for P_0 is found in the proof of implication (iii) $\Rightarrow$ (i) of Proposition 162. The parameterization of all the solutions is a direct consequence of Proposition 161. Claim about the uniqueness of P_0 is proved by noting that for two solutions $P_1, P_2 \in \mathcal{L}(H)$ we have

$$A^{*j}(P_1 - P_2)A^j = P_1 - P_2$$

for all $j > 0$. If both $\mathrm{s}-\lim_{j\to\infty} A^{*j} P_1 A^j = 0$ and $\mathrm{s}-\lim_{j\to\infty} A^{*j} P_2 A^j = 0$, then the left hand side converges to zero pointwise in H, as j grows. The right hand side does not even depend on j. Thus $P_1 = P_2$. The claim involving the strongly stable semigroup is trivial. □

As discussed in the beginning of this section, a fair amount of stability results for DLSs can be given with the aid of the Liapunov equation. The following result is [108, Lemma 21.6], stating that an unstable eigenvector of the semigroup is undetectable.

Proposition 164. *Let $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ be a DLS, and $J \geq 0$ a cost operator. Let $P \in Ric(\phi, J)$, $P \geq 0$ be arbitrary. Assume that $Ax = \lambda x$ for $|\lambda| \geq 1$. Then $J^{\frac{1}{2}} Cx = 0$.*

Proof. If $Ax = \lambda x$, the Liapunov equation takes the form

$$(|\lambda|^2 - 1) \langle Px, x \rangle + \left\langle J^{\frac{1}{2}} Cx, J^{\frac{1}{2}} Cx \right\rangle = 0. \tag{4.22}$$

Now, if $|\lambda|^2 - 1 \geq 0$, then $(|\lambda|^2 - 1) \langle Px, x \rangle \geq 0$ because $P \geq 0$. Because $J \geq 0$, equation (4.22) implies that $J^{\frac{1}{2}} Cx = 0$, and the claim is proved. □

Unfortunately this is too weak to be useful for our purposes. Clearly, this approach is restricted to the cases when the eigenvectors of the semigroup generator A span (the interesting part of) the state space. However, the case when A is a diagonalizable matrix or a Riesz spectral operator is covered, see [18, p. 37]. In order to obtain a more general theory for the operator Riccati equation, a stronger infinite-dimensional Liapunov equation theory is required. In Lemma 166, an essential analogue of Proposition 164 is proved for DLSs with much more complicated semigroups. We start with a result known as the Vigier's theorem in [68, Theorem 4.1.1].

Proposition 165. *Let $\{T_j\}_{j\geq 0} \subset \mathcal{L}(H)$ be a sequence of nonnegative self-adjoint operators such that*

$$0 \leq \langle x, T_j x \rangle \leq \langle x, T_{j-1} x \rangle, \quad j > 0.$$

Then there is a nonnegative self-adjoint operator $T \in \mathcal{L}(H)$ such that $0 \leq T \leq T_j$ for all $j \geq 0$, and

$$\langle x, Tx \rangle = \lim_{j\to\infty} \langle x, T_j x \rangle.$$

Proof. Define $a_j(x, y) := \langle x, T_j y \rangle_H$, for all $j \geq 0$. It is easy to see that $a_j(x, y)$ is a bounded conjugate symmetric sesquilinear form on $H \times H$. Now, because

$\{\langle x, T_j x\rangle\}_{j\geq 0}$ is a nonincreasing sequence of nonnegative real numbers, the limit exists for all $x \in H$. The polarization identity

$$\begin{aligned}
&4a_j(x,y) = 4 \cdot \langle x, T_j y\rangle \\
&= \langle x+y, T_j(x+y)\rangle - \langle x-y, T_j(x-y)\rangle + \\
&i\,\langle x+iy, T_j(x+iy)\rangle - i\,\langle x-iy, T_j(x-iy)\rangle .
\end{aligned}$$

implies that the limit $a(x,y) := \lim_{j\to\infty} a_j(x,y)$ exists, for all $x, y \in H$. It remains to show that $a(x,y)$ is a bounded conjugate symmetric sesquilinear form on $H \times H$.

The linearity in the first argument x and the conjugate linearity in the second argument y is a trivial consequence of the limit process, because this is true for each $a_j(x,y)$ by the properties of the inner product. The same is true about the conjugate symmetricity of $a(x,y)$. To show the boundedness, we see that

$$|a(x,y)| = \lim_{j\to\infty} |a_j(x,y)| = \lim_{j\to\infty} |\,\langle x, T_j y\rangle\,| \leq \lim_{j\to\infty} ||T_j||\,||x||\,||y||.$$

Now, the family $\{T_j\}_{j\geq 0}$ is uniformly bounded by $||T_0||$, because the norms $||T_j||$ are in fact a nonincreasing sequence

$$||T_j|| = \sup_{||x||=1} \langle x, T_j x\rangle \leq \sup_{||x||=1} \langle x, T_{j-1} x\rangle = ||T_{j-1}||,$$

where we have used the assumption that $0 \leq \langle x, T_j x\rangle \leq \langle x, T_{j-1}x\rangle$, for all $x \in H$. As a bounded sesquilinear form, $a(x,y)$ can be written in form $a(x,y) = \langle x, Ty\rangle$, for a unique operator $T \in \mathcal{L}(H)$ (see [79, Theorem 12.8]). T is self-adjoint because $\langle x, Ty\rangle = a(x,y) = \overline{a(y,x)} = \overline{\langle y, Tx\rangle} = \overline{\langle T^*y, x\rangle} = \langle x, T^*y\rangle$. Because the nonnegativity of T is trivial, T satisfies the claims of this proposition. $\square$

By claim (ii) of Proposition 162, we saw that if the Liapunov equation has one solution $\tilde{P}$ such that the residual cost operator $L_{A,P}$ exists, then a number of nice results followed. Now we use Proposition 165 to give an existence of such $L_{A,P}$ for a given nonnegative solution P.

Lemma 166. *Let $\phi = \left(\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right)$ be DLS, and $J \geq 0$ a self-adjoint cost operator. Assume that the Liapunov equation*

$$A^*PA - P + C^*JC = 0,$$

has a nonnegative solution $P \in \mathcal{L}(H)$. Then

(i) The DLS $\phi' := \begin{pmatrix} A & * \\ J^{\frac{1}{2}}C & * \end{pmatrix}$ is output stable, and the residual cost operator $L_{A,P} := \mathrm{s}-\lim_{j\to\infty} A^{*j}PA^j$ exists.

(ii) The operator P_0 is the minimal nonnegative solution of the Liapunov equation (4.18), where $P_0 := \mathcal{C}_{\phi'}^* \mathcal{C}_{\phi'}$, and $L_{A,P_0} = 0$.

The assumption $J \geq 0$ can be replaced by the assumption $C^*JC \geq 0$, if ϕ' is replaced by $\left(\begin{smallmatrix} A & * \\ (C^*JC)^{\frac{1}{2}} & * \end{smallmatrix}\right)$.

Proof. Let $P \geq 0$ be the nonnegative solution whose existence is assumed. By Proposition 161, we have for all $x \in H$ and $n \geq 1$

$$\langle x, Px \rangle - \sum_{j=0}^{n-1} ||J^{\frac{1}{2}} C A^j x||^2 = \langle x, A^{*n} P A^n x \rangle \, ,$$

because $J \geq 0$ by assumption. Define $T_n := A^{*n} P A^n$. It immediately follows that $\langle x, T_n x \rangle$ is a nonincreasing sequence of nonnegative real numbers, because $P \geq 0$. We can apply Proposition 165, and obtain the largest lower bound operator T, such that $0 \leq T \leq A^{*n} P A^n$ for all $n \geq 0$. We proceed show that $T = \mathrm{s} - \lim_{n \to \infty} A^{*n} P A^n =: L_{A,P}$. We have, because $\langle x, Tx \rangle = \lim_{n \to \infty} \langle x, A^{*n} P A^n x \rangle$ for all $x \in H$:

$$0 = \lim_{n \to \infty} \langle x, (A^{*n} P A^n - T) x \rangle = \lim_{n \to \infty} ||(A^{*n} P A^n - T)^{\frac{1}{2}} x||^2.$$

So $(A^{*n} P A^n - T)^{\frac{1}{2}} \to 0$ in the strong operator topology, and $\{(A^{*n} P A^n - T)^{\frac{1}{2}}\}_{n \geq 0}$ is thus a uniformly bounded family, by the Banach–Steinhaus theorem. It follows that $(A^{*n} P A^n - T)x \to 0$ for all $x \in H$, and so we have $T = L_{A,P}$ which, in particular, exists. We conclude that the equivalent conditions of Proposition 162 hold. Furthermore, because $J \geq 0$, ϕ' is output stable.

The proof of the second claim (ii) goes as follows. Because ϕ' is output stable, it follows from Proposition 163 that $P_0 = \mathcal{C}_{\phi'}^* \mathcal{C}_{\phi'}$ is a bounded solution of the Liapunov equation, satisfying $L_{A,P_0} = 0$. It is nonnegative because $J \geq 0$. To show that P_0 is minimal nonnegative, let $P_1 \in \mathcal{L}(H)$ is another nonnegative solution of the Liapunov equation. Then the strong limit L_{A,P_1} exists, by Proposition 162, and because $P_1 \geq 0$, it follows that $L_{P_1} \geq 0$. By Proposition 163, $P_1 = P_0 + L_{A,P_1} \geq P_0$. So P_0 is a minimal nonnegative solution of the Liapunov equation. The final comment follows by replacing C by $(C^*JC)^{\frac{1}{2}}$, and J by I. The proof is now complete. □

We now consider the special case when the Liapunov equation is connected to DARE $Ric(\phi, J)$ for $J \geq 0$, and its nonnegative solution $P \in Ric(\phi, J)$, as in Proposition 160. By applying Lemma 166 with A_P in place for A and C_P in place for C, we get an important results that is used several times in Section 4.4.

Corollary 167. *Let $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ be a DLS, and $J \geq 0$ a cost operator. Let $P \in Ric(\phi, J)$ such that $P \geq 0$. Then the DLS $\phi' := \left(\begin{smallmatrix} A_P & * \\ J^{\frac{1}{2}} C_P & * \end{smallmatrix}\right)$ is output stable, and the (closed loop) residual cost operator $L_{A_P,P}$ exists. Furthermore, $P_0 := \mathcal{C}_{\phi'}^* \mathcal{C}_{\phi'}$ is a minimal nonnegative solution of the Liapunov equation*

$$A_P^* \tilde{P} A_P - \tilde{P} + C_P^* J C_P = 0,$$

where $A_P := A + BK_P$, $C_P := C + DK_P$, and $\tilde{P}$ is the operator to be solved. Also $L_{A_P,P_0} = 0$.

We conclude that not bad instabilities of A_P are seen through the operator C_P, as a dimension independent analogy to Proposition 164. We remark that P_0 does not necessarily solve the DARE $Ric(\phi, J)$. Under stronger conditions, it is shown in Lemma 192 that $L_{A_P,P} = 0$ and then $P = P_0$, by Proposition 163.

We complete this section by considering a case when the Liapunov equation technique is applicable to a nonnegative solution of DARE $Ric(\phi, J)$, even if the cost operator J could be indefinite. In Corollary 167, the closed loop residual condition of P was considered. A conclusion about the open loop residual cost operator $L_{A,P}$ is considered in the following.

Corollary 168. *Let $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ be a DLS, and $J \in \mathcal{L}(Y)$ a self-adjoint cost operator. Let $P \in Ric(\phi, J)$ such that $P \geq A^* P A \geq 0$. Then $P \in Ric_{00}(\phi, J)$.*

Proof. Because $P \in Ric(\phi, J)$, we have the Liapunov equation

$$A^* P A - P + \begin{bmatrix} C^* & K_P^* \end{bmatrix} \begin{bmatrix} J & 0 \\ 0 & -\Lambda_P \end{bmatrix} \begin{bmatrix} C \\ K_P \end{bmatrix} = 0.$$

Now $P \geq A^* P A$ if and only if $\left[\begin{smallmatrix} C^* & K_P^* \end{smallmatrix}\right] \left[\begin{smallmatrix} J & 0 \\ 0 & -\Lambda_P \end{smallmatrix}\right] \left[\begin{smallmatrix} C \\ K_P \end{smallmatrix}\right] \geq 0$. Now claim (i) of Lemma 166 (in its modified form for the indefinite cost operator) shows that the residual cost operator $L_{A,P}$ exists. $\square$

Note that the condition $P \geq A^* P A \geq 0$ implies that $\ker(P)$ is A-invariant, and the orthogonal complement $\ker(P)^\perp$ is A^*-invariant but not necessarily A-invariant. For this reason, we have to introduce the compression of the semigroup generator.

Definition 169. *Let $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ be a DLS and J self-adjoint. Let $P \in Ric(\phi, J)$. Define the closed subspace $H^P := \ker(P)^\perp \subset H$, the orthogonal projection Π_P onto H^P, and the compression of the semigroup $A^P := \Pi_P A | H^P \in \mathcal{L}(H^P)$.*

A nonnegative solution $P \in Ric(\phi, J)$ induces an inner product space structure into $H^P := \ker(P)^\perp$. Everything goes in the same way as discussed in connection with equation (4.19) for the Liapunov equations, with the exception that now the (generally nontrivial) null space of P must be divided away. It is easy to see that $P \geq A^*PA \geq 0$ is equivalent to

$$(4.23) \qquad ||A^P x||_P := ||P^{\frac{1}{2}}Ax|| \leq ||P^{\frac{1}{2}}x|| =: ||x||_P \quad \text{for all} \quad x \in H^P.$$

In this case, we say that the compression A^P is a $||\cdot||_P$-contraction. If $P^{\max} \in Ric(\phi, J)$ was nonnegative and injective, then $H^{P^{\max}} = H$ but the norm $||\cdot||_{P^{\max}}$ could give weaker topology that the original norm of H. More generally, H^P need not be complete, when equipped with the norm $||\cdot||_P$.

Proposition 170. *Let $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ be a DLS and J self-adjoint. Let $P \in Ric(\phi, J)$, $P \geq 0$ such that the compression $A^P = \Pi_P A | H^P$ is a $||\cdot||_P$-contraction, where the objects are given in Definition 169. Then the following holds*

(i) $P \in Ric_{00}(\phi, J)$. *If, in addition, ϕ is output stable and $\Lambda_P > 0$, then ϕ_P is output stable.*

(ii) Assume, in addition, that ϕ is output stable and I/O stable, the input operator B is Hilbert–Schmidt, and the input space U is separable.

If $P \in Ric_{uw}(\phi, J)$ and $\Lambda_P > 0$, then $P \in ric_{00}(\phi, J) \cap ric_{uw}(\phi, J)$. If $\overline{\text{range}\,(\mathcal{B}_\phi)} = H$, then

$$(4.24) \qquad \{P \in Ric_{uw}(\phi, J) \mid P \geq 0, \Lambda_P > 0\} \subset ric_0(\phi, J).$$

Proof. The first part of claim (i) is Corollary 168. The rest follows from claim (i) of Proposition 136. Claim (ii) follows from Corollary 140 and equation (3.27). □

The reader is instructed to compare equations (3.26) and (3.27), and equation (4.24). They all characterize subsets $ric_0(\phi, J)$, where J can be indefinite but the indicators Λ_P must be positive.

The P-contractivity condition $P \geq A^*PA \geq 0$ can be given a game theoretic interpretation. Let $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ be output stable and I/O stable, and let $P_0^{\text{crit}} \in ric_0(\phi, J)$ be a regular critical solution which is assumed nonnegative. If the cost operator J is indefinite, the special case of the minimax cost optimization problem, associated to (ϕ, J), can be seen as a (full information, state feedback) minimax game, where the minimizing and maximizing players are given an initial state x_0 and their task to do the best they can. Some additional information structure of the game itself must be imposed; e.g. the input space U must be divided into two parts, and one player must not have access to the other players input space, but we now disregard all the details. Now, each non-critical solution $P \in ric_0(\phi, J)$ is associated to a strategy where both players have, in a rough sense, made an agreement that the game is played (i.e. the cost is measured by P) only inside the restricted state space H^P.

Let now $P \in ric_0(\phi, J)$ be such that $P \geq A^*PA \geq 0$. Now the open loop trajectories $x_j = A^j x_0$ (with zero input from both players) are nonnegative and nonincreasing, in the sense of the cost functional $\langle x_j, Px_j \rangle$. Thus, the maximizing player "loses money" if he does not do anything, but the future game always has a nonnegative cost if the feedback loop is closed (by the maximizing player) at some later moment. In fact, the maximizing player wins the game also in the open loop, and the final cost at infinite future is $\lim_{j\to\infty} \left\langle A^{*j} P A^j x_0, x_0 \right\rangle = \langle L_{A,P} x_0, x_0 \rangle \geq 0$, because $P \geq 0$ is assumed.

4.4 Factorization of the I/O map

In this section we study the natural partial ordering of the solution set of the H^∞DARE, induced by the cone of nonnegative self-adjoint operators. We work under the assumption that the cost operator $J \geq 0$, and the equivalent conditions of Theorem 114 hold. In this case, we have a nonnegative regular critical solution $P_0^{\mathrm{crit}} = (\mathcal{C}^{\mathrm{crit}})^* J \mathcal{C}^{\mathrm{crit}} \in ric_0(\Phi, J)$.

In Theorem 114, we have indicated that the critical solution $P_0^{\mathrm{crit}} \in ric_0(\Phi, J)$ gives a $(J, \Lambda_{P_0^{\mathrm{crit}}})$-inner-outer factorization of the I/O map. The (generally noncritical) solutions $P \in ric_{uw}(\Phi, J)$ induce other factorizations of the Popov operator $\mathcal{D}^* J \mathcal{D} = \mathcal{D}_{\phi_P}^* \Lambda_P \mathcal{D}_{\phi_P}$ with I/O stable $\mathcal{D}_{\phi_P}$, see Theorem 142. However, these do not necessarily lead to a factorization of the I/O map $\mathcal{D}$ as a composition of two I/O stable operators, in the same way as the spectral factorization leads to the $(J, \Lambda_{P_0^{\mathrm{crit}}})$ inner-outer factorization of $\mathcal{D}$. The task of this section is to describe which solutions P actually do give a factorization of the I/O map $\mathcal{D}$ into compositions of I/O stable I/O maps.

Consider the following. Let $P \in ric(\Phi, J)$, where Φ is output stable and I/O stable. The operator pair $(K_P, 0)$ is a perfectly valid state feedback pair for Φ in the sense of Definition 18. However, if P is not a critical solution, this feedback pair is not I/O stable in the sense of Definition 44. This means that even if the open loop DLS, extended with the feedback pair $(K_P, 0) = [-\mathcal{C}_{\phi_P}, \mathcal{I} - \mathcal{D}_{\phi_P}]$

$$(\Phi, (K_P, 0)) = \left(\begin{matrix} A & B \\ \begin{bmatrix} C \\ K_P \end{bmatrix} & \begin{bmatrix} D \\ 0 \end{bmatrix} \end{matrix} \right) = \left[\begin{matrix} A^j & \mathcal{B}\tau^{*j} \\ \begin{bmatrix} \mathcal{C} \\ -\mathcal{C}_{\phi_P} \end{bmatrix} & \begin{bmatrix} \mathcal{D} \\ \mathcal{I} - \mathcal{D}_{\phi_P} \end{bmatrix} \end{matrix} \right],$$

is output stable and I/O stable, the closed loop extended system

$$\begin{aligned} (4.25) \qquad & (\Phi, (K_P, 0))_\diamond \\ & = \left(\begin{matrix} A_P & B \\ \begin{bmatrix} C_P \\ K_P \end{bmatrix} & \begin{bmatrix} D \\ 0 \end{bmatrix} \end{matrix} \right) = \left[\begin{matrix} A^j - \mathcal{B}\mathcal{D}_{\phi_P}^{-1}\tau^{*j}\mathcal{C}_{\phi_P} & \mathcal{B}\mathcal{D}_{\phi_P}^{-1}\tau^{*j} \\ \begin{bmatrix} \mathcal{C} - \mathcal{D}\mathcal{D}_{\phi_P}^{-1}\mathcal{C}_{\phi_P} \\ -\mathcal{D}_{\phi_P}^{-1}\mathcal{C}_{\phi_P} \end{bmatrix} & \begin{bmatrix} \mathcal{D}\mathcal{D}_{\phi_P}^{-1} \\ \mathcal{D}_{\phi_P}^{-1} - \mathcal{I} \end{bmatrix} \end{matrix} \right] \end{aligned}$$

need not be, where $A_P = A + BK_P$ and $C_P = C + DK_P$. This is the bad news. However, if $P \geq 0$, together with proper technical assumptions, it follows that the upper two rows of the closed loops DLS (4.25) give an I/O stable DLS. Furthermore, this partial DLS is exactly $\phi^P = \left(\begin{smallmatrix} A_P & B \\ C_P & D \end{smallmatrix}\right)$; the inner DLS (of Φ and J) of Definition 150, centered at P. Note that $\mathcal{D}_{\phi^P} := \mathcal{D}\mathcal{D}_{\phi_P}^{-1}$ for the I/O map of ϕ^P, and this algebraic fact does not depend on the stability properties of the systems, apart from the boundedness of the static operators A, B, C, D, and K_P.

Let us review some analogous results of the matrix theory when all the spaces U, H and Y of the DLS $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ are finite dimensional. If the pair (A, B) is stabilizable, $J \geq 0$ and D^*JD coercive, there is a unique maximal positive solution $P^{\max}$ of the Riccati equation such that the closed loop spectrum $\sigma(A_{P^{\max}}) \subset \overline{\mathbf{D}}$, see [49, Corollary 12.1.2]. If $J = I$, $D^*D = I$, $D^*C = 0$ and (C, A) detectable, then the power stability $\sigma(A_{P^{\max}}) \subset \mathbf{D}$ follows, see [49, Corollary 13.5.3]. Such $P^{\max}$ is called the (power) stabilizing solution of $Ric(\Phi, J)$. If the open loop semigroup generator A is power stable and (A, B) is controllable, then $P^{\max}$ clearly equals the unique critical solution (which is defined only for DAREs associated to I/O stable DLSs) in the sense of Theorem 114. Indeed, the semigroup generators of both $\phi_{P^{\max}}$ and $\phi_{P^{\max}}^{-1}$ are power stable, by the formulae given in claim (iii) of Proposition 147.

To obtain a matrix H^∞DARE example, let $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix} \right)$ be a DLS whose spaces U, H and Y are finite dimensional, and the semigroup generator A is power stable; $\sigma(A) \subset \mathbf{D}$. We take $J = I$ to be the cost operator, and assume that the transfer function $\mathcal{D}_\phi(z)$ has no zeroes on the unit circle $\mathbf{T}$. By the assumed finite dimensionality of all the spaces, the last condition can always be achieved, if necessary, by a small perturbation of the DLS ϕ. Then the Popov operator $\mathcal{D}^*\mathcal{D}$ is coercive, and the nonnegative regular critical solution $P_0^{crit} = \left(\mathcal{C}_\phi^{\text{crit}}\right)^* \mathcal{C}_\phi^{\text{crit}} \in ric_0(\phi, J)$ exists, by Corollary 118. It follows that $A_{P_0^{crit}}$ is power stable, by claim (i) of Theorem 50 and the finite dimensionality of the state space H. If there was another power stabilizing solution P^{stab}, it would also be a critical solution in $ric_0(\phi, J)$. Thus, if ϕ, in addition, is controllable range$(\mathcal{B}_\phi) = H$, then P_0^{crit} is the unique power stabilizing solution of H^∞DARE $ric(\phi, J)$, see claim (i) of Corollary 116. In fact, P_0^{crit} is the maximal nonnegative solution in $Ric(\phi, J)$, by Corollary 186 and the fact that the power stability of A implies the equality of solution sets $Ric(\phi, J) = ric_0(\phi, J)$. It is easy to see by a numerical example, using the matrix DARE theory given in [49, Corollary 12.1.2], that it is possible (and even a generic case) that DARE $Ric(\phi, J)$ has long increasing chains of self-adjoint solutions. By using Lemma 156, we can, if necessary, replace $Ric(\phi, J)$ by its spectral DARE $Ric(\phi_{\tilde{P}}, \Lambda_{\tilde{P}})$ for $\tilde{P}$ "small". So there exists a H^∞DARE $ric(\phi, J)$ (with a power stable semigroup generator) that has an arbitrarily long increasing chain of nonnegative solutions, if $\dim H$ is increased sufficiently. We conclude that the power stabilizing solution P_0^{crit} need not be the only nonnegative H^∞ solution of a (matrix) H^∞DARE. For the other nonmaximal $P \in Ric(\phi, J)$, $P_0^{\text{crit}} \geq P \geq 0$, the inner DLS

$$\phi^P := \begin{pmatrix} A_P & B \\ C_P & D \end{pmatrix} = \begin{bmatrix} A^j - \mathcal{B}\mathcal{D}_{\phi_P}^{-1}\tau^{*j}\mathcal{C}_{\phi_P} & \mathcal{B}\mathcal{D}_{\phi_P}^{-1}\tau^{*j} \\ \mathcal{C} - \mathcal{D}_{\phi^P}\mathcal{C}_{\phi_P} & \mathcal{D}_{\phi^P} \end{bmatrix}, \tag{4.26}$$

is nevertheless I/O stable by the following Lemma 171 and the assumption that $J = I$ has a bounded inverse. However, the closed loop semigroup generators A_P are not power stable. In this sense, all the nonnegative solutions of the Riccati equation are I/O-stabilizing, but only the maximal nonnegative P_0^{crit} gives a power stable semigroup generator in the closed loop, under the indicated additional assumptions.

This phenomenon can be viewed from two directions. The first "state space" view is that the DLS ϕ^P is I/O stable because the unstable part of A_P is not "seen" through the output operator C_P of ϕ^P. The second view is the input/output view; that a kind of zero-pole-cancellation process is involved when the feedback loop is closed. In the language of the transfer functions $\mathcal{D}_{\phi^P}(z) = \mathcal{D}(z)\mathcal{D}_{\phi_P}(z)^{-1}$, some of the zeroes of $\mathcal{D}(z)$ get canceled by the poles of $\mathcal{D}_{\phi_P}(z)^{-1}$, at least in the cases when the transfer functions are complex-valued ($U = Y = \mathbf{C}$). We remark that the condition $\dim H < \infty$ amounts to the fact that the inner factors of both $\mathcal{D}(z)$ and $\mathcal{D}_{\phi_P}(z)$ are finite Blaschke products, and the zero–pole cancellation idea makes perfect sense. We remark that using a nonnegative but nonmaximal solution $P \in Ric(\Phi, J)$ for feedback control leads to a partial stabilization of the (unstable) open loop DLS, see [23] and the references therein.

In the following lemma we show that if $P \geq 0$, then $J^{\frac{1}{2}}\mathcal{D}_{\phi^P}$ is an I/O map from $\ell^1(\mathbf{Z}_+; U)$ into $\ell^2(\mathbf{Z}_+; Y)$; i.e. the transfer function $\mathcal{D}_{\phi^P}(z) \in \mathrm{sH}^2(\mathbf{D}; \mathcal{L}(U; Y))$. Step by step, we finally conclude that $J^{\frac{1}{2}}\mathcal{D}_{\phi^P}$ is I/O stable under stronger assumptions. If J has a bounded inverse, the same conclusions clearly hold for the I/O map $\mathcal{D}_{\phi^P}$, too.

Lemma 171. *Let $J \geq 0$ be a cost operator. Let $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ be an I/O stable and output stable DLS. Assume that the regular critical solution $P_0^{\mathrm{crit}} := \left(\mathcal{C}^{\mathrm{crit}}\right)^* J\mathcal{C}^{\mathrm{crit}} \in ric_0(\Phi, J)$ exists. Let $P \in ric(\Phi, J)$, such that $P \geq 0$. By ϕ_P and ϕ^P denote the spectral and inner DLS of Definition 150, both centered at P.*

Then the following holds:

(i) We have

$$\mathcal{D} = \mathcal{D}_{\phi^P}\mathcal{D}_{\phi_P}, \tag{4.27}$$

where ϕ_P is I/O stable and output stable. The DLS $J^{\frac{1}{2}}\phi^P$ is output stable, and the impulse response operator $J^{\frac{1}{2}}\mathcal{D}_{\phi^P}\bar{\pi}_0$ is bounded. The Toeplitz operator $J^{\frac{1}{2}}\mathcal{D}_{\phi^P}\bar{\pi}_+ : \ell^1(\mathbf{Z}_+; U) \to \ell^2(\mathbf{Z}_+; U)$ is bounded, and $J^{\frac{1}{2}}\mathcal{D}_{\phi^P}\bar{\pi}_+ : \ell^2(\mathbf{Z}_+; U) \to \ell^2(\mathbf{Z}_+; U)$ is a densely defined closed operator.

(ii) The transfer function $J^{\frac{1}{2}}\mathcal{D}_{\phi^P}(z)$ is analytic in the whole unit disk $\mathbf{D}$. For each $u_0 \in U$, the analytic function $J^{\frac{1}{2}}\mathcal{D}_{\phi^P}(z)u_0 \in H^2(\mathbf{D}; Y)$. We can write

$$J^{\frac{1}{2}}\mathcal{D}(z) = J^{\frac{1}{2}}\mathcal{D}_{\phi^P}(z)\mathcal{D}_{\phi_P}(z) \quad \text{for all} \quad z \in \mathbf{D}. \tag{4.28}$$

If, in addition, $P \in ric_{uw}(\Phi, J)$, then

$$J^{\frac{1}{2}}\mathcal{N}(z) = J^{\frac{1}{2}}\mathcal{D}_{\phi^P}(z)\mathcal{N}_P(z) \quad \text{for all} \quad z \in \mathbf{D}, \tag{4.29}$$

where $\mathcal{N}$, ($\mathcal{N}_P$) are the $(J, \Lambda_{P_0^{\mathrm{crit}}})$- inner, ($(\Lambda_P, \Lambda_{P_0^{\mathrm{crit}}})$-inner) factors of $\mathcal{D}$, ($\mathcal{D}_{\phi_P}$, respectively).

Assume, in addition, that the input operator B of Φ is Hilbert–Schmidt, and both the spaces U and Y are separable. Then:

(iii) *Then $J^{\frac{1}{2}}\mathcal{D}_{\phi^P}(z) \in H^2(\mathbf{D};\mathcal{L}(U;Y))$. The boundary trace function*

$$J^{\frac{1}{2}}\mathcal{D}_{\phi^P}(e^{i\theta}) := \mathrm{s}-\lim_{z\to e^{i\theta}} J^{\frac{1}{2}}\mathcal{D}_{\phi^P}(z)$$

exists as a nontangential strong limit, a.e. (modulo Lebesque measure of $\mathbf{T}$) on $e^{i\theta}\in\mathbf{T}$.

(iv) *For $P \in ric_{uw}(\Phi, J)$, the boundary trace $\Lambda_P^{\frac{1}{2}}\mathcal{N}_P(e^{i\theta})\Lambda_{P_0^{\mathrm{crit}}}^{-\frac{1}{2}}$ is unitary a.e. $e^{i\theta}\in\mathbf{T}$. In particular, $\mathcal{N}_P(e^{i\theta})$ has a bounded inverse a.e. $e^{i\theta}\in\mathbf{T}$, and the nontangential strong limit $J^{\frac{1}{2}}\mathcal{D}_{\phi^P}(e^{i\theta})$ satisfies*

$$J^{\frac{1}{2}}\mathcal{D}_{\phi^P}(e^{i\theta}) = J^{\frac{1}{2}}\mathcal{N}(e^{i\theta})\mathcal{N}_P(e^{i\theta})^{-1} \quad \text{a.e. on} \quad e^{i\theta}\in\mathbf{T}, \tag{4.30}$$

Furthermore, $J^{\frac{1}{2}}\mathcal{D}_{\phi^P}(z)\Lambda_P^{-\frac{1}{2}} \in H^\infty(\mathbf{D};\mathcal{L}(U;Y))$, and it is inner from the left. The I/O map $J^{\frac{1}{2}}\mathcal{D}_{\phi^P}$ is (I,Λ_P)-inner (but $\mathcal{D}_{\phi^P}$ need not be I/O stable if J is not coercive).

We remark that the function $J^{\frac{1}{2}}\mathcal{D}_{\phi^P}(e^{i\theta})$ means the boundary trace of $(J^{\frac{1}{2}}\mathcal{D}_{\phi^P})(z)$. As an analytic transfer function $\mathcal{D}_{\phi^P}(z)$, $P\geq 0$ makes perfect sense for $z\in\mathbf{D}$, but it need not be of bounded type.

Proof. Claim (i) is proved as follows. The equality (4.27) of the I/O maps is given by formula (4.25), in form $\mathcal{D}_{\phi^P} = \mathcal{D}\mathcal{D}_{\phi_P}^{-1}$. We see that the $J^{\frac{1}{2}}\mathcal{D}_{\phi^P}$ is the I/O map of DLS

$$\phi'' = \begin{pmatrix} A_P & B \\ J^{\frac{1}{2}}C_P & J^{\frac{1}{2}}D \end{pmatrix},$$

which is output stable, by Corollary 167 and the assumption $P \geq 0$. Also the (closed loop) residual cost operator $L_{A_P,P}$ exists, but this is not needed here. But then, if $H \ni x = Bu_0$, with $u_0\in U$, we have $J^{\frac{1}{2}}\mathcal{D}_{\phi^P}\pi_0 u_0 = J^{\frac{1}{2}}D\pi_0u_0 + \tau\mathcal{C}_{\phi''}Bu_0 = D\pi_0 u_0 + \tau\mathcal{C}_{\phi''}x \in \ell^2(\mathbf{Z}_+;Y)$ because $\mathrm{dom}\,(\mathcal{C}_{\phi''}) = H$, by the output stability of ϕ''. Thus $\mathcal{D}_{\phi^P}\pi_0 : U = \mathrm{range}\,(\pi_0) \to \ell^2(\mathbf{Z}_+;U)$, i.e. $\mathrm{dom}\,\big(\mathcal{D}_{\phi^P}\pi_0\big) = U$ is complete, see Definition 29.

Because the impulse response operator $\mathcal{D}_{\phi^P}\pi_0$ is closed by Lemma 31, it follows from the Closed Graph Theorem that $\mathcal{D}_{\phi^P}\pi_0$ is bounded. It immediately follows that $J^{\frac{1}{2}}\mathcal{D}_{\phi^P} \in \mathcal{L}(\ell^1(\mathbf{Z}_+;U),\ell^2(\mathbf{Z}_+;U))$ by the triangle inequality, and the shift invariance of $\mathcal{D}_{\phi^P}$. The Toeplitz operator $\mathcal{D}_{\phi^P}\bar{\pi}_+$ is thus densely defined on $\ell^2(\mathbf{Z}_+;U)$ and closed, by Lemma 31. This completes the proof of claim (i).

Consider now claim (ii). $J^{\frac{1}{2}}\mathcal{D}_{\phi^P}(z)$ is analytic in the whole of $\mathbf{D}$ by Proposition 57 because it is a transfer function of an output stable system ϕ''. Also $J^{\frac{1}{2}}\mathcal{D}_{\phi^P}(z) \in sH^2(\mathbf{D};Y)$, by Definition 56 and Proposition 57.

Because $\mathcal{D}_{\phi^P} = \mathcal{D}\mathcal{D}_{\phi_P}^{-1}$, then also $\mathcal{D}_{\phi^P}\mathcal{D}_{\phi_P} = \mathcal{D}$ on $Seq(U)$. For the transfer functions, we have $\mathcal{D}_{\phi^P}(z)\mathcal{D}_{\phi_P}(z) = (\mathcal{D}_{\phi^P}\mathcal{D}_{\phi_P})(z) = \mathcal{D}(z)$ for all $z \in N_0$, by Corollary 54. Here N_0 is a nonempty open neighborhood of the origin. In fact, $\mathcal{D}(z), \mathcal{D}_{\phi_P}(z) \in H^\infty(\mathbf{D};\mathcal{L}(U;Y))$, by Proposition 55 and the assumed I/O stability of Φ and ϕ_P. As indicated above, also $J^{\frac{1}{2}}\mathcal{D}_{\phi^P}(z)$ is analytic in $\mathbf{D}$. By using a basic analytic continuation technique we conclude that $\mathcal{D}_{\phi^P}(z)\mathcal{D}_{\phi_P}(z) = \mathcal{D}(z)$ for all $z \in \mathbf{D}$, which is equation (4.28).

To prove equation (4.29), proceed as follows. Because the existence of the regular critical solution $P_0^{\mathrm{crit}} \in ric_0(\Phi, J)$ is assumed, the equivalent conditions of Theorem 114 hold, we can write $\mathcal{D} = \mathcal{N}\mathcal{X}$, where $\mathcal{X}$ is outer with a bounded inverse, and $\mathcal{N}$ is $(J, \Lambda_{P_0^{\mathrm{crit}}})$-inner. Furthermore, because $P \in ric_{uw}(\Phi, J)$ we can also write $(\Lambda_P, \Lambda_{P_0^{\mathrm{crit}}})$-inner-outer factorization $\mathcal{D}_{\phi_P} = \mathcal{N}_P\mathcal{X}$, by Proposition 147. By Corollary 54, $\mathcal{D}(z) = \mathcal{N}(z)\mathcal{X}(z)$ and $\mathcal{D}_{\phi_P}(z) = \mathcal{N}_P(z)\mathcal{X}(z)$, for all $z \in \mathbf{D}$. Because $\mathcal{X}$ is outer with a bounded inverse, i.e. $\mathcal{X}^{-1} \in \mathcal{L}(\ell^2(\mathbf{Z};U))$, both $\mathcal{X}$ and $\mathcal{X}^{-1}$ are I/O maps of I/O stable systems. It follows from Corollary 54 that the transfer function $\mathcal{X}(z) \in \mathcal{L}(U)$ has a bounded inverse for all $z \in \mathbf{D}$. Now equation (4.29) follows.

We proceed to prove claim (iii). The Hilbert–Schmidt property of the input operator B admits us to apply Corollary 131 to the output stable DLS ϕ'', defined above. It follows that $J^{\frac{1}{2}}\mathcal{D}_{\phi^P}(z) \in H^2(\mathbf{D};\mathcal{L}(U;Y))$, and this is a function of bounded type. The existence of the nontangential strong limit $J^{\frac{1}{2}}\mathcal{D}_{\phi^P}(e^{i\theta})$ is from [77, Theorem 4.6A], as discussed in Section 1.10.

It remains to prove the final claim (iv). We first note that because $J \geq 0$, then $\Lambda_P > 0$ for all $P \geq 0$. This makes is possible to define the normalized operators $\mathcal{N}^\circ := J^{\frac{1}{2}}\mathcal{N}\Lambda_{P_0^{\mathrm{crit}}}^{-\frac{1}{2}}$ and $\mathcal{N}_P^\circ := \Lambda_P^{\frac{1}{2}}\mathcal{N}_P\Lambda_{P_0^{\mathrm{crit}}}^{-\frac{1}{2}}$. Then both $\mathcal{N}^\circ$ and $\mathcal{N}_P^\circ$ are inner from the left (i.e. (I, I)-inner). We have

$$\mathcal{N}^\circ = J^{\frac{1}{2}}\mathcal{D}_{\phi^P}\mathcal{N}_P\Lambda_{P_0^{\mathrm{crit}}}^{-\frac{1}{2}} = J^{\frac{1}{2}}\mathcal{D}_{\phi^P}\Lambda_P^{-\frac{1}{2}} \cdot \Lambda_P^{\frac{1}{2}}\mathcal{N}_P\Lambda_{P_0^{\mathrm{crit}}}^{-\frac{1}{2}} = \mathcal{M}_P^\circ\mathcal{N}_P^\circ,$$

where $\mathcal{M}_P^\circ := J^{\frac{1}{2}}\mathcal{D}_{\phi^P}\Lambda_P^{-\frac{1}{2}}$. For the corresponding transfer functions and their nontangential limits, we can write

$$\mathcal{N}^\circ(e^{i\theta}) = \mathcal{M}_P^\circ(e^{i\theta})\mathcal{N}_P^\circ(e^{i\theta}), \tag{4.31}$$

a.e. $e^{i\theta} \in \mathbf{T}$. This is legal because all the transfer functions are of bounded type in the sense of Definition 58 and the discussion associated to it. By claim (ii) of Proposition 147, the normalized I/O map $\mathcal{N}_P^\circ = \Lambda_P^{\frac{1}{2}}\mathcal{N}_P\Lambda_{P_0^{\mathrm{crit}}}^{-\frac{1}{2}}$ is inner from both sides in the sense of Definition 120. So the values of the boundary trace $\mathcal{N}_P^\circ(e^{i\theta}) \in \mathcal{L}(U)$ are a unitary operators for a.e. $e^{i\theta} \in \mathbf{T}$. Applying this on equation (4.31) gives

$$\mathcal{N}^\circ(e^{i\theta})\mathcal{N}_P^\circ(e^{i\theta})^* = \mathcal{M}_P^\circ(e^{i\theta})$$

a.e. $e^{i\theta} \in \mathbf{T}$. Because $\mathcal{N}_P^\circ(e^{i\theta})^*$ is unitary and $\mathcal{N}^\circ(e^{i\theta})$ is an isometry, it follows that $\mathcal{M}_P^\circ(e^{i\theta})$ is an isometry a.e. $e^{i\theta} \in \mathbf{T}$. But now $\mathcal{M}_P^\circ(e^{i\theta}) \in L^\infty(\mathbf{T};\mathcal{L}(U;Y)) \cap H^2(\mathbf{T};\mathcal{L}(U;Y))$, and by Lemma 122, $\mathcal{M}_P^\circ(e^{i\theta}) \in H^\infty(\mathbf{T};\mathcal{L}(U;Y))$ is inner from the left. This completes the proof. $\square$

The following normalization, presented in the proof of Lemma 171, will be used throughout the rest of this paper. By Corollary 146, it makes sense even for indefinite solutions $P \in ric_{uw}(\phi, J)$, as far as $\bar\pi_+\mathcal{D}_\phi^* J\mathcal{D}_\phi \geq \epsilon\bar\pi_+$ for some $\epsilon > 0$.

Corollary 172. *Make the same assumptions as in claim (iii) of Lemma 171. By $P_0^{\mathrm{crit}} \in ric_0(\Phi, J)$ denote the regular critical solution. Let $P \in ric_{uw}(\Phi, J)$, $P \geq 0$ be arbitrary. Denote*

$$\begin{aligned}
&\mathcal{D}^\circ := J^{\frac{1}{2}}\mathcal{D}, \quad \mathcal{D}_P^\circ := \Lambda_P^{\frac{1}{2}}\mathcal{D}_{\phi_P},\\
&\mathcal{M}_P^\circ := J^{\frac{1}{2}}\mathcal{D}_{\phi^P}\Lambda_P^{-\frac{1}{2}}, \quad \mathcal{N}_P^\circ := \Lambda_P^{\frac{1}{2}}\mathcal{N}_P\Lambda_{P_0^{\mathrm{crit}}}^{-\frac{1}{2}}, \quad \mathcal{X}^\circ = \Lambda_{P_0^{\mathrm{crit}}}^{\frac{1}{2}}\mathcal{X}.
\end{aligned}$$

Then

$$\mathcal{D}^\circ = \mathcal{M}_P^\circ\mathcal{D}_P^\circ = \mathcal{M}_P^\circ\mathcal{N}_P^\circ\mathcal{X}_P^\circ, \tag{4.32}$$

where $\mathcal{M}_P^\circ : \ell^2(\mathbf{Z};U) \to \ell^2(\mathbf{Z};Y)$ is inner from the left, $\mathcal{N}_P^\circ : \ell^2(\mathbf{Z};U) \to \ell^2(\mathbf{Z};U)$ is two-sided inner, and $\mathcal{X}^\circ : \ell^2(\mathbf{Z};U) \to \ell^2(\mathbf{Z};U)$ is outer with a bounded inverse.

The following Theorem is a variation of Lemma 171. Now, a solution $P \in Ric(\Phi, J)$, $P \geq 0$ gives a factorization of a H^∞-transfer function, such that both the factors are in H^∞. However, the solution is not in $ric_{uw}(\phi, J)$ by an explicit assumption, and ϕ_P is not *a priori* required to be output stable or I/O stable as has been required in Lemma 171.

Theorem 173. *Let $J \geq 0$ be a cost operator. Let $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D}\end{smallmatrix}\right]$ be an I/O stable and output stable DLS, such that both the spaces U and Y are separable. Assume that the input operator $B \in \mathcal{L}(U;H)$ of Φ is Hilbert–Schmidt. Assume that the regular critical solution $P_0^{\mathrm{crit}} := (\mathcal{C}^{\mathrm{crit}})^*J\mathcal{C}^{\mathrm{crit}} \in ric_0(\Phi, J)$ exists. Let $P \in Ric_{00}(\Phi, J) \cap Ric_{uw}(\Phi, J)$, $P \geq 0$.*

Then both the DLSs ϕ_P and $J^{\frac{1}{2}}\phi^P$ are output stable and I/O stable. Furthermore, we have the factorization $J^{\frac{1}{2}}\mathcal{D} = J^{\frac{1}{2}}\mathcal{D}_{\phi^P}\cdot\mathcal{D}_{\phi_P} = J^{\frac{1}{2}}\mathcal{D}_{\phi^P}\cdot\mathcal{N}_P\cdot\mathcal{X}$ where all factors are I/O stable. Here $J^{\frac{1}{2}}\mathcal{D}_{\phi^P}$ is (I, Λ_P)-inner, $\mathcal{N}_P$ is $(\Lambda_P, \Lambda_{P_0^{\mathrm{crit}}})$-inner, and $\mathcal{X}$ is outer with a bounded inverse.

Proof. Because $J \geq 0$ and $P \geq 0$, it follows that $D^*JD + B^*PB = \Lambda_P \geq 0$, and then $\Lambda_P > 0$ because the indicator has a bounded inverse, by definition. Because $P \in Ric_{00}(\Phi, J)$, the residual cost operator $L_{A,P}$ exists and Proposition 136 implies that ϕ_P is output stable. Because $P \in Ric_{uw}(\Phi, J)$, Corollary 140 implies that ϕ_P is I/O stable. Now $P \in ric_{uw}(\Phi, J)$ as in equation (3.27), and we can apply all claims of Lemma 171. In particular, this gives the output stability and I/O stability of the normalized inner DLS $J^{\frac{1}{2}}\phi^P$. The proof is now complete. □

If A is strongly stable, then $Ric(\Phi, J) = Ric_0(\Phi, J) = Ric_{00}(\Phi, J) = Ric_{uw}(\Phi, J)$. But now $Ric(\Phi, J) = Ric_{00}(\Phi, J) \cap Ric_{uw}(\Phi, J)$, and all nonnegative solutions $P \in Ric(\Phi, J)$ give a factorization of Theorem 173. The following lemma is more general than Lemma 171, and it refers to something we might call "generalized factorizations" of an unstable $\mathcal{D}$. Now the spectral DLS ϕ_P need not be I/O stable.

Lemma 174. *Let $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ be output stable and $J \geq 0$. Let $P \in Ric_{00}(\Phi, J)$, $P \geq 0$. Then the following holds:*

(i) The I/O maps satisfy $\mathcal{D} = \mathcal{D}_{\phi^P}\mathcal{D}_{\phi_P}$ on $Seq(U)$, and both ϕ_P and $J^{\frac{1}{2}}\phi^P$ are output stable.

(ii) Assume, in addition, that the input operator B is Hilbert–Schmidt, and both U and Y are separable. Then we have the factorization

$$J^{\frac{1}{2}}\mathcal{D} = J^{\frac{1}{2}}\mathcal{D}_{\phi^P}\mathcal{D}_{\phi_P}, \tag{4.33}$$

where $J^{\frac{1}{2}}\mathcal{D}(z), J^{\frac{1}{2}}\mathcal{D}_{\phi^P}(z) \in H^2(\mathbf{D}; \mathcal{L}(U;Y))$ and $\mathcal{D}_{\phi_P}(z) \in H^2(\mathbf{D}; \mathcal{L}(U))$.

Proof. As before, $\Lambda_P > 0$ for any nonnegative solution. Proposition 136 implies that ϕ_P is output stable. Corollary 167 implies that $J^{\frac{1}{2}}\phi^P$ is output stable. This proves claim (i) because the (algebraic) factorization of the well-posed I/O maps of DLSs does not require any kind of stability. Claim (ii) is a consequence of Corollary 131. □

In particular, Lemma 174 gives H^2 factorizations to H^∞ transfer functions. Note that the existence of a critical regular solution $P_0^{\text{crit}} \in ric_0(\phi, J)$ is not required. Under stronger assumptions, such generalized factorizations easily become ordinary H^∞ factorizations, by Theorem 173. We complete this section by showing that the finite increasing chains of solutions $P_i \in ric_{uw}(\Phi, J)$ behave expectedly.

Theorem 175. *Let $J \in \mathcal{L}(Y)$ be a self-adjoint cost operator. Let $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ be an I/O stable and output stable DLS. Assume that the input operator $B \in$*

$\mathcal{L}(U;H)$ is Hilbert–Schmidt, and both the spaces U and Y are separable. Assume that the regular critical solution $P_0^{\mathrm{crit}} = \left(\mathcal{C}^{\mathrm{crit}}\right)^* J\mathcal{C}^{\mathrm{crit}} \in ric_0(\Phi, J)$ exists.

Let $P_i \in ric_{uw}(\Phi, J)$, $i = 1, \ldots, n+1$ be a sequence of solutions such that $P_i \leq P_{i+1}$ and $\Lambda_{P_i} > 0$ for all $i = 1, \ldots, n$. Denote by $\mathcal{D}_{\phi_{P_i}} = \mathcal{N}_{P_i}\mathcal{X}$ the $(\Lambda_P, \Lambda_{P_0^{\mathrm{crit}}})$ -inner-outer factorization of $\mathcal{D}_{\phi_{P_i}}$ where $\mathcal{X} = \mathcal{D}_{P_0^{\mathrm{crit}}}$ and $\mathcal{N}_{P_i} := \mathcal{D}_{\phi_{P_i}}\mathcal{X}^{-1}$. Then the following holds:

(i) *Then there is a sequence of causal shift-invariant operators $\mathcal{N}_{P_i,P_{i+1}} := \mathcal{D}_{\phi_{P_i}}\mathcal{D}^{-1}_{\phi_{P_{i+1}}}$ on $Seq(U)$ such that*

$$\mathcal{N}_{P_i} = \mathcal{N}_{P_i,P_{i+1}}\mathcal{N}_{P_{i+1}} \quad \text{for all} \quad i = 1, \ldots, n. \tag{4.34}$$

The operator $\mathcal{N}_{P_i,P_{i+1}}$ is the I/O map of the I/O stable DLS

$$\phi_{P_i,P_{i+1}} = \begin{pmatrix} A_{P_{i+1}} & B \\ K_{P_{i+1}} - K_{P_i} & I \end{pmatrix}. \tag{4.35}$$

Furthermore, each $\mathcal{N}_{P_i,P_{i+1}}$ is $(\Lambda_{P_i}, \Lambda_{P_{i+1}})$-inner.

(ii) *We have the factorization*

$$\mathcal{N}_{P_1} = \left(\prod_{i=1}^{n} \mathcal{N}_{P_i,P_{i+1}}\right)\mathcal{N}_{P_{n+1}}, \tag{4.36}$$

where the elements with increasing i enter the product from the left. If, in addition, $J \geq 0$ and $P_{n+1} = P_0^{\mathrm{crit}}$, then

$$J^{\frac{1}{2}}\mathcal{D} = J^{\frac{1}{2}}\mathcal{D}_{\phi^{P_1}}\left(\prod_{i=1}^{n} \mathcal{N}_{P_i,P_{i+1}}\right)\mathcal{X}, \tag{4.37}$$

where $J^{\frac{1}{2}}\mathcal{D}_{\phi^{P_1}}$ is I/O stable and (I, Λ_{P_1}) -inner, and $\mathcal{X} = \mathcal{D}_{\phi_{P_0^{\mathrm{crit}}}}$ is outer with a bounded inverse.

Proof. In order to prove claim (i), note that $(\Lambda_P, \Lambda_{P_0^{\mathrm{crit}}})$-inner-outer factorization $\mathcal{D}_{\phi_{P_i}} = \mathcal{N}_{P_i}\mathcal{X}$ exists for all i, by Proposition 147. Because the feed-through operator of all spectral DLSs is identity, we can speak about the inverse $\mathcal{D}^{-1}_{\phi_{P_i}}$ as a causal shift-invariant operator on $Seq(U)$, see Proposition 17. Because the outer factor (with a bounded inverse) is common for all $\mathcal{D}_{\phi_{P_i}}$, we see that equation (4.34) holds.

Fix the arbitrary two consecutive elements $P_i \leq P_{i+1}$ in the sequence $\{P_i\}$, define $\Delta P_i := P_{i+1} - P_i \geq 0$. Then $\Delta P_i \in Ric(\phi_{P_i}, \Lambda_{P_i})$, by Lemma 156. Now, $Ric(\phi_{P_i}, \Lambda_{P_i})$ is a H^∞DARE with a nonnegative cost operator Λ_{P_i}, but we do not know whether ΔP_i is its H^∞ solution. To see that this is the case, we must consider the spectral DLS $(\phi_{P_i})_{\Delta P_i}$, centered at the solution ΔP_i and relative to the cost operator $\Lambda_{P_i} > 0$ of the spectral DARE. We have for the minimax nodes

$$(\phi_{P_i}, \Lambda_{P_i})_{\Delta P_i} \equiv (\phi_{P_i+\Delta P_i}, \Lambda_{P_i+\Delta P_i}) \equiv \left(\phi_{P_{i+1}}, \Lambda_{P_{i+1}}\right), \tag{4.38}$$

see equation (4.4) of Proposition 151. So, the spectral DLS $(\phi_{P_i})_{\Delta P_i}$ of ΔP_i equals $\phi_{P_{i+1}}$ which is an I/O stable and output stable DLS because $P_{i+1} \in ric(\Phi, J)$, by assumption. We conclude that $\Delta P \in ric(\phi_{P_i}, \Lambda_{P_i})$. The indicator $\tilde{\Lambda}_{\Delta P}$ of $\Delta P \in ric(\phi_{P_i}, \Lambda_{P_i})$ equals $\Lambda_{P_{i+1}}$, by equation (4.38).

Trivially $\operatorname{range}(\mathcal{B}) = \operatorname{range}\left(\mathcal{B}_{\phi_{P_i}}\right)$ because $\mathcal{B} = \mathcal{B}_{\phi_{P_i}}$. Because both P_i and P_{i+1} satisfy the ultra weak residual cost condition with the same semigroup generator A, so does $\Delta P_i = P_{i+1} - P_i$, and we have $\Delta P_i \in ric_{uw}(\phi_{P_i}, \Lambda_{P_i})$.

Now we have reached the situation described in Lemma 171. We see that the operator $\mathcal{N}_{P_i,P_{i+1}} := \mathcal{D}_{\phi_{P_i}} \mathcal{D}^{-1}_{\phi_{P_{i+1}}} = \mathcal{D}_{\phi_{P_i}} \mathcal{D}^{-1}_{(\phi_{P_i})_{\Delta P_i}}$ actually plays the part of the operator $\mathcal{D}_{\phi^P}$ in Lemma 171, when the DLS Φ is replaced by ϕ_{P_i}, the cost operator J is replaced by Λ_{P_i}, the solution P is replaced by ΔP_i, the spectral DLS ϕ_P is replaced by $(\phi_{P_i})_{\Delta P_i} = \phi_{P_{i+1}}$ and the indicator Λ_P is replaced by $\Lambda_{P_{i+1}}$.

Because the input operator B of ϕ_{P_i} is Hilbert–Schmidt, we conclude that $\mathcal{N}_{P_i,P_{i+1}}$ is I/O stable and $(\Lambda_{P_i}, \Lambda_{P_{i+1}})$-inner, by claim (iv) of Lemma 171, and the fact that Λ_{P_i} (used as the cost operator) has a bounded inverse. Realization (4.35) is valid because $\mathcal{N}_{P_i,P_{i+1}} = \mathcal{N}_{P_i}\mathcal{N}^{-1}_{P_{i+1}}$, by equation (4.35) and claim (iii) of Proposition 148. This completes the proof of claim (i).

The factorization in (4.36) is clearly obtained by applying the first part of this theorem n times. The second factorization (4.37) is obtained by first factorizing $J^{\frac{1}{2}}\mathcal{D} = J^{\frac{1}{2}}\mathcal{D}_{\phi^{P_1}}\mathcal{D}_{\phi_{P_1}}$, where $J^{\frac{1}{2}}\mathcal{D}_{\phi^{P_1}}$ is I/O stable and (I, Λ_{P_1}) -inner, by claim (iv) of Lemma 171. This is the only place where we have used the nonnegativity of J. Then the $(\Lambda_{P_1}, \Lambda_{P_0^{\mathrm{crit}}})$-inner factor $\mathcal{N}_{P_1}$ of $\mathcal{D}_{\phi_{P_1}}$ is factorized as in (4.36), noting that the last factor $\mathcal{N}_{P_{n+1}} = \mathcal{I}$ because $P_{n+1} = P_0^{\mathrm{crit}}$, by claim (iii) of Proposition 147. After multiplying from the right by the common outer factor $\mathcal{X}$ of $\mathcal{D}$ and $\mathcal{D}_{\phi_{P_1}}$, the claim follows. □

By Lemma 145, it is sufficient to require $\Lambda_P > 0$ only for one solution $P \in ric_{uw}(\Phi, J)$ that need not be an element of the chain $\{P_i\}$. Clearly, the order of the operator products in claim (ii) is significant, if $\dim U > 1$. The transfer function $\mathcal{N}_{P_i,P_{i+1}}(z)$ can be normalized to $\mathcal{N}^\circ_{P_i,P_{i+1}}(z) := \Lambda^{\frac{1}{2}}_{P_i}\mathcal{N}_{P_i,P_{i+1}}(z)\Lambda^{-\frac{1}{2}}_{P_{i+1}}$ which is inner from both sides. The zero evaluation $\mathcal{N}^\circ_{P_i,P_{i+1}}(0) = \Lambda^{\frac{1}{2}}_{P_i}\Lambda^{-\frac{1}{2}}_{P_{i+1}}$ satisfies the spectral condition $\sigma(\Lambda^{\frac{1}{2}}_{P_i}\Lambda^{-\frac{1}{2}}_{P_{i+1}}) \subset (0, 1)$, as an immediate consequence of the fact that $\Lambda_{P_{i+1}} \geq \Lambda_{P_i}$. However, $\Lambda_{P_i}^{\frac{1}{2}} \Lambda_{P_{i+1}}^{-\frac{1}{2}}$ is generally not normal and, in particular, self-adjoint. In Theorem 175, we have considered only finite increasing chains of solutions. To cover the case of the (countably) infinite chains, one would be lead to consider a limit process, not totally different from the one involved in the study of the Blaschke–Potapov representations for the (matrix-valued) bounded analytic functions. Several applications, references and historical remarks about the Blaschke–Potapov factorizations can be found in the survey article [40].

4.5 I/O stability of inner DLS

In this section, we consider converse results to those given in Section 4.4. Roughly, we show that for $P \in ric_{uw}(\phi, J)$, the I/O stability of ϕ^P implies $P \geq 0$. The nonnegativity of the cost operator $J \geq 0$ is assumed in the main results.

We start by considering solutions $P \in ric(\phi, J)$ such that ϕ^P is I/O stable. Out of such solutions, those that have (J, Λ_P)-inner I/O maps satisfy the minimax condition of Definition 176, by Proposition 178. In particular, all solutions in $ric_{uw}(\phi, J)$ with an I/O stable inner DLS ϕ^P are of this kind, by Proposition 177. In Propositions 179 and 180, the minimax condition of P is connected to an associated Liapunov equation and the DARE $ric(\Phi, J)$. The main result of this section is Lemma 181, which is a partial converse Lemma 171. An equivalence result is finally given in Theorem 182, under stronger assumptions.

Definition 176. *Let $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ be an I/O stable and output stable DLS, and $J \in \mathcal{L}(Y)$ a cost operator. Let $P \in ric(\Phi, J)$ such that the inner DLS ϕ^P is I/O stable. We say that P satisfies the minimax condition if*

$$\bar{\pi}_+ \mathcal{D}^*_{\phi^P} J \mathcal{C}_{\phi^P} = 0, \tag{4.39}$$

where $\mathcal{C}_{\phi^P} = \mathcal{C} - \mathcal{D}_{\phi^P} \mathcal{C}_{\phi_P}$ it the observability map of inner DLS ϕ^P.

The regular critical solution $P_0^{\text{crit}} := (\mathcal{C}^{\text{crit}})^* J \mathcal{C}^{\text{crit}}$ (as discussed in connection with Theorem 114) always satisfies the minimax condition. This is because in this case $\mathcal{D} = \mathcal{N}\mathcal{X}$ (where $\mathcal{N} = \mathcal{D}_{\phi^{P_0^{\text{crit}}}}$ and $\mathcal{X} = \mathcal{D}_{\phi_{P_0^{\text{crit}}}}$) is the $(J, \Lambda_{P_0^{\text{crit}}})$-inner-outer factorization, and $\mathcal{C}_{\phi^{P_0^{\text{crit}}}} = \mathcal{C}^{\text{crit}}$ is the critical (closed loop) observability map. By Lemma 67, $\bar{\pi}_+ \mathcal{D}^* J \mathcal{C}^{\text{crit}} = \bar{\pi}_+ \mathcal{X}^* \bar{\pi}_+ \mathcal{N}^* J \mathcal{C}^{\text{crit}} = 0$, and the minimax condition holds.

In fact, the orthogonality of $\overline{\text{range}\,(\mathcal{D}\bar{\pi}_+)} = \overline{\text{range}\,(\mathcal{N}\bar{\pi}_+)}$ and the range of the desired closed loop observability map $\mathcal{C}_{\phi^{P_0^{\text{crit}}}} = \mathcal{C}^{\text{crit}}$ can be used to find the critical P_0^{crit} without explicitly solving the DARE, see Section 2.2. For a noncritical P, however, one should *a priori* know the (range of the) partial inner factor $\mathcal{D}_{\phi^P} \bar{\pi}_+$ of $\mathcal{D}\bar{\pi}_+$ associated to the yet unknown P, before the correct minimax formulation could be written in the first place.

We proceed to show that quite many interesting solutions $P \in ric(\Phi, J)$ (such that $\mathcal{D}_{\phi^P}$ is I/O stable) satisfy the minimax condition. This will be used as a technical tool to obtain Lemma 181, a rough converse of Lemma 171.

Proposition 177. *Let $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ be an I/O stable and output stable DLS. Assume that the input operator $B \in \mathcal{L}(U; H)$ of Φ is Hilbert–Schmidt, and the* *spaces U and Y are separable. Let $J \geq 0$ be a cost operator. Assume that the regular critical solution $P_0^{\rm crit} = \left(\mathcal{C}^{\rm crit}\right)^* J\mathcal{C}^{\rm crit} \in ric_0(\Phi, J)$ exists. Let $P \in ric_{uw}(\Phi, J)$ such that the inner DLS ϕ^P is I/O stable. Then $\mathcal{D}_{\phi^P}^* J\mathcal{D}_{\phi^P} = \Lambda_P$; i.e. the I/O map $\mathcal{D}_{\phi^P}$ is (J, Λ_P)-inner.*

Proof. We have the familiar factorization of the I/O maps $\mathcal{D} = \mathcal{D}_{\phi^P}\mathcal{D}_{\phi_P}$. Because $P_0^{\rm crit}$ exists, the conditions of Theorem 114 hold, and we can factorize $\mathcal{D} = \mathcal{N}\mathcal{X}$, $\mathcal{D}_{\phi_P} = \mathcal{N}_P\mathcal{X}$, where $\mathcal{N}$, $(\mathcal{N}_P)$ is $(J, \Lambda_{P_0^{\rm crit}})$-inner, $((\Lambda_P, \Lambda_{P_0^{\rm crit}})$-inner, respectively). Here we have used the residual cost assumption $P \in ric_{uw}(\Phi, J)$ and claim (i) of Theorem 142. The operator $\mathcal{X}$ is a common outer factor with a bounded inverse; for details, see Proposition 147. This gives us the factorization

$$\mathcal{N} = \mathcal{D}_{\phi^P}\mathcal{N}_P \tag{4.40}$$

where all the factors I/O stable, the I/O map $\mathcal{D}_{\phi^P}$ by our explicit assumption. Let us consider the factor $\mathcal{N}_P$ more carefully. By Corollary 146, $\Lambda_P > 0$ for all $P \in ric_{uw}(\Phi, J)$, because the conditions of Theorem 114 hold, and $J \geq 0$ implies that $P_0^{\rm crit} \geq 0$ and $\Lambda_{P_0^{\rm crit}} > 0$. So we can normalize $\mathcal{N}_P^\circ := \Lambda_P^{\frac{1}{2}}\mathcal{N}_P\Lambda_{P_0^{\rm crit}}^{-\frac{1}{2}}$ which is (I, I)-inner. By claim (ii) of Proposition 147, $\mathcal{N}_P^\circ$ is in fact inner from both sides, and its boundary trace $\mathcal{N}_P^\circ(e^{i\theta})$ takes unitary values a.e. $e^{i\theta} \in \mathbf{T}$. We remark that here the Hilbert-Schmidt compactness of the input operator B and the separability of U is used.

Because also Y is separable, equation (4.40) implies for the boundary traces

$$\mathcal{D}_{\phi^P}(e^{i\theta}) = \mathcal{N}(e^{i\theta})\mathcal{N}_P(e^{i\theta})^{-1}$$

a.e. $e^{i\theta} \in \mathbf{T}$, as in the proof of claim (iv) of Lemma 171. But now for almost all $e^{i\theta} \in \mathbf{T}$

$$J^{\frac{1}{2}}\mathcal{D}_{\phi^P}(e^{i\theta})\Lambda_P^{-\frac{1}{2}} = \mathcal{N}^\circ(e^{i\theta})\mathcal{N}_P^\circ(e^{i\theta})^*,$$

where $\mathcal{N}^\circ(e^{i\theta}) := J^{\frac{1}{2}}\mathcal{N}(e^{i\theta})\Lambda_{P_0^{\rm crit}}^{-\frac{1}{2}}$ is isometric a.e. $e^{i\theta} \in \mathbf{T}$. It follows that $J^{\frac{1}{2}}\mathcal{D}_{\phi^P}(e^{i\theta})\Lambda_P^{-\frac{1}{2}}$ is isometric a.e. $e^{i\theta} \in \mathbf{T}$, and thus $\mathcal{D}_{\phi^P}$ is (J, Λ_P)-inner. This completes the proof of the proposition. ☐

Proposition 178. *Let $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ be an I/O stable and output stable DLS. Assume that the regular critical solution $P_0^{\rm crit} = \left(\mathcal{C}^{\rm crit}\right)^* J\mathcal{C}^{\rm crit} \in ric_0(\Phi, J)$ exists. Let $P \in ric(\Phi, J)$ such that the inner DLS ϕ^P is I/O stable and its I/O map is (J, Λ_P)-inner. If $\overline{\text{range}\,(\mathcal{B})} = H$, then P satisfies the minimax condition; i.e. $\bar{\pi}_+\mathcal{D}_{\phi^P}^* J\mathcal{C}_{\phi^P} = 0$.*

Proof. Let $\tilde{u} \in Seq_-(U)$ be arbitrary. Because $\mathcal{D}_{\phi^P}^* J\mathcal{D}_{\phi^P} = \Lambda_P$ and $\bar{\pi}_+\mathcal{D}_{\phi^P}^*\pi_- = 0$, we have

$$\bar{\pi}_+\mathcal{D}_{\phi^P}^* J(\bar{\pi}_+\mathcal{D}_{\phi^P}\pi_-\tilde{u}) = \bar{\pi}_+\mathcal{D}_{\phi^P}^* J\mathcal{D}_{\phi^P}\pi_-\tilde{u} = \bar{\pi}_+\Lambda_P\pi_-\tilde{u} = 0.$$

because $\mathcal{D}_{\phi^P}$ is (J, Λ_P)-inner. Define $x = \mathcal{B}_{\phi^P}\pi_-\tilde{u}$. Now $\mathcal{C}_{\phi^P}x = \mathcal{C}_{\phi^P}\mathcal{B}_{\phi^P}\pi_-\tilde{u} = \bar{\pi}_+\mathcal{D}_{\phi^P}\pi_-\tilde{u}$, it follows that $\bar{\pi}_+\mathcal{D}^*_{\phi^P}J\mathcal{C}_{\phi^P}x = 0$. Because $\tilde{u} \in Seq_-(U)$ is arbitrary, we have $\bar{\pi}_+\mathcal{D}^*_{\phi^P}J\mathcal{C}_{\phi^P}x = 0$ for all $x \in \operatorname{range}(\mathcal{B}_{\phi^P})$.

It remains to show that $\overline{\operatorname{range}(\mathcal{B}_{\phi^P})} = H$. Because $\mathcal{B}_{\phi^P} = \mathcal{B}\mathcal{D}^{-1}_{\phi_P}$, we show that $\operatorname{range}\left(\mathcal{B}\mathcal{D}^{-1}_{\phi_P}\right) = \operatorname{range}(\mathcal{B})$. To see this, let $x \in \operatorname{range}(\mathcal{B})$ be arbitrary. Then $x = \mathcal{B}\pi_-\tilde{u}$ for some $\tilde{u} \in Seq_-(U)$. Define $\tilde{w} = \mathcal{D}_{\phi_P}\tilde{u} \in Seq(U)$. Then $\pi_-\tilde{w} \in Seq_-(U)$ has only finitely many nonzero components, and $\mathcal{B}\mathcal{D}^{-1}_{\phi_P}\pi_-\tilde{w} = \mathcal{B}\pi_-\mathcal{D}^{-1}_{\phi_P}\pi_-\tilde{w} = \mathcal{B}\pi_-\mathcal{D}^{-1}_{\phi_P}\tilde{w} = \mathcal{B}\pi_-\mathcal{D}^{-1}_{\phi_P}\mathcal{D}_{\phi_P}\pi_-\tilde{u} = \mathcal{B}\pi_-\tilde{u}$, where we have used the causality of $\mathcal{D}^{-1}_{\phi_P}$. This proves the inclusion $\operatorname{range}(\mathcal{B}) \subset \operatorname{range}\left(\mathcal{B}\mathcal{D}^{-1}_{\phi_P}\right)$. The other inclusion follows similarly by interchanging the causal shift-invariant operators $\mathcal{D}^{-1}_{\phi_P}$, $\mathcal{D}_{\phi_P}$ on $Seq(U)$, and noting that nothing in the proof depends upon the boundedness of neither of these operators. We have now proved that a feedback does not change the reachable subspace.

Because $\operatorname{range}(\mathcal{B}_{\phi^P}) = \operatorname{range}(\mathcal{B})$ and $\overline{\operatorname{range}(\mathcal{B})} = H$, it follows that $\bar{\pi}_+\mathcal{D}^*_{\phi^P}J\mathcal{C}_P = 0$, provided $\bar{\pi}_+\mathcal{D}^*_{\phi^P}J\mathcal{C}_P$ is bounded. Now $\mathcal{D}^*_{\phi^P}$ is bounded because $\mathcal{D}_{\phi^P}$ is assumed to be. Also $\mathcal{C}_{\phi^P} = \mathcal{C} - \mathcal{D}_{\phi^P}\mathcal{C}_{\phi_P}$ is bounded because both Φ and ϕ_P are assumed to be output stable. The proof is now complete. □

Proposition 179. *Let* $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ *be an I/O stable and output stable DLS, and J be a cost operator. Let $P \in ric(\Phi, J)$ such that the inner DLS ϕ^P is I/O stable, and its I/O map is (J, Λ_P)-inner. Then the following are equivalent:*

(i) P satisfies the minimax condition; i.e. $\bar{\pi}_+\mathcal{D}^*_{\phi^P}J\mathcal{C}_{\phi^P} = 0$.

(ii) $\mathcal{C}_{\phi_P} = \Lambda_P^{-1} \cdot \bar{\pi}_+\mathcal{D}^*_{\phi^P}J\mathcal{C}$

(iii) $-K_P = \Lambda_P^{-1} \cdot \pi_0\mathcal{D}^*_{\phi^P}J\mathcal{C}$, *with the identification of spaces* $\operatorname{range}(\pi_0)$ *and* U.

Proof. Proof of the equivalence (i) ⇔ (ii) is the following equivalence:

$$\begin{aligned} & \bar{\pi}_+\mathcal{D}^*_{\phi^P}J\mathcal{C}_{\phi^P} = \bar{\pi}_+\mathcal{D}^*_{\phi^P}J(\mathcal{C} - \mathcal{D}_{\phi^P}\mathcal{C}_{\phi_P}) = 0 \\ \Leftrightarrow \quad & \bar{\pi}_+\mathcal{D}^*_{\phi^P}J\mathcal{C} = (\bar{\pi}_+\mathcal{D}^*_{\phi^P}J\mathcal{D}_{\phi^P}\bar{\pi}_+)\mathcal{C}_{\phi_P} = \Lambda_P \cdot \mathcal{C}_{\phi_P}. \end{aligned}$$

Because $\mathcal{C}_{\phi_P} = \{-K_P A^j\}_{j\geq 0}$ by Definition 150, the implication (ii) ⇒ (iii) is immediate. For the converse direction, we have to show that $\Lambda_P^{-1} \cdot \bar{\pi}_+\mathcal{D}^*_{\phi^P}J\mathcal{C}$ is an observability map of a DLS whose semigroup generator is A — we already know that the first component $-K_P$ is correct if (iii) holds. It remains to prove

$$\left(\Lambda_P^{-1} \cdot \bar{\pi}_+\mathcal{D}^*_{\phi^P}J\mathcal{C}\right)A = \bar{\pi}_+\tau^*\left(\Lambda_P \cdot \bar{\pi}_+\mathcal{D}^*_{\phi^P}J\mathcal{C}\right).$$

But this is the case:

$$\left(\Lambda_P^{-1}\cdot\bar{\pi}_+\mathcal{D}^*_{\phi^P}J\mathcal{C}\right)A=\Lambda_P^{-1}\cdot\bar{\pi}_+\mathcal{D}^*_{\phi^P}\bar{\pi}_+\tau^*J\mathcal{C}$$
$$=\bar{\pi}_+\tau^*\left(\Lambda_P^{-1}\cdot\mathcal{D}^*_{\phi^P}\pi_+J\mathcal{C}\right)=\bar{\pi}_+\tau^*\left(\Lambda_P\cdot\mathcal{D}^*_{\phi^P}\bar{\pi}_+J\mathcal{C}\right),$$

where the last equality follows because $\pi_+\mathcal{D}^*_{\phi^P}\pi_0=0$, by the anti-causality of $\mathcal{D}^*_{\phi^P}$. This completes the proof. □

In claim (ii) of the following proposition, the minimax condition is connected to a Liapunov equation that is almost the Riccati equation.

Proposition 180. *Let $\Phi=\left[\begin{smallmatrix}A^j & \mathcal{B}\tau^{*j}\\ \mathcal{C} & \mathcal{D}\end{smallmatrix}\right]$ be an I/O stable and output stable DLS, and J be a cost operator. Let $P\in ric(\Phi,J)$ such that $\mathcal{D}_{\phi^P}$ is I/O stable and (J,Λ_P)-inner. Define $P_0:=\mathcal{C}^*_{\phi^P}J\mathcal{C}_{\phi^P}\in\mathcal{L}(H)$. Then*

(i) P_0 satisfies the Liapunov equation

$$\begin{aligned}(4.41)\qquad & A^*P_0A-P_0+C^*JC\\ &=-K_P^*\Lambda_PK_P+K_P^*\Lambda_P\left(-\Lambda_P^{-1}\cdot\pi_0\mathcal{D}^*_{\phi^P}J\mathcal{C}\right)\\ &+\left(-\Lambda_P^{-1}\cdot\pi_0\mathcal{D}^*_{\phi^P}J\mathcal{C}\right)^*\Lambda_PK_P,\end{aligned}$$

and the residual cost operator satisfies $L_{A,P_0}=0$.

(ii) Assume, in addition, P satisfies the minimax condition $\bar{\pi}_+\mathcal{D}^*_{\phi^P}J\mathcal{C}_{\phi^P}=0$. Then P_0 satisfies the Liapunov equation

$$(4.42)\qquad A^*P_0A-P_0+C^*JC=K_P^*\Lambda_PK_P.$$

Furthermore, $A^*(P-P_0)A=P-P_0$, and if $P\in ric_{00}(\Phi,J)$, then $P-P_0=L_{A,P}$. If $P\in ric_0(\Phi,J)$ then $P=P_0$.

Proof. We first remark that is ϕ^P is output stable because $\mathcal{C}_{\phi^P}=\mathcal{C}-\mathcal{D}_{\phi^P}\mathcal{C}_{\phi_P}$, and all the operators $\mathcal{C}$, $\mathcal{D}_{\phi^P}$, $\mathcal{C}_{\phi_P}$ are assumed to be bounded. So $\mathcal{C}^*_{\phi^P}$ makes sense, and P_0 is well defined. The proof of claim (i) is the following technical calculation. Because $\mathcal{C}_{\phi^P}=\mathcal{C}-\mathcal{D}_{\phi^P}\mathcal{C}_{\phi_P}$, we obtain

$$P_0:=\mathcal{C}^*J\mathcal{C}-\mathcal{C}^*J\mathcal{D}_{\phi^P}\mathcal{C}_{\phi_P}-\mathcal{C}^*_{\phi_P}\mathcal{D}^*_{\phi^P}J\mathcal{C}+\mathcal{C}^*_{\phi_P}\mathcal{D}^*_{\phi^P}J\mathcal{D}_{\phi^P}\mathcal{C}_{\phi_P}$$
$$=\mathcal{C}^*J\mathcal{C}-\mathcal{C}^*J\mathcal{D}_{\phi^P}\mathcal{C}_{\phi_P}-\mathcal{C}^*_{\phi_P}\mathcal{D}^*_{\phi^P}J\mathcal{C}+\mathcal{C}_{\phi_P}\Lambda_P\mathcal{C}_{\phi_P},$$

where the latter equality is because $\mathcal{D}_{\phi^P}$ is assumed to be (J,Λ_P)-inner. But then

$$\begin{aligned} &A^*P_0A - P_0 + C^*JC \\ &= \overbrace{(A^*\mathcal{C}^*J\mathcal{C}A - \mathcal{C}^*J\mathcal{C} + C^*JC)}^{(i)} + \overbrace{\left(-A^*\mathcal{C}^*J\mathcal{D}_{\phi^P}\mathcal{C}_{\phi_P}A + \mathcal{C}^*J\mathcal{D}_{\phi^P}\mathcal{C}_{\phi_P}\right)}^{(ii)} \\ &+ \overbrace{\left(-A^*\mathcal{C}^*_{\phi_P}\mathcal{D}^*_{\phi^P}J\mathcal{C}A + \mathcal{C}^*_{\phi_P}\mathcal{D}^*_{\phi^P}J\mathcal{C}\right)}^{(iii)} + \overbrace{(A^*\mathcal{C}_{\phi_P}\Lambda_P\mathcal{C}_{\phi_P}A - \mathcal{C}_{\phi_P}\Lambda_P\mathcal{C}_{\phi_P})}^{(iv)}. \end{aligned}$$

Part *(i)* vanishes trivially. Parts *(ii)* and *(iii)* are adjoints of each other, and because A is the semigroup generator of both ϕ and ϕ_P, we have

$$\begin{aligned} &- A^*\mathcal{C}^*J\mathcal{D}_{\phi^P}\mathcal{C}_{\phi_P}A + \mathcal{C}^*J\mathcal{D}_{\phi^P}\mathcal{C}_{\phi_P} = -\mathcal{C}^*J\pi_+(\tau\mathcal{D}_{\phi^P}\tau^*)\pi_+\mathcal{C}_{\phi_P} + \mathcal{C}^*J\mathcal{D}_{\phi^P}\mathcal{C}_{\phi_P} \\ &= -\mathcal{C}^*J\pi_+\mathcal{D}_{\phi^P}\pi_+\mathcal{C}_{\phi_P} + \mathcal{C}^*J\mathcal{D}_{\phi^P}\mathcal{C}_{\phi_P} = \mathcal{C}^*J(\bar{\pi}_+\mathcal{D}_{\phi^P}\bar{\pi}_+ - \pi_+\mathcal{D}_{\phi^P}\pi_+)\mathcal{C}_{\phi_P} \\ &= \mathcal{C}^*J\mathcal{D}_{\phi^P}\pi_0 \cdot \pi_0\mathcal{C}_{\phi_P}, \end{aligned}$$

where the last equality is by the causality of $\mathcal{D}_{\phi^P}$. But $\pi_0\mathcal{C}_{\phi_P} = -K_P$ with the natural identification of the spaces U and range(π_0). So part *(ii)* equals $-\mathcal{C}^*J\mathcal{D}_{\phi^P}\pi_0 \cdot K_P$, and part *(iii)* equals $-K_P \cdot^* \pi_0\mathcal{D}^*_{\phi^P}J\mathcal{C}$. A similar calculation as required for part *(i)* shows that part *(iv)* equals $-K_P^*\Lambda_P K_P$. Collecting out results together, we have (4.41).

Because both $\mathcal{C}$ and $\mathcal{C}_{\phi_P}$ are bounded by assumptions, and A is the semigroup generator of both Φ and ϕ_P, trivially $\mathcal{C}A^j = \bar{\pi}_+\tau^j\mathcal{C} \to 0$ and $\mathcal{C}_{\phi_P}A^j = \bar{\pi}_+\tau^j\mathcal{C}_{\phi_P} \to 0$ in the strong operator topology. Because $\mathcal{C}_{\phi^P} = \mathcal{C} - \mathcal{D}_{\phi^P}\mathcal{C}_{\phi_P}$ where $\mathcal{D}_{\phi^P}$ is bounded, it follows that $\mathcal{C}_{\phi^P}A^j \to 0$ in the strong operator topology. By the Banach–Steinhaus Theorem, the family of operators $\{\mathcal{C}_{\phi^P}A^j\}_{j\geq 0}$ is uniformly bounded, and so is the family of their adjoints. It now follows that for all $x \in H$

$$||A^{*j}P_0A^jx|| \leq \sup_{j\geq 0} ||A^{*j}\mathcal{C}^*_{\phi^P}J|| \cdot ||\mathcal{C}_{\phi^P}A^jx|| \to 0$$

as $j \to \infty$. This completes the proof of claim (i).

In order to prove claim (ii), we use the equivalence of (i) and (iii) in Proposition 179; now P is, in addition, assumed to satisfy the minimax condition. Replacing $-\Lambda_P^{-1} \cdot \pi_0\mathcal{D}^*_{\phi^P}J\mathcal{C}$ by K_P in (4.41) gives (4.42). Note that the Riccati equation solution P, by definition, satisfies the Liapunov equation (4.42) with P in place of P_0, and then $A^*(P-P_0)A = P - P_0$. This completes the proof. $\square$

In the following Lemma, the main result of this section, we give a partial converse result to Lemma 171.

Lemma 181. *Let* $\Phi = \left[\begin{smallmatrix} \mathcal{A}^j & \mathcal{B}_{\mathcal{T}}^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ *be an I/O stable and output stable DLS. Assume that the input operator* $B \in \mathcal{L}(U; H)$ *is Hilbert–Schmidt, and the spaces* U *and* Y *are separable. Assume that* $\overline{\mathrm{range}\,(\mathcal{B})} = H$. *Let* $J \in \mathcal{L}(Y)$ *be a self-adjoint cost operator,* $J \geq 0$. *Assume that the regular critical solution* $P_0^{\mathrm{crit}} = \left(\mathcal{C}^{\mathrm{crit}}\right)^* J\mathcal{C}^{\mathrm{crit}} \in ric_0(\Phi, J)$ *exists.*

If $P \in ric_0(\Phi, J)$ *such that the inner DLS* ϕ^P *is I/O stable, then* $P \geq 0$.

Proof. Let $P \in ric_0(\Phi, J)$ such that the inner DLS ϕ^P is I/O stable. By Proposition 177, $\mathcal{D}_{\phi^P}$ is (J, Λ_P)-inner because $P \in ric_0(\Phi, J) \subset ric_{uw}(\Phi, J)$. By Proposition 178, P satisfies the minimax condition $\bar{\pi}_+ \mathcal{D}_{\phi^P}^* J \mathcal{C}_{\phi^P} = 0$. Define $P_0 := \mathcal{C}_{\phi^P}^* J \mathcal{C}_{\phi^P}$ as in Proposition 180. Because $J \geq 0$, then $P_0 \geq 0$. Because $P \in ric_0(\Phi, J)$, it follows that $P = P_0$ by claim (ii) of Proposition 180. Thus $P \geq 0$, and the proof is complete. □

The following theorem states that the exactly those state feedback laws that associated to nonnegative solutions of DARE, are I/O-stabilizing. We could also say that such solutions partially stabilize the closed loop semigroup generator A_P, and hide the unstable part of A_P to the unobservable (undetectable) subspace.

Theorem 182. *Let* $\Phi = \left[\begin{smallmatrix} \mathcal{A}^j & \mathcal{B}_{\mathcal{T}}^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right] = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ *be an I/O stable and output stable DLS, such that* $\overline{\mathrm{range}\,(\mathcal{B})} = H$. *Assume that the input operator* $B \in \mathcal{L}(U; H)$ *is Hilbert–Schmidt, and the spaces* U *and* Y *are separable. Let* $J \in \mathcal{L}(Y)$ *be a self-adjoint cost operator,* $J \geq 0$. *Assume that the regular critical solution* $P_0^{\mathrm{crit}} = \left(\mathcal{C}^{\mathrm{crit}}\right)^* J\mathcal{C}^{\mathrm{crit}} \in ric_0(\Phi, J)$ *exists. Let* $P \in ric_0(\Phi, J)$ *be arbitrary.*

Then $J^{\frac{1}{2}}\mathcal{D}_{\phi^P}$ *is I/O stable if and only if* $P \geq 0$.

Proof. If $P \geq 0$, then claim (iv) of Lemma 171 implies that $J^{\frac{1}{2}}\mathcal{D}_{\phi^P}$ is I/O stable. The converse direction is an application of Lemma 181. However, we first have to "absorb" the cost operator J into the DLS Φ by replacing the feed-through operator D by $J^{\frac{1}{2}}D$, and the output operator C by $J^{\frac{1}{2}}C$. Call this modified DLS ϕ'. Finally replace the cost operator J by I. Clearly the assumptions of Φ and ϕ' correspond to each other one-to-one, the DARE remains unchanged, and Lemma 181 implies that $P \geq 0$. □

4.6 Partial ordering and factorization

Assume that Φ is an output stable and I/O stable DLS, and the cost operator J is nonnegative. Furthermore, assume that the regular critical solution $P_0^{\text{crit}} \in ric_0(\Phi, J)$ exists. In this section, we consider the partial ordering of the solution set $ric_0(\Phi, J)$ as self-adjoint operators. Recall that for $P \in ric_0(\Phi, J)$, the closed ranges $\text{range}\left(\widetilde{\mathcal{D}}_{\phi_P}\bar{\pi}_+\right) \subset \ell^2(\mathbf{Z}_+; U)$ of the Toeplitz operators $\widetilde{\mathcal{D}}_{\phi_P}\bar{\pi}_+$ are shift-invariant, see Lemma 183 and Corollary 184. Here $\widetilde{\mathcal{D}}_{\phi_P}$ denotes the adjoint of the I/O map $\mathcal{D}_{\phi_P}$ of the spectral DLS ϕ_P. Inclusions of the subspaces $\text{range}\left(\widetilde{\mathcal{D}}_{\phi_P}\bar{\pi}_+\right)$ are considered in Lemma 185. In Corollary 186, the maximality property of the regular critical solution $P_0^{\text{crit}} = \left(\mathcal{C}^{\text{crit}}\right)^* J\mathcal{C}^{\text{crit}} \in ric_0(\Phi, J)$ is proved. The order-preserving equivalence

$$ric_0(\Phi, J) \ni P \mapsto \text{range}\left(\widetilde{\mathcal{N}}_P\bar{\pi}_+\right) \subset \ell^2(\mathbf{Z}_+; U)$$

is considered in Theorem 187. Here $\widetilde{\mathcal{N}}_P$ denotes the adjoint I/O map of $\mathcal{N}_P$, the $(\Lambda_P, \Lambda_{P_0^{\text{crit}}})$-inner factor of $\mathcal{D}_{\phi_P} = \mathcal{N}_P\mathcal{X}$.

We start with reminding some classical results. The Beurling–Lax–Halmos Theorem on the shift-invariant subspaces is the following:

Lemma 183. *Let U be a separable Hilbert space. The following are equivalent*

- *(i) H_1 be a shift-invariant subspace of $\ell^2(\mathbf{Z}_+; U)$,*
- *(ii) $H_1 = \text{range}\,(\Theta\bar{\pi}_+) = \Theta\ell^2(\mathbf{Z}_+; U')$, where $U' \subset U$ is a Hilbert subspace, and $\Theta : \ell^2(\mathbf{Z}; U') \to \ell^2(\mathbf{Z}; U)$ is a causal, shift-invariant and bounded operator, which is inner from the left.*

Furthermore, if $\text{range}\,(\Theta_1\bar{\pi}_+) = \text{range}\,(\Theta_2\bar{\pi}_+)$ *then there is a unitary (static) operator $V \in \mathcal{L}(U)$ such that $\Theta_1 = \Theta_2 V$.*

For proofs, see e.g. [70, Lecture 9, Corollary 9] or [27, Chapter IX, Theorem 2.1]. We can get rid of indexing over the subspaces $U' \subset U$ if we modify the definition of the inner (from the left) operator. This convention is taken in [77], where the inner operators are defined to be such that $\Theta(e^{i\theta})$ is a partial isometry, a.e. $e^{i\theta} \in \mathbf{T}$. Actually this indexing is only over all the cardinalities of the subspaces U, because two Hilbert subspaces of the same dimension can be unitarily identified. For the following corollary, see e.g. [70, Lecture I, Corollary 8]:

Corollary 184. *Let Θ_1, Θ_2 be inner from both sides. Then* $\text{range}\,(\Theta_2\bar{\pi}_+) \subset \text{range}\,(\Theta_1\bar{\pi}_+)$ *if and only if there is an inner operator Θ_3 such that $\Theta_2 = \Theta_1\Theta_3$.*

We now consider the inclusions of the shift-invariant subspaces range $\left(\tilde{\mathcal{D}}_{\phi_P}\bar{\pi}_+\right)$. Under the J-coercivity assumption $\bar{\pi}_+\mathcal{D}^*J\mathcal{D}\bar{\pi}_+ \geq \epsilon\bar{\pi}_+$ for some $\epsilon > 0$, these subspaces are closed, see Proposition 135.

Lemma 185. *Let $J \in \mathcal{L}(Y)$ be a cost operator, and $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right] = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ be an I/O stable and output stable DLS. Assume that the input space U and the output space Y are separable, and the input operator $B \in \mathcal{L}(U;H)$ is Hilbert–Schmidt. Assume that $\bar{\pi}_+\mathcal{D}^*J\mathcal{D}\bar{\pi}_+ \geq \epsilon\bar{\pi}_+$ for some $\epsilon > 0$.*

Let $P_1, P_2 \in ric_{uw}(\Phi, J)$ such that $P_1 \leq P_2$. Then

$$\text{range}\left(\widetilde{\mathcal{D}}_{\phi_{P_1}}\bar{\pi}_+\right) \subset \text{range}\left(\widetilde{\mathcal{D}}_{\phi_{P_2}}\bar{\pi}_+\right).$$

Proof. We begin the proof by centering the problem at the smaller of the solutions P_1. Define $\Delta P := P_2 - P_1 \geq 0$. Then we have $P_2 = P_1 + \Delta P$ where $\Delta P \in Ric(\phi_{P_1}, \Lambda_{P_1})$, by Lemma 156. The spectral DARE $Ric(\phi_{P_1}, \Lambda_{P_1})$ is a H^∞DARE because $P_1 \in ric(\Phi, J)$, by assumption. Also $0 \in ric_0(\phi_{P_1}, \Lambda_{P_1})$ is a trivial solution, corresponding to the solution of the original DARE P_1 itself. By Corollary 146, both the indicators satisfy $\Lambda_{P_1} > 0$ and $\Lambda_{P_2} > 0$.

Note that we have not written $\Delta P \in ric(\phi_{P_1}, \Lambda_{P_1})$ because we do not know *a priori* the output stability and I/O stability of the spectral DLS $(\phi_{P_1})_{\Delta P}$. However, a computation with the minimax nodes reveals that the spectral DLS $(\phi_{P_1})_{\Delta P}$ is a spectral DLS associated to the original Φ and J

$$\left((\phi_{P_1})_{\Delta P}, \tilde{\Lambda}_{\Delta P}\right) \equiv (\phi_{P_1}, \Lambda_{P_1})_{\Delta P} \equiv (\phi_{P_1+\Delta P}, \Lambda_{P_1+\Delta P}) \equiv (\phi_{P_2}, \Lambda_{P_2}), \tag{4.43}$$

see equation (4.4) of Proposition 151. Because $P_2 \in ric(\Phi, J)$ by assumption, it follows that the spectral DLS $(\phi_{P_1})_{\Delta P}$ is output stable and I/O stable. Thus $\Delta P \in ric(\phi_{P_1}, \Lambda_{P_1})$. For all $x_0 \in \text{range}(\mathcal{B})$, we have

$$\left\langle \Delta P A^j x_0, A^j x_0 \right\rangle = \left\langle P_2 A^j x_0, A^j x_0 \right\rangle - \left\langle P_1 A^j x_0, A^j x_0 \right\rangle \to 0 \tag{4.44}$$

as $j \to \infty$, because both P_1 and P_2 are assumed to satisfy the ultra weak residual cost condition of Definition 108. Because the DLSs Φ and ϕ_{P_1} have the common controllability map, we have $\text{range}(\mathcal{B}) = \text{range}(\mathcal{B}_{\phi_P})$, and then equation (4.44) implies that $\Delta P \in ric_{uw}(\phi_{P_1}, \Lambda_{P_1})$. From equation (4.43) we also see that $\Delta P \in ric_{uw}(\phi_{P_1}, \Lambda_{P_1})$ has a positive indicator $\tilde{\Lambda}_{\Delta P} = \Lambda_{P_2} > 0$.

Now we want to apply claim (iii) of Lemma 171 with $(\phi_{P_1}, \Lambda_{P_1})$ in place for (Φ, J), and $\Delta P \in ric_{uw}(\phi_{P_1}, \Lambda_{P_1})$ in place of $P \in ric_{uw}(\Phi, J)$. We have to check that the DLS ϕ_{P_1}, cost operator Λ_{P_1} and solution ΔP satisfy the additional conditions. Firstly, the equivalent conditions of Theorem 114 hold for the pair $(\phi_{P_1}, \Lambda_{P_1})$ because they hold for (Φ, J), by the coercivity assumption $\bar{\pi}_+\mathcal{D}^*J\mathcal{D}\bar{\pi}_+ \geq \epsilon\bar{\pi}_+$ and Corollary 146. For details see Proposition 147 and the discussion following it. We conclude that there is a regular critical solution $\tilde{P}_0^{\mathrm{crit}} \in ric_0(\phi_{P_1}, \Lambda_{P_1})$.

The input operator B is common for both Φ and ϕ_{P_1}, and so the Hilbert–Schmidt assumption holds for ϕ_{P_1}. The same is true for the separability of the Hilbert space U, which is the input and the output space of ϕ_{P_1}. Now claim (iii) of Lemma 171 gives

$$\mathcal{D}_{\phi_{P_1}} = \mathcal{D}_{(\phi_{P_1})^{\Delta P}} \mathcal{D}_{(\phi_{P_1})_{\Delta P}}, \tag{4.45}$$

where $(\phi_{P_1})^{\Delta P}$ is the inner DLS, and $(\phi_{P_1})_{\Delta P}$ is the spectral DLS of ϕ_{P_1}, centered at ΔP. Both $(\phi_{P_1})^{\Delta P}$ and $(\phi_{P_1})_{\Delta P}$ are output stable and I/O stable; the former by claim (iii) of Lemma 171, and the latter because $\Delta P \in ric_{uw}(\phi_P, \Lambda_P)$. It also follows from Lemma 171 that the I/O map $\mathcal{D}_{(\phi_{P_1})^{\Delta P}}$ is in fact $(\Lambda_{P_1}, \Lambda_{P_2})$-inner, because $\tilde{\Lambda}_{\Delta P} = \Lambda_{P_2}$ is the indicator of $\Delta P \in ric(\phi_{P_1}, \Lambda_{P_1})$, as discussed above. Note that because the nonnegative cost operator Λ_{P_1} has a bounded inverse, we do not need to include the square root of it into equation (4.45), as has been done in Lemma 171 for possibly noncoercive cost operator J.

It follows from equation (4.43) that $\mathcal{D}_{(\phi_{P_1})_{\Delta P}} = \mathcal{D}_{\phi_{P_2}}$. By Corollary 118, the regular critical solution $P_0^{\mathrm{crit}} := \left(\mathcal{C}^{\mathrm{crit}}\right)^* J \mathcal{C}^{\mathrm{crit}} \in ric_0(\Phi, J)$ exists because $\bar{\pi}_+ \mathcal{D}^* J \mathcal{D} \bar{\pi}_+ \geq \epsilon \bar{\pi}_+$ is assumed for some $\epsilon > 0$. We now obtain from equation (4.45)

$$\mathcal{N}_{P_1}\mathcal{X} = \mathcal{D}_{\phi_{P_1}} = \mathcal{D}_{(\phi_{P_1})^{\Delta P}} \mathcal{D}_{\phi_{P_2}} = \mathcal{D}_{(\phi_{P_1})^{\Delta P}} \mathcal{N}_{P_2}\mathcal{X}, \tag{4.46}$$

where $\mathcal{D}_{\phi_{P_1}} = \mathcal{N}_{P_1}\mathcal{X}$ ($\mathcal{D}_{\phi_{P_2}} = \mathcal{N}_{P_2}\mathcal{X}$) are $(\Lambda_{P_1}, \Lambda_{P_0^{\mathrm{crit}}})$ (($\Lambda_{P_2}, \Lambda_{P_0^{\mathrm{crit}}}$))-inner-outer factorizations, respectively. The outer factor $\mathcal{X}$ has a bounded inverse, and it is common for both the I/O maps $\mathcal{D}_{\phi_{P_1}}$ and $\mathcal{D}_{\phi_{P_2}}$, see Proposition 147. As noted earlier, $\mathcal{D}_{(\phi_{P_1})^{\Delta P}}$ is bounded and $(\Lambda_{P_1}, \Lambda_{P_2})$-inner. We proceed to prove that it can be normalized to an I/O map that is inner from both sides.

Divide the outer factor away from (4.46), to obtain $\mathcal{N}_{P_1} = \mathcal{D}_{(\phi_{P_1})^{\Delta P}} \cdot \mathcal{N}_{P_2}$. Normalize, as in Corollary 172, to obtain $\mathcal{N}_{P_1}^{\circ} = \mathcal{M}_{P_1,\Delta P}^{\circ} \mathcal{N}_{P_2}^{\circ}$, where $\mathcal{N}_{P_1}^{\circ} := \Lambda_{P_1}^{\frac{1}{2}} \mathcal{N}_{P_1} \Lambda_{P_0^{\mathrm{crit}}}^{-\frac{1}{2}}$, $\mathcal{N}_{P_2}^{\circ} := \Lambda_{P_2}^{\frac{1}{2}} \mathcal{N}_{P_2} \Lambda_{P_0^{\mathrm{crit}}}^{-\frac{1}{2}}$, and

$$\mathcal{M}_{P_1,\Delta P}^{\circ} := \Lambda_{P_1}^{\frac{1}{2}} \mathcal{D}_{(\phi_{P_1})^{\Delta P}} \Lambda_{P_2}^{-\frac{1}{2}} : \ell^2(\mathbf{Z}; U) \to \ell^2(\mathbf{Z}; U).$$

By claim (ii) of Proposition 147, both the I/O maps $\mathcal{N}_{P_1}^{\circ}$ and $\mathcal{N}_{P_2}^{\circ}$ are inner from both sides, and their boundary traces are unitary-valued functions. Because $\mathcal{D}_{(\phi_{P_1})^{\Delta P}}$ is $(\Lambda_{P_1}, \Lambda_{P_2})$-inner, it follows that normalized I/O map $\mathcal{M}_{P_1,\Delta P}^{\circ}$ is inner from the left. Now the boundary traces satisfy

$$\mathcal{M}_{P_1,\Delta P}^{\circ}(e^{i\theta}) = \mathcal{N}_{P_1}^{\circ}(e^{i\theta}) \mathcal{N}_{P_2}^{\circ}(e^{i\theta})^*,$$

and it follows that the boundary trace evaluation $\mathcal{M}_{P_1,\Delta P}^{\circ}(e^{i\theta})$ is unitary for almost all $e^{i\theta} \in \mathbf{T}$. Thus $\mathcal{M}_{P_1,\Delta P}^{\circ}$ is, in fact, inner from both sides.

By using the adjoint I/O maps, we change the order of factors

$$\widetilde{\mathcal{N}}^{\circ}_{P_1} = \widetilde{\mathcal{N}}^{\circ}_{P_2}\widetilde{\mathcal{M}}^{\circ}_{P_1,\Delta P},$$

where all the factors are inner from the both sides. Now Corollary 184 implies that

$$\begin{aligned}(4.47)\qquad \operatorname{range}\left(\Lambda^{-\frac{1}{2}}_{P_0^{\mathrm{crit}}}\widetilde{\mathcal{N}}_{P_1}\bar{\pi}_+\right) &= \operatorname{range}\left(\widetilde{\mathcal{N}}^{\circ}_{P_1}\bar{\pi}_+\right)\\ \subset \operatorname{range}\left(\widetilde{\mathcal{N}}^{\circ}_{P_2}\bar{\pi}_+\right) &= \operatorname{range}\left(\Lambda^{-\frac{1}{2}}_{P_0^{\mathrm{crit}}}\widetilde{\mathcal{N}}_{P_2}\bar{\pi}_+\right).\end{aligned}$$

By considering the outer transfer functions as in claim (ii) of Proposition 127, it is easy to see that $\widetilde{\mathcal{X}}$ is outer with a bounded inverse if and only if $\mathcal{X}$ is outer with a bounded inverse. In particular, $\widetilde{\mathcal{X}}\Lambda^{\frac{1}{2}}_{P_0^{\mathrm{crit}}}$ is outer with a bounded inverse, and the Toeplitz operator $\tilde{\mathcal{X}}\Lambda^{\frac{1}{2}}_{P_0^{\mathrm{crit}}}\bar{\pi}_+$ is a bounded bijection on $\ell^2(\mathbf{Z}_+;U)$. Thus the inclusion of ranges in (4.47) remains valid if we multiply the operators from the left by $\tilde{\mathcal{X}}\Lambda^{\frac{1}{2}}_{P_0^{\mathrm{crit}}}\bar{\pi}_+$. Now the claim follows. □

The following corollary is somewhat analogous to [49, Theorem 13.5.2].

Corollary 186. *Let $J \geq 0$ be a cost operator. Let $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D}\end{smallmatrix}\right]$ be an I/O stable and output stable DLS. Assume that the input space U and the output space Y are separable Hilbert spaces, and the input operator $B \in \mathcal{L}(U;H)$ of Φ is Hilbert–Schmidt. Assume that the regular critical solution $P_0^{\mathrm{crit}} := \left(\mathcal{C}^{\mathrm{crit}}\right)^* J\mathcal{C}^{\mathrm{crit}} \in ric_0(\Phi, J)$ exists.*

(i) Let $P_0 \in ric_0(\Phi,J)$ be such that $P_1^{\mathrm{crit}} \leq P_0$ where $P_1^{\mathrm{crit}} \in ric_0(\Phi,J)$ is any regular critical solution. Then P_0 is a regular critical solution.

(ii) If, in addition, $\overline{\operatorname{range}(\mathcal{B})} = H$, then the unique critical solution $P_0^{\mathrm{crit}} := (\mathcal{C}^{\mathrm{crit}})^* J\mathcal{C}^{\mathrm{crit}}$ is maximal in the set $ric_0(\Phi,J)$.

Proof. By Lemma 185, equation (4.47) gives for the ranges of the adjoined operators, because $P_0 \geq P_1^{\mathrm{crit}}$

$$\ell^2(\mathbf{Z}_+;U) = \operatorname{range}\left(\tilde{\mathcal{D}}_{\phi_{P_1^{\mathrm{crit}}}}\bar{\pi}_+\right) \subset \operatorname{range}\left(\tilde{\mathcal{D}}_{\phi_{P_0}}\bar{\pi}_+\right) \subset \ell^2(\mathbf{Z}_+;U),$$

and immediately $\operatorname{range}\left(\tilde{\mathcal{D}}_{\phi_{P_0}}\bar{\pi}_+\right) = \ell^2(\mathbf{Z}_+;U)$. By $\mathcal{D}_{\phi_{P_0}} = \mathcal{N}_{P_0}\mathcal{X}$ denote the $(\Lambda_{P_0}, \Lambda_{P_0^{\mathrm{crit}}})$-inner-outer factorization, and normalize the inner part as before: $\widetilde{\mathcal{N}}^{\circ}_{P_0} = \Lambda^{\frac{1}{2}}_{P_0}\widetilde{\mathcal{N}}_{P_0}\Lambda^{-\frac{1}{2}}_{P_0^{\mathrm{crit}}}$. Then $\operatorname{range}\left(\widetilde{\mathcal{N}}_{P_0}\bar{\pi}_+\right) = \ell^2(\mathbf{Z}_+;U)$, as in the last part of the proof of Lemma 185. Now the uniqueness part of Lemma 183 shows that $\tilde{\mathcal{N}}^{\circ}_{P_0}$ is a static unitary constant operator $V \in \mathcal{L}(U)$. By canceling the normalization, we obtain $\mathcal{D}_{\phi_{P_0}} = \Lambda_{P_0}^{-\frac{1}{2}} V^* \Lambda_{P_0^{\mathrm{crit}}}^{\frac{1}{2}} \mathcal{X}$. Because the static part of both $\mathcal{D}_{\phi_{P_0}}$ and $\mathcal{X}$ is the identity operator $I \in \mathcal{L}(U)$, it follows that $\Lambda_{P_0}^{-\frac{1}{2}} V^* \Lambda_{P_0^{\mathrm{crit}}}^{\frac{1}{2}} = I$ and hence $\mathcal{D}_{\phi_{P_0}} = \mathcal{X}$. Because $P_0 \in ric_0(\phi, J)$, it is a regular critical solution, and the first claim (i) is verified. Under the approximate controllability $\overline{\mathrm{range}\,(\mathcal{B})} = H$, an application of claim (i) of Corollary 116 proves the remaining claim. □

We remark that the solution $P_0^{\mathrm{crit}} := (\mathcal{C}^{\mathrm{crit}})^* J \mathcal{C}^{\mathrm{crit}}$ is not generally maximal in the full solution set $Ric(\Phi, J)$. A plenty of examples about this are provided by Lemma 193 in Section 4.7. Even if $\overline{\mathrm{range}\,(\mathcal{B})} = H$ is assumed, we do not yet know whether P_0^{crit} is the largest element of $ric_0(\Phi, J)$ — there could be a solution $P \in ric_0(\Phi, J)$ that is not comparable to P_0^{crit}. However, this is not the case, as shown in Theorem 188. This result is based on the following equivalence of the two order relations.

Theorem 187. *Let $J \geq 0$ be a cost operator. Let $\Phi = \left[\begin{smallmatrix} \mathcal{A}^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right] = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ be an I/O stable and output stable DLS, such that $\overline{\mathrm{range}\,(\mathcal{B})} = H$. Assume that the input space U and the output space Y are separable, and the input operator $B \in \mathcal{L}(U; H)$ is Hilbert–Schmidt. Assume that the regular critical solution $P_0^{\mathrm{crit}} := \left(\mathcal{C}^{\mathrm{crit}}\right)^* J \mathcal{C}^{\mathrm{crit}} \in ric_0(\Phi, J)$ exists.*

For $P_1, P_2 \in ric_0(\Phi, J)$, the following are equivalent

(i) $P_1 \leq P_2$.

(ii) $\mathrm{range}\left(\widetilde{\mathcal{N}}_{P_1} \bar{\pi}_+\right) \subset \mathrm{range}\left(\widetilde{\mathcal{N}}_{P_2} \bar{\pi}_+\right)$, *where $\mathcal{N}_P$ is the $(\Lambda_P, \Lambda_{P_0^{\mathrm{crit}}})$-inner factor of $\mathcal{D}_{\phi_P}$.*

In other words, the mapping

$$ric_0(\Phi, J) \ni P \mapsto \mathrm{range}\left(\widetilde{\mathcal{N}}_P \bar{\pi}_+\right) \subset \ell^2(\mathbf{Z}_+; U)$$

is order-preserving from the POSET $ric_0(\Phi, J)$ (ordered by the natural partial ordering of self-adjoint operators) into the sub-POSET $\{\mathrm{range}\left(\widetilde{\mathcal{N}}_P \bar{\pi}_+\right)\}_{P \in ric_0(\Phi, J)}$ of the shift-invariant subspaces of $\ell^2(\mathbf{Z}_+; U)$ (ordered by the inclusion of subspaces).

Proof. The implication (i) ⇒ (ii) is Lemma 185. We just remark that if $J \geq 0$, the existence of the regular critical solution P_0^{crit} is equivalent to $\bar{\pi}_+ \mathcal{D}^* J \mathcal{D} \bar{\pi}_+ \geq \epsilon \bar{\pi}_+$ for $\epsilon > 0$, see Theorem 114 and Corollary 117. For the converse direction (ii) ⇒ (i), note that $\mathrm{range}\left(\widetilde{\mathcal{N}}_{P_1} \bar{\pi}_+\right) \subset \mathrm{range}\left(\widetilde{\mathcal{N}}_{P_2} \bar{\pi}_+\right)$ is equivalent to range $\left(\widetilde{\mathcal{N}}^{\circ}_{P_1}\bar{\pi}_+\right) \subset$ range $\left(\widetilde{\mathcal{N}}^{\circ}_{P_2}\bar{\pi}_+\right)$, where the normalization is as in Corollary 172. This normalization is possible because both the indicators Λ_{P_1}, Λ_{P_2} and $\Lambda_{P_0^{\rm crit}}$ are positive, by Corollary 146. By Corollary 184, there is an inner (from both sides) operator Θ such that $\widetilde{\mathcal{N}}^{\circ}_{P_2}\Theta = \widetilde{\mathcal{N}}^{\circ}_{P_1}$, or equivalently

$$\mathcal{D}_{\phi_{P_1}} = \Lambda_{P_1}^{-\frac{1}{2}}\widetilde{\Theta}\Lambda_{P_2}^{\frac{1}{2}} \cdot \mathcal{D}_{\phi_{P_2}}, \tag{4.48}$$

because we can factorize $\mathcal{D}_{\phi_P} = \mathcal{N}_P\mathcal{X}$ for $P \in ric_{uw}(\phi, J)$, by Proposition 147.

Now we continue as in proof of Lemma 185, and center the problem around the smaller solution P_1. As in the proof of Lemma 185, we have the solution $\Delta P := P_2 - P_1 \in ric(\phi_{P_1}, \Lambda_{P_1})$ whose nonnegativity is to be shown. We have $(\phi_{P_1})_{\Delta P} = \phi_{P_2}$ and

$$\mathcal{D}_{\phi_{P_1}} = \mathcal{D}_{(\phi_{P_1})^{\Delta P}}\mathcal{D}_{(\phi_{P_1})_{\Delta P}} = \mathcal{D}_{(\phi_{P_1})^{\Delta P}}\mathcal{D}_{\phi_{P_2}}, \tag{4.49}$$

as in the proof of Lemma 185.

We have to check that ϕ_{P_1}, Λ_{P_1} and ΔP satisfy the assumptions of Lemma 181. Firstly, the separable U is the input space and the output space of the output stable and I/O stable DLS ϕ_{P_1}. Also $\overline{\text{range}\left(\mathcal{B}_{\phi_{P_1}}\right)} = H$, because $\mathcal{B}_{\phi_{P_1}} = \mathcal{B}$. The indicator Λ_{P_1}, serving as the cost operator, is nonnegative as already has been discussed. The H^∞DARE $ric(\phi_{P_1}, \Lambda_{P_1})$ has a regular critical solution because the original H^∞DARE $ric(\Phi, J)$ has, see Theorem 114 and claim (i) of Proposition 147. Because $\Delta P = P_2 - P_1$ and $P_1, P_2 \in ric_0(\Phi, J)$ by assumption, the residual cost operator $L_{A,\Delta P}$ exists. Furthermore, $L_{A,\Delta P} = L_{A,P_2} - L_{A,P_1} = 0$, and it follows that $\Delta P \in ric_0(\phi_{P_1}, \Lambda_{P_1})$ because A is the common semigroup generator of all the DLSs Φ, ϕ_{P_1} and $(\phi_{P_1})_{\Delta P}$. Now we see that the assumptions of Lemma 181 are satisfied.

By comparing (4.48) and (4.49), we see that the inner DLS $(\phi_{P_1})^{\Delta P}$ is I/O stable. Compare, for example, the transfer functions in a small neighborhood of the origin, to convince yourself that $\Lambda_{P_1}^{-\frac{1}{2}}\tilde{\Theta}\Lambda_{P_2}^{\frac{1}{2}} = \mathcal{D}_{(\phi_{P_1})^{\Delta P}}$. Also claim (ii) of Proposition 135 can be used, to see that the I/O map $\mathcal{D}_{\phi_{P_2}}$ has a bounded, shift-invariant but generally noncausal inverse in $\ell^2(\mathbf{Z}; U)$. By Lemma 181, $\Delta P \geq 0$ and the proof is completed. □

We proceed to give an order-theoretic characterization of the set of nonnegative regular H^∞ solutions of the H^∞DARE $ric(\phi, J)$. Under approximate controllability, these are exactly those that give H^∞ factorizations in Lemma 171, see Corollary 137.

Theorem 188. *Let $J \geq 0$ be a cost operator. Let $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right] = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ be an I/O stable and output stable DLS, such that $\overline{\text{range}(\mathcal{B})} = H$. Assume that the input space U and the output space Y are separable, and the input operator*

$B \in \mathcal{L}(U;H)$ is Hilbert–Schmidt. Assume that there is a (unique) regular critical solution $P_0^{\mathrm{crit}} := (\mathcal{C}^{\mathrm{crit}})^* J \mathcal{C}^{\mathrm{crit}} \in ric_0(\Phi, J)$. Then

$$\{P \in ric_0(\Phi, J) \quad | \quad P \geq 0\} = \{P \in Ric(\Phi, J) \quad | \quad 0 \leq P \leq P_0^{\mathrm{crit}}\}.$$

Proof. The inclusion $\supset$ has already been established in claim (ii) of Corollary 141. For the converse inclusion, let a nonnegative $P \in ric_0(\Phi, J)$ be arbitrary. Because $\widetilde{\mathcal{N}}_{P_0^{\mathrm{crit}}} = \mathcal{I}$, it follows that the range of the Toeplitz operator $\widetilde{\mathcal{N}}_{P_0^{\mathrm{crit}}} \bar{\pi}_+$ is all of $\ell^2(\mathbf{Z}_+; U)$. In particular, range $\left(\widetilde{\mathcal{N}}_P \bar{\pi}_+\right) \subset$ range $\left(\widetilde{\mathcal{N}}_{P_0^{\mathrm{crit}}} \bar{\pi}_+\right)$, and it follows that $P \leq P_0^{\mathrm{crit}}$, by Theorem 187. The proof is complete. $\square$

4.7 H^∞ solutions of the inner and spectral DAREs

We start with a motivation of the contents of this section. For simplicity, assume for a while that the nonnegative cost operator J is boundedly invertible. In claim (iv) Lemma 171, we introduce the factorization of the I/O map as a composition of two I/O stable I/O maps

$$\mathcal{D}_\phi = \mathcal{D}_{\phi^{\tilde{P}}} \cdot \mathcal{D}_{\phi_{\tilde{P}}}, \tag{4.50}$$

for any nonnegative $\tilde{P} \in ric_0(\phi, J)$. As a conclusion of the same lemma, it follows that the inner DLS $\phi^{\tilde{P}}$ is output stable and I/O stable. The technical assumptions of Lemma 171, such as the separability of the Hilbert spaces and the Hilbert–Schmidt compactness of the common input operator $B \in \mathcal{L}(U; H)$, are inherited from ϕ by $\phi^{\tilde{P}}$. This makes it possible to apply claim (iv) of Lemma 171 to inner DLS ϕ^P and the associated inner H^∞DARE $ric(\phi^P, J)$. In this way, the $(J, \Lambda_{\tilde{P}})$-inner factor $\mathcal{D}_{\phi^{\tilde{P}}}$ can be further factorized by the nonnegative solutions $P \in ric_0(\phi^{\tilde{P}}, J)$. A similar consideration can be given for the right factor $\mathcal{D}_{\phi_{\tilde{P}}}$, which is the I/O map of the spectral DLS ϕ_P, and a stable spectral factor of the Popov operator $\mathcal{D}_\phi^* J \mathcal{D}_\phi$, too. The nonnegative solutions $P \in ric_0(\phi_{\tilde{P}}, \Lambda_{\tilde{P}})$ of the spectral DARE factorize $\mathcal{D}_{\phi_{\tilde{P}}}$ into I/O stable factors.

Because of the possibility of a recursive factorization of factors in equation (4.50), we conclude that both the solutions sets

$$\begin{aligned} &ric_0(\phi_{\tilde{P}}, \Lambda_{\tilde{P}}), \quad \text{for all} \quad \tilde{P} \in ric_0(\phi, J), \\ &\left\{ P \in ric_0(\phi^{\tilde{P}}, J) \quad | \quad P \geq 0 \right\}, \quad \text{for all} \quad \tilde{P} \in ric_0(\phi, J), \quad \tilde{P} \geq 0 \end{aligned}$$

are quite interesting. So it is desirable to characterize them in terms of the original data, namely the DLS $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix} \right)$, the cost operator J, and the solution sets $Ric(\phi, J)$ and $ric_0(\phi, J)$ of the original DARE. This is the subject of the present section.

We start with considering the spectral DARE, as it is quite easy. In fact, the result on the spectral DLSs has already been used in the proof of Theorem 187.

Lemma 189. *Let $J \geq 0$ a cost operator. Let $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix} \right)$ be an output stable and I/O stable DLS. Assume that the input operator $B \in \mathcal{L}(U; H)$ is Hilbert–Schmidt and the input space U is separable. Let $\tilde{P} \in ric_0(\phi, J)$ be arbitrary.*

Then the following are equivalent:

(i) $\Delta P \in ric_0(\phi_{\tilde{P}}, \Lambda_{\tilde{P}})$,

(ii) $\tilde{P} + \Delta P \in ric_0(\phi, J)$.

Proof. To prove the implication (i) $\Rightarrow$ (ii), let $\Delta P \in ric_0(\phi_{\tilde{P}}, \Lambda_{\tilde{P}})$ be arbitrary. Then, because A is the semigroup generator of both ϕ and $\phi_{\tilde{P}}$, it follows that the residual cost operator $L_{A,\tilde{P}+\Delta P}$ exists and satisfies $L_{A,\tilde{P}+\Delta P} = L_{A,\tilde{P}} + L_{A,,\Delta P} = 0$. By Lemma 156, $\tilde{P} + \Delta P \in Ric_0(\phi, J)$.

Because $J \geq 0$, it follows that $P_0^{\mathrm{crit}} = \left(\mathcal{C}_\phi^{\mathrm{crit}}\right)^* J \mathcal{C}_\phi^{\mathrm{crit}} \geq 0$ and also $\Lambda_{P_0^{\mathrm{crit}}} > 0$. By Theorem 114 and Lemma 145, it follows that $\Lambda_{\tilde{P}} > 0$ because $\tilde{P} \in ric_0(\phi, J)$. The spectral H^∞DARE $ric(\phi_{\tilde{P}}, \Lambda_{\tilde{P}})$ has a regular critical solution $\tilde{P}_0^{\mathrm{crit}} \in ric_0(\phi_{\tilde{P}}, \Lambda_{\tilde{P}})$ because $P_0^{\mathrm{crit}} \in ric_0(\phi, J)$ is assumed to exist, see Proposition 147. Because the cost operator of DARE $ric(\phi_{\tilde{P}}, \Lambda_{\tilde{P}})$ is nonnegative, the indicator $\tilde{\Lambda}_{\tilde{P}_0^{\mathrm{crit}}}$ is nonnegative and the same is true for the indicator $\tilde{\Lambda}_{\Delta P}$, by Lemma 145 and the assumption $\Delta P \in ric_0(\phi_{\tilde{P}}, \Lambda_{\tilde{P}})$. Now, by equation (4.4) of Proposition 151, $\Lambda_{\tilde{P}+\Delta P} = \tilde{\Lambda}_{\Delta P} > 0$.

Now we have concluded that $\tilde{P} + \Delta P \in Ric_0(\phi, J)$, and its indicator is positive. It follows that $\tilde{P} + \Delta P \in ric_0(\phi, J)$, by Corollary 140. This completes the proof of the first implication.

To prove the other direction (ii) $\Rightarrow$ (i), assume that $P_2 := \tilde{P} + \Delta P \in ric_0(\phi, J)$. Then $\Delta P = P_2 - \tilde{P} \in Ric(\phi_{\tilde{P}}, \Lambda_{\tilde{P}})$ by Lemma 156, and also $L_{A,\Delta P} = 0$. Thus $\Delta P \in Ric_0(\phi_{\tilde{P}}, \Lambda_{\tilde{P}})$ because the same A is the semigroup generator of all spectral DLSs. The indicator $\tilde{\Lambda}_\Delta$ of $\Delta P \in Ric(\phi_{\tilde{P}}, \Lambda_{\tilde{P}})$ satisfies $\tilde{\Lambda}_\Delta = \Lambda_{P_2}$, by equation (4.4) of Proposition 151. But the latter is positive because $P_2 \in ric_0(\phi, J)$, by the same argument that is presented in the first part of the proof for $\Lambda_{\tilde{P}}$.

We have proved that $\Delta P \in Ric_0(\phi_{\tilde{P}}, \Lambda_{\tilde{P}})$, and its indicator $\tilde{\Lambda}_{\Delta P}$ is positive. Now, because the Hilbert–Schmidt class input operator B and the separable input space U is common for all spectral DLSs, an application of Corollary 140 completes the proof. $\square$

A similar results can be given for other residual cost conditions introduced in Definition 108. The case of the ultra weak residual cost condition has been considered in the proof of Lemma 185. We proceed to characterize a regular critical solution of the spectral DARE.

Corollary 190. *Make the same assumption as in Lemma 189. By $P_0^{\mathrm{crit}} := (\mathcal{C}_\phi^{\mathrm{crit}})^* J \mathcal{C}_\phi^{\mathrm{crit}} \in ric_0(\phi, J)$ denote the regular critical solution.*

Then $P_0^{\mathrm{crit}} - \tilde{P} \in ric_0(\phi_{\tilde{P}}, \Lambda_{\tilde{P}})$ is a regular critical solution. If, in addition, $\overline{\mathrm{range}\,(\mathcal{B}_\phi)} = H$, then it is the unique regular critical solution.

Proof. By Lemma 189, we see that $\Delta P := P_0^{\mathrm{crit}} - \tilde{P} \in ric_0(\phi_{\tilde{P}}, \Lambda_{\tilde{P}})$. By equation (4.4) of Proposition 151, we have for $(\phi_{\tilde{P}})_{P_0^{\mathrm{crit}} - \tilde{P}} = \phi_{P_0^{\mathrm{crit}}}$, whose I/O map is the outer factor $\mathcal{X}$ of $\mathcal{D}_\phi$, by the definition of the critical solution P_0^{crit}. It follows that $P_0^{\mathrm{crit}} - \tilde{P} \in ric(\phi_{\tilde{P}}, \Lambda_{\tilde{P}})$ is a regular critical solution of the spectral H^∞DARE $ric(\phi_{\tilde{P}}, \Lambda_{\tilde{P}})$. If $\overline{\mathrm{range}\,(\mathcal{B}_\phi)} = H$, then also $\overline{\mathrm{range}\,(\mathcal{B}_{\phi_{\tilde{P}}})} = H$ because the controllability maps of ϕ and $\phi_{\tilde{P}}$ coincide. The uniqueness of the regular critical solution of $ric(\phi_{\tilde{P}}, \Lambda_{\tilde{P}})$ follows from Corollary 116. $\square$

The spectral DLS and DARE can be used to show that the solution set $ric_0(\phi, J)$ is order-convex:

Lemma 191. *Let $J \geq 0$ be a cost operator. Let $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ be an output stable and I/O stable DLS. Assume that the input space U is separable, and the input operator $B \in \mathcal{L}(U; H)$ is Hilbert–Schmidt. By $P_0^{\mathrm{crit}} := (\mathcal{C}_\phi^{\mathrm{crit}})^* J \mathcal{C}_\phi^{\mathrm{crit}} \in ric_0(\phi, J)$ denote the regular critical solution.*

Then $ric_0(\phi, J)$ is order-convex in the following sense: if $\tilde{P} \in ric_0(\phi, J)$ is such that $\tilde{P} \leq P_0^{\mathrm{crit}}$, then all $P \in Ric(\phi, J)$ such that $\tilde{P} \leq P \leq P_0^{\mathrm{crit}}$ satisfy $P \in ric_0(\phi, J)$.

Proof. Because $\tilde{P} \leq P \leq P_0^{\mathrm{crit}}$, then $0 \leq P - \tilde{P} \leq P_0^{\mathrm{crit}} - \tilde{P}$. By Lemma 156, $P - \tilde{P} \in Ric(\phi_{\tilde{P}}, \Lambda_{\tilde{P}})$. By Corollary 190, $P_0^{\mathrm{crit}} - \tilde{P} \in ric_0(\phi_{\tilde{P}}, \Lambda_{\tilde{P}})$ is a regular critical solution. By claim (ii) of Corollary 141, $P - \tilde{P} \in ric_0(\phi_{\tilde{P}}, \Lambda_{\tilde{P}})$. The proof is now complete. $\square$

Now we have dealt with the spectral DLSs and DAREs. We proceed to study the regular H^∞ solutions for the inner H^∞DARE $ric(\phi^{\tilde{P}}, J)$, centered at $\tilde{P} \geq 0$. We need to assume that the nonnegative cost operator J has a bounded inverse. By Lemma 171, this guarantees that $\phi^{\tilde{P}}$ is output stable and I/O stable, when questions about H^∞ solutions become meaningful.

Lemma 192. *Let $J > 0$ a boundedly invertible cost operator. Let $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ be an output stable and I/O stable DLS, such that $\overline{\mathrm{range}\,(\mathcal{B}_\phi)} = H$. Assume that the input operator $B \in \mathcal{L}(U; H)$ is Hilbert–Schmidt, and the input space U and the output space Y of ϕ are separable. Assume that the regular critical solution $P_0^{\mathrm{crit}} := (\mathcal{C}_\phi^{\mathrm{crit}})^* J \mathcal{C}_\phi^{\mathrm{crit}} \in ric_0(\phi, J)$ exists. Let $\tilde{P} \in ric_0(\phi, J)$, $\tilde{P} \geq 0$, be arbitrary.*

Then the inner DLS $\phi^{\tilde{P}}$ is output stable and I/O stable. The inner DARE $Ric(\phi^{\tilde{P}}, J)$ is a H^∞DARE. Furthermore, $\tilde{P}$ is the unique regular critical solution of its own inner DARE $ric_0(\phi^{\tilde{P}}, J)$. In particular, $L_{A_{\tilde{P}}, \tilde{P}} = 0$.

Proof. Let $\tilde{P} \in ric_0(\phi, J)$, $\tilde{P} \geq 0$, be arbitrary. By claim (iv) of Lemma 171, $\phi^{\tilde{P}}$ is output stable and I/O stable, because $J > 0$ has a bounded inverse. Thus $Ric(\phi^{\tilde{P}}, J)$ is a H^∞DARE, and it makes sense to ask about the regular H^∞ solutions $P \in ric_0(\phi^{\tilde{P}}, J)$.

By claim (iv) of Lemma 171, $\mathcal{D}_{\phi^{\tilde{P}}}$ is $(J, \Lambda_{\tilde{P}})$-inner. Because $\tilde{P} \geq 0$ and $J \geq 0$, it follows that $\Lambda_{\tilde{P}} > \epsilon I$ for some $\epsilon > 0$. Thus the Popov operator satisfies $\mathcal{D}^*_{\phi^{\tilde{P}}} J \mathcal{D}_{\phi^{\tilde{P}}} = \Lambda_P \cdot \mathcal{I} \geq \epsilon \mathcal{I}$, and by Corollary 146, there is a regular critical solution $\tilde{P}_0^{\mathrm{crit}} \in ric_0(\phi^{\tilde{P}}, J)$. It follows from the approximate controllability assumption $\overline{\mathrm{range}\,(\mathcal{B})} = H$ of ϕ that the inner DLS $\phi^{\tilde{P}}$ is approximately controllable, too, because $\mathrm{range}\left(\mathcal{B}_{\phi^{\tilde{P}}}\right) = \mathrm{range}\,(\mathcal{B}_\phi)$ as in the proof of Proposition 178. Now claim (i) of Corollary 116 implies that $\tilde{P}_0^{\mathrm{crit}}$ is the unique regular critical solution of H^∞DARE $ric(\phi^{\tilde{P}}, J)$. Furthermore, $\tilde{P}_0^{\mathrm{crit}}$ is nonnegative, because $J > 0$. Expectedly, the outer factor of $\mathcal{D}_{\phi^{\tilde{P}}}$ is the static identity operator $\mathcal{I}$, which equals the I/O map $(\phi^{\tilde{P}})_{\tilde{P}_0^{\mathrm{crit}}}$ of the corresponding spectral DLS (associated to pair $(\phi^{\tilde{P}}, J)$).

Let $P \in Ric(\phi^{\tilde{P}}, J) = Ric(\phi, J)$, $P \geq 0$, be arbitrary. Then the spectral DLS $(\phi^{\tilde{P}})_P$ can be put into form

$$\left((\phi^{\tilde{P}})_P, \tilde{\Lambda}_P\right) :\equiv \left(\phi^{\tilde{P}}, J\right)_P \equiv \left(\begin{pmatrix} A_{\tilde{P}} & B \\ K_{\tilde{P}} - K_P & I \end{pmatrix}, \Lambda_P\right), \tag{4.51}$$

see equation (4.2) of Proposition 151. Here $A_{\tilde{P}} := A + BK_{\tilde{P}}$, $\Lambda_Q = D^*JD + B^*QB$, and $\Lambda_Q K_Q = -D^*JC - B^*QA$ for $Q = \tilde{P}, P$ are the closed loop semigroup generator, indicator and feedback operator, relative to the original DLS ϕ and the cost operator J.

By setting $P = \tilde{P}$ in equation (4.51), we get

$$(\phi^{\tilde{P}})_{\tilde{P}} = \begin{pmatrix} A_{\tilde{P}} & B \\ -\tilde{K}_{\tilde{P}} & I \end{pmatrix} = \begin{pmatrix} A_{\tilde{P}} & B \\ 0 & I \end{pmatrix},$$

and the feedback operator $\tilde{K}_{\tilde{P}}$, associated to pair $(\phi^{\tilde{P}}, J)$, satisfies $\tilde{K}_{\tilde{P}} = 0$.

However, the same is true for the unique regular critical solution $\tilde{P}_0^{\mathrm{crit}} \in ric_0(\phi^{\tilde{P}}, J)$ if $\overline{\mathrm{range}\,(\mathcal{B}_\phi)} = H$. It follows that $\mathrm{range}\left(\mathcal{B}_{\phi^{\tilde{P}}}\right) = \mathrm{range}\,(\mathcal{B}_\phi)$ as in the proof of Proposition 178. But now assumption $\overline{\mathrm{range}\,(\mathcal{B}_\phi)} = H$ implies $\overline{\mathrm{range}\left(\mathcal{B}_{\phi^{\tilde{P}}}\right)} = H$. Furthermore, because the controllability maps of a DLS and any of its spectral DLSs are equal, the approximate controllability $\overline{\mathrm{range}\left(\mathcal{B}_{(\phi^{\tilde{P}})_P}\right)} = H$

follows for all $P \in Ric(\phi, J)$. Now, for $P = \tilde{P}_0^{\rm crit}$ equation (4.51) gives

$$(\phi^{\tilde{P}})_{\tilde{P}_0^{\rm crit}} = \begin{pmatrix} A_{\tilde{P}} & B \\ -\tilde{K}_{\tilde{P}_0^{\rm crit}} & I \end{pmatrix} = \begin{pmatrix} A_{\tilde{P}} & B \\ K_{\tilde{P}} - K_{\tilde{P}_0^{\rm crit}} & I \end{pmatrix}.$$

By the definition of the critical solution, the I/O map of the spectral DLS $(\phi^{\tilde{P}})_{\tilde{P}_0^{\rm crit}}$ is the outer factor of $\mathcal{D}_{\phi^{\tilde{P}}}$. But this is the static identity operator $\mathcal{I}$, as discussed above. Thus $\tilde{K}_{\tilde{P}_0^{\rm crit}}|\text{range}\left(\mathcal{B}_{(\phi^{\tilde{P}})_P}\right) = 0$, and by the approximate controllability assumption, it follows that $\tilde{K}_{\tilde{P}_0^{\rm crit}} = 0$.

By the definition of the inner DARE $ric(\phi^{\tilde{P}}, J)$, the following Liapunov equations are satisfied

$$\begin{aligned} &A_{\tilde{P}}^* \tilde{P} A_{\tilde{P}} - \tilde{P} + C_{\tilde{P}}^* J C_{\tilde{P}} = \tilde{K}_{\tilde{P}}^* \tilde{\Lambda}_{\tilde{P}} \tilde{K}_{\tilde{P}} = 0, \\ &A_{\tilde{P}}^* \tilde{P}_0^{\rm crit} A_{\tilde{P}} - \tilde{P}_0^{\rm crit} + C_{\tilde{P}}^* J C_{\tilde{P}} = \tilde{K}_{\tilde{P}_0^{\rm crit}}^* \tilde{\Lambda}_{\tilde{P}_0^{\rm crit}} \tilde{K}_{\tilde{P}_0^{\rm crit}} = 0. \end{aligned}$$

But now $\tilde{P}_0^{\rm crit} - \tilde{P} = A_{\tilde{P}}^* (\tilde{P}_0^{\rm crit} - \tilde{P}) A_{\tilde{P}}$ and by iterating

$$\tilde{P}_0^{\rm crit} - \tilde{P} - A_{\tilde{P}}^{*j} \tilde{P}_0^{\rm crit} A_{\tilde{P}}^j = -A_{\tilde{P}}^{*j} \tilde{P} A_{\tilde{P}}^j.$$

Because $\tilde{P}_0^{\rm crit}$ is the regular critical solution of $ric(\phi^{\tilde{P}}, J)$, it follows that $A_{\tilde{P}}^{*j} \tilde{P}_0^{\rm crit} A_{\tilde{P}}^j$ converges strongly to zero as $j \to \infty$. But then $L_{A_{\tilde{P}}, \tilde{P}} := \text{s} - \lim_{j\to\infty} A_{\tilde{P}}^{*j} \tilde{P} A_{\tilde{P}}^j$ exists, and

$$\tilde{P}_0^{\rm crit} - \tilde{P} = -L_{A_{\tilde{P}}, \tilde{P}}. \tag{4.52}$$

A similar kind of calculation can be carried out with the open loop operators. Because $\tilde{K}_{\tilde{P}_0^{\rm crit}} = 0$ as shown above, and by formula (4.51), $\tilde{K}_{\tilde{P}_0^{\rm crit}} = K_{\tilde{P}} - K_{\tilde{P}_0^{\rm crit}}$, it follows that $K_{\tilde{P}} = K_{\tilde{P}_0^{\rm crit}}$. For the indicators we have $\Lambda_{\tilde{P}} = \Lambda_{\tilde{P}_0^{\rm crit}}$, too. To see this equality, consider first the solution $\tilde{P} \in ric_0(\phi, J)$. The I/O map of its inner DLS $\phi^{\tilde{P}}$ is $(J, \Lambda_{\tilde{P}})$-inner, as has already been mentioned. The critical solution $\tilde{P}_0^{\rm crit} \in ric_0(\phi^{\tilde{P}}, J)$ gives the $(J, \tilde{\Lambda}_{\tilde{P}_0^{\rm crit}})$-inner-outer factorization

$$\mathcal{D}_{\phi^{\tilde{P}}} = \mathcal{D}_{(\phi^{\tilde{P}})^{\tilde{P}_0^{\rm crit}}} \cdot \mathcal{D}_{(\phi^{\tilde{P}})_{\tilde{P}_0^{\rm crit}}} = \mathcal{D}_{(\phi^{\tilde{P}})^{\tilde{P}_0^{\rm crit}}} \cdot \mathcal{I} = \mathcal{D}_{(\phi^{\tilde{P}})^{\tilde{P}_0^{\rm crit}}}$$

by claim (iv) of Lemma 171, and the uniqueness of the (J, S)-inner-outer factorizations of an I/O map if the feed-through part of the outer factor is normalized to identity, see Proposition 83. We conclude that $\mathcal{D}_{\phi^{\tilde{P}}}$ is $(J, \tilde{\Lambda}_{\tilde{P}_0^{\rm crit}})$-inner. So, $\mathcal{D}_{\phi^{\tilde{P}}}$ is simultaneously both $(J, \Lambda_{\tilde{P}})$-inner and $(J, \tilde{\Lambda}_{\tilde{P}_0^{\rm crit}})$-inner. This implies that $\Lambda_{\tilde{P}_0^{\rm crit}} = \tilde{\Lambda}_{\tilde{P}_0^{\rm crit}} = \Lambda_{\tilde{P}}$ because the indicator of a solution is not changed under transition to any inner DARE.

Because $K_{\tilde{P}} = K_{\tilde{P}_0^{\mathrm{crit}}}$ and $\Lambda_{\tilde{P}_0^{\mathrm{crit}}} = \Lambda_{\tilde{P}}$ holds, the open loop DARE $Ric(\phi, J)$ gives us the equality

$$A^* \tilde{P}_0^{\mathrm{crit}} A - \tilde{P}_0^{\mathrm{crit}} = A^* \tilde{P} A - \tilde{P},$$

because both the operator $\tilde{P}_0^{\mathrm{crit}}$ and $\tilde{P}$ are solutions of the original DARE $Ric(\phi, J)$, and the right hand sides of the DARE at these solutions coincide.

Thus $\tilde{P}_0^{\mathrm{crit}} - \tilde{P} = A^*(\tilde{P}_0^{\mathrm{crit}} - \tilde{P})A$ and in the same way as proving equation (4.52) we obtain

$$\tilde{P}_0^{\mathrm{crit}} - \tilde{P} = L_{A,\tilde{P}_0^{\mathrm{crit}}} - L_{A,\tilde{P}} = L_{A,\tilde{P}_0^{\mathrm{crit}}}. \tag{4.53}$$

Here the strong limit exists and equality holds because $L_{A,\tilde{P}} = 0$, by assumption $P \in ric_0(\phi, J)$.

Comparing equations (4.52) and (4.53), we see that $-L_{A_{\tilde{P}},\tilde{P}} = L_{A,\tilde{P}_0^{\mathrm{crit}}}$. Both the residual cost operators are nonnegative, as strong limits of sequences of nonnegative operators. It immediately follows that $L_{A_{\tilde{P}},\tilde{P}} = L_{A,\tilde{P}_0^{\mathrm{crit}}} = 0$. Thus $\tilde{P} \in ric_0(\phi^{\tilde{P}}, J)$ is the critical regular solution of its own inner DARE. This completes the proof. □

In the following Lemma 193 we characterize the regular H^∞ solutions of the inner DARE $Ric(\phi^{\tilde{P}}, J)$ for nonnegative $\tilde{P} \in ric_0(\phi, J)$. As in Lemma 192, we have to be a little careful to see that $Ric(\phi^{\tilde{P}}, J)$ is a H^∞DARE. For this reason, we assume again that the cost operator $J > 0$ has a bounded inverse. It is important that the particular case when $\tilde{P} = P_0^{\mathrm{crit}} = (\mathcal{C}_\phi^{\mathrm{crit}})^* J \mathcal{C}_\phi^{\mathrm{crit}}$ can be solved for general $J \geq 0$, see Theorem 197.

Lemma 193. *Let $J > 0$ a boundedly invertible cost operator. Let $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ be an output stable and I/O stable DLS, such that $\overline{\mathrm{range}\,(\mathcal{B}_\phi)} = H$. Assume that the input operator $B \in \mathcal{L}(U; H)$ is Hilbert–Schmidt, and the input space U and the output space Y of ϕ are separable. Assume that the regular critical solution $P_0^{\mathrm{crit}} := (\mathcal{C}_\phi^{\mathrm{crit}})^* J \mathcal{C}_\phi^{\mathrm{crit}} \in ric_0(\phi, J)$ exists.*

Then for all $\tilde{P} \in ric_0(\phi, J)$, $\tilde{P} \geq 0$, the DLS $\phi^{\tilde{P}}$ is output stable and I/O stable. Furthermore, we have the following equality of the solution sets of H^∞DAREs

$$\left\{ P \in ric_0(\phi, J) \quad | \quad P \leq \tilde{P} \right\} = ric_0(\phi^{\tilde{P}}, J).$$

Proof. The output stability and I/O stability of $\phi^{\tilde{P}}$ follow from Lemma 171 and the assumption that J has a bounded nonnegative inverse. We conclude that the inner DARE $Ric(\phi^{\tilde{P}}, J)$ is a H^∞DARE, and the claim about the solution sets $ric_0(\phi, J)$ and $ric_0(\phi^{\tilde{P}}, J)$ is meaningful. We proceed to prove the equality of the solution sets. Fix $\tilde{P} \in ric_0(\phi, J)$ such that $\tilde{P} \geq 0$.

To prove inclusion "$\subset$", let $P \in ric_0(\phi, J)$ be arbitrary, such that $P \leq \tilde{P}$. By Lemma 189, $\Delta P := \tilde{P} - P \in ric_0(\phi_P, \Lambda_P)$ and we can consider the inner DARE of $ric(\phi_P, \Lambda_P)$, centered at $\Delta P \geq 0$. Because the input operator B of ϕ_P is Hilbert–Schmidt, the input space U is separable, the cost operator $\Lambda_P > 0$ is boundedly invertible, and the H^∞ solution $\Delta P \in ric_0(\phi_P, \Lambda_P)$ is nonnegative, Lemma 192 implies that ΔP is the unique regular critical solution of its own inner DARE $ric((\phi_P)^{\Delta P}, \Lambda_P)$.

By Corollary 152, the minimax nodes have the "commutation" relation

$$\left((\phi_P)^{\Delta P}, \Lambda_P\right) \equiv \left((\phi^{\tilde{P}})_P, \Lambda_P\right). \tag{4.54}$$

Because the semigroup generator of $(\phi_P)^{\Delta P} = (\phi^{\tilde{P}})_P$, equaling that of $\phi^{\tilde{P}}$, is $A_{\tilde{P}}$, it follows

$$0 = L_{A_{\tilde{P}}, \Delta P} = L_{A_{\tilde{P}}, (\tilde{P} - P)} = L_{A_{\tilde{P}}, \tilde{P}} - L_{A_{\tilde{P}}, P} = -L_{A_{\tilde{P}}, P},$$

where the first equality is because $\Delta P \in ric_0((\phi_P)^{\Delta P}, \Lambda_P)$ as the unique regular critical solution, and the last follows from the last claim of Lemma 192. This implies the existence of $L_{A_{\tilde{P}}, P}$ as a strong limit and also $L_{A_{\tilde{P}}, P} = 0$. Because $A_{\tilde{P}}$ is also the semigroup generator of $\phi^{\tilde{P}}$, it remains to prove that $P \in ric(\phi^{\tilde{P}}, J)$.

By identity (4.54), we conclude that $(\phi^{\tilde{P}})_P$ is output stable and I/O stable, because this DLS equals $(\phi_P)^{\Delta P}$, which is I/O stable and output stable by claim (iv) of Lemma 171 and the fact that $\Delta P \in ric_0(\phi_P, \Lambda_P)$ is nonnegative, as discussed earlier.

Here we have used the fact that the cost operator Λ_P of DARE $ric(\phi_P, \Lambda_P)$ is nonnegative with a bounded inverse, by Lemma 145, because $P \in ric_0(\phi, J)$ and the regular critical solution $P_0^{\text{crit}} := (\mathcal{C}_\phi^{\text{crit}})^* J \mathcal{C}_\phi^{\text{crit}} \geq 0$ surely has a positive indicator, by the nonnegativity of J. This completes the first part of the proof.

For the converse inclusion "$\supset$", let $P \in ric_0(\phi^{\tilde{P}}, J)$ be arbitrary, and define $\Delta P = \tilde{P} - P$. Now our task is to show that ϕ_P is output stable and I/O stable, and $\tilde{P} \geq P$. To clarify things, we first write the observability map of $\phi^{\tilde{P}}$ in I/O-form, by using formula (4.25), with ϕ_P in place of ϕ, ΔP in place of P, and so on. Recall that this formula does not require any stability properties of any of the DLSs involved (apart from the boundedness of the generating operators), because is solely based on the equivalence of DLSs (and their feedbacks) in I/O-form and difference equation form, presented in the sense of Lemmas 24 and 26. We obtain

$$\mathcal{C}_{(\phi^{\tilde{P}})_P} = \mathcal{C}_{(\phi_P)^{\Delta P}} = \mathcal{C}_{\phi_P} - \mathcal{D}_{\phi_P} \mathcal{D}^{-1}_{(\phi_P)_{\Delta P}} \mathcal{C}_{(\phi_P)_{\Delta P}},$$

where the first equality is because $(\phi_P)^{\Delta P} = (\phi^{\tilde{P}})_P$, by equation (4.54). Furthermore, $\mathcal{D}_{\phi_P}\mathcal{D}^{-1}_{(\phi_P)_{\Delta P}} = \mathcal{D}_{(\phi_P)^{\Delta P}}$, as causal, shift invariant operators in the sequence space $Seq(U)$, by formulae (4.25) and (4.26). But now $(\phi_P)^{\Delta P} = (\phi^{\tilde{P}})_P$ implies that $\mathcal{D}_{(\phi_P)^{\Delta P}} = \mathcal{D}_{\left(\phi^{\tilde{P}}\right)_P}$ in $Seq(U)$. Because $(\phi_P)_{\Delta P} = \phi_{\tilde{P}}$ by equation (4.4) of Proposition 151, we get

$$\mathcal{C}_{\phi_P} = \mathcal{C}_{(\phi^{\tilde{P}})_P} + \left(\mathcal{D}_{\left(\phi^{\tilde{P}}\right)_P}\right) \cdot \mathcal{C}_{\phi_{\tilde{P}}}. \tag{4.55}$$

Because $P \in ric_0(\phi^{\tilde{P}}, J)$ by assumption, both $\mathcal{C}_{(\phi^{\tilde{P}})_P} : H \to \ell^2(\mathbf{Z}_+; U)$ and $\mathcal{D}_{\left(\phi^{\tilde{P}}\right)_P} : \ell^2(\mathbf{Z}; U) \to \ell^2(\mathbf{Z}; U)$ are bounded. Similarly $\mathcal{C}_{\phi_{\tilde{P}}} : H \to \ell^2(\mathbf{Z}_+; U)$ is bounded because $\tilde{P} \in ric_0(\phi, J)$, by assumption. We now conclude that ϕ_P is output stable, because all the operators in equation (4.55) are bounded between the corresponding (dense subspaces of the) Hilbert spaces H, $\ell^2(\mathbf{Z}_+; U)$, and $\ell^2(\mathbf{Z}; U)$.

We proceed to show the I/O stability of ϕ_P. As above, $\mathcal{D}_{\phi_P}\mathcal{D}^{-1}_{(\phi_P)_{\Delta P}} = \mathcal{D}_{(\phi_P)^{\Delta P}} = \mathcal{D}_{\left(\phi^{\tilde{P}}\right)_P}$ in $Seq(U)$. Also, $\mathcal{D}_{(\phi_P)_{\Delta P}} = \mathcal{D}_{\phi_{\tilde{P}}}$ because $(\phi_P)_{\Delta P} = \phi_{\tilde{P}}$. Because the feed-through operator of the spectral DLS $\phi_{\tilde{P}}$ is always the invertible identity operator, it follows from Proposition 16 that $\mathcal{D}_{\phi_{\tilde{P}}}$ is a causal bijection in $Seq(U)$. It follows that $\mathcal{D}_{\phi_P} = \mathcal{D}_{\left(\phi^{\tilde{P}}\right)_P}\mathcal{D}_{\phi_{\tilde{P}}}$ in $Seq(U)$. From assumptions $\tilde{P} \in ric_0(\phi, J)$ and $P \in ric_0(\phi^{\tilde{P}}, J)$ it follows that both $\mathcal{D}_{\phi_{\tilde{P}}}$ and $\mathcal{D}_{\left(\phi^{\tilde{P}}\right)_P}$ are bounded in $\ell^2(\mathbf{Z}; U)$, and so is $\mathcal{D}_{\phi_P}$. We have now proved that $P \in ric(\phi, J)$, and thus $Ric(\phi_P, \Lambda_P)$ is a H^∞DARE.

Because $P \in ric_0(\phi^{\tilde{P}}, J)$, it follows from claim (iii) of Lemma 171 that the I/O map of the inner DLS $\left(\phi^{\tilde{P}}\right)^P$ is I/O stable and $(J, \tilde{\Lambda}_P)$-inner. The indicator $\tilde{\Lambda}_P$ of P, as a solution of the inner DARE $Ric(\phi^{\tilde{P}}, J)$, equals the indicator Λ_P of P, as a solution of the original DARE $Ric(\phi, J)$. Because $\left(\phi^{\tilde{P}}\right)^P = \phi^P$ by equation (4.3) of Proposition 151, it follows that $\mathcal{D}_{\phi^P}$ is (J, Λ_P)-inner.

Thus $\mathcal{D}_\phi = \mathcal{D}_{\phi^P}\mathcal{D}_{\phi_P}$ where both the factors are bounded. For the Popov operator we get

$$\mathcal{D}^*_\phi J \mathcal{D}_\phi = \left(\mathcal{D}_{\phi^P}\mathcal{D}_{\phi_P}\right)^* J \mathcal{D}_{\phi^P}\mathcal{D}_{\phi_P} = \mathcal{D}^*_{\phi_P} \cdot \mathcal{D}^*_{\phi^P} J \mathcal{D}_{\phi^P} \cdot \mathcal{D}_{\phi_P} = \mathcal{D}^*_{\phi_P}\Lambda_P\mathcal{D}_{\phi_P}.$$

Because we already know that $P \in ric(\phi, J)$, it follows that the residual cost operator in I/O-form satisfies $\mathcal{L}_{\phi,P} = 0$, by claim (ii) of Lemma 144. Because $\overline{\text{range}\,(\mathcal{B}_\phi)} = H$ is assumed, it follows that $L_{A,P} = 0$, by claim (iii) of Lemma 144. We have now shown that $P \in ric_0(\phi, J)$.

Because $\tilde{P}, P \in ric_0(\phi, J)$, Lemma 189 implies that that $\Delta P := \tilde{P} - P \in ric_0(\phi_P, \Lambda_P)$. Because $(\phi_P)^{\Delta P} = (\phi^{\tilde{P}})_P$ and $P \in ric_0(\phi^{\tilde{P}}, J)$, it follows that the inner DLS $(\phi_P)^{\Delta P}$ at solution ΔP is I/O stable. Because the DLS ϕ_P, the cost operator Λ_P, and the solution $\Delta P \in ric_0(\phi_P, \Lambda_P)$ satisfy the conditions of Theorem 188, it follows that $\Delta P \geq 0$ and thus $\tilde{P} \geq P$. This completes the proof. $\square$

4.8 Reduction of H^∞DARE to an inner DARE

In this section, we consider the H^∞DARE $ric(\phi, J)$ that has a regular critical solution $P_0^{\rm crit} := (\mathcal{C}_\phi^{\rm crit})^* J \mathcal{C}_\phi^{\rm crit} \in ric_0(\phi, J)$, where

$$\mathcal{C}_\phi^{\rm crit} := (\mathcal{I} - \bar\pi_+ \mathcal{D}_\phi (\bar\pi_+ \mathcal{D}_\phi^* J \mathcal{D}_\phi \bar\pi_+)^{-1} \bar\pi_+ \mathcal{D}_\phi^* J)\mathcal{C}_\phi. \tag{4.56}$$

In essence, we show under technical assumptions that $ric(\phi, J)$ and $ric(\phi^{P_0^{\rm crit}}, J)$ are practically equivalent, as H^∞DAREs. Many of these results hold for general cost operator J; the nonnegativity assumption $J \geq 0$ is required only when the sets $ric_0(\phi, J)$ and $ric_0(\phi^{P_0^{\rm crit}}, J)$ of regular H^∞ solutions are related to each other.

Suppose we are interested in the H^∞ solutions of H^∞DARE $ric(\phi, J)$. If we know *some* solution $\tilde P \in ric(\phi, J)$, we can study the (possibly non-H^∞) inner DARE $Ric(\phi^{\tilde P}, J)$ in place of the original $ric(\phi, J)$. Furthermore, under the conditions of claim (iv) Lemma 171, if we can find a *nonnegative* solution $\tilde P \in ric_{uw}(\phi, J)$ for $J \geq 0$, then the inner DARE $Ric(\phi^{\tilde P}, J)$ is essentially the H^∞DARE $ric(J^{\frac12}\phi^{\tilde P}, I)$, with an $(I, \Lambda_{\tilde P})$-inner I/O map $J^{\frac12}\mathcal{D}_{\phi^{\tilde P}}$. If, in addition, the nonnegative cost operator J has a bounded inverse, then $Ric(\phi^{\tilde P}, J)$ itself is a H^∞DARE. We remark that an inner DLS $\phi^{\tilde P}$ is generally not observable (i.e. $\ker\left(\mathcal{C}_{\phi^{\tilde P}}\right) \neq \{0\}$), and the semigroup generator A_P is generally not even power bounded.

In Lemmas 192 and 193 we have considered the solution set $ric_0(\phi^{\tilde P}, J)$ for $\tilde P \geq 0$ and boundedly invertible, nonnegative cost operator J. In this section, we give stronger results in the particular case $\tilde P = P_0^{\rm crit}$. The I/O map $\mathcal{D}_{\phi^{P_0^{\rm crit}}}$ is now the $(J, \Lambda_{P_0^{\rm crit}})$-inner factor $\mathcal{N}$ of the I/O map $\mathcal{D}_\phi = \mathcal{N}\mathcal{X}$, and there is no need to assume a bounded inverse for J to make $\phi^{P_0^{\rm crit}}$ output stable and I/O stable. The outer factor $\mathcal{X}$ of the I/O map $\mathcal{D}_\phi$ is not very important from the Riccati equation point of view, as implied by Theorem 197, the main result of this section. An important application of these results is in Section 5.7.

We start by answering the uniqueness questions associated to various critical operators.

Proposition 194. *Let $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ be an output stable and I/O stable DLS, and $J \in \mathcal{L}(Y)$ a self-adjoint cost operator. Assume that the regular critical solution $P_0^{\rm crit} := (\mathcal{C}_\phi^{\rm crit})^* J \mathcal{C}_\phi^{\rm crit} \in ric_0(\phi, J)$ exists. Then*

(i) the critical indicators satisfy $\Lambda_{P^{\rm crit}} = \Lambda_{P_0^{\rm crit}}$ for all critical $P^{\rm crit} \in Ric_{uw}(\phi, J)$,

(ii) *If* $\overline{\mathrm{range}\,(\mathcal{B}_\phi)} = H$, *then the critical feedback operators satisfy* $K_{P^{\mathrm{crit}}} = K_{P_0^{\mathrm{crit}}}$ *for all critical* $P^{\mathrm{crit}} \in Ric_{uw}(\phi, J)$. *Furthermore, the closed loop operators* $A_{P^{\mathrm{crit}}} = A_{P_0^{\mathrm{crit}}}$ *and* $C_{P^{\mathrm{crit}}} = C_{P_0^{\mathrm{crit}}}$, *where critical* $P^{\mathrm{crit}} \in Ric_{uw}(\phi, J)$ *is arbitrary.* P_0^{crit} *is the unique critical solution in the set* $ric_{00}(\phi, J)$.

(iii) *If* $\overline{\mathrm{range}\,(\mathcal{B}_\phi)} = H$, *and the open loop semigroup* A *is strongly stable, then there is only one critical solution* $P^{\mathrm{crit}} \in Ric_{uw}(\phi, J)$, *and it equals* P_0^{crit}.

We conclude that if $\overline{\mathrm{range}\,(\mathcal{B})} = H$, it makes sense to speak about the critical (closed loop) feedback operator K^{crit}, the critical semigroup A^{crit} and critical output operator C^{crit}, because these are now independent of the choice of the critical solution. In Definitions 70 and 73, we defined the objects K^{crit}, A^{crit} and C^{crit} differently. We proceed to show that under approximate controllability $\overline{\mathrm{range}\,(\mathcal{B}_\phi)} = H$, both these definitions coincide. This makes it possible to write the inner DLS $\phi^{P_0^{\mathrm{crit}}} = \begin{pmatrix} A_{P_0^{\mathrm{crit}}} & B \\ C_{P_0^{\mathrm{crit}}} & D \end{pmatrix}$ in I/O-form, without explicit reference to the solution P_0^{crit}.

Proposition 195. *Let* $J \in \mathcal{L}(Y)$ *be a self-adjoint cost operator. Let* $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ *be an output stable and I/O stable DLS, such that* $\overline{\mathrm{range}\,(\mathcal{B}_\phi)} = H$. *Assume that there exists a regular critical solution* $P_0^{\mathrm{crit}} \in ric_0(\phi, J)$.

Define the critical (closed loop) feedback operator

$$\mathcal{K}^{\mathrm{crit}} := -(\bar{\pi}_+ \mathcal{D}_\phi^* J \mathcal{D}_\phi \bar{\pi}_+)^{-1} \bar{\pi}_+ \mathcal{D}_\phi^* J \mathcal{C}_\phi$$

and the critical (closed loop) observability map $\mathcal{C}_\phi^{\mathrm{crit}} := \mathcal{C}_\phi + \mathcal{D}_\phi \mathcal{K}^{\mathrm{crit}}$. *By* $\mathcal{X}$ *and* $\mathcal{N}$ *denote the* $(J, \Lambda_{P_0^{\mathrm{crit}}})$*-inner and outer factors in the* $(J, \Lambda_{P_0^{\mathrm{crit}}})$*-inner-outer factorization* $\mathcal{D}_\phi = \mathcal{N}\mathcal{X}$.

Then

(i) $K_{P_0^{\mathrm{crit}}} = K^{\mathrm{crit}}$, *where* $K^{\mathrm{crit}} := \pi_0 \mathcal{K}^{\mathrm{crit}}$ *with the natural identification of spaces* $\mathrm{range}\,(\pi_0)$ *and* U,

(ii) *the observability map of the spectral DLS satisfies* $\mathcal{C}_{\phi_{P_0^{\mathrm{crit}}}} = \mathcal{X}\mathcal{K}^{\mathrm{crit}}$,

(iii) $A_{P_0^{\mathrm{crit}}} := A + BK_{P_0^{\mathrm{crit}}} = A^{\mathrm{crit}}$, *where* $A^{\mathrm{crit}} := A + \mathcal{B}_\phi \tau^* \mathcal{K}^{\mathrm{crit}}$,

(iv) $C_{P_0^{\mathrm{crit}}} := C + DK_{P_0^{\mathrm{crit}}} = C^{\mathrm{crit}}$, *where* $C^{\mathrm{crit}} := \pi_0 \mathcal{C}_\phi^{\mathrm{crit}}$ *with the natural identification of spaces* $\mathrm{range}\,(\pi_0)$ *and* Y.

(v) *In particular, the inner DLS* $\phi^{P_0^{\mathrm{crit}}}$ *is given in I/O-form by the critical (closed loop) DLS*

$$\Phi^{P_0^{\mathrm{crit}}} = \begin{bmatrix} (A^{\mathrm{crit}})^j & \mathcal{B}_\phi \mathcal{X}^{-1} \tau^{*j} \\ \mathcal{C}_\phi^{\mathrm{crit}} & \mathcal{N} \end{bmatrix}. \tag{4.57}$$

Proof. Let $\mathcal{D}_\phi = \mathcal{N}\mathcal{X}$ be the $(J, \Lambda_{P_0^{\rm crit}})$-inner-outer factorization, where the outer part $\mathcal{X}$ has a bounded inverse, and the feed-through operator is normalized $\pi_0 \mathcal{X} \pi_0 = I$. The existence of such factorization follows from the assumption that the critical solution $P_0^{\rm crit}$ exists, by Theorem 114. It also follows that the Popov operator $\bar{\pi}_+ \mathcal{D}_\phi^* J \mathcal{D}_\phi \bar{\pi}_+$ has a bounded inverse, and it follows that all the operators $\mathcal{K}^{\rm crit}$, $K^{\rm crit}$, $A^{\rm crit}$, $\mathcal{C}_\phi^{\rm crit}$ and $C^{\rm crit}$ are well defined.

Then, as in the proof of Lemma 87, it follows that the outer factor $\mathcal{X}$ has the realization, written in I/O-form

$$\Phi_{\mathcal{X}} = \begin{bmatrix} A^j & \mathcal{B}_\phi \tau^{*j} \\ -\mathcal{K} & \mathcal{X} \end{bmatrix}, \tag{4.58}$$

where $\mathcal{K} := -\Lambda_{P_0^{\rm crit}}^{-1} \mathcal{N}^* J \mathcal{C}_\phi$. On the other hand, the critical (closed loop) feedback operator $\mathcal{K}^{\rm crit} := -(\bar{\pi}_+ \mathcal{D}_\phi^* J \mathcal{D}_\phi \bar{\pi}_+)^{-1} \bar{\pi}_+ \mathcal{D}_\phi^* J \mathcal{C}_\phi$ can be written in form $\mathcal{K}^{\rm crit} = \mathcal{X}^{-1} \cdot \mathcal{K}$, by Lemma 84. We have now enough information to translate the DLS $\Phi_{\mathcal{X}}$ in formula (4.58) into difference equation form; we have

$$\phi_{\mathcal{X}} = \begin{pmatrix} A & B \\ -K^{\rm crit} & I \end{pmatrix}, \quad K^{\rm crit} := \pi_0 \mathcal{K}^{\rm crit}, \tag{4.59}$$

because $\pi_0 \mathcal{X} \pi_0 = I$ implies that $\pi_0 \mathcal{X}^{-1} \pi_0 = I$, and then $\pi_0 \mathcal{K} = \pi_0 \mathcal{K}^{\rm crit}$. Note that we have identified the spaces range$\,(\pi_0)$ and U in the natural way.

Now, because $P_0^{\rm crit} \in ric_0(\phi, J)$ is a critical solution, the outer factor $\mathcal{X}$ can be expressed also as the I/O map of the spectral DLS $\phi_{P_0^{\rm crit}} = \left(\begin{smallmatrix} A & B \\ -K_{P_0^{\rm crit}} & I \end{smallmatrix} \right)$. Because the controllability maps of $\phi_{P_0^{\rm crit}}$ and $\phi_{\mathcal{X}}$ coincide with $\mathcal{B}_\phi$, we conclude that $K^{\rm crit} | \text{range}\,(\mathcal{B}_\phi) = K_{P_0^{\rm crit}} | \text{range}\,(\mathcal{B}_\phi)$. By approximate controllability, $K^{\rm crit} = K_{P_0^{\rm crit}}$, because both the operators are bounded. This proves now claim (i), and claim (ii) immediately follows because $\mathcal{K} = \mathcal{C}_{\phi_{P_0^{\rm crit}}}$ and $\mathcal{K}^{\rm crit} = \mathcal{X}^{-1} \cdot \mathcal{K}$, as discussed above.

Claims (iii), (iv) and (v) are consequences of Lemma 26, where it is shown that the state feedback structures of DLSs in I/O-form and difference equation form are equivalent. More precisely, the pairs $[\mathcal{K}, \mathcal{I} - \mathcal{X}]$ and $(K^{\rm crit}, 0)$ are corresponding state feedback pairs for the (open loop) DLS ϕ in I/O-form and difference equation form, respectively. It follows that the closed loop DLSs $[\phi, [\mathcal{K}, \mathcal{I} - \mathcal{X}]]_\diamond$ in I/O-form and $\left(\phi, \left(K^{\rm crit}, 0\right)\right)_\diamond$ in difference equation form are equal, by Lemma 26. But these equal $\Phi^{P_0^{\rm crit}}$ and $\phi^{P_0^{\rm crit}}$, extended by the equal feedback pairs. □

Now we have tools to find out how the continuity properties of ϕ are inherited by the inner DLS $\phi^{P_0^{\rm crit}}$.

Proposition 196. *$J \in \mathcal{L}(Y)$ a self-adjoint cost operator. Let $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ be an output stable and I/O stable DLS. Assume that $\overline{\operatorname{range}(\mathcal{B}_\phi)} = H$, and the (unique) regular critical solution $P_0^{\mathrm{crit}} \in ric_0(\phi, J)$ exists. Then*

(i) $\phi^{P_0^{\mathrm{crit}}}$ is output stable and I/O stable. The I/O map of $\phi^{P_0^{\mathrm{crit}}}$ is the $(J, \Lambda_{P_0^{\mathrm{crit}}})$-inner factor $\mathcal{N}$ of $\mathcal{D}_\phi = \mathcal{N}\mathcal{X}$. Furthermore, ϕ is input stable if and only if $\phi^{P_0^{\mathrm{crit}}}$ is.

(ii) We have $\overline{\operatorname{range}\left(\mathcal{B}_{\phi^{P_0^{\mathrm{crit}}}}\right)} = H$. If ϕ is input stable, then $\mathcal{B}_\phi\, \ell^2(\mathbf{Z}_-; U) = H$ if and only if $\mathcal{B}_{\phi^{P_0^{\mathrm{crit}}}}\, \ell^2(\mathbf{Z}_-; U) = H$.

Proof. In claim (i), the output stability and I/O stability of $\phi^{P_0^{\mathrm{crit}}}$ follows directly from equation (4.57) in Proposition 195. More precisely, the observability map $\mathcal{C}_\phi^{\mathrm{crit}}$ is bounded because all operators in (4.56) are bounded by our explicit assumptions; in particular, the inverse of the Popov operator $\bar{\pi}_+ \mathcal{D}_\phi^* J \mathcal{D}_\phi \bar{\pi}_+$ is bounded because P_0^{crit} exists, see Theorem 114. Also the I/O map of $\phi^{P_0^{\mathrm{crit}}}$ is $(J, \Lambda_{P_0^{\mathrm{crit}}})$-inner factor $\mathcal{N}$ of $\mathcal{D}_\phi$, by equation (4.57).

To complete the proof, we first show that show that the bounded, anti-causal Toeplitz operator $\pi_- \mathcal{X}^{-1} \pi_- : \ell^2(\mathbf{Z}_-; U) \to \ell^2(\mathbf{Z}_-; U)$ with a causal symbol $\mathcal{X}^{-1}$ is a bijection in this space. Let us start with the surjectivity. Let $\pi_- \tilde{u} \in \ell^2(\mathbf{Z}_-; U)$ be arbitrary. Because $\mathcal{X}$ is outer with a bounded inverse, it follows that $\mathcal{X}^{-1} : \ell^2(\mathbf{Z}; U) \to \ell^2(\mathbf{Z}; U)$ is a bounded, shift-invariant and causal bijection. Thus there is a $\tilde{v} \in \ell^2(\mathbf{Z}; U)$ such that $\pi_- \tilde{u} = \mathcal{X}^{-1} \tilde{v}$. But now

$$\pi_- \tilde{u} = \mathcal{X}^{-1} \pi_- \tilde{v} + \mathcal{X}^{-1} \bar{\pi}_+ \tilde{v} = \pi_- \mathcal{X}^{-1} \pi_- \tilde{v} + \pi_- \mathcal{X}^{-1} \bar{\pi}_+ \tilde{v}.$$

The causality of $\mathcal{X}^{-1}$ implies that $\pi_- \mathcal{X}^{-1} \bar{\pi}_+ \tilde{v} = 0$ and so $\pi_- \tilde{u} = \pi_- \mathcal{X}^{-1} \pi_- \cdot \pi_- \tilde{v}$. The surjectivity of $\pi_- \mathcal{X}^{-1} \pi_-$ follows because $\pi_- \tilde{v} \in \ell^2(\mathbf{Z}_-; U)$.

We show the injectivity of $\pi_- \mathcal{X}^{-1} \pi_-$. Assume $\pi_- \tilde{v} \in \ell^2(\mathbf{Z}_-; U)$ is such that $\pi_- \mathcal{X}^{-1} \pi_- \tilde{v} = 0$. Then

$$0 = \mathcal{X} \pi_- \mathcal{X}^{-1} \pi_- \tilde{v} = \mathcal{X} \mathcal{X}^{-1} \pi_- \tilde{v} - \mathcal{X} \bar{\pi}_+ \mathcal{X}^{-1} \pi_- \tilde{v} = \pi_- \tilde{v} - \mathcal{X} \bar{\pi}_+ \mathcal{X}^{-1} \pi_- \tilde{v},$$

or equivalently $\pi_- \tilde{v} = \mathcal{X} \bar{\pi}_+ \mathcal{X}^{-1} \pi_- \tilde{v} = \pi_- \mathcal{X} \bar{\pi}_+ \mathcal{X}^{-1} \pi_- \tilde{v}$. The causality of $\mathcal{X}$ implies that $\pi_- \mathcal{X} \bar{\pi}_+ = 0$, and so $\pi_- \tilde{v} = 0$. We conclude that the Toeplitz operator $\pi_- \mathcal{X}^{-1} \pi_-$ in injective, and thus a bounded bijection. It then follows from the Open Mapping Theorem, that $\pi_- \mathcal{X}^{-1} \pi_-$ has a bounded inverse in $\ell^2(\mathbf{Z}_-; U)$. Because $\mathcal{B}_{\phi^{P_0^{\mathrm{crit}}}} = \mathcal{B}_\phi \mathcal{X}^{-1} = \mathcal{B}_\phi \cdot \pi_- \mathcal{X}^{-1} \pi_-$ by equation (4.57) in Proposition 195, the equivalence of the input stabilities of ϕ and $\phi^{P_0^{\mathrm{crit}}}$ follows.

It remains to consider claims (ii) about the range of $\mathcal{B}_{\phi^{P_0^{\mathrm{crit}}}}$. Again, we have $\mathcal{B}_{\phi^{P_0^{\mathrm{crit}}}} = \mathcal{B}_\phi \pi_- \cdot \pi_- \mathcal{X}^{-1} \pi_-$. As a causal operator, $\pi_- \mathcal{X}^{-1}$ maps the domain of any controllability map (consisting of the sequences $Seq_-(U) \subset \ell^2(\mathbf{Z}_-;U)$ that have only finitely many nonzero components) onto itself. This implies that $\mathrm{range}\,(\mathcal{B}_\phi) = \mathrm{range}\left(\mathcal{B}_{\phi^{P_0^{\mathrm{crit}}}}\right)$, and the approximate controllability claim follows. The (infinite time) exact controllability claim follows because the Toeplitz operator $\pi_- \mathcal{X}^{-1} \pi_-$ is boundedly invertible. The proof is now complete. $\square$

Now that we have related the DLSs ϕ and $\phi^{P_0^{\mathrm{crit}}}$, we proceed to consider the inner DARE $ric(\phi^{P_0^{\mathrm{crit}}}, J)$ and give the main result of this section. The significance of the following theorem is that the structure of a H^∞DARE does not essentially depend on the outer factor of $\mathcal{D}_\phi$ if the cost operator J is nonnegative. It is then possible, under proper technical assumptions, to replace an original H^∞DARE $ric(\phi, J)$ by the inner H^∞DARE $ric(\phi^{P_0^{\mathrm{crit}}}, J)$ that has a $(J, \Lambda_{P_0^{\mathrm{crit}}})$-inner I/O map. This result has an application in Section 5.7.

Theorem 197. *Let $J \in \mathcal{L}(Y)$ be a self-adjoint cost operator. Let $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ be an output stable and I/O stable DLS, such that $\overline{\mathrm{range}\,(\mathcal{B}_\phi)} = H$. Assume that the regular critical solution $P_0^{\mathrm{crit}} \in ric_0(\phi, J)$ exists. Then the following holds:*

(i) The inner DARE $Ric(\phi^{P_0^{\mathrm{crit}}}, J)$ is a H^∞DARE. The full solution sets satisfy $Ric(\phi, J) = Ric(\phi^{P_0^{\mathrm{crit}}}, J)$. The I/O map $\mathcal{D}_{\phi^{P_0^{\mathrm{crit}}}}$ is the $(J, \Lambda_{P_0^{\mathrm{crit}}})$-inner factor $\mathcal{N}$ of $\mathcal{D}_\phi = \mathcal{N}\mathcal{X}$.

(ii) The unique regular critical solution $\tilde{P}_0^{\mathrm{crit}} := \left(\mathcal{C}^{\mathrm{crit}}_{\phi^{P_0^{\mathrm{crit}}}}\right)^* J \mathcal{C}^{\mathrm{crit}}_{\phi^{P_0^{\mathrm{crit}}}} \in ric_0(\phi^{P_0^{\mathrm{crit}}}, J)$ satisfies $\tilde{P}_0^{\mathrm{crit}} = P_0^{\mathrm{crit}}$.

(iii) Assume, in addition, the input space U and output space Y are separable, the input operator B is Hilbert–Schmidt, and $J \geq 0$. Then

$$ric_0(\phi, J) = ric_0(\phi^{P_0^{\mathrm{crit}}}, J). \tag{4.60}$$

Proof. By claim (i) of Proposition 196, $\phi^{P_0^{\mathrm{crit}}}$ is output stable and I/O stable. It follows that $Ric(\phi^{P_0^{\mathrm{crit}}}, J)$ is a H^∞DARE. By claim (v) of Proposition 195, the I/O map of $\phi^{P_0^{\mathrm{crit}}}$ is $(J, \Lambda_{P_0^{\mathrm{crit}}})$-inner. The full solution sets satisfy $Ric(\phi, J) = Ric(\phi^{P_0^{\mathrm{crit}}}, J)$, by Lemma 157.

We prove claim (ii) by calculating an expression for the critical (closed loop) observability map $\mathcal{C}^{\mathrm{crit}}_{\phi^{P_0^{\mathrm{crit}}}}$ for the inner DLS $\phi^{P_0^{\mathrm{crit}}}$ and the cost operator J. Clearly, $\mathcal{D}_{\phi^{P_0^{\mathrm{crit}}}} = \mathcal{N} = \mathcal{N}\mathcal{I}$ is the unique $(J, \Lambda_{P_0^{\mathrm{crit}}})$-inner-outer factorization, where $\mathcal{I}$ is the unique outer factor whose feed-through operator is the identity of U. By claim (iii) Lemma 84, we obtain

$$\mathcal{C}^{\text{crit}}_{\phi^{P_0^{\text{crit}}}} = \mathcal{C}_{\phi^{P_0^{\text{crit}}}} - \mathcal{N}\Lambda^{-1}_{P_0^{\text{crit}}}\bar{\pi}_+\mathcal{N}^* J\mathcal{C}_{\phi^{P_0^{\text{crit}}}}. \tag{4.61}$$

By claim (v) of Proposition 195, $\mathcal{C}_{\phi^{P_0^{\text{crit}}}} = \mathcal{C}^{\text{crit}}_\phi$, and again, by (iii) Lemma 84

$$\mathcal{C}^{\text{crit}}_\phi = \mathcal{C}_\phi - \mathcal{N}\Lambda^{-1}_{P_0^{\text{crit}}}\bar{\pi}_+\mathcal{N}^* J\mathcal{C}_\phi, \tag{4.62}$$

because $\mathcal{D}_\phi = \mathcal{N}\mathcal{X}$ is the unique $(J, \Lambda_{P_0^{\text{crit}}})$-inner-outer factorization, where $\mathcal{X}$ is the unique outer factor whose feed-through operator is the identity of U. By combining equations (4.61) and (4.62), we obtain

$$\begin{aligned}
\mathcal{C}^{\text{crit}}_{\phi^{P_0^{\text{crit}}}} &= \left(\mathcal{C}_\phi - \mathcal{N}\Lambda^{-1}_{P_0^{\text{crit}}}\bar{\pi}_+\mathcal{N}^* J\mathcal{C}_\phi\right) \\
&- \mathcal{N}\Lambda^{-1}_{P_0^{\text{crit}}}\bar{\pi}_+\mathcal{N}^* J\left(\mathcal{C}_\phi - \mathcal{N}\Lambda^{-1}_{P_0^{\text{crit}}}\bar{\pi}_+\mathcal{N}^* J\mathcal{C}_\phi\right) \\
&= \mathcal{C}_\phi - \mathcal{N}\Lambda^{-1}_{P_0^{\text{crit}}}\bar{\pi}_+\mathcal{N}^* J\mathcal{C}_\phi - \mathcal{N}\Lambda^{-1}_{P_0^{\text{crit}}}\bar{\pi}_+\mathcal{N}^* J\mathcal{C}_\phi \\
&+ \mathcal{N}\Lambda^{-1}_{P_0^{\text{crit}}}\bar{\pi}_+ \cdot \left(\mathcal{N}^* J\mathcal{N}\Lambda^{-1}_{P_0^{\text{crit}}}\right) \cdot \bar{\pi}_+\mathcal{N}^* J\mathcal{C}_\phi.
\end{aligned}$$

Because $\mathcal{N}^* J\mathcal{N} = \Lambda_{P_0^{\text{crit}}}$, the last two terms on the right hand side cancel each other, and it follows

$$\mathcal{C}^{\text{crit}}_{\phi^{P_0^{\text{crit}}}} = \mathcal{C}_\phi - \mathcal{N}\Lambda^{-1}_{P_0^{\text{crit}}}\bar{\pi}_+\mathcal{N}^* J\mathcal{C}_\phi = \mathcal{C}^{\text{crit}}_\phi,$$

where the last equality is by (iii) Lemma 84. Now claim (ii) is verified.

We prove now the inclusion "$\subset$" of claim (iii). In fact, the inclusion "$\subset$" of Lemma 193 is almost what we need, if we set $\tilde{P} = P_0^{\text{crit}} \in ric_0(\phi, J)$. In the proof of this lemma, the bounded inverse of the cost operator $J > 0$ was only needed to show that $\phi^{\tilde{P}}$ is output stable and I/O stable. In the special case when $\tilde{P} = P_0^{\text{crit}}$, we know by Proposition 196 that $\phi^{\tilde{P}}$ is output stable and I/O stable, even if $J \geq 0$ is not boundedly invertible. We now conclude that

$$\left\{P \in ric_0(\phi, J), \quad | \quad P \leq P_0^{\text{crit}}\right\} \subset ric_0(\phi^{P_0^{\text{crit}}}, J).$$

as in the proof of Lemma 193. By Theorem 188, P_0^{crit} is the largest element of the set $ric_0(\phi, J)$, and $P \leq P_0^{\text{crit}}$ need not be explicitly written. The claimed inclusion now follows.

The proof of the converse inclusion "$\supset$" is identical to that given in Lemma 193 for $\tilde{P} = P_0^{\text{crit}}$. We remark that the invertibility of the cost operator J is never used in the proof of this converse inclusion "$\supset$". The proof is now complete. □

The statement on Theorem 197 is in a perfect harmony with the following intuitive observation of this paper: finding solutions for the H^∞ Riccati equation

$ric(\phi, J)$ is related to moving in the lattice of the inner factors of $\mathcal{D}_\phi$. We remark that the input operator $B \in \mathcal{L}(U : H)$ is required to be Hilbert–Schmidt and the cost operator J nonnegative only in claim (iii) of Theorem 197. All the other results in this section hold for arbitrary B and self-adjoint J.

Under the assumptions of claim (iii) of Theorem 197, it is enough to be able to solve (numerically) H^∞DAREs with an inner I/O map. To transform ϕ into $\phi^{P_0^{\mathrm{crit}}}$, we need not directly solve the original DARE $ric(\phi, J)$; the regular critical solution P_0^{crit} can be computed from $\mathcal{C}_\phi^{\mathrm{crit}}$ by using formula (4.56). We remark that in this process, the most requiring thing is to calculate the inverse of the (Toeplitz) Popov operator $\bar{\pi}_+ \mathcal{D}_\phi^* J \mathcal{D}_\phi \bar{\pi}_+$. At least when U is finite dimensional, and there is some smoothness in the Popov function $e^{i\theta} \mapsto \mathcal{D}_\phi(e^{i\theta})^* J \mathcal{D}_\phi(e^{i\theta})$, we can efficiently solve the required Toeplitz systems of equations iteratively, see [58], [53], and [63]. We conclude that we have some hope in this direction, even from the numerical analysis point of view.

So as to the numerical solution of the resulting H^∞DARE with an inner I/O map, things seem to be wide open. It is not even clear what a nice solver would have to do, in order to be nice. Particularly interesting would be algorithms that would not require the dimensionality of the state space, and would not reduce the computation into some type of generalized eigenvalue problem. Such a solver could possibly be an iterative process, formulated for infinite dimensional objects and without any discretization. State space isomorphism techniques could be helpful, so that convenient (minimal) realizations of $\mathcal{D}_{\phi^{P_0^{\mathrm{crit}}}}$ could be used instead. Some additional functionality would have to be required, to enable such solver to move in the solution set of DARE and to find a particular solution of interest. It is not clear, how the natural lattice operations of the set $ric_0(\phi^{P_0^{\mathrm{crit}}}, J)$ can be realized, without replacing them by intersections and spans of subspaces. These problems we leave open for the future research.

4.9 Notes and references

Different DAREs appearing in literature

It is well known that the algebraic Riccati equations are associated to optimal control problems, more general critical control problems and game theoretic problems. The information structure of such a problem is reflected by the form of the associated DARE. Because of the general nature of the critical control problem formulation, presented in Section 2.2, we are lead to use the DARE

$$\begin{cases} A^*PA - P + C^*JC = K_P^*\Lambda_P K_P, \\ \Lambda_P = D^*JD + B^*PB, \\ \Lambda_P K_P = -D^*JC - B^*PA, \end{cases} \tag{4.63}$$

instead of the conventional LQDARE

$$\begin{cases} A^*PA - P + C^*JC = A^*PB \cdot \Lambda_P^{-1} \cdot B^*PA \\ \Lambda_P = D^*JD + B^*PB, \end{cases} \tag{4.64}$$

that appears e.g. in the linear quadratic control problems when a direct cost is applied on the input of the system. It is the latter equation (4.64) that is traditionally discussed in the literature, together with its continuous time analogue. As the reader can see, the difference between DAREs (4.63) and (4.64) is the absence of the cross term D^*JC in (4.64). The more general matrix DARE (4.63) is considered in [49, Chapter 12 and 13]. Furthermore, in the continuous time works [82] (Staffans, 1995), [83] (Staffans, 1997), [103] (G. Weiss and M. Weiss, 1997) and [64] (Mikkola, 1997), the presented CAREs for the regular WPLSs generally have nontrivial cross terms.

Reduction of DARE to LQDARE

It is well known that by the preliminary static state feedback

$$u_j = -(D^*JD)^{-1}D^*JCx_j, \tag{4.65}$$

(if it makes sense) equation (4.63) can always be cast in the form of (4.64) without changing the full solution set, see [49, Proposition 12.1.1]. This can be used to check that the DARE theory presented here is in harmony with the LQDARE and LQCARE theories presented in the literature.

However, there is a number of reasons why this reduction is not always desirable. Suppose we are given some critical control problem whose DARE is of the general form (4.63). Because the preliminary feedback (4.65) changes (one might even say: confuses) the information structure of the original DARE, it is no longer possible to conclude what the original critical control problem is, by looking at the modified cross term free DARE alone.

We remark that the feedback in (4.65) can be "formally" associated to an artificial zero solution of DARE (4.63), and this feedback can be given a optimization theoretic interpretation: it minimizes the cost of the first step. The closed loop trajectories starting from some initial states $x_0 \in H$ generally grows very wildly. If the feed-through operator D of the original DLS $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ has a bounded inverse, then the zero operator $0 \in \mathcal{L}(H)$, indeed, solves DARE $Ric(\phi, J)$, and the inner DARE $Ric(\phi^0, J)$ is of the form (4.64). Now the closed loop I/O map $\mathcal{D}_{\phi^0}$ is a static constant operator D, and the inner DARE $Ric(\phi^0, J)$ "lives" in the unobservable subspace, equaling all of the state space H. If the semigroup generator A of the original DLS $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ is e.g. strongly stable, the same is not true for the semigroup $A_0 = A - B(D^*JD)^{-1}D^*JC$ of ϕ^0, unless $\mathcal{D}_\phi$ is outer. Then the semigroup of DLS ϕ^0 would have "undetectable unstable modes" even if both the open loop semigroup A and the critical closed loop semigroup $A_{P_0^{\mathrm{crit}}}$ are very nice, e.g. strongly stable.

Let us outline the control theoretic meaning of the inner DLSs for various solutions of the DARE. For simplicity, assume that the cost operator J is nonnegative and coercive, and the nonnegative regular critical solution P_0^{crit} exists. Then, each solution $0 \leq P \leq P_0^{\mathrm{crit}}$ of the H^∞DARE $ric(\phi, J)$ gives a feedback control strategy, which is a compromise between the desired internal stability of the closed loop semigroup generator A_P, and the desired performance of the closed loop system, corresponding to the inner DLS ϕ^P. Internal semigroup stability of the closed loop is enhanced when P is chosen larger, and then the closed loop transfer function $\mathcal{D}_{\phi^P}(z)$ will have more "zeroes" in $\mathbf{D}$. However, because $\mathcal{D}_{\phi^P}^* J \mathcal{D}_{\phi^P} = \Lambda_P$ and Λ_P increases together with P, the closed loop systems have a larger "power gain" for larger solutions P. For larger P, a larger portion of the state space is penalized, and the closed loop cost $||P^{\frac{1}{2}} x_0||_H^2$ of a given initial state $x_0 \in H$ is higher.

The non-H^∞ solutions $P \geq P_0^{\mathrm{crit}}$ can be used for feedback control, too, but then we have introduced additional zeroes to the closed loop transfer function $\mathcal{D}_{\phi^P}(z)$ that are not possessed by the transfer function $\mathcal{D}_\phi(z)$ of the open loop DLS. For the corresponding (closed loop) inner DLSs ϕ^P we do not have a guarantee of the I/O stability without extra assumptions, even though the open loop DLS ϕ is assumed to be I/O stable.

Now, if the LQDARE $Ric(\phi^0, J)$ describes completely the solution set $Ric(\phi, J)$, why do we not always normalize the cross term to zero by the preliminary feedback (4.65)? We first remark that as a H^∞DARE, $ric(\phi^0, J)$ is trivial because it has no nontrivial nonnegative H^∞ solutions, by Lemma 193. The same comment holds for the general cross term free LQDARE (4.64). This is, of course, to be expected, because a nontrivial H^∞ solution would have to factorize the static I/O map D, see Lemma 171. We conclude that the LQDARE $Ric(\phi^0, J)$ is no longer directly connected to a factorization of any I/O map into I/O stable factors. This is somewhat unfortunate if our interest in DAREs comes from such factorizations.

Internal self-similarity of the DARE theory

In claim (iv) of Lemma 171 we introduce the factorization of the I/O map as a composition of two I/O stable I/O maps

$$J^{\frac{1}{2}}\mathcal{D}_\phi = J^{\frac{1}{2}}\mathcal{D}_{\phi^P} \cdot \mathcal{D}_{\phi_P},$$

for any $P \in ric_0(\phi, J)$, $P \geq 0$. The left (I, Λ_P)-inner factor $J^{\frac{1}{2}}\mathcal{D}_{\phi^P}$ is related to the inner DLS ϕ^P, and this inner factor can be further factorized by nonnegative solutions $\tilde{P} \in ric_0(\phi^P, J)$ of the inner H^∞DARE, at least if J is boundedly invertible. We remark that even if the whole solution set satisfies $Ric(\phi^P, J) = Ric(\phi, J)$, the set of regular H^∞ solutions $ric_0(\phi^P, J)$ is smaller than the original $ric_0(\phi, J)$ by Lemma 193. This is roughly related to the fact that the transfer function $J^{\frac{1}{2}}\mathcal{D}_{\phi^P}$ has less "zeroes" than $J^{\frac{1}{2}}\mathcal{D}_\phi(z)$ because some of them belong to the factor $\mathcal{D}_{\phi_P}$.

A similar consideration can be given for the right factor $\mathcal{D}_{\phi_P}$, which is a spectral factor of the Popov operator $\mathcal{D}_\phi^* J \mathcal{D}_\phi$: nonnegative solutions of the spectral DARE $P \in ric_0(\phi_P, \Lambda_P)$ factorize $\mathcal{D}_{\phi_P}$ into stable factors. We remark that the "cardinality" of nonnegative solutions in $ric_0(\phi_{\tilde{P}}, \Lambda_P)$ is diminished from that of the original $ric_0(\phi, J)$ because a "shift" by $P \geq 0$ appears, as described in Lemma 189. We further remark that each inner and spectral DARE $ric_0(\phi^P, J)$, $ric_0(\phi_P, \Lambda_P)$ is associated to a critical control problem in a natural way. This gives a system theoretic interpretation to each of the various DAREs.

We conclude that our DARE theory and factorization theory are fully recursive in the sense explained above. It is clear that the multiplicative factorization in any associative algebra (or factorial monoid) is recursive in the following sense: One would like to go on factoring the previous factors, until an irreducible element has been reached. Because the algebraic Riccati equation is related to such multiplicative factorization, we feel that the algebraic Riccati equation theory should be presented in a way that does not hide the recursive nature of things. For this to be possible, we need to have a class of DAREs that is large enough to be closed under passage to inner and spectral DAREs at solutions of interest. In fact, many of our proofs rely on a recursive application of the same DARE theory to inner or spectral DLSs and DAREs.

We complete this discussion by looking at the chains of inner and spectral DAREs $Ric(\phi^P, J)$ and $Ric(\phi_P, \Lambda_P)$. If D^*JD is boundedly invertible, it follows that $Ric(\phi^P, J)$ is LQDARE if and only if $D^* J C_P = 0$ if and only if $K_P = -(D^* J D)^{-1} D^* J C$. If, in addition, D is boundedly invertible, then the previous is equivalent to $K_P = K_0$, $\Lambda_P = \Lambda_0$ and $P = A^* P A$. If, in addition, $L_{A,P} = 0$, then $P = 0$ and we conclude that an inner DARE is LQDARE if and essentially only if $P = 0$. Because the feed-through operator of ϕ_P is the identity and the indicator Λ_P is invertible, a spectral DARE is a LQDARE if and only if $K_P = 0$ if and only if $\mathcal{D}_{\phi_P} = \mathcal{I}$. But this is equivalent with the fact that the original I/O map $\mathcal{D}_\phi$ is (J, Λ_P)-inner, and if, in addition, $L_{A,P} = 0$, the solution P must equal the regular critical solution P_0^{crit}. We conclude that it is very exceptional that an inner or spectral DARE has a vanishing cross term, and that the cross term free class of LQDAREs (4.64) is not large enough to accommodate a recursive DARE theory. Introducing the preliminary feedback would destroy the overall recursive picture, and confuse the meaning of the various inner and spectral DAREs.

Chapter 5

Invariant subspaces

5.1 Introduction

Let $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix} \right)$ be an output stable and I/O stable DLS and $J \in \mathcal{L}(Y)$ a nonnegative cost operator. In this chapter, we consider the connection of the solution subset $ric_0(\phi, J)$ of a H^∞DARE to the invariant subspaces of an associated linear operator. More precisely, we seek answers to the following two main questions:

A. Is there a bounded linear operator T, a model operator, such that the natural partial ordering of the solution set $ric_0(\phi, J)$ (under some restrictive, but technical assumptions) gets encoded into the invariant (or co-invariant) subspace structure of T?

B. If such a T exists, can it be expressed in simple and practical terms of the given original data, namely the quadruple $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix} \right)$ together with the cost operator J? Furthermore, can we obtain system theoretic information about the DLS ϕ and the associated H^∞DARE $ric(\phi, J)$, by looking at the structure of such an operator T?

It is well know that several variants of both these question can be and have been given a positive answer, under some particular restrictive assumptions that vary from paper to paper. Several existing approaches provide different descriptions of the partial ordering of the solutions set of the DARE. A brief survey of this literature can be found in Section 5.8.

In this chapter, we first give two ways to construct a candidate for the model operator T, and then we show that these approaches are intimately connected to each other. Our starting point is Theorem 187. It relates, under technical assumptions, the partial ordering of the self-adjoint solutions $P \in ric_0(\phi, J)$ to the partial ordering of certain chains of (adjoined) partial inner factors $\widetilde{\mathcal{N}}_P$ of the I/O map $\mathcal{D}_{\phi_\phi}$. More precisely, the following claims

(i) $P_1 \leq P_2$ and

(ii) $\operatorname{range}\left(\widetilde{\mathcal{N}}_{P_1}\bar{\pi}_+\right) \subset \operatorname{range}\left(\widetilde{\mathcal{N}}_{P_2}\bar{\pi}_+\right)$

are equivalent for $P_1, P_2 \in ric_0(\phi, J)$. The inclusion of ranges is connected to the factorization of inner operator-valued functions by the Beurling–Lax–Halmos Theorem. By using the tools of shift operator models and characteristic functions, these factorizations are associated to backward shift invariant subspaces in an order preserving way. Finally, a further connection to the invariant subspaces of semigroup of a certain DLS is given.

The following standing assumptions are used throughout the paper: The basic DLS $\phi := \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ is I/O stable and output stable, so that $\operatorname{dom}(\mathcal{C}_\phi) := \{x \in H \mid \mathcal{C}x \in \ell^2(\mathbf{Z}_+; Y)\}$ is the whole state space H. Furthermore, ϕ is assumed to be approximately controllable in the sense the $\overline{\operatorname{range}(\mathcal{B}_\phi)} = H$ where $\operatorname{dom}(\mathcal{B}_\phi) := Seq_-(U)$. The input space U, the state space H, and the output space Y are separable Hilbert spaces. The input operator $B \in \mathcal{L}(U; H)$ is Hilbert–Schmidt. The H^∞DARE $ric(\phi, J)$ has a unique regular critical solution P_0^{crit} whose indicator $\Lambda_{P_0^{\mathrm{crit}}}$ is nonnegative. It given by $P_0^{\mathrm{crit}} := (\mathcal{C}_\phi^{\mathrm{crit}})^* J \mathcal{C}_\phi^{\mathrm{crit}} \in ric_0(\phi, J)$.

We also assume that the I/O map $\mathcal{D}_\phi$ is $(J, \Lambda_{P_0^{\mathrm{crit}}})$-inner, but this technical assumption is lifted in the final Section 5.7. To obtain the full results of this paper, the DLS $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ is assumed to be input stable, and the cost operator J is nonnegative. In this case, the regular critical solution P_0^{crit} is nonnegative, and its indicator is, of course, positive.

We give a technical outline of this chapter. In Section 5.2, we give basic results for a DLS ϕ whose I/O map $\mathcal{D}_\phi$ is (J, S)-inner, i.e.

$$\mathcal{D}_\phi^* J \mathcal{D}_\phi = S$$

for some self-adjoint boundedly invertible $S \in \mathcal{L}(U)$, regarded above as a static operator on $\ell^2(\mathbf{Z}; U)$. It appears that the H^∞DARE $ric(\phi, J)$ has a critical regular solution P_0^{crit} and in fact $\Lambda_{P_0^{\mathrm{crit}}} = S$, see Proposition 198. In claim (iii) of Lemma 202, we show that $P_0^{\mathrm{crit}} = \mathcal{C}_\phi^* J \mathcal{C}_\phi$. In claim (iv) of Lemma 202, we show that the null space $\ker\left(P_0^{\mathrm{crit}} - P\right)$ is A-invariant, for any $P \in ric_0(\phi, J)$ with a positive indicator. The rest of Section 5.2 is devoted to proving that the null spaces of type $\ker\left(\tilde{P} - P\right)$ are $A_{\tilde{P}}$-invariant, provided that $P, \tilde{P} \in$

$ric_0(\phi, J)$ are comparable to each other, see Lemma 205 and Corollary 206. The reason to study a DLS with a $(J, \Lambda_{P_0^{\mathrm{crit}}})$-inner I/O map is the following. If we consider the critical control problem of Section 2.2, associated to the pair (ϕ, J), many formulae will simplify considerably. The same comment holds also for the H^∞DARE theory that has been presented in Chapters 3 and 4. This is due to the fact that the outer factor $\mathcal{X}$ in the $(J, \Lambda_{P_0^{\mathrm{crit}}})$-inner-outer factorization $\mathcal{D} = \mathcal{N}\mathcal{X}$ is identity, because we normalize $S = \Lambda_{P_0^{\mathrm{crit}}}$ and $\pi_0 \mathcal{X} \pi_0 = I$. We take the full advantage of this triviality. In the final Section 5.7, we generalize the results to DLSs having a nontrivial outer factor $\mathcal{X} \neq \mathcal{I}$, by using the results of Section 4.8.

In Proposition 208, the null space of the observability map $\mathcal{C}_\phi$ is "divided away" from the state space H, to obtain an observable DLS ϕ^{red} that has the same I/O map as ϕ but a smaller state space. We remark that the I/O map $\mathcal{D}_\phi$ is not required to be $(J, \Lambda_{P_0^{\mathrm{crit}}})$-inner in Proposition 208. In Definition 209, we associate the characteristic DLS $\phi(P)$ to each $P \in ric_0(\phi, J)$. The characteristic DLS $\phi(P)$ is simply the reduced, observable version of the spectral DLS ϕ_P in the sense of Proposition 208. The basic properties of $\phi(P)$ are given in Lemma 210. In particular, $\mathcal{D}_{\phi(P)} = \mathcal{D}_{\phi_P} = \mathcal{N}_P$, where $\mathcal{D}_{\phi_P} = \mathcal{N}_P \mathcal{X} = \mathcal{N}_P \mathcal{I}$ is the $(\Lambda_P, \Lambda_{P_0^{\mathrm{crit}}})$-inner-outer factorization, see Proposition 147. The semigroup generator of $\phi(P)$ is the compression $\Pi_P A | H^P$, where Π_P is the orthogonal projection of H onto $\ker \left(P_0^{\mathrm{crit}} - P\right)^\perp$, and $H^P := \mathrm{range}\,(\Pi_P)$ is the state space of $\phi(P)$. Because $\Pi_P A = \Pi_P A \Pi_P$ by Lemma 202, $\left(\Pi_P A | H^P\right)^*$ equals the restriction $A^* | H^P$. Trivially, if $P_0^{\mathrm{crit}} \geq P_1 \geq P_2$ for $P_1, P_2 \in ric_0(\phi, J)$, then $\{0\} = H^{P_0^{\mathrm{crit}}} \subset H^{P_1} \subset H^{P_2} \subset H$. This connects the partial ordering of the solution set $ric_0(\phi, J)$ to the partial ordering of the A^*-invariant subspaces H^P, for the DLS ϕ with a (J, S)-inner I/O map. We conclude that the operator A^* can be seen as a model operator, as discussed in the beginning of this section.

We consider also another description of the solution set $ric_0(\phi, J)$, with the aid of the shift operator model of contractions and their characteristic functions. In order to accomplish this, we must first deal with some technicalities. In Section 5.4, an orthogonality result is given for DLSs whose transfer functions are inner. In claim (iii) of Proposition 211, it is shown that $\mathrm{range}\,(\mathcal{C}_\phi) = \mathrm{range}\,(\bar{\pi}_+ \mathcal{D}_\phi \pi_-)$ if $\mathrm{range}\,(\bar{\pi}_+ \mathcal{D}_\phi \pi_-)$ is closed and proper technical assumptions hold. An application of this result is Lemma 213, where the orthogonal direct sum decomposition

$$\ell^2(\mathbf{Z}_+; U) = \mathrm{range}\left(\widetilde{\mathcal{N}}_P^\circ \bar{\pi}_+\right) \oplus \mathrm{range}\left(\mathcal{C}_{\widetilde{\phi^\circ(P)}}\right) \tag{5.1}$$

is proved for DLSs ϕ whose I/O map is $(J, \Lambda_{P_0^{\mathrm{crit}}})$-inner and $P \in ric_0(\phi, J)$ is arbitrary. We remark that the statement that $\mathrm{range}\left(\mathcal{C}_{\widetilde{\phi^\circ(P)}}\right)$ is closed is a conclusion, not an assumption of Lemma 213. The operator $\widetilde{\mathcal{N}}_P^\circ$ and the DLS $\widetilde{\phi^\circ(P)}$ are connected to the characteristic DLS $\phi(P)$ by the defining equations (5.7) and (5.8). In Section 5.5, we give a brief overview of a particular case of the Sz.Nagy–Foias shift operator model. The inner characteristic functions for the C_{00}-contractions are introduced, and necessary results from the spectral function theory are presented. Some work is done to translate the frequency space notions, commonly used in the literature, to the time domain notions used in this book.

In Section 5.6 we give our first main results. For arbitrary $P \in ric_0(\phi, J)$, we study the normalized and adjoint version $\widetilde{\phi^\circ(P)}$ of the characteristic DLS. The inner transfer function $\mathcal{D}_{\widetilde{\phi^\circ(P)}}(z) = \widetilde{\mathcal{N}}_P^\circ(z)$ is the characteristic function of the truncated shift operator $S^*|K_{\widetilde{\phi^\circ(P)}}$ in the sense of Sz.Nagy–Foias. Here $K_{\widetilde{\phi^\circ(P)}} := \ell^2(\mathbf{Z}_+; U) \ominus \operatorname{range}\left(\mathcal{D}_{\widetilde{\phi^\circ(P)}}\right)$ is the S^*-invariant subspace, as given in Definition 217. This gives another candidate S^* for the model operator whose invariant subspaces $K_{\widetilde{\phi^\circ(P)}}$ encode the partial ordering of $ric_0(\phi, J)$. The spectral function theory, presented in Section 5.5, connects effectively the operator theoretic properties of the C_{00}-contraction $S^*|K_{\widetilde{\phi^\circ(P)}}$ to the function theory of the normalized transfer function $\widetilde{\mathcal{N}}_P^\circ(z)$, without assuming any finite dimensionality in any of the spaces or the operators. It remains to connect the model operators $S^*|K_{\widetilde{\phi^\circ(P)}}$ to the state spaces and semigroup generators of the DLSs $\widetilde{\phi^\circ(P)}$.

Because $\mathcal{D}_{\phi(P)} = \mathcal{D}_{\phi_P} = \mathcal{N}_P$ by our standing assumption on the triviality of the outer factor $\mathcal{X} = \mathcal{I}$, we conclude from equation (5.1) the equality $K_{\widetilde{\phi^\circ(P)}} = \operatorname{range}\left(\mathcal{C}_{\widetilde{\phi^\circ(P)}}\right)$, by Lemma 213. This gives the similarity transform

$$\left(S^*|K_{\widetilde{\phi^\circ(P)}}\right)\mathcal{C}_{\widetilde{\phi^\circ(P)}} = \bar{\pi}_+\tau^*\mathcal{C}_{\widetilde{\phi^\circ(P)}} = \mathcal{C}_{\widetilde{\phi^\circ(P)}}\left(A^*|H^P\right)$$

by the basic formula $\bar{\pi}_+\tau^*\mathcal{C}_\phi = \mathcal{C}_\phi A$ that describes the interaction of the backward time shift and the semigroup generator A for any DLS ϕ. When the observability map $\mathcal{C}_{\widetilde{\phi^\circ(P)}}$ is a bounded bijection with a bounded inverse, the two descriptions of the set $ric_0(\phi, J)$, the former by restricted adjoint semigroup generators $A^*|H^P$ and the latter by restricted shifts $S^*|\operatorname{range}\left(\mathcal{C}_{\widetilde{\phi^\circ(P)}}\right)$, are connected by a similarity equivalence, see Lemma 218 and Theorem 219. In particular, the restrictions $A^*|H^P$ are similar to a C_{00}-contractions, whose characteristic functions are causal, shift-invariant and stable partial inner factors of the I/O map $\mathcal{D}_\phi$, see Theorems 173 and 175. This connection is analogous to the connection of the zeroes and poles of a rational inner function to the eigenvalues of the semigroup generator of its matrix-valued realization. However, we use neither the notion of zeroes, nor the generalized eigenspaces of the semigroups.

So far we have considered only DLSs $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ whose I/O maps are $(J, \Lambda_{P_0^{\mathrm{crit}}})$-inner. The general case, when $\mathcal{D}_\phi$ is assumed to be only I/O stable, is considered in Section 5.7. Instead of requiring an inner I/O map, we now require only that the regular critical solution $P_0^{\mathrm{crit}} \in ric_0(\phi, J)$ exists. It is shown in Section 4.8, that the structure of the H^∞DARE $ric(\phi, J)$ remains unchanged, if a preliminary critical feedback associated to $P_0^{\mathrm{crit}} \in ric_0(\phi, J)$ is applied. The resulting (closed loop) inner DLS has a $(J, \Lambda_{P_0^{\mathrm{crit}}})$ -inner I/O map, and the results of the previous sections can be applied on the pair $(\phi^{P_0^{\mathrm{crit}}}, J)$ instead of the original pair (ϕ, J). For details, see Theorem 223. Clearly, now the co-invariant subspace results are for the critical closed loop semigroup generator $A^{\mathrm{crit}} = A_{P_0^{\mathrm{crit}}}$ of the inner DLS $\phi^{P_0^{\mathrm{crit}}}$, rather than the open loop semigroup generator A of the original DLS ϕ.

The results of this chapter appeared in [60] (Malinen, 1999). A preliminary version [57] has been presented in MMAR98 conference (Poland, August, 1998).

5.2 DLSs with inner I/O maps

As discussed in Section 5.1, we start this paper by considering first DLSs $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ whose I/O map $\mathcal{D}_\phi$ is (J, S)-inner for two self-adjoint operators $J \in \mathcal{L}(Y)$ and $S \in \mathcal{L}(U)$. Basic results for such DLSs are given in this section. In particular, we are interested in the invariant subspaces of the semigroup generator A that are of the form $\ker\left(P_0^{\text{crit}} - P\right)$. Here $P_0^{\text{crit}} := \left(\mathcal{C}_\phi^{\text{crit}}\right)^* J \mathcal{C}_\phi^{\text{crit}} \in ric_0(\phi, J)$ is a regular critical solution, the closed loop critical observability map is given by

$$\mathcal{C}_\phi^{\text{crit}} := (\mathcal{I} - \bar{\pi}_+ \mathcal{D}_\phi (\bar{\pi}_+ \mathcal{D}_\phi^* J \mathcal{D}_\phi \bar{\pi}_+)^{-1} \bar{\pi}_+ \mathcal{D}_\phi^* J) \mathcal{C}_\phi,$$

and $P \in ric_0(\phi, J)$ is another solution that is comparable to P_0^{crit}. Such invariant subspaces are considered in Corollary 206. The A-co-invariant orthogonal complements $H^P := \ker\left(P_0^{\text{crit}} - P\right)^\perp$ in H are central in the later developments of this work.

In order to be able to speak about the spaces $\ker\left(P_0^{\text{crit}} - P\right)$, the regular critical solution P_0^{crit} must, of course, exist. Clearly, for an (J, S)-inner I/O map $\mathcal{D}_\phi$, the Popov operator is a static constant: $\mathcal{D}_\phi^* J \mathcal{D}_\phi = S$. Then the sufficient and necessary conditions for the existence of a critical solution of DARE are easy to give. The following result is a consequence of Theorem 114 and Proposition 115.

Proposition 198. *Let $J \in \mathcal{L}(Y)$ be a self-adjoint cost operator, and $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ an output stable and I/O stable DLS, such that $\mathcal{D}_\phi$ is (J, S)-inner.*

Then S has a bounded inverse if and only if a regular critical solution $P_0^{\text{crit}} \in ric_0(\phi, J)$ exists. When this equivalence holds, $S = \Lambda_{P_0^{\text{crit}}}$, $\mathcal{D}_\phi$ is $(J, \Lambda_{P_0^{\text{crit}}})$-inner and $\mathcal{D}_\phi = \mathcal{D}_\phi \mathcal{I}$ is the unique $(J, \Lambda_{P_0^{\text{crit}}})$-inner-outer factorization, where the outer factor has a bounded inverse.

For later reference, we give somewhat trivial and technical results about DLSs with an inner I/O map. If a DLS has an inner I/O map, so has its adjoint DLS:

Proposition 199. *Let $\mathcal{N} : \ell^2(\mathbf{Z}; U) \to \ell^2(\mathbf{Z}; U)$ be an I/O map of an output stable and I/O stable DLS ϕ, whose input operator $B \in \mathcal{L}(U; H)$ is Hilbert–Schmidt and U is a separable Hilbert space. Assume that $S_1, S_2 \in \mathcal{L}(U)$ are boundedly invertible and positive. If $\mathcal{N}$ is (S_1, S_2)-inner, and the feed-through operator $\mathcal{N}(0)$ of ϕ is identity, then the adjoint I/O map $\widetilde{\mathcal{N}}$ is (S_2^{-1}, S_1^{-1})-inner.*

Proof. By Proposition 198, $\mathcal{D}_\phi = \mathcal{D}_\phi \mathcal{I}$ is the unique $(S_1, \Lambda_{P_0^{\text{crit}}})$-inner-outer factorization, where $S_2 = \Lambda_{P_0^{\text{crit}}}$ and the trivial outer factor $\mathcal{I}$ has a bounded inverse. It follows from claim (ii) of Lemma 134 that the normalized transfer function $\mathcal{N}^\circ(z)$ is inner from both sides, and the boundary trace $\mathcal{N}^\circ(e^{i\theta})$ is unitary a.e. $e^{i\theta} \in \mathbf{T}$. So the boundary trace of the adjoint function satisfies $\widetilde{\mathcal{N}}^\circ(e^{i\theta}) := S_2^{-\frac{1}{2}}\widetilde{\mathcal{N}}(e^{i\theta})S_1^{\frac{1}{2}} = \mathcal{N}^\circ(e^{-i\theta})^*$ which is unitary a.e. $e^{i\theta} \in \mathbf{T}$. But now $\widetilde{\mathcal{N}}$ is (S_2^{-1}, S_1^{-1})-inner. □

The following corollary is about the I/O map $\widetilde{\mathcal{N}}_P$ whose Toeplitz operator appears in Theorem 187.

Corollary 200. *Let $J \in \mathcal{L}(Y)$ a self-adjoint cost operator. Let $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ be an output stable and I/O stable DLS, whose input operator $B \in \mathcal{L}(U;H)$ is Hilbert–Schmidt and the input space U is separable. Assume that a critical $P_0^{\mathrm{crit}} \in ric_0(\phi, J)$ exists, such that $\Lambda_{P_0^{\mathrm{crit}}} > 0$. For any $P \in ric_0(\phi, J)$, let $\mathcal{N}_P$ denote the $(\Lambda_P, \Lambda_{P_0^{\mathrm{crit}}})$-inner factor of $\mathcal{D}_{\phi_P}$. Then the adjoint I/O map $\widetilde{\mathcal{N}}_P$ is $(\Lambda^{-1}_{P_0^{\mathrm{crit}}}, \Lambda_P^{-1})$-inner.*

Proof. By claim (i) of Proposition 147, $\mathcal{D}_{\phi_P}$ has the $(\Lambda_P, \Lambda_{P_0^{\mathrm{crit}}})$-inner factor $\mathcal{N}_P$. The static part of $\mathcal{N}_P$ is identity, by claim (iii) of Proposition 147. The inertia result, Lemma 145 implies that $\Lambda_P > 0$ for all $P \in ric_0(\phi, J)$. An application of Proposition 199 completes the proof. □

If $J \geq 0$, there are plenty of examples of DLS with (J, S)-inner I/O maps. If the conditions of claim (iii) of Lemma 171 are satisfied, the (normalized) inner DLS $J^{\frac{1}{2}}\phi^P$ has a (I, Λ_P)-inner I/O map, for each nonnegative $P \in ric_0(\phi, J)$. We also remark that, under restrictive assumptions, the family of DLSs with inner I/O maps is sufficiently rich to carry the structure of all H^∞DAREs that have a critical solution, in the sense of Theorem 197. This will be exploited in Section 5.7 where the results of this paper are extended to the general DLSs that do not have an inner I/O map.

The rest of this section is devoted to the study the Riccati equation, and semigroup invariant subspaces of the state space. We start with a technical proposition that only marginally depends on the structure of DARE.

Proposition 201. *Let $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ be a DLS and J a self-adjoint cost operator. Let $P_1, P_2 \in Ric(\phi, J)$. Then $K_{P_2} - K_{P_1} = \Lambda_{P_2}^{-1}B^*(P_2 - P_1)A_{P_1}$ and $\Lambda_{P_1}^{-1}B^*(P_2 - P_1)A_{P_1} = \Lambda_{P_2}^{-1}B^*(P_2 - P_1)A_{P_2}$.*

Proof. To prove the first equation, we calculate

$$K_{P_1} - K_{P_2} = \Lambda_{P_1}^{-1}Q_{P_1} - \Lambda_{P_2}^{-1}Q_{P_2} = (\Lambda_{P_1}^{-1} - \Lambda_{P_2}^{-1})Q_{P_1} + \Lambda_{P_2}^{-1}(Q_{P_1} - Q_{P_2}),$$

where $Q_P := -D^*JC - B^*PA$. Because $x^{-1} - y^{-1} = y^{-1}(y-x)x^{-1}$, we have $\Lambda_{P_1}^{-1} - \Lambda_{P_2}^{-1} = \Lambda_{P_2}^{-1}B^*(P_2 - P_1)B\Lambda_{P_1}^{-1}$. Now we obtain, because $Q_{P_1} - Q_{P_2} = B^*(P_2 - P_1)A$

$$\begin{aligned} K_{P_1} - K_{P_2} &= \Lambda_{P_2}^{-1}\left(B^*(P_2 - P_1)B\,K_{P_1} + B^*(P_2 - P_1)A\right) \\ &= \Lambda_{P_2}^{-1}B^*(P_2 - P_1)(A + BK_{P_1}). \end{aligned}$$

This gives the first equation of the claim. The second equation is obtained by interchanging P_1 and P_2 in the first equation, and comparing these two equations. $\square$

Basic properties of DLSs with $(J, \Lambda_{P^{\rm crit}})$-inner I/O map are given below.

Lemma 202. *Let $J \in \mathcal{L}(Y)$ be a self-adjoint cost operator. Let $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ be an output stable and I/O stable DLS, such that $\overline{\operatorname{range}(\mathcal{B}_\phi)} = H$. Assume that the regular critical solution $P_0^{\rm crit} := \left(\mathcal{C}_\phi^{\rm crit}\right)^* J\mathcal{C}_\phi^{\rm crit} \in ric_0(\phi, J)$ exists, and the I/O map $\mathcal{D}_\phi$ is $(J, \Lambda_{P_0^{\rm crit}})$-inner.*

Then for any $P \in Ric(\phi, J)$ the following holds:

(i) The feedback operators satisfy $K_{P_0^{\rm crit}} = 0$ and $K_P = -\Lambda_P^{-1}B^*(P_0^{\rm crit} - P)A$. Furthermore, $A_{P_0^{\rm crit}} = A$ and $C_{P_0^{\rm crit}} = C$. The operator $Q = P_0^{\rm crit} - P$ satisfies the following Riccati equation

$$\left\{ \begin{array}{l} A^*QA - Q + A^*QB \cdot \Lambda_P^{-1} \cdot B^*QA = 0, \\ \Lambda_P = D^*JD + B^*PB. \end{array} \right. \tag{5.2}$$

(ii) The spectral DLS ϕ_P can be written in the following equivalent forms:

(5.3)
$$\phi_P = \begin{pmatrix} A & B \\ -K_P & I \end{pmatrix} = \begin{pmatrix} A_{P_0^{\rm crit}} & B \\ K_{P_0^{\rm crit}} - K_P & I \end{pmatrix} = \begin{pmatrix} A & B \\ \Lambda_P^{-1}B^*(P_0^{\rm crit} - P)A & I \end{pmatrix}.$$

(iii) We have $\mathcal{C}_\phi = \mathcal{C}_{\phi_{P_0^{\rm crit}}} = \mathcal{C}_\phi^{\rm crit}$ and $P_0^{\rm crit} = \mathcal{C}_\phi^* J\mathcal{C}_\phi$.

(iv) Assume, in addition, that $P \in ric_0(\phi, J)$ and $\Lambda_P > 0$. Then $\ker\left(P_0^{\rm crit} - P\right) = \ker\left(\mathcal{C}_{\phi_P}\right)$. In particular, $\ker\left(P_0^{\rm crit} - P\right)$ is A-invariant.

Proof. Because $\mathcal{D}_\phi$ is assumed to be $(J, \Lambda_{P_0^{\rm crit}})$-inner, the outer factor $\mathcal{X}$ in the unique $(J, \Lambda_{P_0^{\rm crit}})$-inner-outer factorization $\mathcal{D}_\phi = \mathcal{N}\mathcal{X}$ equals the identity $\mathcal{I}$. The outer factor $\mathcal{X} = \mathcal{I}$ is the I/O map of the spectral DLS $\phi_{P_0^{\rm crit}} = \left(\begin{smallmatrix} A & B \\ -K_{P_0^{\rm crit}} & I \end{smallmatrix}\right)$, whence we conclude that $-K_{P_0^{\rm crit}}|\operatorname{range}(\mathcal{B}_\phi) = 0$. Because $K_{P_0^{\rm crit}}$ is a bounded operator and $\overline{\text{range}\,(\mathcal{B}_\phi)} = H$, by explicit assumption, it follows that the critical feedback operator $K_{P_0^{\text{crit}}} = 0$. Immediately $A_{P_0^{\text{crit}}} = A + BK_{P_0^{\text{crit}}} = A$, $C_{P_0^{\text{crit}}} = C + DK_{P_0^{\text{crit}}} = C$, and the second equality in (5.3) is proved.

By applying Proposition 201 to $K_P = K_P - K_{P_0^{\text{crit}}}$ we obtain $K_P = -\Lambda_P^{-1} B^*(P_0^{\text{crit}} - P)A$, for any $P \in Ric(\phi, J)$. This gives the third equality in (5.3), and completes the proof of claim (ii).

To complete the proof of claim (i), the Riccati equation (5.2) must be verified. Because $A^*_{P_0^{\text{crit}}} P_0^{\text{crit}} A_{P_0^{\text{crit}}} - P_0^{\text{crit}} + C^*_{P_0^{\text{crit}}} J C_{P_0^{\text{crit}}} = 0$ by Proposition 160 and $A_{P_0^{\text{crit}}} = A$, $C_{P_0^{\text{crit}}} = C$, we have

$$A^* P_0^{\text{crit}} A - P_0^{\text{crit}} + C^* J C = 0.$$

By rewriting the original DARE $Ric(\phi, J)$ with the aid of the already proved $K_P = -\Lambda_P^{-1} B^*(P_0^{\text{crit}} - P)A$, we obtain for any $P \in Ric(\phi, J)$

$$A^* P A - P + C^* J C = A^*(P_0^{\text{crit}} - P)B \cdot \Lambda_P^{-1} \cdot B^*(P_0^{\text{crit}} - P)A.$$

Subtracting these equations will give give the Riccati equation (5.2).

We now consider claim (iii). Because $K_{P_0^{\text{crit}}} = 0$, the inner DLS at P_0^{crit} satisfies

$$\phi^{P_0^{\text{crit}}} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \phi,$$

and so $\mathcal{C}_\phi = \mathcal{C}_{\phi^{P_0^{\text{crit}}}}$. Now claim (v) of Proposition 195 gives $\mathcal{C}_{\phi^{P_0^{\text{crit}}}} = \mathcal{C}\phi^{\text{crit}}$, where

$$\mathcal{C}_\phi^{\text{crit}} := (\mathcal{I} - \bar{\pi}_+ \mathcal{D}_\phi (\bar{\pi}_+ \mathcal{D}_\phi^* J \mathcal{D}_\phi \bar{\pi}_+)^{-1} \bar{\pi}_+ \mathcal{D}_\phi^* J) \mathcal{C}_\phi.$$

Thus $P_0^{\text{crit}} := \left(\mathcal{C}_\phi^{\text{crit}}\right)^* J \mathcal{C}_\phi^{\text{crit}} = \mathcal{C}_\phi^* J \mathcal{C}_\phi$, and claim (iii) follows.

Because $P \in ric(\phi, J)$, both ϕ and ϕ_P are output stable. As in the proof of Proposition 110, we conclude from DARE $A^*PA - P + C^*JC = K_P^* \Lambda_P K_P$ that

$$P = P - L_{A,P} = \mathcal{C}_\phi^* J \mathcal{C}_\phi - \mathcal{C}_{\phi_P}^* \Lambda_P \mathcal{C}_{\phi_P}, \tag{5.4}$$

where the residual cost $L_{A,P} = s - lim_{n\to\infty} A^*PA$ exists and vanishes because $P \in ric_0(\phi, J)$, by assumption. Inserting $P_0^{\text{crit}} = \mathcal{C}_\phi^* J \mathcal{C}_\phi$ into equation (5.4) gives

$$P_0^{\text{crit}} - P = \mathcal{C}_{\phi_P}^* \Lambda_P \mathcal{C}_{\phi_P}$$

where $P \in ric_0(\phi, J)$ is arbitrary. Because $\Lambda_P > 0$, claim (iv) immediately follows because $\ker(\mathcal{C}_{\phi_P})$ is A-invariant. $\square$

Actually, we now have all the results on invariant subspaces of the semigroup that we need to complete this work. For academic interest, we continue to study the subspaces $\ker\left(P_0^{\mathrm{crit}} - P\right)$. We begin with another variant for the result of claim (iv) of Lemma 202 is the following:

Corollary 203. *Make the same assumptions as in Lemma 202. Let $P \in Ric(\phi, J)$ be arbitrary, such that $\Lambda_P > 0$ and $P \leq P_0^{\mathrm{crit}}$.*

Then $A \ker\left(P_0^{\mathrm{crit}} - P\right) \subset \ker\left(P_0^{\mathrm{crit}} - P\right)$.

Proof. Now $Q := P_0^{\mathrm{crit}} - P \geq 0$ satisfies DARE (5.2). Furthermore, this equation can be put into form

$$A^* Q^{\frac{1}{2}} \cdot R \cdot Q^{\frac{1}{2}} A = Q, \quad R = I + Q^{\frac{1}{2}} B \Lambda_P^{-1} B^* Q^{\frac{1}{2}}.$$

Now, because $\Lambda_P > 0$ and the indicator is always invertible, $\Lambda_P^{-1} > 0$. It now follows that $R \geq I$. For any $x \in H$ we can now write the balance equation

$$||R^{\frac{1}{2}} \cdot Q^{\frac{1}{2}} A x|| = ||Q^{\frac{1}{2}} x||.$$

Because $\ker\left(Q^{\frac{1}{2}}\right) = \ker(Q) = \ker\left(P_0^{\mathrm{crit}} - P\right)$, and $R^{\frac{1}{2}}$ has a bounded inverse, the claim follows. ☐

The case when $P_0^{\mathrm{crit}} \leq P$ instead of $P_0^{\mathrm{crit}} \geq P$ is investigated similarly:

Corollary 204. *Make the same assumptions as in Lemma 202, but assume, in addition, that $0 \in Ric(\phi, J)$, $\Lambda_0 > 0$, and $P_0^{\mathrm{crit}} \geq 0$. Let $P \in Ric(\phi, J)$ be arbitrary, such that $\Lambda_P > 0$ and $P_0^{\mathrm{crit}} \leq P$.*

Then $A \ker\left(P_0^{\mathrm{crit}} - P\right) \subset \ker\left(P_0^{\mathrm{crit}} - P\right)$.

Proof. Again, we use the DARE (5.2). This time we write $Q := P - P_0^{\mathrm{crit}} \geq 0$. By claim (i) of Lemma 202, Q satisfies

$$A^* Q^{\frac{1}{2}} \cdot R \cdot Q^{\frac{1}{2}} A = Q, \quad R = I - Q^{\frac{1}{2}} B \Lambda_P^{-1} B^* Q^{\frac{1}{2}}.$$

This is exactly the same as the corresponding equation in Corollary 203, except that one $+$ has changed into $-$. The claim is proved when we can show, under the additional assumption, that nevertheless $R > 0$ is boundedly invertible.

Because $P_0^{\mathrm{crit}} \geq 0$, we have $0 < \Lambda_{P - P_0^{\mathrm{crit}}} = \Lambda_P - B^* P_0^{\mathrm{crit}} B \leq \Lambda_P$. Because the indicator operator always has a bounded inverse, it follows that $0 < \Lambda_P^{-1} \leq \Lambda_{P - P_0^{\mathrm{crit}}}^{-1} = \Lambda_Q^{-1}$. Now, clearly $R > 0$ has a bounded inverse, if in equation

$$R \geq I - Q^{\frac{1}{2}} B \Lambda_{P - P_0^{\mathrm{crit}}}^{-1} B^* Q^{\frac{1}{2}} = I - Q^{\frac{1}{2}} B \Lambda_Q^{-1} B^* Q^{\frac{1}{2}}$$

the right hand side is strictly positive. Because $0 \in Ric(\phi, J)$, is follows that $\Lambda_0 = D^*JD > 0$ has a bounded inverse. We have

$$Q^{\frac{1}{2}} B \Lambda_Q^{-1} B^* Q^{\frac{1}{2}} = Q^{\frac{1}{2}} B \left(\Lambda_0 + B^* Q B\right)^{-1} B^* Q^{\frac{1}{2}}$$
$$= Q^{\frac{1}{2}} B \Lambda_0^{-\frac{1}{2}} \left(I + \Lambda_0^{-\frac{1}{2}} B^* Q B \Lambda_0^{-\frac{1}{2}}\right)^{-1} \Lambda_0^{-\frac{1}{2}} B^* Q^{\frac{1}{2}} = Q^{\frac{1}{2}} \tilde{B} \left(I + \tilde{B}^* Q \tilde{B}\right)^{-1} \tilde{B}^* Q^{\frac{1}{2}},$$

where $\tilde{B} := B\Lambda_0^{-\frac{1}{2}}$. Now, by a straightforward calculation (e.g. with the aid of the Neumann series),

$$(I + Q^{\frac{1}{2}} \tilde{B} \tilde{B}^* Q^{\frac{1}{2}})^{-1} = I - Q^{\frac{1}{2}} \tilde{B} (I + \tilde{B}^* Q \tilde{B})^{-1} \tilde{B}^* Q^{\frac{1}{2}} = R,$$

because $Q^{\frac{1}{2}} \tilde{B} \tilde{B}^* Q^{\frac{1}{2}} \geq 0$ and thus $I + Q^{\frac{1}{2}} \tilde{B} \tilde{B}^* Q^{\frac{1}{2}}$ is boundedly invertible. It follows that $R > 0$ with a bounded inverse, and the claim is proved. □

An immediate consequence of Corollaries 203 and 204 is the following:

Lemma 205. *Let $J \in \mathcal{L}(Y)$ be a self-adjoint cost operator. Let $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ be an output stable and I/O stable DLS, such that $\overline{\operatorname{range}(\mathcal{B}_\phi)} = H$. Assume that the regular critical solution $P_0^{\text{crit}} := \left(\mathcal{C}_\phi^{\text{crit}}\right)^* J \mathcal{C}_\phi^{\text{crit}} \in ric_0(\phi, J)$ exists, and $P_0^{\text{crit}} \geq 0$. Assume that the I/O map $\mathcal{D}_\phi$ is $(J, \Lambda_{P_0^{\text{crit}}})$-inner. Assume that $0 \in Ric(\phi, J)$ and $D^*JD = \Lambda_0 > 0$,*

Let $P \in Ric(\phi, J)$ be arbitrary, such that $\Lambda_P > 0$, and P is comparable to P_0^{crit}. Then $A \ker\left(P_0^{\text{crit}} - P\right) \subset \ker\left(P_0^{\text{crit}} - P\right)$.

The closed loop semigroup generators $A_{\tilde{P}} = A = BK_{\tilde{P}}$ have the following invariance properties, for $\tilde{P} \in ric_0(\phi, J)$, $\tilde{P} \geq 0$. Recall that these solutions are exactly those that satisfy $0 \leq \tilde{P} \leq P_0^{\text{crit}}$, if the conditions of Theorem 188 hold.

Corollary 206. *Let $J > 0$ be a coercive self-adjoint cost operator in $\mathcal{L}(Y)$. Let $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ be an output stable and I/O stable DLS, such that $\overline{\operatorname{range}(\mathcal{B}_\phi)} = H$. Assume that the input space U and the output space Y are separable, and the input operator $B \in \mathcal{L}(U; H)$ is Hilbert–Schmidt. Assume that the regular critical solution $P_0^{\text{crit}} := \left(\mathcal{C}_\phi^{\text{crit}}\right)^* J \mathcal{C}_\phi^{\text{crit}} \in ric_0(\phi, J)$ exists, $0 \in Ric(\phi, J)$. Let $\tilde{P} \in ric_0(\phi, J)$, $\tilde{P} \geq 0$, be arbitrary.*

Let $P \in Ric(\phi, J)$ be arbitrary, such that $\Lambda_P > 0$ and P is comparable to $\tilde{P}$. Then $A_{\tilde{P}} \ker\left(\tilde{P} - P\right) \subset \ker\left(\tilde{P} - P\right)$.

Proof. By claim (iii) of Lemma 171 and the assumption that J has a bounded inverse, the inner DLS

$$\phi^{\tilde{P}} = \begin{pmatrix} A_{\tilde{P}} & B \\ C_{\tilde{P}} & D \end{pmatrix}$$

is output stable and I/O stable, and the I/O map $\mathcal{D}_{\phi^{\tilde{P}}}$ is $(J, \Lambda_{\tilde{P}})$-inner. Thus $\underline{Ric(\phi^{\tilde{P}}, J)}$ is a H^∞DARE. Because $\overline{\text{range}\,(\mathcal{B}_\phi)} = H$, it also follows that $\overline{\text{range}\left(\mathcal{B}_{\phi^{\tilde{P}}}\right)} = H$, as in the proof of Proposition 178. By Proposition 198, there is a regular critical solution $\tilde{P}_0^{\text{crit}} \in ric_0(\phi^{\tilde{P}}, J)$, and by Lemma 192, $\tilde{P}_0^{\text{crit}} = \tilde{P} \geq 0$. Because the full solution sets of DAREs satisfy $Ric(\phi, J) = Ric(\phi^{\tilde{P}}, J)$ by Lemma 157, it follows that $0 \in Ric(\phi^{\tilde{P}}, J)$. Because $J \geq 0$, it follows that the indicator $\tilde{\Lambda}_0 = \Lambda_0 = D^*JD > 0$. An application of Lemma 205 on DLS $\phi^{\tilde{P}}$ and cost operator J proves the claim. $\square$

5.3 Characteristic DLS $\phi(P)$

In this section, we first develop tools that are required to "divide" the unobservable subspace $\ker(\mathcal{C}_\phi)$ away from the state space. This gives us a reduced DLS. With the aid of this construction, we define the characteristic DLS $\phi(P)$ for each solution $P \in ric(\phi, J)$, see Definition 209. The basic properties of $\phi(P)$ are given in Lemma 210.

Proposition 207. *Let $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ be an output stable DLS. Then $\widetilde{\phi} = \left(\begin{smallmatrix} A^* & C^* \\ B^* & D^* \end{smallmatrix}\right)$ is input stable, and $\mathcal{C}_\phi^* = \mathcal{B}_{\widetilde{\phi}} \cdot \mathrm{flip}$. Here $\mathrm{flip} = \mathrm{flip}^2 = \mathrm{flip}^*$ is the unitary mapping on $\tilde{y} \in \ell^2(\mathbf{Z}; Y)$, given by*

$$(\mathrm{flip}\, \tilde{y})_j = y_{-j-1}.$$

Proof. Let $\tilde{y} \in \ell^2(\mathbf{Z}_+; Y)$, $x_0 \in H$ be arbitrary. Then

$$\begin{aligned} \langle \tilde{y}, \mathcal{C} x_0 \rangle &= \sum_{j=0}^{\infty} \left\langle y_j, C A^j x_0 \right\rangle = \sum_{j=0}^{\infty} \left\langle A^{*j} C^* y_j, x_0 \right\rangle \\ &= \sum_{j=0}^{\infty} \left\langle A^{*j} C^* (\mathrm{flip}\, \tilde{y})_{-j-1}, x_0 \right\rangle = \left\langle \mathcal{B}_{\widetilde{\phi}} (\mathrm{flip}\, \tilde{y}), x_0 \right\rangle = \left\langle \mathcal{C}_\phi^* \tilde{y}, x_0 \right\rangle. \end{aligned}$$

Actually the previous is (at first) true only for $\tilde{y}$ with finitely many nonzero components. Only in this case $\mathrm{flip}\, \tilde{y} \in \mathrm{dom}\left(\mathcal{B}_{\widetilde{\phi}}\right)$, but then because $\mathrm{dom}\left(\mathcal{B}_{\widetilde{\phi}}\right) := Seq_-(Y)$ is dense in $\ell^2(\mathbf{Z}_-; Y)$, it follows that $\mathcal{B}_{\widetilde{\phi}} \cdot \mathrm{flip}$ coincides with the bounded operator $\mathcal{C}^*$ in a dense set. Because $flip$ is unitary, it follows that $\mathcal{B}_{\widetilde{\phi}}$ is bounded and $\widetilde{\phi}$ is input stable. Recall that $\mathrm{dom}(\mathcal{B}) := Seq_-(U)$ consist of finitely long input sequences for all controllability maps. The input stable controllability map $\mathcal{B}$ can always be extended by continuity from $\mathrm{dom}(\mathcal{B})$ to all of $\ell^2(\mathbf{Z}_-; U)$. □

For a quite general DLS ϕ, the kernel $\ker(\mathcal{C}_\phi)$ can be divided away from the state space, without changing the I/O map $\mathcal{D}_\phi$.

Proposition 208. *Let $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ be an output stable and I/O stable DLS, with state space H. Assume that $H_0 := \ker(\mathcal{C}_\phi)$ is nontrivial.*

(i) Then there is a reduced DLS ϕ^{red} with a smaller state space $H^{red} := \ker(\mathcal{C}_\phi)^\perp \subset H$, $H = H_0 \oplus H^{red}$, such that $\mathcal{D}_\phi = \mathcal{D}_{\phi^{red}}$ and $\ker\left(\mathcal{C}_{\phi^{red}}\right) = \{0\}$. The DLS ϕ^{red} is given by

$$\phi^{red} := \begin{pmatrix} \Pi^{red} A | H^{red} & \Pi^{red} B \\ C | H^{red} & D \end{pmatrix},$$

where Π^{red} is the orthogonal projection of H onto H^{red}. In particular, ϕ^{red} is I/O stable and output stable.

(ii) *We have* $\Pi^{red}A = \Pi^{red}A\Pi^{red}$, $\mathcal{B}_{\phi^{red}} = \Pi^{red}\mathcal{B}_\phi$ *and* $\mathcal{C}_{\phi^{red}} = \mathcal{C}_\phi|H^{red}$. *Thus* ϕ^{red} *written in I/O-form is*

$$\Phi^{red} = \begin{bmatrix} (\Pi^{red}A|H^{red})^j & \Pi^{red}\mathcal{B}_\phi\tau^{*j} \\ \mathcal{C}_\phi|H^{red} & \mathcal{D}_\phi \end{bmatrix}.$$

(iii) *The adjoint DLS* $\widetilde{\phi^{red}}$ *is I/O stable and input stable. Furthermore,* $\operatorname{range}\left(\mathcal{B}_{\widetilde{\phi^{red}}}\right) = H^{red}$.

(iv) *If, in addition,* ϕ *is input stable, then* ϕ^{red} *is input stable and* $\widetilde{\phi^{red}}$ *is output stable.*

Proof. Trivially $H_0 := \ker(\mathcal{C}_\phi) = \cap_{j\geq 0}\ker\left(CA^j\right)$ is A-invariant. By Proposition 207, $\mathcal{C}_\phi^* = \mathcal{B}_{\tilde\phi}\cdot\text{flip}$, where $flip$ is the unitary flip reflecting $\ell^2(\mathbf{Z}_+;Y)$ onto $\ell^2(\mathbf{Z}_-;Y)$. We have $\ker(\mathcal{C}_\phi) = \operatorname{range}\left(\mathcal{C}_\phi^*\right)^\perp = \operatorname{range}\left(\mathcal{B}_{\tilde\phi}\right)^\perp$, where $\tilde\phi = \left(\begin{smallmatrix} A^* & C^* \\ B^* & D^*\end{smallmatrix}\right)$ is the adjoint DLS of ϕ.

Because the semigroup generator of $\widetilde{\phi}$ is A^*, it follows that the controllable subspace of $\widetilde{\phi}$, given by $H^{red} := \operatorname{range}\left(\mathcal{B}_{\tilde\phi}\right) = \ker(\mathcal{C}_\phi)^\perp$ is A^*-invariant, and we have the orthogonal direct sum decomposition $H_0 \oplus H^{red} = H$. If Π^{red} is the orthogonal projection onto H^{red}, then $A^*\Pi^{red} = \Pi^{red}A^*\Pi^{red}$ because the range of the observability map is always semigroup invariant.

Define the bounded operators via their adjoints as follows: $(A^{red})^* := A^*|H^{red} : H^{red} \to H^{red}$, $(C^{red})^* := \Pi^{red}C^* : Y \to H^{red}$ and $(B^{red})^* := B^*|H^{red} : H^{red} \to U$. Define the DLSs

$$\phi^{red} := \begin{pmatrix} A^{red} & B^{red} \\ C^{red} & D \end{pmatrix}, \quad \widetilde{\phi^{red}} = \begin{pmatrix} (A^{red})^* & (C^{red})^* \\ (B^{red})^* & D^* \end{pmatrix}.$$

These DLSs are adjoints of each other, and the state space of both ϕ^{red} and $\widetilde{\phi^{red}}$ is, by definition, $H^{red} \subset H$. It is easy to see that ϕ^{red} equals the one given in claim (i).

Because $A^*\Pi^{red} = \Pi^{red}A^*\Pi^{red}$, it follows that $(A^{red})^{*j}(C^{red})^* = (A^*)^j\Pi^{red}C^*$. Now, because C^* is the input operator of $\tilde\phi$, we have $\operatorname{range}(C^*) \subset \operatorname{range}\left(\mathcal{B}_{\tilde\phi}\right)$, and thus $\Pi^{red}C^* = C^*$. This shows that $\mathcal{B}_{\widetilde{\phi^{red}}} = \mathcal{B}_{\tilde\phi} = \Pi^{red}\mathcal{B}_{\tilde\phi}$ where H^{red} is regarded as a subspace of H and the projection Π^{red} serves only as a reminder of this. In particular, because ϕ is output stable, then $\widetilde{\phi}$ is input stable together with $\widetilde{\phi^{red}}$. But then, ϕ^{red} is output stable. From definition of H^{red}, it immediately follows that $\operatorname{range}\left(\mathcal{B}_{\widetilde{\phi^{red}}}\right)$ is dense in H^{red}, and then $\ker\left(\mathcal{C}_{\phi^{red}}\right) = \{0\}$, where $\mathcal{C}_{\phi^{red}} : H^{red} \to \ell^2(\mathbf{Z}_+;Y)$.

Claim (i) is proved, once we show that the I/O maps coincide $\mathcal{D}_{\widetilde{\phi}} = \mathcal{D}_{\widetilde{\phi^{red}}}$. Because $A^*\Pi^{red} = \Pi^{red}A^*\Pi^{red}$, then $(A|H^{red})^j = A^j|H^{red}$. Now

$$(B^{red})^* \left(A^{red}\right)^{*j} (C^{red})^* = B^*(A^*)^j|H^{red} \cdot \Pi^{red}C^*$$

As above, from the inclusion $\mathrm{range}\,(C^*) \subset \mathrm{range}\left(\mathcal{B}_{\widetilde{\phi}}\right)$ it follows that $(B^{red})^* \left(A^{red}\right)^{*j} (C^{red})^* = B^*(A^*)^jC^*$ for all $j \geq 0$. Because also the static parts coincide, we have $\mathcal{D}_{\widetilde{\phi}} = \mathcal{D}_{\widetilde{\phi^{red}}}$, and equivalently $\mathcal{D}_{\phi} = \mathcal{D}_{\phi^{red}}$.

We consider the second claim (ii). The claim about the semigroup is already settled. We have already shown $\mathcal{B}_{\widetilde{\phi^{red}}} = \Pi^{red}\mathcal{B}_{\widetilde{\phi}}$, and adjoining this gives flip $\cdot$ $\mathcal{C}_{\phi}\Pi^{red} = \text{flip} \cdot \mathcal{C}_{\phi^{red}}$, or $\mathcal{C}_{\phi}|H^{red} = \mathcal{C}_{\phi^{red}}$, because flip is unitary.

It remains to consider the controllability map of ϕ^{red}. Because $\Pi^{red}A = \Pi^{red}A\Pi^{red}$, $(A^{red})^jB^{red} = (\Pi^{red}A\Pi^{red})^j\Pi^{red}B = \Pi^{red}A^jB$. Thus $\mathcal{B}_{\phi^{red}}\tilde{u} = \Pi^{red}\mathcal{B}_{\phi}\tilde{u}$ for all $\tilde{u} \in \mathrm{dom}\,(\mathcal{B}_{\phi})$. Consequently, if ϕ is input stable, so is ϕ^{red}. This proves claims (ii) and (iv). The claim (iii) follows by adjoining the previous results. $\square$

We make an additional remark on the controllability properties of ϕ^{red}. Because $\mathcal{B}_{\phi^{red}} = \Pi^{red}\mathcal{B}_{\phi}$, it follows from the boundedness of the orthogonal projection that $\Pi^{red}\,\overline{\mathrm{range}\,(\mathcal{B}_{\phi})} \subset \overline{\Pi^{red}\mathrm{range}\,(\mathcal{B}_{\phi})} = \overline{\mathrm{range}\left(\mathcal{B}_{\phi^{red}}\right)}$. Because the range of the projection $\Pi^{red} : H \to H^{red}$ is of the second category in H^{red}, $\Pi^{red}\,\overline{\mathrm{range}\,(\mathcal{B}_{\phi})}$ is, by the Open Mapping Theorem, a closed subspace of $\overline{\mathrm{range}\left(\mathcal{B}_{\phi^{red}}\right)}$, in the norm of H^P. If ϕ is approximately controllable, then $\Pi^{red}\,\mathrm{range}\,(\mathcal{B}_{\phi})$ is dense in H^{red}, because a continuous surjective mapping maps dense sets onto dense sets. It then follows that $\overline{\mathrm{range}\left(\mathcal{B}_{\phi^{red}}\right)} = H^{red}$; i.e. ϕ^{red} is approximately controllable.

Similar results as Proposition 208 for continuous time well-posed linear systems are given in [89]. There, the state space of the reduced system is a factor space of type $H/\ker\,(\mathcal{C}_{\phi})$. If H is a Hilbert space, we can identify this with the Hilbert subspace $\ker\,(\mathcal{C}_{\phi})^{\perp}$.

We are ready to define the main object of this section, namely the characteristic DLS $\phi(P)$, for $P \in ric(\phi, J)$.

Definition 209. *Let $J \in \mathcal{L}(Y)$ be a self-adjoint cost operator. Let $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ be an output stable and I/O stable DLS. Assume that there exists a regular critical solution $P_0^{\mathrm{crit}} \in ric_0(\phi, J)$ and the I/O map $\mathcal{D}$ is $(J, \Lambda_{P_0^{\mathrm{crit}}})$-inner. Let $P \in ric(\phi, J)$ be arbitrary.*

(i) *Define the closed subspaces*

$$H_P := \ker(\mathcal{C}_{\phi_P}), \quad H^P := \ker(\mathcal{C}_{\phi_P})^{\perp},$$

of the state space H. By Π_P denote the orthogonal projection onto H^P.

(ii) *The reduced DLS $(\phi_P)^{red}$ of ϕ_P (as given in Proposition 208) is denoted by*

$$\phi(P) := \begin{pmatrix} \Pi_P A|H^P & \Pi_P B \\ -K_P|H^P & I \end{pmatrix},$$

The DLS $\phi(P)$ is called the characteristic DLS (of pair (ϕ, J)), centered at P

The following lemma collects the results we have obtained in a useful form.

Lemma 210. *Let $J \in \mathcal{L}(Y)$ be a self-adjoint cost operator. Let $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ be an output stable and I/O stable DLS. Assume that there exists a regular critical solution $P_0^{\mathrm{crit}} \in ric_0(\phi, J)$, and the I/O map $\mathcal{D}_\phi$ is $(J, \Lambda_{P_0^{\mathrm{crit}}})$-inner. Let $P \in ric_0(\phi, J)$ be arbitrary. Then the following holds:*

(i) *The state space of $\phi(P)$ is H^P. The DLS $\phi(P)$ is I/O stable, output stable, and $\ker\left(\mathcal{C}_{\phi(P)}\right) = \{0\}$. The I/O map of $\phi(P)$ satisfies $\mathcal{D}_{\phi(P)} = \mathcal{D}_{\phi_P}$.*

The adjoint DLS $\widetilde{\phi(P)}$ is input stable and approximately controllable: $\overline{\mathrm{range}\left(\mathcal{B}_{\widetilde{\phi(P)}}\right)} = H^P$.

(ii) *If, in addition, ϕ is input stable, then $\phi(P)$ is input stable and $\widetilde{\phi(P)}$ is output stable.*

(iii) *Assume, in addition, that $\overline{\mathrm{range}\,(\mathcal{B}_\phi)} = H$, and $\Lambda_P > 0$. Then $H_P = \ker\left(P_0^{\mathrm{crit}} - P\right)$, where $P_0^{\mathrm{crit}} := \left(\mathcal{C}_\phi^{\mathrm{crit}}\right)^* J\mathcal{C}_\phi^{\mathrm{crit}} \in ric_0(\phi, J)$ is the unique regular critical solution.*

Proof. Claim (i) follows from claims (i) and (iii) of Proposition 208. If ϕ is input stable, so are all spectral DLSs ϕ_P, $P \in ric(\phi, J)$ because they have the same controllability map. Claim (ii) follows now from claim (iv) of Proposition 208. Claim (iii) is a consequence of claim (iv) of Lemma 202. ☐

We remark that only the last claim (iii) required the I/O map of ϕ to be $(J, \Lambda_{P_0^{\mathrm{crit}}})$-inner. Because we can write H_P in terms of the solutions P and P_0^{crit}, we can actually calculate the projection Π_P and also the operators appearing in $\phi(P)$.

5.4 Hankel and Toeplitz operators, and the characteristic DLS $\phi(P)$

Let $J \in \mathcal{L}(Y)$ be a self-adjoint cost operator, and $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ be an I/O stable and output stable DLS, such that a regular critical $P_0^{\mathrm{crit}} \in ric_0(\phi, J)$ exists. Furthermore, assume that ϕ has a $(J, \Lambda_{P_0^{\mathrm{crit}}})$-inner I/O map. In Definition 209 and Lemma 210, we associate the characteristic DLS $\phi(P)$ to each $P \in ric_0(\phi, J)$. The I/O map $\mathcal{D}_{\phi(P)}$ equals the $(\Lambda_P, \Lambda_{P_0^{\mathrm{crit}}})$-inner operator $\mathcal{N}_P$, where $\mathcal{N}_P$ is the inner factor in the $(\Lambda_P, \Lambda_{P_0^{\mathrm{crit}}})$-inner-outer factorization of the spectral factor $\mathcal{D}_{\phi_P} = \mathcal{N}_P \mathcal{X}$. If $\mathcal{D}_\phi$ itself is $(J, \Lambda_{P_0^{\mathrm{crit}}})$-inner, then $\mathcal{D}_{\phi_P} = \mathcal{N}_P$ and the outer factor is trivially $\mathcal{X} = \mathcal{I}$, see Proposition 147. However, we use the symbol $\mathcal{N}_P$ in place for $\mathcal{D}_{\phi_P}$, because in the final Section 5.7, we allow $\mathcal{D}_{\phi_P}$ to have a nontrivial outer factor $\mathcal{X}$.

In the main result of this section, Lemma 213, we consider the ranges of the observability map $\mathcal{C}_{\widetilde{\phi(P)}}$ and the Hankel operator $\bar{\pi}_+ \widetilde{\mathcal{N}}_P \pi_-$ of the adjoint characteristic DLS given by

$$\widetilde{\phi(P)} := \begin{pmatrix} A^*|H^P & -\Pi_P K_P^* \\ B^*|H^P & I \end{pmatrix}.$$

Naturally, the I/O map of $\widetilde{\phi(P)}$ equals $\widetilde{\mathcal{N}}_P$. If the input operator $B \in \mathcal{L}(U; H)$ is Hilbert–Schmidt and the input space U is separable, it follows that $\widetilde{\mathcal{N}}_P$ is $(\Lambda_{P_0^{\mathrm{crit}}}^{-1}, \Lambda_P^{-1})$-inner because $\mathcal{N}_P$ is $(\Lambda_P, \Lambda_{P_0^{\mathrm{crit}}})$-inner, by Corollary 200.

The DLS $\widetilde{\phi(P)}$ is interesting because the ranges of the Toeplitz operators $\widetilde{\mathcal{N}}_P \bar{\pi}_+$ code the partial ordering of the solution set $ric_0(\phi, J)$, even if $\mathcal{D}_{\phi_P}$ contains a nontrivial outer factor. For details, see Theorem 187 and the discussion associated to it. We remark that because Theorem 187 deals with the adjoint operators $\widetilde{\mathcal{N}}_P$ rather than the original $\mathcal{N}_P$, the adjoint DLS $\widetilde{\phi(P)}$ must be considered instead of $\phi(P)$.

In order to prove Lemma 213, we again need auxiliary Propositions 211 and 212 that have some interest in themselves. Let ϕ be a quite general I/O stable and output stable DLS. In Proposition 211, we consider the inclusions of the ranges $\mathrm{range}\,(\mathcal{C}_\phi)$ and $\mathrm{range}\,(\bar{\pi}_+ \mathcal{D}_\phi \pi_-)$. In the particular case, when the $\mathrm{range}\,(\bar{\pi}_+ \mathcal{D}_\phi \pi_-)$ is closed, equality of the ranges appears.

Proposition 211. *Let $\phi := \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ be an output stable and I/O stable DLS, with input space U, state space H and output space Y. Define the domains and ranges as follows:* $\mathrm{range}\,(\bar{\pi}_+ \mathcal{D}_\phi \pi_-) := \bar{\pi}_+ \mathcal{D}_\phi \, \ell^2(\mathbf{Z}_-; U)$, $\mathrm{dom}\,(\mathcal{B}_\phi) := Seq_-(U)$, $\mathrm{range}\,(\mathcal{B}_\phi) := \mathcal{B}_\phi \, \mathrm{dom}\,(\mathcal{B}_\phi)$, *and* $\mathrm{range}\,(\mathcal{C}_\phi) := \mathcal{C}_\phi \, H$.

(i) If ϕ is input stable, then

$$\mathrm{range}\,(\bar{\pi}_+ \mathcal{D}_\phi \pi_-) \subset \mathrm{range}\,(\mathcal{C})\,.$$

(ii) If ϕ is approximately controllable, i.e. $\overline{\mathrm{range}\,(\mathcal{B}_\phi)} = H$, then

$$\mathrm{range}\,(\mathcal{C}_\phi) \subset \overline{\mathrm{range}\,(\bar{\pi}_+ \mathcal{D}_\phi \pi_-)}.$$

(iii) If ϕ is input stable and approximately controllable, and the Hankel operator $\bar{\pi}_+ \mathcal{D}_\phi \pi_-$ has closed range, then

$$\mathrm{range}\,(\mathcal{C}_\phi) = \mathrm{range}\,(\bar{\pi}_+ \mathcal{D}_\phi \pi_-)\,.$$

Proof. We start by establishing claim (i). Let $\tilde{y} \in \mathrm{range}\,(\bar{\pi}_+ \mathcal{D}_\phi \bar{\pi}_-)$ be arbitrary. Then there exists a (possibly nonunique) $\tilde{u} \in \ell^2(\mathbf{Z}_-;U)$ such that $\tilde{y} = \bar{\pi}_+ \mathcal{D}_\phi \pi_- \tilde{u}$. Because $\mathrm{dom}\,(\mathcal{B}_\phi) := \mathit{Seq}_-(U)$ is dense in $\ell^2(\mathbf{Z}_-;U)$, we can choose a sequence $\{\tilde{u}_j\}_{j\geq 0} \subset \mathrm{dom}\,(\mathcal{B}_\phi)$ such that $\tilde{u}_j \to \tilde{u}$ in the norm of $\ell^2(\mathbf{Z}_-;U)$. Then, because $\mathcal{D}_\phi$ is bounded,

$$\bar{\pi}_+ \mathcal{D}_\phi \pi_- \tilde{u}_j \to \tilde{y} \quad \text{as} \quad j \to \infty, \tag{5.5}$$

in the norm of $\ell^2(\mathbf{Z}_+;Y)$. Because $\mathcal{B}_\phi$ is bounded, there is $x \in H$, such that $\mathcal{B}_\phi \pi_- \tilde{u}_j \to x$. Because $\mathcal{C}_\phi$ is bounded,

$$\mathcal{C}_\phi \mathcal{B}_\phi \pi_- \tilde{u}_j \to \mathcal{C}_\phi x \quad \text{as} \quad j \to \infty, \tag{5.6}$$

in the norm of $\ell^2(\mathbf{Z}_+;Y)$. Because $\bar{\pi}_+ \mathcal{D}_\phi \pi_- = \mathcal{C}_\phi \mathcal{B}_\phi$ on $\mathrm{dom}\,(\mathcal{B}_\phi)$, we have $\mathcal{C}_\phi x = \tilde{y}$ and $\tilde{y} \in \mathrm{range}\,(\mathcal{C}_\phi)$, by equations (5.5), (5.6), and the uniqueness of the limit. Because $\tilde{y} \in \mathrm{range}\,(\bar{\pi}_+ \mathcal{D}_\phi \bar{\pi}_-)$ was arbitrary, claim (i) follows.

The proof of claim (ii) is straightforward. Trivially $\mathcal{C}_\phi\, \mathrm{range}\,(\mathcal{B}_\phi) \subset \mathrm{range}\,(\bar{\pi}_- \mathcal{D} \pi_-)$. But then, the continuity of $\mathcal{C}_\phi$ implies the inclusions

$$\mathrm{range}\,(\mathcal{C}) := \mathcal{C}\, H = \mathcal{C}\, \overline{\mathrm{range}\,(\mathcal{B}_\phi)} \subset \overline{\mathcal{C}\, \mathrm{range}\,(\mathcal{B}_\phi)} \subset \overline{\mathrm{range}\,(\bar{\pi}_- \mathcal{D} \pi_-)},$$

because $H = \overline{\mathrm{range}\,(\mathcal{B}_\phi)}$ as claimed. The last claim (iii) is an easy consequence of the previous claims. $\square$

Proposition 212. *Let H be a Hilbert Space, and H_1 its closed subspace. Let H_2 be a (possibly nonclosed) vector subspace of H, such that $H_1 \perp H_2$ and $H = H_1 + H_2$.*

Then H_2 is closed, and we have the orthogonal direct sum decomposition $H = H_1 \oplus H_2$.

Proof. If $x \in H_1 \cap H_2$, then the orthogonality of H_1 and H_2 implies that $0 = \langle x, x\rangle = ||x||^2$, whence $x = 0$. Thus $H_1 \cap H_2 = \{0\}$, and $H = H_1 + H_2$ is an algebraic direct sum. Assume $x \in \overline{H_2}$, and let $H_2 \ni x_j \to x$ in the norm of H. Then $x = \tilde{x}_1 + \tilde{x}_2$ for unique $\tilde{x}_1 \in H_1$ and $\tilde{x}_2 \in H_2$. Let P be the orthogonal projection onto H_1. Then $Px_j = 0$ for all j because $x_j \in H_2 \subset H_1^\perp$. Now we can estimate

$$||Px|| = ||Px - Px_j|| \leq ||x - x_j|| \to 0 \quad \text{as} \quad j \to \infty.$$

It follows that $0 = Px = P\tilde{x}_1 + P\tilde{x}_2$. Because $\tilde{x}_1 \in H_1$, then $P\tilde{x}_1 = \tilde{x}_1$. Because $\tilde{x}_2 \in H_2 \subset H_1^\perp$, then $P\tilde{x}_2 = 0$. Thus $\tilde{x}_1 = 0$ and $x = \tilde{x}_2 \in H_2$. This implies that H_2 is (sequentially) closed. □

Now we have obtained necessary preliminary results, and it remains to apply Propositions 211 and 212 to the adjoint characteristic DLS $\widetilde{\phi(P)}$. We work under the assumption that a regular critical $P_0^{\text{crit}} \in ric_0(\phi, J)$ exists, the critical indicator $\Lambda_{P_0^{\text{crit}}}$ is positive, the input operator $B \in \mathcal{L}(U; H)$ is Hilbert–Schmidt and the input space U is separable. Then all indicators Λ_P for $P \in ric_0(\phi, J)$ are positive by Lemma 145, applied as in Corollary 200. So we can define the normalized I/O maps

$$\mathcal{N}_P^\circ := \Lambda_P^{\frac{1}{2}} \mathcal{N}_P \Lambda_{P_0^{\text{crit}}}^{-\frac{1}{2}}, \quad \widetilde{\mathcal{N}}_P^\circ := \Lambda_{P_0^{\text{crit}}}^{-\frac{1}{2}} \widetilde{\mathcal{N}}_P \Lambda_P^{\frac{1}{2}} \tag{5.7}$$

where $\mathcal{N}_P^\circ$ is inner from the left, (i.e. (I, I)-inner). In fact, the transfer functions of both these normalized DLSs are inner from both sides. If the input space U is finite dimensional, this is a trivial fact because all isometries are unitary in a finite dimensional space. The general case, when U is just a separable Hilbert space, has been dealt in claim (ii) of Proposition 147. The normalized DLSs are defined analogously:

$$\phi^\circ(P) := \Lambda_P^{\frac{1}{2}} \phi(P) \Lambda_{P_0^{\text{crit}}}^{-\frac{1}{2}}, \quad \text{and} \quad \widetilde{\phi^\circ(P)} := \Lambda_{P_0^{\text{crit}}}^{-\frac{1}{2}} \widetilde{\phi(P)} \Lambda_P^{\frac{1}{2}}. \tag{5.8}$$

In the following lemma, we consider the adjoint characteristic DLS $\widetilde{\phi^\circ(P)}$. We show that the range of the Toeplitz operator $\widetilde{\mathcal{N}}_P \bar{\pi}_+$ is "complemented" in $\ell^2(\mathbf{Z}_+; U)$ by the state space H^P of $\widetilde{\phi(P)}$, through the observability map $\mathcal{C}_{\widetilde{\phi^\circ(P)}}$.

Lemma 213. *Let $J \in \mathcal{L}(Y)$ be a self-adjoint cost operator. Let $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ be an input stable, output stable and I/O stable DLS, whose input operator B is Hilbert–Schmidt and input space U is separable. Assume that a regular critical $P_0^{\text{crit}} \in ric_0(\phi, J)$ exists, and $\Lambda_{P_0^{\text{crit}}} > 0$. Assume that the I/O map $\mathcal{D}_\phi$ is $(J, \Lambda_{P_0^{\text{crit}}})$-inner.*

For all $P \in ric_0(\phi, J)$, we have an orthogonal direct sum decomposition

$$\ell^2(\mathbf{Z}_+; U) = \text{range}\left(\widetilde{\mathcal{N}}_P^\circ \bar{\pi}_+\right) \oplus \text{range}\left(\mathcal{C}_{\widetilde{\phi^\circ(P)}}\right),$$

where the symbols are defined as in equations (5.7) and (5.8). In fact, $\operatorname{range}\left(\mathcal{C}_{\widetilde{\phi^\circ(P)}}\right) = \operatorname{range}\left(\bar{\pi}_+ \widetilde{\mathcal{N}}_P^\circ \pi_-\right)$*, where both subspaces are closed.*

Proof. We first show that

$$\ell^2(\mathbf{Z}_+; U) = \operatorname{range}\left(\widetilde{\mathcal{N}}_P^\circ \bar{\pi}_+\right) \oplus \operatorname{range}\left(\bar{\pi}_+ \widetilde{\mathcal{N}}_P \bar{\pi}_-\right), \tag{5.9}$$

where both the spaces are closed in $\ell^2(\mathbf{Z}_+; U)$. Because $\mathcal{N}_P^\circ(e^{i\theta})$ is inner from both sides, also $\widetilde{\mathcal{N}}_P^\circ(e^{i\theta})$ is inner from both sides as has been discussed before the statement of this lemma. We conclude that $\widetilde{\mathcal{N}}_P^\circ : \ell^2(\mathbf{Z}; U) \to \ell^2(\mathbf{Z}; U)$ is a bounded bijection, with $\operatorname{range}\left(\widetilde{\mathcal{N}}_P^\circ\right) = \ell^2(\mathbf{Z}; U)$ and a bounded, shift-invariant (but noncausal) inverse. Thus, for each $\tilde{w} \in \ell^2(\mathbf{Z}_+; U)$, there is a $\tilde{u} \in \ell^2(\mathbf{Z}; U)$ such that

$$\tilde{w} = \bar{\pi}_+ \widetilde{\mathcal{N}}_P^\circ \tilde{u} = \bar{\pi}_+ \widetilde{\mathcal{N}}_P^\circ \bar{\pi}_+ \tilde{u} + \bar{\pi}_+ \widetilde{\mathcal{N}}_P^\circ \pi_- \tilde{u}.$$

So the algebraic direct sum of the (yet possibly nonclosed) vector spaces $\operatorname{range}\left(\bar{\pi}_+ \widetilde{\mathcal{N}}_P^\circ \bar{\pi}_+\right)$ and $\operatorname{range}\left(\bar{\pi}_+ \widetilde{\mathcal{N}}_P^\circ \pi_-\right)$ is all of $\ell^2(\mathbf{Z}_+; U)$.

We prove the orthogonality of these spaces. $\widetilde{\mathcal{N}}_P^\circ$ is a causal isometry on $\ell^2(\mathbf{Z}; U)$, by [27, part (a) Theorem 1.1]; here we have used the fact that $\mathcal{N}_P^\circ(e^{i\theta})$ is unitary a.e. $e^{i\theta} \in \mathbf{T}$, as discussed before this lemma. We have

$$\begin{aligned}
&(\bar{\pi}_+ \widetilde{\mathcal{N}}_P^\circ \bar{\pi}_+)^* \cdot \bar{\pi}_+ \widetilde{\mathcal{N}}_P^\circ \pi_- = \bar{\pi}_+ (\widetilde{\mathcal{N}}_P^\circ)^* \bar{\pi}_+ \cdot \bar{\pi}_+ \widetilde{\mathcal{N}}_P^\circ \pi_- \\
&= \bar{\pi}_+ (\widetilde{\mathcal{N}}_P^\circ)^* \widetilde{\mathcal{N}}_P^\circ \pi_- - (\pi_- \widetilde{\mathcal{N}}_P^\circ \bar{\pi}_+)^* \pi_- \widetilde{\mathcal{N}}_P^\circ \pi_- \\
&= \bar{\pi}_+ \pi_- - (\pi_- \widetilde{\mathcal{N}}_P^\circ \bar{\pi}_+)^* \pi_- \widetilde{\mathcal{N}}_P^\circ \pi_- = 0,
\end{aligned}$$

because $\pi_- \widetilde{\mathcal{N}}_P^\circ \bar{\pi}_+ = 0$ by causality. The range of the Toeplitz operator $\widetilde{\mathcal{N}}_P^\circ \bar{\pi}_+$ is closed, because its symbol is inner from both sides. The range of the Hankel operator $\operatorname{range}\left(\bar{\pi}_+ \widetilde{\mathcal{N}}_P^\circ \pi_-\right)$ is closed, by Proposition 212 where the spaces are $H = \ell^2(\mathbf{Z}_+; U)$, $H_1 = \operatorname{range}\left(\widetilde{\mathcal{N}}_P^\circ\right)$ and $H_2 = \operatorname{range}\left(\bar{\pi}_+ \widetilde{\mathcal{N}}_P^\circ \pi_-\right)$. This verifies that we have the orthogonal direct sum decomposition (5.9), and it remains to show that the same is essentially true when the Hankel operator is replaced by the observability map $\mathcal{C}_{\widetilde{\phi^\circ(P)}}$.

As discussed before the statement of this Lemma, $\Lambda_P > 0$ for all $P \in ric_0(\phi, J)$, and the adjoint characteristic DLS is described by Lemma 210. Clearly $\widetilde{\phi^\circ(P)}$ is I/O stable, because its I/O map is even inner. By claim (i) of Lemma 210, $\widetilde{\phi^\circ(P)}$ is input stable, and approximately controllable $\overline{\operatorname{range}\left(\mathcal{B}_{\widetilde{\phi^\circ(P)}}\right)} = H^P$. Finally, by claim (ii) of Lemma 210, $\widetilde{\phi^\circ(P)}$ is output stable, because ϕ is assumed to be input stable. Now, claim (iii) of Proposition 211 implies that range $\left(\bar{\pi}_+\widetilde{\mathcal{N}}^\circ_P\pi_-\right)$ = range $\left(\mathcal{C}_{\widetilde{\phi^\circ(P)}}\right)$, and, in particular, they are closed subspaces. The proof is now complete. $\square$

For the closedness of the range of a Hankel operator, see [35, p. 258-259]. In Theorem 219 it is important that the observability map $\mathcal{C}_{\widetilde{\phi^\circ(P)}}$ is coercive. To have this under the conditions of Lemma 213, it is enough to establish the injectivity.

5.5 Truncated shifts and operator models

In this section, we recall some notions from the Sz.Nagy–Foias operator model for later use in Section 5.6. Good references are e.g. [27, Chapter IX, Section 5], [70], and [90]. In this section, all Hilbert spaces are assumed to be separable. This makes it possible to work in terms of the boundary traces because our transfer functions are always of bounded type. As before, if Θ denotes an I/O map, then $\Theta(z)$ is its transfer function, and $\Theta(e^{i\theta})$ is the nontangential boundary trace. We identify the spaces $H^2(\mathbf{T};U)$, $(L^2(\mathbf{T};U))$ and $\ell^2(\mathbf{Z}_+;U)$, $(\ell^2(\mathbf{Z};U)$, respectively), by Fourier transform. With this identification, the unilateral shift operator $S = \tau\bar{\pi}_+$ denotes the forward shift on $\ell^2(\mathbf{Z}_+;U)$ as well as multiplication by $e^{i\theta}$ on $H^2(\mathbf{T};U)$. The adjoint backward shift $S^* = \bar{\pi}_+\tau^*$ is understood in the analogous way. Finally, the symbol Θ denotes the multiplication operator by $\Theta(e^{i\theta})$ on $L^2(U)$, as well as the corresponding I/O map on $\ell^2(\mathbf{Z};U)$.

As before, an analytic function $\Theta(z) \in H^\infty(\mathcal{L}(U))$ is called inner (inner from the left), if the boundary trace function $\Theta(e^{i\theta})$ is unitary (isometry, respectively) a.e. $e^{i\theta} \in \mathbf{T}$. If $\Theta(z)$ is an inner from the left, the closed subspace is defined by

$$K_\Theta := H^2(\mathbf{T};U) \ominus \Theta H^2(\mathbf{T};U). \tag{5.10}$$

By P_Θ we denote the orthogonal projection onto K_Θ. Because $\Theta H^2(\mathbf{T};U)$ is S-invariant, K_Θ is S^*-invariant, or equivalently, S-co-invariant. By the Beurling–Lax–Halmos Theorem, all S^*-invariant subspaces of $H^2(\mathbf{T};U)$ are of the form $H^2(\mathbf{T};U) \ominus \Theta H^2(\mathbf{T};U')$, where $\Theta(z) \in H^\infty(\mathcal{L}(U;U'))$ is inner from the left, and $U' \subset U$ is a Hilbert subspace.

We now consider the restriction $S^*|K_\Theta$ and its adjoint, the compression $P_\Theta S|K_\Theta$. The restriction $S^*|K_\Theta$ is a contractive linear operator on the Hilbert subspace $K_\Theta \subset H^2(\mathbf{T};U)$. It is well known that various properties of $S^*|K_\Theta$ are coded into the function $\Theta(e^{i\theta})$; for this reason it is called the characteristic function of $S^*|K_\Theta$. In a more general case, the characteristic function $\Theta(e^{i\theta}) \in H^\infty(\mathbf{T};U)$ can be allowed to be just contractive in the sense that $||\Theta(e^{i\theta})|| \leq 1$ a.e. $e^{i\theta} \in \mathbf{T}$. In this case, the set of operators $\{S^*|K_\Theta\}$ is rich enough to model all contractive linear operators. This is the famous Sz.Nagy–Foias operator model of contractions. For a lucid introduction, see [27, Chapter IX, Section 5]. The special case, appropriate to this work, is when the characteristic function $\Theta(e^{i\theta})$ is inner. Then the contraction $S^*|K_\Theta$ has a number of interesting properties and we now look at some of them. The following proposition is [70, Corollary, p. 43]:

Proposition 214. *Let $\Theta(e^{i\theta})$ be a contractive analytic function. Then $\Theta(e^{i\theta})$ is inner (from both sides) if and only if $S^*|K_\Theta \in C_{00}$. Here C_{00} denotes the class of contractions T on a Hilbert space, such that*

$$s - lim_{j\to\infty} T^j = 0, \quad s - lim_{j\to\infty} T^{*j} = 0.$$

We clearly see that class of C_{00}-contractions is invariant under unitary similarity, and closed under taking the Hilbert space adjoint. Actually [70, Corollary on p. 43] says more than Proposition 214: all C_{00}-contractions are unitarily equivalent to some $S^*|K_\Theta$, for some inner $\Theta(z)$. The adjoint $(S^*|K_\Theta)^* = P_\Theta S|K_\Theta$ is a C_{00}-contraction, and it is unitarily equivalent to $S^*|K_{\widetilde{\Theta}}$, where $\widetilde{\Theta}(z) = \Theta(\bar{z})^*$ is the adjoint inner function. For proof, see [70, Lemma on p. 75].

The spectrum of $S^*|K_\Theta \in C_{00}$ is studied in Lemma 216 with the aid of spectrum of the function $\Theta(z)$, defined as follows:

Definition 215. *Let $\Theta(z)$ be an inner function. Its spectrum $\sigma(\Theta)$ is defined to be the complement of the set of $z \in \overline{\mathbf{D}}$, such that an open neighborhood $N_z \subset \mathbf{C}$ of z exists with*

(i) *$\Theta(z)^{-1}$ exists in $N_z \cap \overline{\mathbf{D}}$,*

(ii) *$\Theta(z)^{-1}$ can be analytically continued to a full neighborhood N_z.*

For the proof of the following Livsic-Möller -type result, [70, Theorem on p. 75].

Lemma 216. *Let U be a separable Hilbert space, and $\Theta(z) \in H^\infty(\mathcal{L}(U))$ be inner. Define $T_\Theta := P_\Theta S|K_\Theta \in \mathcal{L}(K_\Theta)$. Then*

(i) *$\sigma(T_\Theta) = \sigma(\Theta)$, where $\sigma(\Theta) \subset \overline{\mathbf{D}}$ is the spectrum of the characteristic function $\Theta(z)$.*

(ii) *The point spectrum of T_Θ and $T_\Theta^* = S^*|K_\Theta$ satisfies*

$$\begin{aligned}\sigma_p(T_\Theta) &= \{z \in \mathbf{D} \quad | \quad \ker(\Theta(z)) \neq \{0\}\}\\ \sigma_p(T_\Theta^*) &= \{z \in \mathbf{D} \quad | \quad \ker\left(\widetilde{\Theta}(z)\right) \neq \{0\}\}\end{aligned}$$

We remark that $\sigma_P(T_\Theta) \subset \sigma(T_\Theta)$, and the inclusion can be proper. The dimension $\dim U$ is the multiplicity of the shift that models T_Θ. If $\dim U < \infty$, then $\sigma_p(T_\Theta^*) = \overline{\sigma_p(T_\Theta)}$, by dimension counting. Also, $\dim\ker(z - T_\Theta) \leq \dim U$ for all $z \in D$. Much more is known about the truncated shift $S^*|K_\Theta$ if we know its characteristic function $\Theta(z)$, and conversely. For example, the invariant subspace structure of $S^*|K_\Theta$ and the left inner factors of $\Theta(z)$ are connected. To apply these descriptions to DARE, we need to translate these notions into the time domain and state space language.

Definition 217. *Let $\phi = \left(\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right)$ be an I/O stable and output stable DLS. We define the following subspaces*

$$\begin{aligned}K_\phi &:= \ell^2(\mathbf{Z}_+; Y) \ominus \operatorname{range}(\mathcal{D}_\phi \bar{\pi}_+)\\ \widetilde{K}_\phi &:= \operatorname{range}(\mathcal{C}_\phi) \subset \ell^2(\mathbf{Z}_+; Y).\end{aligned}$$

Both K_ϕ and $\widetilde{K}_\phi$ are S^*-invariant. If the transfer function $\mathcal{D}_\phi(z)$ is inner, we see that the closed subspace K_ϕ corresponds, via Fourier transform, to the co-invariant subspace $K_{\mathcal{D}_\phi} \subset H^2(\mathbf{T};Y)$, as defined in equation (5.10). In this paper, the spaces $K_{\widetilde{\phi^\circ(P)}}$ is investigated. Under the assumptions of Lemma 213, we have the equality of the spaces range$\left(\mathcal{C}_{\widetilde{\phi^\circ(P)}}\right) = \widetilde{K}_{\widetilde{\phi^\circ(P)}} = K_{\widetilde{\phi^\circ(P)}}$, where $\mathcal{D}_{\widetilde{\phi^\circ(P)}} = \widetilde{\mathcal{N}}^\circ_P$. The model operator $S^*|K_{\widetilde{\mathcal{N}}^\circ_P}$ is the truncated unilateral shift $(\bar{\pi}_+\tau^*)\,|K_{\widetilde{\phi^\circ(P)}}$ in space $\ell^2(\mathbf{Z}_+;U)$. Actually, we shall write S^* instead of $\bar{\pi}_+\tau^*$ also in the time domain. Stated in other words, the backward shift $S^* = \bar{\pi}_+\tau^*$, restricted to $K_{\widetilde{\phi^\circ(P)}} = $ range$\left(\mathcal{C}_{\widetilde{\phi^\circ(P)}}\right)$ is a contractive linear operator whose characteristic function is $\widetilde{\mathcal{N}}^\circ_P(z) \in H^\infty(\mathcal{L}(U))$. In the next section, we shall make a connection to the state space and semigroup of $\widetilde{\phi^\circ(P)}$.

5.6 Invariant subspaces of the semigroup

It is now time to combine the results of previous sections, and produce the first of our main results. We start by reminding the main lines of previous sections. Let $J \in \mathcal{L}(Y)$ be a cost operator, and $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ be an output stable and I/O stable DLS, such that $\overline{\text{range}\,(\mathcal{B}_\phi)} = H$. We assume that the regular critical solution $P_0^{\text{crit}} := (\mathcal{C}_\phi^{\text{crit}})^* J \mathcal{C}_\phi^{\text{crit}} \in ric_0(\phi, J)$ exists and $\Lambda_{P_0^{\text{crit}}} > 0$. It then follows that all $P \in ric_0(\phi, J)$ have a positive indicator, see Corollary 146. In this section, we still make the technical assumption that the I/O map $\mathcal{D}_\phi$ is $(J, \Lambda_{P_0^{\text{crit}}})$-inner, as in Lemma 210. This assumption will be removed in the final Section 5.7 of this work.

Under these assumptions, we associate two mutually orthogonal subspaces $H_P := \ker\,(\mathcal{C}_{\phi_P}) \subset H$ and $H^P := H \ominus H_P$ to each solution $P \in ric_0(\phi, J)$. Here, as always before, $\phi_P := \left(\begin{smallmatrix} A & B \\ -K_P & I \end{smallmatrix}\right)$ denotes the spectral DLS, centered at P. In claim (iv) Lemma 202 it is shown that H_P is A-invariant. By the same lemma, the subspace H_P is related to the solution $P \in ric_0(\phi, J)$ in the following simple way: Because $\mathcal{D}_\phi$ is $(J, \Lambda_{P_0^{\text{crit}}})$-inner, $H_P = \ker\,(\mathcal{C}_{\phi_P}) = \ker\left(P_0^{\text{crit}} - P\right)$. Now we see that the solutions $P \in ric_0(\phi, J)$ are immediately associated to a family $\{H^P\}$ of A^*-invariant subspaces. This makes it possible to define the restricted operators $A^*|H^P$ and their adjoints, the compressions $\Pi_P A|H^P$ of the semigroup generator.

In this section, we study the structure of the restriction $A^*|H^P \in \mathcal{L}(H^P)$ in terms of the characteristic (transfer) function $\widetilde{\mathcal{N}}_P^\circ(z)$, for arbitrary $P \in ric_0(\phi, J)$. This is done with the aid of the (normalized) adjoint characteristic DLS $\widetilde{\phi^\circ(P)}$ whose semigroup generator is $A^*|H^P$, and I/O maps is $\mathcal{D}_{\widetilde{\phi^\circ(P)}}(z) = \widetilde{\mathcal{N}}_P^\circ(z)$. The DLS $\widetilde{\phi^\circ(P)}$ is the conveniently normalized adjoint DLS of $\phi(P)$ which has been introduced in the following way: By Proposition 208, the null space $H_P := \ker\,(\mathcal{C}_{\phi_P}) \subset H$ is divided away from the state space H of the spectral DLS ϕ_P. We obtain another DLS, the characteristic $\phi(P) := (\phi_P)^{red}$ whose state space is H^P — it is the reduced DLS whose I/O map equals that of the spectral DLS ϕ_P. Furthermore, the DLS $\phi(P)$ is output stable and observable: $\ker\left(\mathcal{C}_{\phi(P)}\right) = \{0\}$. The adjoint DLS $\widetilde{\phi(P)}$ is input stable and approximately controllable: $\overline{\text{range}\left(\mathcal{B}_{\widetilde{\phi(P)}}\right)} = H^P$. A simple normalization is now required to turn $\widetilde{\phi(P)}$ into $\widetilde{\phi^\circ(P)}$.

Under the above assumptions, the I/O map $\mathcal{N}_P$ of $\phi(P)$ is $(\Lambda_P, \Lambda_{P_0^{\text{crit}}})$-inner, where both Λ_P and $\Lambda_{P_0^{\text{crit}}}$ are positive. The normalization of formulae (5.7) and (5.8), gives us $\phi^\circ(P)$ and its adjoint DLS $\widetilde{\phi^\circ(P)}$. The latter is particularly interesting to us, and already considered in Section 5.4. The DLS $\phi(P)$ and its normalized version $\phi^\circ(P)$ is given by

$$\phi(P) := \begin{pmatrix} \Pi_P A|H^P & \Pi_P B \\ -K_P|H^P & I \end{pmatrix}, \quad \phi^\circ(P) := \begin{pmatrix} \Pi_P A|H^P & \Pi_P B \Lambda_{P^{\mathrm{crit}}}^{-\frac{1}{2}} \\ -\Lambda_P^{\frac{1}{2}} K_P|H^P & \Lambda_P^{\frac{1}{2}} \Lambda_{P^{\mathrm{crit}}}^{-\frac{1}{2}} \end{pmatrix}.$$

The state space of the DLSs $\phi(P)$, $\phi^\circ(P)$, $\widetilde{\phi(P)}$ and $\widetilde{\phi^\circ(P)}$ is H^P, which is regarded as a subspace of H. The properties of $\widetilde{\phi^\circ(P)}$ and its semigroup generator $A^*|H^P$ are described in the following.

Lemma 218. *Let $J \in \mathcal{L}(Y)$ be a self-adjoint cost operator. Let $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ be an output stable and I/O stable DLS, whose input operator $B \in \mathcal{L}(U;H)$ is Hilbert–Schmidt and input space U is separable. Assume that the regular critical solution $P_0^{\mathrm{crit}} := (\mathcal{C}_\phi^{\mathrm{crit}})^* J \mathcal{C}_\phi^{\mathrm{crit}} \in ric_0(\phi, J)$ exists, and $\Lambda_{P_0^{\mathrm{crit}}} > 0$. Assume that the I/O map $\mathcal{D}_\phi$ is $(J, \Lambda_{P_0^{\mathrm{crit}}})$-inner.*

For arbitrary $P \in ric_0(\phi, J)$, the following holds:

(i) The normalized adjoint characteristic DLS $\widetilde{\phi^\circ(P)}$ is input stable and $\overline{\mathrm{range}\left(\mathcal{B}_{\widetilde{\phi^\circ(P)}}\right)} = H^P$. The observability map $\mathcal{C}_{\widetilde{\phi^\circ(P)}}$ is densely defined and closed in H^P. We have the commutant equation

$$\left(S^*|\tilde{K}_{\widetilde{\phi^\circ(P)}}\right) \cdot \mathcal{C}_{\widetilde{\phi^\circ(P)}} x_0 = \mathcal{C}_{\widetilde{\phi^\circ(P)}} \cdot (A^*|H^P)\, x_0, \quad S^* := \bar{\pi}_+ \tau^*, \tag{5.11}$$

for all $x_0 \in \mathrm{dom}\left(\mathcal{C}_{\widetilde{\phi^\circ(P)}}\right)$, where the possibly nonclosed subspace $\tilde{K}_{\widetilde{\phi^\circ(P)}} \subset \ell^2(\mathbf{Z}_+;U)$ is given in Definition 217.

(ii) Assume, in addition, that ϕ is input stable. Then the DLS $\widetilde{\phi^\circ(P)}$ is output stable and $\mathrm{dom}\left(\mathcal{C}_{\widetilde{\phi^\circ(P)}}\right) = H^P$. The range of $\mathcal{C}_{\widetilde{\phi^\circ(P)}}$ is closed, and equals $K_{\widetilde{\phi^\circ(P)}}$, given in Definition 217. The following similarity transform holds

$$\left(S^*|K_{\widetilde{\phi^\circ(P)}}\right) \cdot \mathcal{C}_{\widetilde{\phi^\circ(P)}} = \mathcal{C}_{\widetilde{\phi^\circ(P)}} \cdot (A^*|H^P), \tag{5.12}$$

where all the operators are bounded.

(iii) Assume, in addition, that ϕ is input stable and approximately controllable: $\overline{\mathrm{range}\,(\mathcal{B}_\phi)} = H$. Then $\ker\left(\mathcal{C}_{\widetilde{\phi^\circ(P)}}\right) = \{0\}$, and the observability map $\mathcal{C}_{\widetilde{\phi^\circ(P)}} : H^P \to K_{\widetilde{\phi^\circ(P)}}$ is a bounded bijection with a bounded inverse.

Proof. We start with claim (i). The DLS $\widetilde{\phi^\circ(P)}$ is input stable and approximately controllable, by claim (i) of Lemma 210, because the normalization by the boundedly invertible indicator operators $\Lambda_{P_0^{\mathrm{crit}}}$ and Λ_P plays no essential role. For any I/O stable DLS ϕ, $\mathrm{range}\,(\mathcal{B}_\phi) \subset \mathrm{dom}\,(\mathcal{C}_\phi)$, by Lemma 35. It follows that the observability map $\mathcal{C}_{\widetilde{\phi^\circ(P)}}$ is densely defined in H^P, because $\overline{\mathrm{range}\left(\widetilde{\phi^\circ(P)}\right)} = H^P$. The closedness of $\mathcal{C}_{\widetilde{\phi^\circ(P)}}$ has been proved in Lemma 31. Equation (5.11) is a basic property of the DLS, and claim (i) is now proved.

We proceed to prove claim (ii). Claim (ii) of Lemma 210 implies the output stability of $\widetilde{\phi^\circ(P)}$, if it is assumed that ϕ is input stable. By the Closed Graph theorem, we see that $\mathrm{dom}\left(\mathcal{C}_{\widetilde{\phi^\circ(P)}}\right) = H^P$. The range of $\mathcal{C}_{\widetilde{\phi^\circ(P)}}$ is closed, and equals $K_{\widetilde{\phi^\circ(P)}}$, by Lemma 213. Now the similarity transform (5.12) follows now from equation (5.11).

To prove the final claim (iii), we show that approximately controllability $\overline{\mathrm{range}\,(\mathcal{B}_\phi)} = H$ implies the injectivity of the observability map $\mathcal{C}_{\widetilde{\phi^\circ(P)}}$. We first show that if $\overline{\mathrm{range}\,(\mathcal{B}_\phi)} = \overline{\mathrm{range}\,(\mathcal{B}_{\phi_P})} = H$, then $\overline{\mathrm{range}\left(\mathcal{B}_{\phi(P)}\right)} = H^P = \mathrm{range}\,(\Pi_P)$. For contradiction, assume that $x_0 \in \mathrm{range}\,(\Pi_P) \ominus \overline{\mathrm{range}\left(\mathcal{B}_{\phi(P)}\right)}$. Because $\mathcal{B}_{\phi(P)} = \Pi_P \mathcal{B}_{\phi_P} = \Pi_P \mathcal{B}_\phi$ by claim (ii) of Proposition 208, we would have for such x_0 and all $\tilde{u} \in \ell^2(\mathbf{Z}_-; U)$:

$$0 = \langle x_0, \Pi_P \mathcal{B}_\phi \tilde{u} \rangle = \langle \Pi_P x_0, \mathcal{B}_\phi \tilde{u} \rangle = \langle x_0, \mathcal{B}_\phi \tilde{u} \rangle .$$

But then $x_0 = 0$ because $\mathrm{range}\,(\mathcal{B}_\phi)$ is dense in H. So $\overline{\mathrm{range}\left(\mathcal{B}_{\phi(P)}\right)} = H^P$, or equivalently, $\ker\left(\mathcal{C}_{\widetilde{\phi(P)}}\right) = \{0\}$, by Proposition 207. The proof is completed, by recalling the well known functional analytic fact that a bounded bijection between Hilbert spaces has a bounded inverse. $\square$

We conclude from claim (iii) of Lemma 218 that if the observability map $\mathcal{C}_{\widetilde{\phi^\circ(P)}}$ is injective, then the similarity transform (5.12) effectively combines the properties of $A^*|H^P$ to the properties of the restricted shift $S^*|K_{\widetilde{\phi^\circ(P)}}$. By using the theory of shift operator models as outlined in Section 5.5, the properties of $S^*|K_{\widetilde{\phi^\circ(P)}}$ and its characteristic function $\mathcal{D}_{\widetilde{\phi^\circ(P)}}(z) = \widetilde{\mathcal{N}}_P^\circ(z)$ are tied together in a very strong manner.

Theorem 219. *Let $J \in \mathcal{L}(Y)$ be a self-adjoint cost operator. Let $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ be an input stable, output stable and I/O stable DLS, such that $\overline{\mathrm{range}\,(\mathcal{B}_\phi)} = H$. Assume that the input operator $B \in \mathcal{L}(U; H)$ is Hilbert–Schmidt and the input space U is separable. Assume that the regular critical $P_0^{\mathrm{crit}} := (\mathcal{C}_\phi^{\mathrm{crit}})^* J \mathcal{C}_\phi^{\mathrm{crit}} \in ric_0(\phi, J)$ exists, and $\Lambda_{P_0^{\mathrm{crit}}} > 0$. Assume that the I/O map $\mathcal{D}_\phi$ is $(J, \Lambda_{P_0^{\mathrm{crit}}})$-inner.*

Then for arbitrary $P \in ric_0(\phi, J)$ the following holds:

(i) The restriction $A^*|H^P$ *is similar to a* C_{00}*-contraction, whose inner characteristic function is* $\widetilde{\mathcal{N}}_P^\circ(z) \in H^\infty(\mathcal{L}(U))$. *The similarity transform is given by*

$$\left(S^*|K_{\widetilde{\phi^\circ(P)}}\right) \cdot \mathcal{C}_{\widetilde{\phi^\circ(P)}} = \mathcal{C}_{\widetilde{\phi^\circ(P)}} \cdot (A^*|H^P) \tag{5.13}$$

where $\mathcal{C}_{\widetilde{\phi^\circ(P)}} : H^P \to K_{\widetilde{\phi^\circ(P)}}$ *is a bounded bijection, and the* S^**-invariant subspace* $K_{\widetilde{\phi^\circ(P)}}$ *is given in Definition 217.*

(ii) The spectra satisfy $\sigma(\Pi_P A|H^P) = \sigma(\widetilde{\mathcal{N}}_P^\circ) = \overline{\sigma(A^*|H^P)}$*, where the bar denotes complex conjugation, and the spectrum of the inner function is given in Definition 215.*

In particular, both $\sigma(\Pi_P A|H^P)$ *and* $\sigma(A^*|H^P)$ *are subsets of the closed unit disk* $\overline{\mathbf{D}}$.

(iii) The point spectra satisfy

$$\sigma_p(A^*|H^P) = \{z \in \mathbf{D} \mid \ker(\mathcal{N}_P(z)) \neq \{0\}\} \tag{5.14}$$

and

$$\sigma_p(\Pi_P A|H^P) = \{z \in \mathbf{D} \mid \ker\left(\widetilde{\mathcal{N}}_P(z)\right) \neq \{0\}\}. \tag{5.15}$$

In particular, if $A^*|H^P$ *is compact, then it is power stable (i.e.* $\rho(A^*|H^P) < 1$*).*

(iv) Both $A^*|H^P$ *and its adjoint* $\Pi_P A|H^P$ *are strongly stable.*

Proof. The first claim (i) follows from the similarity transform in equation (5.12), under the assumptions of claim (iii) of Lemma 218, together with the discussion in Section 5.5.

Let us look at claim (ii) of the spectrum. Let $\lambda \in \mathbf{C}$ be arbitrary. Then we have

$$\left(\lambda - S^*|K_{\widetilde{\phi^\circ(P)}}\right) = \mathcal{C}_{\widetilde{\phi^\circ(P)}} \left(\lambda - A^*|H^P\right) \left(\mathcal{C}_{\widetilde{\phi^\circ(P)}}\right)^{-1}. \tag{5.16}$$

where $\left(\mathcal{C}_{\widetilde{\phi^\circ(P)}}\right)^{-1} : K_{\widetilde{\phi^\circ(P)}} \to H^P$ is the bounded inverse of the bounded bijection. Immediately, $\sigma\left(S^*|K_{\widetilde{\phi^\circ(P)}}\right) = \sigma(A^*|H^P)$. By adjoining

$$\sigma\left(P_{\widetilde{\phi^\circ(P)}} S|K_{\widetilde{\phi^\circ(P)}}\right) = \sigma(\left(A^*|H^P\right)^*) = \sigma(\Pi_P A|H^P),$$

where $P_{\widetilde{\phi^\circ(P)}}$ is the orthogonal projection of $\ell^2(\mathbf{Z}_+; U)$ onto $K_{\widetilde{\phi^\circ(P)}}$. Lemma 216 implies now that $\overline{\sigma(A^*|H^P)} = \sigma(\Pi_P A|H^P) = \sigma(\widetilde{\mathcal{N}}_P^\circ)$. This proves claim (ii). Claim (iii) about the point spectra follows similarly from equation (5.16) and the latter claim of Lemma 216. We just remark that if $A^*|H^P$ is compact, then $\sigma(A^*|H^P) \subset D$ because the origin is the only accumulation point that a spectrum of a compact operator can have.

To verify claim (iv), note first that $\left(S^*|K_{\widetilde{\phi^\circ(P)}}\right)$ is a C_{00}-contraction, see Proposition 214. Then we have

$$|| \left(A^*|H^P\right)^j x_0|| \leq || \left(\mathcal{C}_{\widetilde{\phi^\circ(P)}}\right)^{-1} || \cdot || \left(S^*|K_{\widetilde{\phi^\circ(P)}}\right)^j \mathcal{C}_{\widetilde{\phi^\circ(P)}} x_0|| \to 0,$$

as $j \to \infty$. The adjoint part is similar, and the proof is complete. □

Corollary 220. *Make the same assumptions as in Theorem 219, but assume, in addition, that* $\dim U < \infty$. *Then for arbitrary* $P \in ric_0(\phi, J)$

$$\sigma(A^*|H^P) \cap \mathbf{D} = \sigma_p(A^*|H^P) = \overline{\sigma_p(\Pi_P A|H^P)}, \tag{5.17}$$

where the bar denotes complex conjugation. If $\{\lambda_j(A^*|H^P)\}_{j\geq 1}$ *is the enumeration of the eigenvalues* $\sigma_p(A^*|H^P)$ *in the nondecreasing order of absolute values, then the following Blaschke condition is satisfied*

$$\sum_{j\geq 1} (1 - |\lambda_j(A^*|H^P)|) < \infty. \tag{5.18}$$

In particular, both $A^*|H^P$ *and* $\Pi_P A|H^P$ *are injective.*

Proof. From claim (iii) of Theorem 219 we conclude that $\sigma_p(A^*|H^P) = \overline{\sigma_p(\Pi_P A|H^P)}$ because for each $z \in \mathbf{D}$, $\ker\left(\widetilde{\mathcal{N}}_P^\circ(z)\right) \neq \{0\}$ is equivalent to $\ker\left(\widetilde{\mathcal{N}}_P^\circ(z)^*\right) = \ker\left(\mathcal{N}_P^\circ(\bar{z})\right) \neq \{0\}$, by dimension counting in the finite dimensional space U. Because $\sigma_p(A^*|H^P) \subset \sigma(A^*|H^P) \cap \mathbf{D}$ by claim (iii) of Theorem 219, the equality (5.17) is proved once we establish $\sigma(A^*|H^P) \cap \mathbf{D} \subset \sigma_p(A^*|H^P)$.

Because $n := \dim U < \infty$, we can consider the complex function $\det \widetilde{\mathcal{N}}_P^\circ(z)$, for $z \in \mathbf{D}$. By recalling the definition of the determinant as a finite sum of products of the matrix elements, we see that $\det \widetilde{\mathcal{N}}_P^\circ(z)$ is an analytic function. For any $n \times n$ matrix M we have by

$$|\det M| = \prod_{j=1}^n |\lambda_j(M)| \leq \prod_{j=1}^n \sigma_j(M) \leq ||M||^n$$

where $\lambda_j(M)$ are the eigenvalues of H, $\sigma_j(M)$ are the singular values of M, and their inequality is by H. Weyl, see [24, p. 1092]. This makes is possible to conclude that $\det \widetilde{\mathcal{N}}_P^\circ(z) \in H^\infty(\mathbf{D}; \mathbf{C})$, and because $|\det(U)| = 1$ for unitary U, we conclude that $\det \widetilde{\mathcal{N}}_P^\circ(z)$ is an inner function. Of course, the same is true for $\det \mathcal{N}_P^\circ(z)$, too.

We proceed to show that

$$\sigma(\widetilde{\mathcal{N}}_P^\circ) \cap \mathbf{D} = \{z \in \mathbf{D} \mid \det \widetilde{\mathcal{N}}_P^\circ(z) = 0\}. \tag{5.19}$$

By the basic property of the determinant, the open set

$$E := \mathbf{D} \setminus \{z \in \mathbf{D} \mid \det \widetilde{\mathcal{N}}_P^\circ(z) = 0\}$$

is exactly the set of $z \in \mathbf{D}$ where $\widetilde{\mathcal{N}}_P^\circ(z)$ is invertible. To show (5.19), we must additionally show that the mapping $z \mapsto \widetilde{\mathcal{N}}_P^\circ(z)^{-1}$ is analytic in the set $E \subset \mathbf{D}$. This follows from the following outline of an argument: Assume $f(z)$ is a matrix-valued analytic function in $E \subset \mathbf{C}$, such that $\det f(z_0) \neq 0$ for some $z_0 \in E$. Then $f(z_0)$ has an inverse, and we can assume that $f(z_0) = I$ without any loss of generality. By developing $f(z)$ into its power series at z_0, we have $||I - f(z)|| \leq 1/2$ if $|z - z_0| < \delta$ for some $\delta > 0$. It then follows that the von Neumann series

$$f(z)^{-1} = (I - (I - f(z)))^{-1} = \sum_{j \geq 0} (I - f(z))^j$$

converges for all $|z - z_0| < \delta$. In fact, the convergence is uniform on the compact subsets of $\{z \mid |z - z_0| < \delta\}$. Because the limit of such a sequence of analytic functions is analytic, $f(z)^{-1}$ is analytic for $|z - z_0| < \delta$. Equation (5.19) follows from this consideration and Definition 215 of $\sigma(\widetilde{\mathcal{N}}_P^\circ)$.

From equality (5.19), we conclude that $\overline{\sigma(A^*|H^P)} \cap \mathbf{D} = \{z \in \mathbf{D} \mid \det \widetilde{\mathcal{N}}_P^\circ(z) = 0\}$, by claim (ii) of Theorem 219. Let $z \in \overline{\sigma(A^*|H^P)} \cap \mathbf{D}$ be arbitrary. Then $\det \widetilde{\mathcal{N}}_P^\circ(z) = 0$, and the matrix $\widetilde{\mathcal{N}}_P^\circ(z)$ fails to be injective. The same is true for $\mathcal{N}_P^\circ(\bar{z}) = \widetilde{\mathcal{N}}_P^\circ(z)^*$ because $\dim U < \infty$. Now claim (iv) of Theorem 219 shows that $\bar{z} \in \sigma_P(A^*|H^P)$, and the converse inclusion $\sigma(A^*|H^P) \cap \mathbf{D} \subset \sigma_p(A^*|H^P)$ follows.

We have now proved that

$$\sigma(A^*|H^P) \cap \mathbf{D} = \{z \in \mathbf{D} \mid \det \mathcal{N}_P^\circ(z) = 0\} = \sigma_P(A^*|H^P),$$

where $\det \mathcal{N}_P^\circ(z)$ is an inner function. By e.g. [78, Theorem 17.9], the zeroes of an inner function can be factorized away by a Blaschke product. Because the zeroes of the Blaschke product satisfy the Blaschke condition, equation (5.18) follows. The final claim about the injectivity of $A^*|H^P$ and $\Pi_P A|H^P$ follows because $\widetilde{\mathcal{N}}_P(0) = I$ is invertible. $\square$

Under particular conditions, we can make conclusions of the unrestricted semigroup generator A itself. The proof of the following corollary is based on Lemma 218 and Corollary 220.

Corollary 221. *Make the same assumptions as in Theorem 219. Assume that there exists a* $P \in ric_0(\phi, J)$ *such that* $H^P = H$. *Then* A *is similar to a* C_{00}*-contraction, and is strongly stable together with its adjoint* A^*. *If* A *is compact, then it is power stable* $\rho(A) < \infty$. *If* $\dim U < \infty$, *then the eigenvalues* $\lambda_j(A)_{j \geq 0} = \sigma(A) \cap \mathbf{D}$ *satisfy the Blaschke condition*

$$\sum_{j \geq 1} (1 - |\lambda_j|) < \infty.$$

In particular, if $P_0^{\text{crit}} > 0$ *and there exists a* $P \in ric_0(\phi, J)$ *such that* $P \leq 0$, *it follows that* $H^P = H$.

We complete this section by considering what happens if the approximate controllability condition in claim (iii) of Lemma 218 is not satisfied, but all the other conditions of the preceding claim (ii) are satisfied. Then all the operators are bounded in the commutant equation

$$\left(S^*|K_{\widetilde{\phi^\circ(P)}}\right) \cdot \mathcal{C}_{\widetilde{\phi^\circ(P)}} = \mathcal{C}_{\widetilde{\phi^\circ(P)}} \cdot (A^*|H^P),$$

and even range $\left(\mathcal{C}_{\widetilde{\phi^\circ(P)}}\right) = K_{\widetilde{\phi^\circ(P)}}$ is closed. However, $\ker\left(\mathcal{C}_{\widetilde{\phi^\circ(P)}}\right)$ can be nontrivial. If we make the decomposition of the state space $H^P = \ker\left(\mathcal{C}_{\widetilde{\phi^\circ(P)}}\right)^\perp \oplus \ker\left(\mathcal{C}_{\widetilde{\phi^\circ(P)}}\right)$ and use the fact the null space of the observability map is semigroup invariant, the commutant equation takes now the form

$$\begin{aligned}
&\left(S^*|K_{\widetilde{\phi^\circ(P)}}\right) \cdot \begin{bmatrix} \mathcal{C}_{\widetilde{\phi^\circ(P)}}|\ker\left(\mathcal{C}_{\widetilde{\phi^\circ(P)}}\right)^\perp & 0 \end{bmatrix} \\
&= \begin{bmatrix} \mathcal{C}_{\widetilde{\phi^\circ(P)}}|\ker\left(\mathcal{C}_{\widetilde{\phi^\circ(P)}}\right)^\perp & 0 \end{bmatrix} \cdot \\
&\cdot \begin{bmatrix} \Pi_1 A^*|\ker\left(\mathcal{C}_{\widetilde{\phi^\circ(P)}}\right)^\perp & 0 \\ (I - \Pi_1)A^*|\ker\left(\mathcal{C}_{\widetilde{\phi^\circ(P)}}\right)^\perp & (I - \Pi_1)A^*|\ker\left(\mathcal{C}_{\widetilde{\phi^\circ(P)}}\right) \end{bmatrix}
\end{aligned}$$

or

$$\begin{aligned}
&\left(S^*|K_{\widetilde{\phi^\circ(P)}}\right) \cdot \mathcal{C}_{\widetilde{\phi^\circ(P)}}|\ker\left(\mathcal{C}_{\widetilde{\phi^\circ(P)}}\right)^\perp \\
&= \mathcal{C}_{\widetilde{\phi^\circ(P)}}|\ker\left(\mathcal{C}_{\widetilde{\phi^\circ(P)}}\right)^\perp \cdot \left(\Pi_1 A^*|\ker\left(\mathcal{C}_{\widetilde{\phi^\circ(P)}}\right)^\perp\right),
\end{aligned}$$

where Π_1 is the orthogonal projection of H^P onto $\ker\left(\mathcal{C}_{\widetilde{\phi^\circ(P)}}\right)^\perp$, and $\mathcal{C}_{\widetilde{\phi^\circ(P)}}|\ker\left(\mathcal{C}_{\widetilde{\phi^\circ(P)}}\right)^\perp$ is now a bounded bijection. What has already been stated about $A^*|H^P$ under the approximate controllability of ϕ, can now be generally stated about the compression $\Pi_1 A^*|\ker\left(\mathcal{C}_{\widetilde{\phi^\circ(P)}}\right)^\perp$, at the cost of increased notational burden.

5.7 Generalization

In this section, we use extensively the tools developed in Section 4.8, and in particular Proposition 196 and Theorem 197. The general goal of this section is to translate the results of previous sections (valid for DLSs ϕ having a (J, S)-inner I/O map) to general output stable and I/O stable DLS ϕ without this restriction. For this to be possible, we must require that a regular critical solution $P_0^{\rm crit} := (\mathcal{C}_\phi^{\rm crit})^* J \mathcal{C}_\phi^{\rm crit} \in ric_0(\phi, J)$ exists, where

$$\mathcal{C}_\phi^{\rm crit} := (\mathcal{I} - \bar{\pi}_+ \mathcal{D}_\phi (\bar{\pi}_+ \mathcal{D}_\phi^* J \mathcal{D}_\phi \bar{\pi}_+)^{-1} \bar{\pi}_+ \mathcal{D}_\phi^* J) \mathcal{C}_\phi.$$

Furthermore, we make it a standing hypothesis that both $J \geq 0$ and $\overline{\operatorname{range}(\mathcal{B}_\phi)} = H$. This implies that $P_0^{\rm crit}$ is the unique critical solution in set $ric(\phi, J) \supset ric_0(\phi, J)$.

We first make the preliminary state feedback, associated to the solution $P_0^{\rm crit}$. This gives the closed loop system

$$\phi^{P_0^{\rm crit}} = \begin{pmatrix} A_{P_0^{\rm crit}} & B \\ C_{P_0^{\rm crit}} & D \end{pmatrix}.$$

This is the inner DLS of ϕ, centered at the regular critical solution $P_0^{\rm crit} \in ric_0(\phi, J)$. The DLS $\phi^{P_0^{\rm crit}}$ carries much of the interesting structure of the original DLS ϕ, see Proposition 196, Even the structure H^∞DAREs $ric(\phi, J)$ and $ric(\phi^{P_0^{\rm crit}}, J)$ is quite similar, see Theorem 197. However, the I/O map of $\phi^{P_0^{\rm crit}}$ is $(J, \Lambda_{P_0^{\rm crit}})$-inner, by Lemma 171. To the inner DLS $\phi^{P_0^{\rm crit}}$ and inner DARE $ric(\phi^{P_0^{\rm crit}}, J)$, we can apply the theory of Section 5.6. The results are then translated back to the original data, namely the DLS ϕ, cost operator J and H^∞DARE $ric(\phi, J)$. This trick gives us information about the invariant and co-invariant subspace structure of the closed loop semigroup generator $A_{P_0^{\rm crit}}$, rather than the open loop semigroup generator A.

The full solution sets of the DAREs $Ric(\phi, J)$ and $Ric(\phi^{P_0^{\rm crit}}, J)$ are equal by Lemma 157. Thus the spectral DLS $(\phi^{P_0^{\rm crit}})_P$ makes sense, for all $P \in Ric(\phi, J)$. It is given by

$$(\phi^{P_0^{\rm crit}})_P = \begin{pmatrix} A_{P_0^{\rm crit}} & B \\ K_{P_0^{\rm crit}} - K_P & I \end{pmatrix}. \tag{5.20}$$

by equation (4.2) of Proposition 151. With the aid of formula (5.20), we enlarge the definition of the characteristic DLS $\phi(P)$ (see Definition 209) to DLSs whose I/O map need not be $(J, \Lambda_{P_0^{\rm crit}})$-inner.

Definition 222. *Let $J \in \mathcal{L}(Y)$ be a self-adjoint cost operator. Let $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ be an output stable and I/O stable DLS, such that the input space U is separable.*

Assume that the regular critical solution $P_0^{\text{crit}} := (\mathcal{C}_\phi^{\text{crit}})^* J \mathcal{C}_\phi^{\text{crit}} \in ric_0(\phi, J)$ *exists.*

For $P \in ric(\phi, J)$*, the characteristic DLS* $\phi(P)$ *of* P *is the reduced DLS (in the sense of Proposition 208) of the spectral DLS* $(\phi^{P_0^{\text{crit}}})_P$*. It is given by*

$$\phi(P) = \begin{pmatrix} \Pi_P A_{P_0^{\text{crit}}} | H^P & \Pi_P B \\ (K_{P_0^{\text{crit}}} - K_P) | H^P & I \end{pmatrix}.$$

where $H^P := \ker \left(P_0^{\text{crit}} - P\right)^\perp$*,* Π_P *is the orthogonal projection of* H *onto* H^P*.*

If $\overline{\operatorname{range}(\mathcal{B}_\phi)} = H$ and ϕ itself has an $(J, \Lambda_{P_0^{\text{crit}}})$-inner I/O map, then $K_{P_0^{\text{crit}}} = 0$, $A_{P_0^{\text{crit}}} = A$ and immediately $\phi^{P_0^{\text{crit}}} = \phi$, see the proof of Lemma 202. In this case, the characteristic DLS $\phi(P)$ coincides with the one given in Definition 209, for DLSs with $(J, \Lambda_{P_0^{\text{crit}}})$-inner I/O map. We now consider restrictions of $A_{P_0^{\text{crit}}}$ to its certain invariant subspaces, for each $P \in ric_0(\phi, J)$.

Theorem 223. *Let* $J \geq 0$ *be a self-adjoint cost operator. Let* $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ *be an input stable, output stable and I/O stable DLS,* $\overline{\operatorname{range}(\mathcal{B}_\phi)} = H$*. Assume the input operator* $B \in \mathcal{L}(U; H)$ *is Hilbert–Schmidt and the input space* U *and output space* Y *are separable. Assume that the regular critical solution* $P_0^{\text{crit}} := (\mathcal{C}_\phi^{\text{crit}})^* J \mathcal{C}_\phi^{\text{crit}} \in ric_0(\phi, J)$ *exists.*

Let $P \in ric_0(\phi, J)$ *be arbitrary. By* $\phi(P)$ *denote its characteristic DLS, given by Definition 222. By* $\mathcal{N}_P$ *denote the* $(\Lambda_P, \Lambda_{P_0^{\text{crit}}})$*-inner factor of* $\mathcal{D}_{\phi_P}$*. Then the following holds:*

(i) The restriction of $\Pi_P A^*_{P_0^{\text{crit}}} | H^P$ *is similar to a* C_{00}*-contraction, whose characteristic function is* $\widetilde{\mathcal{N}}^\circ_P(z)$*. The similarity transform is given by*

$$\left(S^* | K_{\widetilde{\phi^\circ(P)}}\right) \mathcal{C}_{\widetilde{\phi^\circ(P)}} = \mathcal{C}_{\widetilde{\phi^\circ(P)}} \cdot \left(A^{\text{crit}*} | H^P\right) \tag{5.21}$$

where $\mathcal{C}_{\widetilde{\phi^\circ(P)}} : H^P \to K_{\widetilde{\phi^\circ(P)}}$ *is a bounded bijection, and the* S^**-invariant subspace* $K_{\widetilde{\phi^\circ(P)}}$ *is given in Definition 217.*

(ii) The spectra satisfy $\sigma(\Pi_P A_{P_0^{\text{crit}}} | H^P) = \sigma(\widetilde{\mathcal{N}}^\circ_P) = \overline{\sigma(A^*_{P_0^{\text{crit}}} | H^P)}$*, where the bar denotes complex conjugation, and the spectrum of the inner function is given in Definition 215. In particular, both* $\sigma(\Pi_P A_{P_0^{\text{crit}}} | H^P)$ *and* $\sigma(A^*_{P_0^{\text{crit}}} | H^P)$ *are subsets of the closed unit disk* $\overline{\mathbf{D}}$*.*

(iii) The point spectra satisfy

$$\sigma_p(A^*_{P_0^{\text{crit}}} | H^P) = \{z \in \mathbf{D} \mid \ker(\mathcal{N}_P(z)) \neq \{0\}\}$$

and

$$\sigma_p(\Pi_P A_{P_0^{\rm crit}}|H^P) = \{z \in \mathbf{D} \mid \ker\left(\widetilde{\mathcal{N}}_P(z)\right) \neq \{0\}\}.$$

In particular, if $A^*_{P_0^{\rm crit}}|H^P$ *is compact, then it is power stable (i.e.* $\rho(A^*_{P_0^{\rm crit}}|H^P) < 1$*).*

(iv) Both $A^*_{P_0^{\rm crit}}|H^P$ *and its adjoint* $\Pi_P A_{P_0^{\rm crit}}|H^P$ *are strongly stable.*

Proof. We reduce this theorem to Theorem 219 by making a preliminary feedback, associated to the solution $P_0^{\rm crit} := (\mathcal{C}_\phi^{\rm crit})^* J \mathcal{C}_\phi^{\rm crit}$. This amounts to replacing the original pair (ϕ, J) by the pair $(\phi^{P_0^{\rm crit}}, J)$. By claims (i) and (ii) of Proposition 196, the inner DLS $\phi^{P_0^{\rm crit}}$ is input stable, output stable, I/O stable and approximately controllable $\overline{\text{range}\left(\mathcal{B}_{\phi^{P_0^{\rm crit}}}\right)} = H$. Also, the I/O map of $\phi^{P_0^{\rm crit}}$ is $(J, \Lambda_{P_0^{\rm crit}})$-inner. The input and output spaces of ϕ and $\phi^{P_0^{\rm crit}}$ coincide, and are thus separable. The Hilbert–Schmidt input operator $B \in \mathcal{L}(U; H)$ is common to both ϕ and $\phi^{P_0^{\rm crit}}$.

By (ii) of Proposition 196, $P_0^{\rm crit}$ is the unique regular critical solution of its inner DARE $ric(\phi^{P_0^{\rm crit}}, J)$, too. Because $J \geq 0$, it follows that $P_0^{\rm crit} \geq 0$ and its indicator, equaling $\Lambda_{P_0^{\rm crit}}$, is positive. We conclude that the inner DLS $\phi^{P_0^{\rm crit}}$, together with the cost operator J, satisfies the conditions of Theorem 219.

An application of Theorem 219 to the DLS $\phi^{P_0^{\rm crit}}$, the cost operator J and the H^∞DARE $ric(\phi^{P_0^{\rm crit}}, J)$ proves all claims (i), (ii), (iii) and (iv) for arbitrary $P \in ric_0(\phi^{P_0^{\rm crit}}, J)$. But $ric_0(\phi^{P_0^{\rm crit}}, J) = ric_0(\phi, J)$, by claim (iii) of Theorem iii and the fact that the input operator B, common to both ϕ and $\phi^{P_0^{\rm crit}}$, is Hilbert–Schmidt. This completes the proof. □

Under the assumptions of Theorem 223, also the analogous results to Corollaries 220 and 221 hold, if the open loop semigroup generator A is replaced by the closed loop semigroup generator $A_{P_0^{\rm crit}}$. In particular, Corollary 221 gives a stabilization result for the critical closed loop semigroup. We remark that the Hilbert–Schmidt compactness assumption of the input operator B in Theorem 223 is required only to obtain the equality of the solution sets $ric_0(\phi^{P_0^{\rm crit}}, J) = ric_0(\phi, J)$. In particular, if $\dim U < \infty$, this assumption is trivially satisfied.

5.8 Notes and references

Description of $Ric(\phi, J)$ in terms of invariant subspaces of a Hamiltonian operator

The standard theory of a matrix DARE has been presented in great detail in the monograph [49] (Lancaster and Rodman, 1995). The presented algebraic Riccati equation theory provides us with a construction of a model operator in the following way. The solutions of the DARE are shown to be in one-to-one correspondence with the family of maximal, j-neutral invariant subspaces of a j-unitary Hamiltonian operator T. Here the Hermitian matrix $j := \left(\begin{smallmatrix} 0 & -iI \\ iI & 0 \end{smallmatrix}\right)$ induces an indefinite scalar product, and the requirement of j-neutrality is related to the requirement that the solution of the DARE should be self-adjoint. For a particular construction of T from the data of DARE, see [49, Chapter 12]. See also [47] (Ionescu and M. Weiss, 1993) which contains a lot of further references and an account of the history.

Analogous operator approaches have been developed for systems with an infinite-dimensional state space, see the continuous time example [18, Ex. 6.25] (Curtain and Zwart, 1995) for Hamiltonians that are Riesz spectral operators, and its application [23, Lemma 3.0.4] (Dumortier, 1998). The latter two references deal with the LQDARE

$$
\left\{
\begin{array}{l}
A^*PA - P + C^*JC = A^*PB \cdot \Lambda_P^{-1} \cdot B^*PA \\
\Lambda_P = D^*JD + B^*PB.
\end{array}
\right. \tag{5.22}
$$

Description of $Ric(\phi, J)$ in terms of unobservable, unstable semigroup invariant subspaces

The unobservable and unstable subspaces of the semigroup generator A can be used to classify the nonnegative solutions P for LQDARE of type (5.22). These subspaces coincide with (the essential part of) the null spaces $\ker(P)$. In this direction we refer to finite dimensional papers [104], [105], [107], [106] (Wimmer, 1994, 1995, 1996, 1996) and [50] (Langer, Ran and Temme, 1997). Solutions of a special homogeneous algebraic Riccati equation are parameterized by unobservable, semigroup invariant subspaces of the semigroup and by the inner factors of a rational inner I/O map in [36, Theorem 4.3] (Fuhrmann, 1995) in continuous time and [39, Theorem 4.1] (Fuhrmann and Hoffmann, 1997) in discrete time. The continuous time infinite dimensional results in [9] (Callier, Dumortier and Winkin, 1995), [23] (Dumortier, 1998) and [8] (Callier and Dumortier, 1998) are also closely related.

We now consider the discrete time matrix work [107] (Wimmer, 1996) as a representative of this genre. The LQDARE considered is a special case of (5.22), written in our notations as

$$A^*PA - P + C^*C = A^*PB\left(I + B^*PB\right)^{-1}B^*PA. \tag{5.23}$$

The linear system associated to this LQDARE is assumed to output stabilizable, which is a sufficient and necessary condition for the LQDARE to have a nonnegative solution. The state space $\mathbf{C}^n$ is written as a direct sum of two subspaces $\mathbf{C}^n := U_0 \oplus U_r$, where U_0 is a subspace of $V_=(A,C)$, which is the subspace spanned by unobservable generalized eigenvectors associated to the unimodular eigenvalues of A. In [107, Theorem 1.1], it is shown that any nonnegative solution P of LQDARE (5.23) can be decomposed according to this direct sum representation. The part corresponding to U_0, say $P_0 \geq 0$, is a solution of a Liapunov equation. As a source of inconvenience, P_0 is essentially forgotten. The other part, say $P_r \geq 0$, solves a reduced algebraic Riccati equation, and it is interesting enough to be further studied. The nonnegative solutions $P_r \in \mathcal{S}$ of the reduced algebraic Riccati equation can now be classified roughly as follows. Firstly, the family $\mathcal{N}$ of subspaces of $\mathbf{C}^n$

$$\begin{aligned}\mathcal{N} := \big\{ N \subset \mathbf{C}^n \quad | \quad & AN \subset N, \\ & V_{\leq}(A,C) \subset N \subset V(A,C), \quad N + R(A,B) + E_{<}(A) = \mathbf{C}^n \big\}\end{aligned}$$

is introduced where $V(A,C)$ is the unobservable subspace, $V_{\leq}(A,C)$ is the stable unobservable subspace, $R(A,B)$ is the controllable subspace (range of the controllability map) and $E_{<}(A)$ is the stable spectral subspace of the semigroup generator A. The set $\mathcal{N}$ is shown to be in one-to-one order-preserving correspondence with the solutions $P_r \in \mathcal{S}$ of the reduced LQDARE, see [107, Theorem 1.3]. The correspondence is given by the mapping $\gamma : \mathcal{S} \to \mathcal{N}$ is given by $\gamma(P_r) = \ker(P_r)$. We remark that for the class of LQDAREs (5.23), it is quite easy to show that the null spaces $\ker(P)$ are A-invariant. In fact, we use this type of technique in the proof of Lemma 205.

Comparison of existing approaches

In the previous subsection, it was indicated how to parameterize the solution of LQDARE by the A-invariant null spaces of P. In our approach, we seem to have turned everything upside down; we associate $\left(A^{\mathrm{crit}}\right)^*$-invariant subspaces $H^P := \ker\left(P_0^{\mathrm{crit}} - P\right)^{\perp}$ to the solutions $P \leq P_0^{\mathrm{crit}}$ of DARE. We now explain why this is done. For all nonnegative $P \in ric_0(\phi, J)$ we have the stable factorization

$$J^{\frac{1}{2}}\mathcal{D}_{\phi} = J^{\frac{1}{2}}\mathcal{D}_{\phi^P} \cdot \mathcal{D}_{\phi_P}, \tag{5.24}$$

assuming that the technical assumptions of Lemma 171 are satisfied. In principle, either of the factors $J^{\frac{1}{2}}\mathcal{D}_{\phi^P}$ and $\mathcal{D}_{\phi_P}$ could be used to associate chains of inner factors and shift-invariant subspaces to the ordered chains in $ric_0(\phi, J)$. In Chapter 4.2, we have chosen to use spectral DLS ϕ_P because it is an easier object to handle than the normalized inner DLS $J^{\frac{1}{2}}\phi^P$. The first reason for this is that the input space U and the output space Y of $J^{\frac{1}{2}}\phi^P$ are generally different, but for ϕ_P only the space U is used. We have the additional trouble that for noncoercive $J \geq 0$, we can conclude the output stability and I/O stability of only $J^{\frac{1}{2}}\phi^P$ in Lemma 171, and not of the inner DLS ϕ^P. Thus $Ric(\phi^P, J)$ is not necessarily a H^∞DARE, even if $P \in ric_0(\phi, J)$ is nonnegative. Finally, because we make the requirement that any solution $P \in Ric(\phi, J)$ must have a boundedly invertible indicator Λ_P, it has been possible to normalize the spectral DLSs ϕ_P so that they have boundedly invertible feed-through operators — in our case they equal the identity. Thus the inconvenient nonsquareness and possible "zero" of the transfer function $\mathcal{D}_\phi(z)$ at $z = 0$ is always included in the left factor $J^{\frac{1}{2}}\mathcal{D}_{\phi^P}$ in the factorization (5.24).

We now explain why the choice of ϕ_P over ϕ^P "turns everything upside down" in the sense discussed in the beginning of this subsection. Denote the $(\Lambda_P, \Lambda_{P_0^{\mathrm{crit}}})$-inner-outer factorization by $\mathcal{D}_{\phi_P} = \mathcal{N}_P\mathcal{X}$. Because the inner factor in $\mathcal{D}_{\phi_P}$ "decomposes" from the left in the factorization (5.24), and it should "decompose" from the right in order to be in harmony with the Beurling–Lax–Halmos Theorem, we have to adjoin and use $\widetilde{\mathcal{N}}_P$ instead of $\mathcal{N}_P$ in Theorem 187. This is the reason why $\left(A^{\mathrm{crit}}\right)^*$-invariant subspaces H^P must be used, instead of some A^{crit}-invariant subspaces.

In this section, we have discussed two approaches to parameterize the solution set of an algebraic Riccati equation. We remark that, under proper technical assumptions, the two approaches discussed in the previous subsections give a full classification of the solution sets of the DARE by invariant subspaces of a linear operator, at least in the case of a finite dimensional state space. Particularly interesting equivalence results on the factorization of rational inner function are [36, Theorem 4.3] (Fuhrmann, 1995) in continuous time and [39, Theorem 4.1] (Fuhrmann and Hoffmann, 1997) in discrete time. In [36, Theorem 4.3], an equivalence is shown between left and right inner factors of an inner I/O map, nonnegative solutions of a DARE and invariant subspaces of the semigroup generator. However, a special minimal realization and a homogeneous CARE is used. Our corresponding results work only in one direction: to each reasonable solution of the DARE, a restricted backward shift is associated, but not conversely. Much of this apparent weakness could be fixed (under stronger assumptions) if a practical form of a state space isomorphism theorem were available, and equivalence results on the special realizations could be transferred to more general realizations.

Grau, theurer Freund, ist alle Theorie
Und grün des Lebens goldner Baum.

J. W. Goethe

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