POINTWISE BEHAVIOUR OF SOBOLEV FUNCTIONS WHOSE GRADIENT IS INTEGRABLE TO POWER ONE

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Abstract. Our main objective is to study regularity of Sobolev functions on metric measure spaces equipped with a doubling measure. We show that every Sobolev function, whose gradient is integrable to power one, has Lebesgue points outside a set of capacity zero. We also show that every such function coincides with a Hölder continuous Sobolev function outside a set of small Hausdorff content. Our proofs are based on Sobolev space estimates for the maximal functions.

1. Introduction

If the gradient of a function is locally integrable to a power which is higher than the dimension of the underlying space, then by the Sobolev embedding theorem the function is locally Hölder continuous. It is a more delicate question to study the pointwise behaviour of a Sobolev function if the gradient is integrable to a power which is smaller than the dimension. Indeed, in this case a Sobolev function may be discontinuous everywhere. In this paper we focus on two basic questions: Lebesgue points and Hölder quasicontinuity. The advantage of our approach is that it applies in the limiting case when the gradient is integrable to the power one in the context of metric measure spaces.

The case when the gradient is locally integrable to a power which is smaller than the dimension and strictly bigger than one has been studied, for example, in [4], [7], [10], [14], [18]. The purpose of this work is to deal with the case when the gradient is integrable to the power one. In this case we have new challenges and new phenomena. The basic problem is that the Hardy-Littlewood maximal function is not bounded on L^1. We overcome this problem by restricting ourselves to the Sobolev space introduced by Hajłasz in [5] and using appropriate versions of Sobolev-Poincaré inequalities. For the exponents which are strictly greater than one this space coincides with the standard Sobolev space $W^{1,p}(\mathbb{R}^n)$, but for the exponent one we have a strictly smaller class of functions. We show that functions which belong to Hajłasz type Sobolev space with the exponent one have Lebesgue points outside a set of capacity zero and that they coincide with Hölder continuous Sobolev functions outside a set of small Hausdorff content.

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Our proofs are based on maximal function arguments. More precisely, when studying the existence of Lebesgue points, we use a maximal function which is related to discrete convolution approximations of the function. The discrete maximal function is comparable by two-sided pointwise estimates with the Hardy-Littlewood maximal function, but it is smoother than the standard maximal function. Thus it can be used as a test function for the capacity. Indeed, we show that the discrete maximal function is bounded in the Hajłasz type Sobolev space also with the exponent one. This is somewhat unexpected, since the discrete maximal function is not bounded on L^1. The corresponding result in the Euclidean space for exponents which are strictly bigger than one has been studied in [13], see also [14] for the metric case. As far as we know, the corresponding question for the Hardy-Littlewood maximal function in the standard Euclidean Sobolev space with the exponent one is open. Our result applies only for the Hajłasz type Sobolev space with the exponent one and it is not clear to us how to obtain the corresponding result for the standard Sobolev space.

In the last section we prove a Hölder type quasicontinuity result. Again the proof is based on maximal functions. Some parts of the proof are similar to the proof of the case when the exponent in strictly greater than one in [7], but since the proof is rather involved and the modifications are not completely obvious, we decided to present full details here.

2. Notation and preliminaries

2.1. **Basic assumptions.** Throughout the paper, X is a metric measure space equipped with a metric d and a Borel regular outer measure μ. We assume that μ is *doubling*, that is, there is a fixed constant $C_\mu > 0$, *a doubling constant of* μ, such that

$$\mu\big(B(x,2r)\big) \le C_\mu \mu\big(B(x,r)\big)$$

for each $x \in X$, and all $r > 0$. Here $B(x,r) = \{y \in X : \mathrm{d}(y,x) < r\}$ is the open ball of radius r centered at x. If $0 < t < \infty$ and $B = B(x,r)$ is a ball in X, then $tB = B(x,tr)$. We also assume that the measure of every open set is positive, and that the measure of each bounded set is finite. Recall that the doubling condition of μ implies that there exists a constant $C_0 > 0$ such that whenever $B_0 = B(x_0,r_0)$ and $B = B(x,r)$ are balls with $x \in B_0$ and $0 < r \le r_0$, then

$$\frac{\mu(B)}{\mu(B_0)} \ge C_0 \Big(\frac{r}{r_0}\Big)^s, \tag{2.1}$$

where $s = \log_2 C_\mu$, (see for example [8, Lemma 14.6]). In this paper, s denotes the smallest exponent for which (2.1) holds and it is called *the doubling dimension* of μ.

The integral average of a function $u \in L^1(A)$ over a μ-measurable set A with finite and positive measure is

$$u_A = \fint_A u\,d\mu = \frac{1}{\mu(A)} \int_A u\,d\mu.$$

We say that a function u belongs to the local space $L^p_{\mathrm{loc}}(X)$ if it belongs to $L^p(B)$ for each ball $B \subset X$.

We continue by recalling the definitions of two maximal functions. Let $0 \le \alpha < \infty$, $0 < \beta < \infty$, $R > 0$, and $u \in L^1_{\mathrm{loc}}(X)$. The (restricted) *fractional maximal function* of u is

$$\mathcal{M}_{\alpha,R}\, u(x) = \sup_{0<r\le R} r^{\alpha} \fint_{B(x,r)} |u|\,d\mu.$$

If $R = \infty$, then there is no restriction for the radii and we denote $\mathcal{M}_{\alpha,\infty}\, u = \mathcal{M}_{\alpha}\, u$. If $\alpha = 0$ and $R = \infty$, then we obtain the usual Hardy-Littlewood maximal function and write $\mathcal{M}_{0,\infty}\, u = \mathcal{M}\, u$.

The (restricted) *fractional sharp maximal function* of u is

$$u^{\#}_{\beta,R} u(x) = \sup_{0<r\le R} r^{-\beta} \fint_{B(x,r)} |u - u_{B(x,r)}|\,d\mu.$$

Again, if $R = \infty$, we denote $u^{\#}_{\beta,\infty} = u^{\#}_{\beta}$.

The Hausdorff t-content of a set E is the number $\mathcal{H}^t_\infty(E) = \inf \sum_i r_i^t$, where the infimum is taken over all countable coverings $\{B_i\}$ of E by balls B_i of radius r_i.

By χ_E, we denote the characteristic function of a set $E \subset X$. In general, C will denote a positive constant whose value is not necessarily the same at each occurrence. By writing $C = C(\tau, \lambda)$, we indicate that the constant depends only on τ and λ. If there is a positive constant C_1 such that the two-sided estimate $C_1^{-1} u \le v \le C_1 u$ holds, we write $u \approx v$, and say that u and v are comparable.

2.2. **Sobolev spaces $M^{1,p}(X)$ and Poincaré inequalities.** We recall the definition of the Sobolev space $M^{1,p}(X)$, $1 \le p < \infty$, in a metric measure space defined by Hajłasz in [5]. A measurable function $g \ge 0$ is *a generalized gradient* of a measurable function u in X, $g \in \mathrm{D}(u)$, if there is a set $E \subset X$ with $\mu(E) = 0$ such that

$$|u(x) - u(y)| \le \mathrm{d}(x,y)\big(g(x) + g(y)\big) \tag{2.2}$$

for all $x, y \in X \setminus E$. A function $u \in L^p(X)$ belongs to $M^{1,p}(X)$ if there exists a function $g \in L^p(X) \cap \mathrm{D}(u)$. The space $M^{1,p}(X)$, equipped with the norm

$$\|u\|_{M^{1,p}(X)} = \|u\|_{L^p(X)} + \inf \|g\|_{L^p(X)}, \tag{2.3}$$

where the infimum is taken over all functions $g \in L^p(X) \cap \mathrm{D}(u)$, is a Banach space [6, Theorem 8.3]. The space $M^{1,p}(X)$ can be defined for all $0 < p < \infty$, but (2.3) is a norm only when $p \geq 1$. By [6, Theorem 8.4], Lipschitz functions are dense in $M^{1,p}(X)$ if $p \geq 1$.

A pair $u \in L^1_{\mathrm{loc}}(X)$ and a measurable function $g \geq 0$ *satisfies a $(1,p)$-Poincaré inequality in X*, $p > 0$, if there exist constants $C > 0$ and $\tau \geq 1$ such that

$$\fint_B |u - u_B|\, d\mu \leq Cr\Big(\fint_{\tau B} g^p\, d\mu\Big)^{1/p} \tag{2.4}$$

for each ball $B = B(x,r)$ in X.

Each pair $u \in M^{1,p}(X)$, $g \in \mathrm{D}(u)$, satisfies a $(1,q)$-Poincaré inequality for all $q \geq 1$; this follows by integrating inequality (2.2) twice and using the Hölder inequality as in [5, Lemma 2]. For the case $p \leq 1$, see Section 3 and [6].

2.3. **Sobolev capacity.** The Sobolev capacity in $M^{1,p}(X)$ for $1 < p < \infty$ has been studied [15]. The definition of [15] extends for all $1 \leq p < \infty$ in a natural way. *The p-capacity* with $1 \leq p < \infty$ of the set $E \subset X$ is defined by setting

$$\mathcal{C}_p(E) = \inf\big\{\, \|u\|^p_{M^{1,p}(X)} : u \in \mathcal{A}(E)\big\}, \tag{2.5}$$

where

$$\mathcal{A}(E) = \big\{u \in M^{1,p}(X) : u \geq 1 \text{ in an open neighborhood of } E\big\}$$

is the set of *admissible functions (test functions)* for $\mathcal{C}_p(E)$. If $\mathcal{A}(E) = \emptyset$, then we set $\mathcal{C}_p(E) = \infty$. Carefully reading the proofs of [15], we note that most properties of p-capacity hold also for $p = 1$. In particular, the p-capacity is an outer measure [15, Theorem 3.2] and an outer capacity [15, Remark 3.3], that is,

$$\mathcal{C}_p(E) = \inf\big\{\, \mathcal{C}_p(U) : E \subset U,\, U \text{ open}\big\}.$$

It is easy to see that $\mu(E) \leq \mathcal{C}_p(E)$, in particular, sets of zero p-capacity are of zero measure, see [15, Lemma 4.1]. The doubling property of μ gives an upper bound for the p-capacity of a ball B of radius $0 < r \leq 1$,

$$\mathcal{C}_p(B) \leq Cr^{-p}\mu(B), \tag{2.6}$$

where the constant C depends only on the doubling constant of μ and p. For the proof, we observe that $1/r$-Lipschitz function with support in $2B$ is a suitable test function also for 1-capacity, see [15, Theorem 4.6].

A function $u : X \to \overline{\mathbb{R}}$ is *p-quasicontinuous* if for every $\varepsilon > 0$, there is a set $E \subset X$ such that $\mathcal{C}_p(E) < \varepsilon$ and the restriction of u to $X \setminus E$ is continuous.

By the definition, functions of $M^{1,1}(X)$ are defined only up to sets of measure zero. However, it was shown in [15, Corollary 3.7] that each Sobolev function $u \in M^{1,p}(X)$ has a p-quasicontinuous representative, that is, there is a p-quasicontinuous function $u^* \in M^{1,p}(X)$ such that $u = u^*$ μ-almost everywhere in X. The proof remains valid also for $p = 1$ because continuous functions are dense in $M^{1,1}(X)$ and $M^{1,1}(X)$ is complete.

Since the p-capacity is an outer capacity, and the norm of $M^{1,p}(X)$ does not see sets of zero measure, a result of Kilpeläinen [12] implies that the p-quasicontinuous representative u^* of $u \in M^{1,p}(X)$, $1 \le p < \infty$, is unique. Indeed, if two p-quasicontinuous functions f and g coincide μ-almost everywhere, then the p-capacity of the set where $f \neq g$ is zero.

3. Basic tools

In this section, we recall without proofs some results needed in the later sections. We begin by recalling a Sobolev-Poincaré inequality from [6]. Below, the Sobolev exponent $p^* = sp/(s-p)$ for $p < s$, where s is the doubling dimension of μ.

Theorem 3.1. [6, Theorem 8.7] *Let B be a ball of radius r, $\sigma > 1$, and let $s/(s+1) \le p < s$. If $u \in M^{1,p}(\sigma B)$ and $g \in \mathrm{D}(u)$, then $u \in L^{p^*}(B)$ and there is a constant $C = C(p, C_\mu, \sigma)$ such that*

$$\Big(\fint_B |u - u_B|^{p^*}\, d\mu\Big)^{1/p^*} \le Cr\Big(\fint_{\sigma B} g^p\, d\mu\Big)^{1/p}. \tag{3.1}$$

Note that if $u \in M^{1,q}(X)$, $g \in \mathrm{D}(u)$, and $s/(s+1) \le q < 1$, then $u \in M^{1,s/(s+1)}(\sigma B)$ for each ball. Since $(s/(s+1))^* = 1$, inequality (3.1) above and the Hölder inequality imply that

$$\fint_B |u - u_B|\, d\mu \le Cr\Big(\fint_{\sigma B} g^{s/(s+1)}\, d\mu\Big)^{(s+1)/s} \le Cr\Big(\fint_{\sigma B} g^q\, d\mu\Big)^{1/q} \tag{3.2}$$

for all balls $B = B(x,r)$ in X. In particular, the pair u, g satisfies a $(1,p)$-Poincaré inequality for all $p \ge s/(s+1)$ (cf. [6, Theorem 9.2]). We will frequently use (q^*, q)- and $(1,q)$-Poincaré inequalities for $u \in M^{1,1}(X)$ and $s/s+1 \le q < 1$ in the proofs of Lemmas 4.4 and 4.5.

The following well-known lemma, which is proved using the Lebesgue differentiation theorem ([2]) and a telescoping argument, provides a pointwise estimate for the oscillation of an integrable function via the fractional sharp maximal function, see [3], [7, Lemma 3.6], [16].

Lemma 3.2. *Let $u \in L^1_{loc}(X)$ and $0 < \beta < \infty$. Then there is a constant $C = C(\beta, C_\mu) > 0$ such that*

$$|u(x) - u(y)| \le C\, \mathrm{d}(x,y)^\beta \big(u^{\#}_{\beta, 4\,\mathrm{d}(x,y)}(x) + u^{\#}_{\beta, 4\,\mathrm{d}(x,y)}(y)\big) \tag{3.3}$$

for almost all $x, y \in X$.

Next result is a a weak type inequality for the fractional maximal function.

Lemma 3.3. [1, Lemma 3.2], [7, Lemma 2.6] *Let $Y \subset X$ be a bounded set with $\mu(Y) > 0$ and let $0 \le \alpha < s$. Then for all $u \in L^1(X)$ and for every $\lambda > 0$, we have*

$$\mathcal{H}^{s-\alpha}_\infty\big(\{x \in Y : \mathcal{M}_{\alpha, \operatorname{diam} Y}\, u(x) > \lambda\}\big) \le C\lambda^{-1} \int_X |u|\, d\mu,$$

where $C = 5^{s-\alpha}(2\operatorname{diam} Y)^s \mu(Y)^{-1}$.

We continue with two lemmas for generalized gradients. The proofs for $p = 1$ are similar to those for $p > 1$. The first lemma is a version of the Leibniz differentiation rule.

Lemma 3.4. [7, Lemma 5.20] *Let* $u \in M^{1,p}(X)$, $1 \le p < \infty$, *and let* φ *be a bounded L-Lipschitz function. Then* $u\varphi$ *belongs to* $M^{1,p}(X)$. *Moreover, if* $E \subset X$ *such that* $\varphi = 0$ *in* $X \setminus E$, *then*

$$g = \big(g_u \|\varphi\|_\infty + L|u|\big)\chi_E$$

belongs to $\mathrm{D}(u\varphi) \cap L^p(X)$ *whenever* $g_u \in \mathrm{D}(u) \cap L^p(X)$.

The next lemma shows that generalized gradients behave nicely with respect to the increasing convergence.

Lemma 3.5. [14, Lemma 2.6] *Let* (u_i) *be a sequence of measurable functions with a corresponding sequence of generalized gradients* (g_i), *and let* $u = \sup_i u_i$, $g = \sup_i g_i$. *If* u *is finite almost everywhere, then* $g \in \mathrm{D}(u)$.

We close this section by recalling a Whitney type covering lemma for an open set $U \neq X$ of a doubling metric measure space X, see [2, Theorem III.1.3], [17, Lemma 2.9]. In Lemma 3.6, the usual assumption that the set U is bounded is not necessary, see [2, the footnote of Theorem III.1.3]. Namely, in the original proof of Coifman and Weiss boundedness is used only in their version of the $5r$-covering theorem [2, Theorem III.1.2]. In our case, where X is a metric measure space with a doubling measure, the $5r$-covering theorem holds for all subsets of X; for every family $\mathcal{B}$ of balls of uniformly bounded radius, there is a countable subfamily $\{B_i\} \subset \mathcal{B}$ of pairwise disjoint balls such that $\cup_{B\in\mathcal{B}} B \subset \cup_i 5B_i$, see [19, Theorem 2.1], [11, Theorem 1.2].

Lemma 3.6. *Let* $U \subset X$ *be an open set,* $C_W \ge 1$, *let* $r(x) = \mathrm{d}(x, X \setminus U)/(2C_W)$. *There is* $M \in \mathbb{N}$ *and a sequence* (x_i) *of points in* U *with* $r_i = r(x_i)$, *such that*

1. *the balls* $B(x_i, r_i/5)$ *are pairwise disjoint,*
2. $U = \cup_i B(x_i, r_i)$,
3. $B(x_i, C_W r_i) \subset U$,
4. *if* $x \in B(x_i, C_W r_i)$, *then* $C_W r_i \le \mathrm{d}(x, X \setminus U) \le 3C_W r_i$,
5. *there is* $x_i^* \in X \setminus U$ *such that* $\mathrm{d}(x_i, x_i^*) < 3C_W r_i$, *and*
6. $\sum_{i=1}^\infty \chi_{B(x_i, C_W r_i)}(x) \le M$ *for all* $x \in U$.

We need the following technical lemma in the proof of Theorem 5.3. We omit the proof which consists of simple calculations using the properties of the Whitney covering and the doubling property of μ. All constants depend only on the constants of the Whitney covering, on the doubling constant C_μ, or the doubling dimension of μ.

Lemma 3.7. *Let $\mathcal{B} = \{B_i\}$ be a Whitney covering of an open set $U \subset X$, and let $x \in B_{i_0}$, $y \in B_{i_1}$, where $B_{i_0}, B_{i_1} \in \mathcal{B}$.*

(1) *If $x \in 2B_i$, then $2/3r_i \le r_{i_0} \le 3/2r_i$ and $1/5r_{i_0} \le \mathrm{d}(x, X \setminus U) \le 15r_{i_0}$. Moreover, if $\bar{x} \in X \setminus U$ is such that $\mathrm{d}(x, \bar{x}) \le 2\,\mathrm{d}(x, X \setminus U)$, then $2B_i \subset B(\bar{x}, 34r_i)$.*

(2) *Denote $\delta = 1/4 \max\{\mathrm{d}(x, X \setminus U), \mathrm{d}(y, X \setminus U)\}$. Assume that $y \in 2B_i$ and $\mathrm{d}(x, y) \le \delta$. If $\mathrm{d}(y, X \setminus U) \le \mathrm{d}(x, X \setminus U)$, then $y \in 6B_{i_0}$ and $1/2r_i \le r_{i_0} \le 3r_i$. If $\mathrm{d}(x, X \setminus U) \le \mathrm{d}(y, X \setminus U)$, then $x \in 6B_{i_1}$, $2/3r_i \le r_{i_1} \le 3/2r_i$, and $1/2r_{i_0} \le r_{i_1} \le 3r_{i_0}$. In both cases, $r_i \approx r_{i_0} \approx \mathrm{d}(x, X \setminus U)$.*

(3) *If x or y is in $2B_i$ and $\mathrm{d}(x, y) \le \delta$, then*

$$2B_i \subset B(x, 28r_i) \subset B(x, 140r_{i_0}) \subset B(x, 700\,\mathrm{d}(x, X \setminus U))$$

and $\mathrm{d}(x, y) \le 12r_i$. Moreover, $2B_i \subset B(x^*_{i_0}, 80r_i)$, where $x^*_{i_0}$ is the closest point of x_{i_0} in $X \setminus U$ from Lemma 3.6(5).

4. Lebesgue points

One of our main results shows that the 1-quasicontinuous representative of an $M^{1,1}$-function u is obtained as a limit of integral averages of u over small balls. This result implies that almost every point, in the 1-capacity sense, is a Lebesgue point of u.

Theorem 4.1. *Let $u \in M^{1,1}(X)$. Then there is a set $E \subset X$ with $\mathcal{C}_1(E) = 0$ such that*

$$\lim_{r \to 0} \fint_{B(x,r)} u\, d\mu = u^*(x) \tag{4.1}$$

for all $x \in X \setminus E$, where u^* is the 1-quasicontinuous representative of u.

The proof is presented in the end of this section.

Remark 4.2. By using Theorem 4.1 for functions $u - q_k$, where $u \in M^{1,1}(X)$ and (q_k) is an enumeration of rational numbers, and basic properties of capacity, we see that if $u \in M^{1,1}(X)$, then

$$\lim_{r \to 0} \fint_{B(x,r)} |u - u^*(x)|\, d\mu = 0$$

for all $x \in X \setminus F$ with $\mathcal{C}_1(F) = 0$. Hence 1-quasi every point is a Lebesgue point of u, see [10, Remark 2.8(2)].

Using Sobolev-Poincaré inequality (3.1) we obtain a stronger result which corresponds the second part of [14, Theorem 4.5].

Theorem 4.3. *Let $u \in M^{1,1}(X)$ with the 1-quasicontinuous representative u^*, and let $0 < q \le s/(s-1)$. Then there is a set $F \subset X$ with $\mathcal{C}_1(F) = 0$ such that*

$$\lim_{r\to 0} \fint_{B(x,r)} |u - u^*(x)|^q \, d\mu = 0 \tag{4.2}$$

for all $x \in X \setminus F$.

4.1. **The discrete maximal operator.** To prove Theorem 4.1, we will use the maximal operator for a discrete convolution approximation, defined in [14]. We begin with a covering of X and a corresponding partition of unity, see also [2], [11], [19]. For $r > 0$, let $\{B_i\}$ be a covering of X by balls $B_i = B(x_i, r)$ such that the enlarged balls $6B_i$ have bounded overlap, that is, there is a constant $N = N(C_\mu) \in \mathbb{N}$ such that $\sum_{i=1}^\infty \chi_{6B_i}(x) \le N$ for all $x \in X$. Let then (φ_i) be a partition of unity related to the covering $\{B_i\}$ such that $\sum_i \varphi_i(x) = 1$ for all $x \in X$, $0 \le \varphi_i \le 1$ in X, $\varphi_i \ge C$ in $3B_i$, $\operatorname{supp} \varphi_i \subset 6B_i$, and that each φ_i is L/r-Lipschitz. Here the constants $C > 0$ and $L > 0$ depend only on the doubling constant of μ.

We define u_r, *the discrete convolution* of u, by setting

$$u_r(x) = \sum_{i=1}^\infty \varphi_i(x) u_{3B_i}, \quad x \in X.$$

For the definition of the discrete maximal operator, we numerate the positive rationals and choose for each radius r_j a covering consisting of balls $B_i^j = B(x_i, r_j)$ and a corresponding partition of unity as above. Then we define *the discrete maximal function* of $u \in L^1_{\mathrm{loc}}(X)$ related to coverings $\{B_i^j\}$ by

$$\mathcal{M}^* u(x) = \sup_j |u|_{r_j}(x), \quad x \in X. \tag{4.3}$$

The maximal operator $\mathcal{M}^*$ depends on the covering. However, the estimates for $\mathcal{M}^*$ are independent on the covering. The operator $\mathcal{M}^*$ has the useful property of being comparable with the usual Hardy-Littlewood maximal operator,

$$C^{-1} \mathcal{M} u(x) \le \mathcal{M}^* u(x) \le C \mathcal{M} u(x) \tag{4.4}$$

for each $u \in L^1_{\mathrm{loc}}(X)$ and all $x \in X$, where $C = C(C_\mu)$, see [14, Lemma 3.1]. This means that for almost all practical purposes we may use the discrete maximal function instead of the Hardy-Littlewood maximal function. If $p > 1$, then (4.4) together with the boundedness of the maximal operator $\mathcal{M}$ (see [2]) imply that there is a constant $C = C(C_\mu, p)$ such that

$$\| \mathcal{M}^* u\|_{L^p(X)} \le C \| \mathcal{M} u\|_{L^p(X)} \le C \|u\|_{L^p(X)}$$

for all $u \in L^p(X)$.

In Lemmas 4.4 and 4.5 below, we will show that if $u \in M^{1,1}(X)$, then both u_r and $\mathcal{M}^* u$ are in $M^{1,1}(X)$, in particular, the discrete operator is bounded in $M^{1,1}(X)$. Since the Hardy-Littlewood maximal operator is not bounded in $L^1(X)$, the case $p=1$ requires a different proof than the case $p>1$. We will follow the proofs of [14, Lemma 3.3, Theorem 3.6] and use ideas from [9] and the Sobolev-Poincaré inequality to overcome the difficulties caused by the case $p=1$.

The constants C in Lemmas 4.4 and 4.5 depend only on the doubling constant of μ and on the constants of the Sobolev-Poincaré inequality (3.1).

Lemma 4.4. *Let $u \in M^{1,1}(X)$, $g \in \mathrm{D}(u) \cap L^1(X)$, $r>0$, and $s/(s+1) \le q < 1$. Then $u_r \in M^{1,1}(X)$ and there is a constant C such that $C(g + (\mathcal{M}\, g^q)^{1/q})$ belongs to $\mathrm{D}(u_r) \cap L^1(X)$. Moreover, there is a constant C such that*

$$\|u_r\|_{M^{1,1}(X)} \le C\|u\|_{M^{1,1}(X)}.$$

Proof. Let $u \in M^{1,1}(X)$ and $g \in \mathrm{D}(u) \cap L^1(X)$. Fix $1<\sigma<2$ for the Sobolev-Poincaré inequality (3.1). The proof consists of two steps. First we will find a generalized gradient for u_r, and then we will show that both u_r and the gradient are in $L^1(X)$.

Generalized gradient of u_r: For each $x \in X$ we have, since $\sum_i \varphi_i(x) = 1$, that

$$u_r(x) = \sum_{i=1}^{\infty} \varphi_i(x) u_{3B_i} = u(x) + \sum_{i=1}^{\infty} \varphi_i(x)(u_{3B_i} - u(x)),$$

where the sum is over finitely many terms only by the bounded overlap of the balls $6B_i$. Hence, by the definition of generalized gradient, the function

$$g + \sum_{i=1}^{\infty} g_i \in \mathrm{D}(u_r),$$

where g_i is a generalized gradient of $\varphi_i(u_{3B_i} - u)$. To find suitable gradients g_i, we first note that by Lemma 3.4 and the properties of the functions φ_i, the function $(Lr^{-1}|u - u_{3B_i}| + g)\chi_{6B_i}$ belongs to $\mathrm{D}(\varphi_i(u_{3B_i} - u))$.

To estimate $|u - u_{3B_i}|$, let $x \in 6B_i$. Then $3B_i \subset B(x, 9r) \subset 15B_i$, and

$$|u(x) - u_{3B_i}| \le |u(x) - u_{B(x,9r)}| + |u_{B(x,9r)} - u_{3B_i}|. \tag{4.5}$$

The first term in the right-hand side of (4.5) is estimated by a standard telescoping argument. We use the doubling property of μ and the $(1,q)$-Poincaré

inequality (3.2) for the pair u, g, and obtain
(4.6)

$$\begin{aligned}|u(x)-u_{B(x,9r)}| &\le \sum_{j=0}^{\infty} |u_{B(x,3^{2-j}r)}-u_{B(x,3^{1-j}r)}| \\ &\le C\sum_{j=0}^{\infty} \fint_{B(x,3^{2-j}r)} |u-u_{B(x,3^{2-j}r)}|\,d\mu \\ &\le Cr\sum_{j=0}^{\infty} 3^{2-j}\Big(\fint_{B(x,\sigma 3^{2-j}r)} g^q\,d\mu\Big)^{1/q} \le Cr\big(\mathcal{M}\,g^q(x)\big)^{1/q}\end{aligned}$$

whenever x is a Lebesgue point of u. For the second term, we use the doubling property of μ, and the $(1,q)$-Poincaré inequality, and obtain

$$\begin{aligned}|u_{B(x,9r)}-u_{3B_i}| &\le C\fint_{B(x,9r)} |u-u_{B(x,9r)}|\,d\mu \le Cr\Big(\fint_{\sigma B(x,9r)} g^q\,d\mu\Big)^{1/q} \\ &\le Cr\big(\mathcal{M}\,g^q(x)\big)^{1/q}.\end{aligned} \tag{4.7}$$

By (4.5) - (4.7) we have that

$$|u(x)-u_{3B_i}| \le Cr\big(\mathcal{M}\,g^q(x)\big)^{1/q}$$

for all Lebesgue points of u, and hence for almost all $x\in X$. Hence we can select $g_i=(C(\mathcal{M}\,g^q)^{1/q}+g)\chi_{6B_i}$. Using the bounded overlap of the balls $6B_i$, we conclude that the function

$$g_{u_r}=C\big(g+(\mathcal{M}\,g^q)^{1/q}\big)$$

belongs to $D(u_r)$.

Integrability of u_r and g_{u_r}: By the properties $\varphi_i=0$ in $X\setminus 6B_i$, $0\le\varphi_i\le 1$, the bounded overlap of the balls $6B_i$, and the doubling property of μ, we have that

$$\begin{aligned}\int_X |u_r|\,d\mu &\le \int_X \sum_{i=1}^{\infty}\varphi_i|u|_{3B_i}\,d\mu = \sum_{i=1}^{\infty}\int_X \varphi_i|u|_{3B_i}\,d\mu \\ &= \sum_{i=1}^{\infty}\int_{6B_i}|u|_{3B_i}\,d\mu \le C\sum_{i=1}^{\infty}\int_{3B_i}|u|\,d\mu \le C\int_X |u|\,d\mu,\end{aligned} \tag{4.8}$$

and hence $u_r\in L^1(X)$.

For the integrability of g_{u_r}, it suffices to show that $(\mathcal{M}\,g^q)^{1/q}\in L^1(X)$. Since $g\in L^1(X)$ we have $g^q\in L^{1/q}(X)$. The boundedness of the maximal operator for $1/q>1$ implies that $\mathcal{M}\,g^q\in L^{1/q}(X)$ and

$$\int_X (\mathcal{M}\,g^q)^{1/q}\,d\mu \le C\int_X (g^q)^{1/q}\,d\mu = C\int_X g\,d\mu, \tag{4.9}$$

and hence $g_{u_r} \in L^1(X)$ with $\|g_{u_r}\|_{L^1(X)} \le C\|g\|_{L^1(X)}$. By choosing the generalized gradient g of u so that $\|g\|_{L^1(X)} \le 2\|u\|_{M^{1,1}(X)}$ and combining estimates (4.8) and (4.9), the claim $\|u_r\|_{M^{1,1}(X)} \le C\|u\|_{M^{1,1}(X)}$ follows. $\square$

Lemma 4.5. *Let $u \in M^{1,1}(X)$, $g \in \mathrm{D}(u) \cap L^1(X)$ and $s/(s+1) < q < 1$. Then $\mathcal{M}^* u \in M^{1,1}(X)$ and there is a constant C such that $C(g + (\mathcal{M}\, g^q)^{1/q})$ belongs to $\mathrm{D}(\mathcal{M}\, u^*) \cap L^1(X)$. Moreover, there is a constant C such that*

$$\| \mathcal{M}^* u\|_{M^{1,1}(X)} \le C\|u\|_{M^{1,1}(X)}.$$

Proof. We begin with the integrability of $\mathcal{M}^* u$. By (4.4), it is enough to show that $\mathcal{M}\, u \in L^1(X)$. Fix $1 < \sigma < 2$ for the Sobolev-Poincaré inequality (3.1).

Step 1: We show that $\mathcal{M}\, u \in L^1(B)$ for all balls B:

Let B be a ball of radius r_B. Using the (q^*, q)-Poincaré inequality (3.1) for u and g in B we have that

(4.10)
$$\begin{aligned}\Big(\int_B |u|^{q^*}\,d\mu\Big)^{1/q^*} &\le \Big(\int_B |u-u_B|^{q^*}\,d\mu\Big)^{1/q^*} + \Big(\int_B |u|_B^{q^*}\,d\mu\Big)^{1/q^*}\\ &\le C r_B \Big(\fint_{\sigma B} g^q\,d\mu\Big)^{1/q} \mu(B)^{1/q^*} + \mu(B)^{1/q^*-1}\int_B |u|\,d\mu.\end{aligned}$$

Since $q < 1$ and $g \in L^1(X)$, g is in $L^q(\sigma B)$. As also $u \in L^1(X)$, (4.10) shows that $u \in L^{q^*}(B)$. By the assumption $q > s/(s+1)$, we have that $q^* > 1$, and hence $\mathcal{M}\, u \in L^{q^*}(B)$ with

$$\int_B (\mathcal{M}\, u)^{q^*}\,d\mu \le C\int_B |u|^{q^*}\,d\mu.$$

Using the Hölder inequality, (4.10) and the doubling property of μ, we obtain

$$\begin{aligned}\int_B \mathcal{M}\, u\,d\mu &\le \Big(\int_B (\mathcal{M}\, u)^{q^*}\,d\mu\Big)^{1/q^*}\mu(B)^{1-1/q^*}\\ &\le C r_B\Big(\fint_{\sigma B} g^q\,d\mu\Big)^{1/q}\mu(B) + C\int_B |u|\,d\mu \qquad (4.11)\\ &\le C r_B\int_{\sigma B} g\,d\mu + C\int_B |u|\,d\mu,\end{aligned}$$

and conclude that $\mathcal{M}\, u \in L^1(B)$.

Step 2: $\mathcal{M}\, u \in L^1(X)$:

We cover X by balls of radius $1/5$, and use the $5r$-covering theorem to obtain balls $B(y_i, 1)$ such that the balls $B(y_i, 1)$ cover X, the balls $B(y_i, 1/5)$ are pairwise disjoint, and that there is a constant $N = N(C_\mu)$ such that $\sum_i \chi_{B(y_i,2)}(x) \le N$ for all $x \in X$. Using (4.11), the bounded overlap of the balls $B(y_i, 2)$, and the assumption $1 < \sigma < 2$, we have that

$$
\begin{aligned}
\int_X \mathcal{M}\, u \, d\mu &\le \sum_{i=1}^{\infty} \int_{B(y_i,1)} \mathcal{M}\, u \, d\mu \\
&\le C \sum_{i=1}^{\infty} \Big(\int_{B(y_i,\sigma)} g \, d\mu + \int_{B(y_i,1)} |u| \, d\mu \Big) \qquad (4.12) \\
&\le C \Big(\int_X g \, d\mu + \int_X |u| \, d\mu \Big),
\end{aligned}
$$

which shows that $\mathcal{M}\, u$ is in $L^1(X)$.

Now $\mathcal{M}^* u$ is in $L^1(X)$ by (4.12) and (4.4), and hence it is finite almost everywhere in X. Since the function $C(g + (\mathcal{M}\, g^q)^{1/q}) \in L^1(X)$ belongs to $\mathrm{D}(u_{r_j}) \subset \mathrm{D}(|u|_{r_j})$ for all $j \in \mathbb{N}$, the claim that $\mathcal{M}^* u \in M^{1,1}(X)$ with generalized gradient $C(g + (\mathcal{M}\, g^q)^{1/q})$ follows from Lemmas 3.5 and 4.4.

As in the proof of Lemma 4.4, we obtain the desired norm estimate by choosing the generalized gradient g of u so that $\|g\|_{L^1(X)} \le 2\|u\|_{M^{1,1}(X)}$, and combining estimates (4.12), (4.4), and (4.9). $\square$

Remark 4.6. Note that $u_r \le C\, \mathcal{M}\, u$ almost everywhere in X by (4.4). Hence the integrability of u_r follows also from 4.12. Note also that by a similar arguments as in the proof of Lemma 4.4, we see that $u_r(x) \to u(x)$ as $r \to 0$ for almost every $x \in X$, and that $u_r \to u$ in $L^1(X)$. Namely,

$$
\begin{aligned}
|u_r(x) - u(x)| &\le \sum_{i=1}^{\infty} \varphi_i(x) |u(x) - u_{3B_i}| \qquad (4.13) \\
&\le \sum_{i'} |u(x) - u_{3B_i}| \le Cr\big(\mathcal{M}\, g^q(x)\big)^{1/q},
\end{aligned}
$$

where the last sum is taken over all indices i' for which $x \in 6B_i$. The right-hand side of (4.13) tends to zero as $r \to 0$ for almost every $x \in X$ because $(\mathcal{M}\, g^q)^{1/q} \in L^1(X)$.

In [10, Theorem 2.11] the first author and Harjulehto gave several equivalent conditions for a differentiation basis, which give the validity of (4.1) for $u \in M^{1,p}(X)$ p-quasi everywhere. We will use part of this result in the proof of Theorem 4.1 below.

Proof of Theorem 4.1. By [10, Theorem 2.11], the claim follows if we manage to show that there is a constant $C > 0$ such that

$$
\mathcal{C}_1\Big(\Big\{ x \in X : \limsup_{r \to 0} \fint_{B(x,r)} |u| \, d\mu > \lambda \Big\} \Big) \le C\lambda^{-1} \|u\|_{M^{1,1}(X)}
$$

for all $\lambda > 0$ and every $u \in M^{1,1}(X)$ (note that the proof of [10, Theorem 2.11 (ii) $\Rightarrow$ (i)] holds also for $p = 1$).

Since

$$\limsup_{r\to 0} \fint_{B(x,r)} |u|\,d\mu \le \mathcal{M}\,u(x)$$

for all $x \in X$, it suffices to show that

$$\mathcal{C}_1\big(\{x \in X : \mathcal{M}\,u(x) > \lambda\}\big) \le C\lambda^{-1}\|u\|_{M^{1,1}(X)}. \tag{4.14}$$

To show that the weak type estimate (4.14) holds, we proceed as in the proof of [14, Lemma 4.4]. Let $u \in M^{1,1}(X)$, $\lambda > 0$, and let $\mathcal{M}^*\,u$ be the discrete maximal function of u. By (4.4), $\{x \in X : \mathcal{M}\,u(x) > \lambda\} \subset E_\lambda$, where

$$E_\lambda = \{x \in X : C\,\mathcal{M}^*\,u(x) > \lambda\}.$$

The set E_λ is open by the lower semicontinuity of $\mathcal{M}^*\,u$, and the function $C\lambda^{-1}\,\mathcal{M}^*\,u$ is a test function for $\mathcal{C}_1(E_\lambda)$. Hence, using the definition of the p-capacity and Lemma 4.5, we have that

$$\begin{aligned}\mathcal{C}_1(E_\lambda) &\le \|C\lambda^{-1}\,\mathcal{M}^*\,u\|_{M^{1,1}(X)}\\ &= C\lambda^{-1}\|\,\mathcal{M}^*\,u\|_{M^{1,1}(X)} \le C\lambda^{-1}\|u\|_{M^{1,1}(X)},\end{aligned}$$

from which (4.14) follows by the monotonicity of the capacity. □

Note that instead of using the sufficient condition for the existence of the differentiation basis that differentiates $M^{1,1}(X)$, we could have followed the proof of [14, Theorem 4.5]. Namely, the main tools used in the proof; density of continuous functions in $M^{1,1}(X)$, (4.14), and the basic properties of the p-capacity, hold also in our case.

Proof of Theorem 4.3. Let $u \in M^{1,1}(X)$, $g \in \mathrm{D}(u) \cap L^1(X)$, and let u^* be the 1-quasicontinuous representative of u. By Theorem 4.1, there is a set $E \subset X$ with $\mathcal{C}_1(E) = 0$ such that $u_{B(x,r)} \to u^*(x)$ as $r \to 0$ whenever $x \in X \setminus E$. Since $g \ge 0$ is in $L^1(X)$, the proof of [14, Lemma 4.3], which uses the $5r$-covering theorem, the subadditivity of capacity, the doubling condition of μ, estimate (2.6), and the absolutely continuity of the integral, implies that the set

$$D = \Big\{x \in X : \limsup_{r\to 0} r \fint_{B(x,r)} g\,d\mu > 0\Big\}$$

has zero 1-capacity.

Let $B = B(x,r)$ be a ball in X, and let $0 < q \le 1^*$, where $1^* = s/(s-1)$. By the Hölder inequality and the Sobolev-Poincaré inequality (3.1), we have that

$$\Big(\fint_B |u - u_B|^q\,d\mu\Big)^{1/q} \le \Big(\fint_B |u - u_B|^{1^*}\,d\mu\Big)^{1/1^*} \le Cr \fint_{\sigma B} g\,d\mu.$$

Hence

$$\lim_{r\to 0} \fint_{B(x,r)} |u - u_B|^q\,d\mu = 0$$

whenever $x \in X \setminus D$.

The subadditivity of the 1-capacity implies that $\mathcal{C}_1(E\cup D)=0$. For $x\in X\setminus(E\cup D)$ we have that

$$
\begin{aligned}
&\Big(\fint_{B(x,r)}|u-u^*(x)|^q\,d\mu\Big)^{1/q}\\
&\qquad\le\Big(\fint_{B(x,r)}|u-u_{B(x,r)}|^q\,d\mu\Big)^{1/q}+|u_{B(x,r)}-u^*(x)|.
\end{aligned}
\tag{4.15}
$$

By the selection of the sets E and D, the limit as $r\to 0$ of the right-hand side of (4.15) is zero, and hence the claim follows for $F=E\cup D$. $\square$

5. Hölder quasicontinuity

In this section, we show that Hölder continuous functions are dense in $M^{1,1}(X)$ both in norm and Lusin sense, see Theorem 5.3. This is a generalization of the main result of [7] to the case $p=1$.

The first theorem provides a characterization of $M^{1,1}(X)$ using a Poincaré inequality or fractional sharp maximal function, see [7, Theorem 3.4] for the case $p>1$.

Theorem 5.1. *Let $u\in L^1(X)$. The following three conditions are equivalent:*
(1) *$u\in M^{1,1}(X)$,*
(2) *there is a function $g\in L^1(X)$, $g\ge 0$, and $0<q<1$ such that the pair u,g satisfies a $(1,q)$-Poincaré inequality,*
(3) *$u_1^{\#}\in L^1(X)$.*

Proof. If $u\in M^{1,1}(X)$, then a $(1,q)$-Poincaré inequality for $s/(s+1)\le q<1$ follows from the Sobolev-Poincaré inequality (3.2).

Suppose that (2) holds for u and g. Then for $B=B(x,r)$,

$$r^{-1}\fint_B|u-u_B|\,d\mu\le C\Big(\fint_{\sigma B}g^q\,d\mu\Big)^{1/q},$$

and hence $u_1^{\#}(x)\le C(\mathcal{M}\,g^q(x))^{1/q}$. Since $g\in L^1(X)$, the function g^q belongs to $L^{1/q}(X)$. The assumption $0<q<1$ together with the boundedness of the Hardy-Littlewood maximal operator implies that $\mathcal{M}\,g^q\in L^{1/q}(X)$. Thus $(\mathcal{M}\,g^q)^{1/q}$, and hence also $u_1^{\#}$ is in $L^1(X)$.

Suppose then that $u_1^{\#}\in L^1(X)$. Then $u\in M^{1,1}(X)$ by (3.3) and the definition of $M^{1,1}(X)$. Hence (1) follows. $\square$

The next result is a useful tool in the proof of Theorem 5.3, see [7, Corollary 3.10] for the case $p>1$.

Corollary 5.2. *Let $u\in M^{1,1}(X)$, $g\in L^1(X)\cap \mathrm{D}(u)$, $0\le\alpha<1$, $R>0$, $\sigma>1$, and $s/(s+1)\le q<1$. Then*

$$u^{\#}_{1-\alpha,R}(x)\le C\big(\mathcal{M}_{\alpha q,\sigma R}\,g^q(x)\big)^{1/q}$$

for all $x \in X$*.*

Proof. Let $x \in X$, $0 < r < R$, and $B = B(x,r)$. By the Sobolev-Poincaré inequality (3.2) we have that

$$\begin{aligned} r^{\alpha-1} \fint_B |u - u_B|\, d\mu &\le C r^{\alpha-1+1} \Big(\fint_{\sigma B} g^q \, d\mu \Big)^{1/q} \\ &= C \Big(r^{\alpha q} \fint_{\sigma B} g^q \, d\mu \Big)^{1/q} \le C \big(\mathcal{M}_{\alpha q, \sigma R}\, g^q(x) \big)^{1/q}. \end{aligned}$$

The claim follows by taking supremum over r. □

As in [7, Section 5], we define $\tilde{u}$ by setting

$$\tilde{u}(x) = \limsup_{r \to 0} \fint_{B(x,r)} u \, d\mu \tag{5.1}$$

for a function u of $M^{1,1}(X)$. By Theorem 4.1, the limit of the right-hand side of (5.1) exists and equals u^*, the quasicontinuous representative of u, except on a set of 1-capacity zero. In this section, we use the representative $\tilde{u}$ for u, and denote it by u. Then, by the proof of (3.3) ([7, Lemma 3.6]), the inequality

$$|u(x) - u(y)| \le C \, \mathrm{d}(x,y)^\beta \big(u^\#_{\beta, 4\,\mathrm{d}(x,y)}(x) + u^\#_{\beta, 4\,\mathrm{d}(x,y)}(y) \big) \tag{5.2}$$

holds for every $x, y \in X$ and for all $0 < \beta \le 1$, see [7] for the discussion on infinite values of u. Hence u is Hölder continuous with exponent β if $\|u^\#_\beta\|_\infty < \infty$.

Theorem 5.3. *Let* $u \in M^{1,1}(X)$ *be defined pointwise by* (5.1)*, and let* $0 < \beta \le 1$*. Then for each* $\varepsilon > 0$*, there is a function* v *and an open set* O *such that*

(1) $u = v$ *in* $X \setminus O$*,*
(2) $v \in M^{1,1}(X)$ *and it is Hölder continuous with exponent* β *on every bounded set of* X*,*
(3) $\|u - v\|_{M^{1,1}(X)} < \varepsilon$*,*
(4) $\mathcal{H}^{s-(1-\beta)}_\infty(O) < \varepsilon$*.*

Since the Hardy-Littlewood maximal function is not bounded in L^1, the case $p = 1$ requires a different proof than the case $p > 1$. When showing that the approximation of u has a generalized gradient, we use the Sobolev-Poincaré inequality of Theorem 3.1. In the proof we first assume that u vanishes outside a ball. The general case follows by using a localization argument as in [7, Theorem 5.3].

Proof. Let $u \in M^{1,1}(X)$ and $g \in L^1(X) \cap \mathrm{D}(u)$. Let also $s/(s+1) \le q < 1$ and $1 < \sigma < 2$, and recall from Theorem 3.1 that a $(1,q)$-Poincaré inequality holds for the pair u, g. This will be an important tool for us.

Step 1: Suppose that the support of u is in $B(x_0, 1)$ for some $x_0 \in X$.

Let $\lambda > 0$, and denote

$$E_\lambda = \{x \in X : u^\#_\beta(x) > \lambda\}.$$

The set E_λ is open, and by (5.2), u is Hölder continuous with exponent β in $X \setminus E_\lambda$.

We will correct the values of u in the bad set E_λ using discrete convolution. For that, let $\mathcal{B} = \{B_i\}$, $B_i = B(x_i, r_i)$, be the covering of E_λ by Whitney balls from Lemma 3.6 with $C_W = 5$. Let (φ_i) be a partition of unity corresponding to the collection $\mathcal{B}$ such that $\operatorname{supp} \varphi_i \subset 2B_i$, $0 \le \varphi_i \le 1$, each φ_i is K/r_i-Lipschitz, and that $\sum_{i=1}^\infty \varphi_i(x) = \chi_{E_\lambda}(x)$, see for example [17, Lemma 2.16]. For each x_i, let x_i^* be the "closest" point in $X \setminus E_\lambda$ given by Lemma 3.6(5).

Before proving (1)-(4), we study the properties of the set E_λ.

Claim 1: There is $\lambda_0 > 0$ such that $E_\lambda \subset B(x_0, 2)$ for $\lambda > \lambda_0$.

Proof. We will show that there is $\lambda_0 > 0$ such that

$$r^{-\beta} \fint_{B(x,r)} |u - u_{B(x,r)}| \, d\mu < \lambda_0 \tag{5.3}$$

for all $x \in X$ and $r > 1$. Namely, if $B = B(x, r)$ is a ball in X with $r > 1$, and

$$r^{-\beta} \fint_B |u - u_B| \, d\mu = a > 0,$$

then

$$\int_B |u - u_B| \, d\mu > a\mu(B)$$

and, by the assumption that $\operatorname{supp} u \subset B(x_0, 1)$, $B \cap B(x_0, 1) \ne \emptyset$.

By the doubling property of μ and the assumption $r > 1$, we have that $\mu(B(x_0, 1)) \le C_\mu \mu(B)$ and that

$$r^{-\beta} \fint_B |u - u_B| \, d\mu \le 2C_\mu \mu(B(x_0, 1))^{-1} \int_B |u| \, d\mu,$$

and hence we may choose $\lambda_0 = 2C_\mu \mu(B(x_0, 1))^{-1} \|u\|_{L^1(X)}$.

Claim 1 follows now from (5.3) and the assumption $\operatorname{supp} u \subset B(x_0, 1)$. $\square$

Claim 2: $\mu(E_\lambda) \to 0$ as $\lambda \to \infty$.

Proof. Since $0 < q < 1$, the Hölder inequality implies that for all $R > 0$ and $\alpha \ge 0$,

$$\Big(r^{\alpha q} \fint_{B(x,r)} g^q \, d\mu\Big)^{1/q} \le r^\alpha \fint_{B(x,r)} g \, d\mu \le \mathcal{M}_{\alpha,R}\, g(x)$$

whenever $x \in X$ and $r \le R$. Hence $(\mathcal{M}_{\alpha q,R}\, g^q(x))^{1/q} \le \mathcal{M}_{\alpha,R}\, g(x)$. Moreover, if $\alpha \le 1$ and $R \ge 1$, then

$$\mathcal{M}_{\alpha,R}\, g(x) \le R\, \mathcal{M}\, g(x).$$

By Claim 1, Corollary 5.2, and the estimate above, we have for $x \in E_\lambda$, $\lambda > \lambda_0$, that

$$u^{\#}_\beta(x) = u^{\#}_{\beta,1}(x) \le C\big(\mathcal{M}_{(1-\beta)q,\sigma}\, g^q(x)\big)^{1/q} \le C\,\mathcal{M}_{1-\beta,\sigma}\, g(x) \le C\,\mathcal{M}\, g(x). \tag{5.4}$$

By (5.4) and the weak type estimate for the maximal operator $\mathcal{M}$, we have that

$$\mu(E_\lambda) \le \mu\Big(\big\{x \in X : \mathcal{M}\, g(x) > C\lambda\big\}\Big) \le C\lambda^{-1}\int_X g\, d\mu. \tag{5.5}$$

Claim 2 follows because $g \in L^1(X)$ and the right-hand side of (5.5) tends to zero as $\lambda \to \infty$. □

Now we define the function $v = v_\lambda$ as a Whitney type extension of u to the set E_λ by setting

$$v(x) = \begin{cases} u(x), & \text{if } x \in X \setminus E_\lambda, \\ \sum_{i=1}^\infty \varphi_i(x) u_{2B_i}, & \text{if } x \in E_\lambda. \end{cases}$$

We will select the open set O to be E_λ for sufficiently large $\lambda > \lambda_0$. Hence the claim (1) of Theorem 5.3 follows from the definition of v. Since $\operatorname{supp} u \subset B(x_0, 1)$ and $E_\lambda \subset B(x_0, 2)$ for $\lambda > \lambda_0$, the support of v is in $B(x_0, 2)$.

Proof of (2) *- the Hölder continuity of* v. We begin by proving an estimate for $|v(x) - v(\bar{x})|$, where $x \in E_\lambda$ and $\bar{x} \in X \setminus E_\lambda$ is such that $\mathrm{d}(x, \bar{x}) \le 2\,\mathrm{d}(x, X \setminus E_\lambda)$.

Denote $\mathcal{B}_x = \{B_i \in \mathcal{B} : x \in 2B_i\}$. By the bounded overlap of the balls $2B_i$, there is a bounded number of balls in $\mathcal{B}_x$. By the definition of v and the properties of the functions φ_i, we have

$$|v(x) - v(\bar{x})| = \Big|\sum_{i=1}^\infty \varphi_i(x)(u(\bar{x}) - u_{2B_i})\Big| \le \sum_{\mathcal{B}_x} |u(\bar{x}) - u_{2B_i}|. \tag{5.6}$$

Now

$$|u(\bar{x}) - u_{2B_i}| \le |u(\bar{x}) - u_{B(\bar{x},34r_i)}| + |u_{B(\bar{x},34r_i)} - u_{2B_i}|, \tag{5.7}$$

where, by Lemma 3.7(1), $2B_i \subset B(\bar{x}, 34r_i)$. For the first term on the right-hand size we use a telescoping argument as in (4.6) and the doubling property of μ to obtain

$$|u(\bar{x}) - u_{B(\bar{x},34r_i)}| \le C r_i^\beta u^{\#}_{\beta,34r_i}(\bar{x}). \tag{5.8}$$

Since $B(\bar{x}, 34r_i) \subset 66B_i$ and μ is doubling, we have that

$$|u_{B(\bar{x},34r_i)} - u_{2B_i}| \le C \fint_{B(\bar{x},34r_i)} |u - u_{B(\bar{x},34r_i)}|\, d\mu \le C r_i^\beta u^{\#}_{\beta,34r_i}(\bar{x}). \tag{5.9}$$

There is a bounded number of balls in $\mathcal{B}_x$ and $r_i \approx \mathrm{d}(x, X \setminus E_\lambda)$ by Lemma 3.7(1). Hence (5.6)-(5.9) show that

$$|v(x) - v(\bar{x})| \le C\,\mathrm{d}(x, X \setminus E_\lambda)^\beta u_\beta^\#(\bar{x}) \le C\,\mathrm{d}(x, X \setminus E_\lambda)^\beta \lambda, \tag{5.10}$$

where the last inequality follows because $\bar{x} \in X \setminus E_\lambda$.

Let then $x, y \in X$. Our aim is to show that

$$|v(x) - v(y)| \le C\lambda\,\mathrm{d}(x,y)^\beta. \tag{5.11}$$

We will consider four cases that depend on the location of x and y.

(i) If $x, y \in X \setminus E_\lambda$, then (5.11) follows from (5.2) and the definition of E_λ.

(ii) Let $x, y \in E_\lambda$ and $\mathrm{d}(x,y) \ge \delta$, where

$$\delta = \tfrac{1}{4} \max\big\{\mathrm{d}(x, X \setminus E_\lambda), \mathrm{d}(y, X \setminus E_\lambda)\big\}.$$

Let $\bar{x}, \bar{y} \in X \setminus E_\lambda$ be like $\bar{x}$ in the beginning of the proof of (2). Then, by (5.10),

$$\begin{aligned} |v(x) - v(y)| &\le |v(x) - v(\bar{x})| + |v(\bar{x}) - v(\bar{y})| + |v(y) - v(\bar{y})| \\ &\le C\lambda\,\mathrm{d}(x, X \setminus E_\lambda)^\beta + |v(\bar{x}) - v(\bar{y})| + C\lambda\,\mathrm{d}(y, X \setminus E_\lambda)^\beta, \end{aligned}$$

where $|v(\bar{x}) - v(\bar{y})| \le C\lambda\,\mathrm{d}(\bar{x},\bar{y})^\beta$ by (5.2) and the fact that $\bar{x}, \bar{y} \in X \setminus E_\lambda$. Since $\mathrm{d}(x,y) \ge \delta$ and

$$\begin{aligned} \mathrm{d}(\bar{x},\bar{y}) &\le \mathrm{d}(\bar{x},x) + \mathrm{d}(x,y) + \mathrm{d}(\bar{y},y) \\ &\le 2\,\mathrm{d}(x, X \setminus E_\lambda) + \mathrm{d}(x,y) + 2\,\mathrm{d}(y, X \setminus E_\lambda) \le 17\,\mathrm{d}(x,y), \end{aligned}$$

we have that $|v(x) - v(y)| \le C\lambda\,\mathrm{d}(x,y)^\beta$.

(iii) Let then $x, y \in E_\lambda$ and $\mathrm{d}(x,y) \le \delta$. Similarly as $\mathcal{B}_x$ above, we denote $\mathcal{B}_y = \{B_i \in \mathcal{B} : y \in 2B_i\}$. Let $B_{i_0} = B(x_{i_0}, r_{i_0})$ be a Whitney ball such that $x \in B_{i_0}$, and let $x_{i_0}^*$ be the closest point of x_{i_0} in $X \setminus E_\lambda$ given by Lemma 3.6(5). By the properties of the functions φ_i, we have that

$$\begin{aligned} |v(x) - v(y)| &= \Big|\sum_{i=1}^{\infty} \big(\varphi_i(x) - \varphi_i(y)\big)\big(u(x_{i_0}^*) - u_{2B_i}\big)\Big| \\ &\le C\,\mathrm{d}(x,y) \sum_{\mathcal{B}_x \cup \mathcal{B}_y} r_i^{-1} |u(x_{i_0}^*) - u_{2B_i}|. \end{aligned} \tag{5.12}$$

We continue as in (5.7)-(5.9); by Lemma 3.7 we have that $r_i \approx r_{i_0}$ and that $2B_i \subset B(x_{i_0}^*, 80 r_i)$, and obtain

$$|u(x_{i_0}^*) - u_{2B_i}| \le C r_i^\beta u_\beta^\#(x_{i_0}^*) \le C r_i^\beta \lambda. \tag{5.13}$$

Now (5.12) and (5.13) show that

$$|v(x) - v(y)| \le C\lambda\,\mathrm{d}(x,y)^\beta \sum_{\mathcal{B}_x \cup \mathcal{B}_y} \frac{\mathrm{d}(x,y)^{1-\beta}}{r_i^{1-\beta}}.$$

The desired estimate follows because $\mathrm{d}(x,y) \le 12 r_i$ by Lemma 3.7(3).

(iv) Finally, let $x \in E_\lambda$ with $\bar{x}$ as above and $y \in X \setminus E_\lambda$. Then $v(y) = u(y)$, and using (5.10) and (5.2) we have that

$$\begin{aligned}|v(x) - v(y)| &= |v(x) - u(y)| \le |v(x) - v(\bar{x})| + |u(\bar{x}) - u(y)| \\ &\le C\lambda \,\mathrm{d}(x, X \setminus E_\lambda)^\beta + C\lambda \,\mathrm{d}(y, \bar{x})^\beta \le C\lambda \,\mathrm{d}(x, y)^\beta.\end{aligned}$$

The last inequality follows from the selection of $\bar{x}$, and the fact that $\mathrm{d}(x, X \setminus E_\lambda) \le \mathrm{d}(x, y)$.

The Hölder continuity of v with estimate (5.11) follows from the four cases above. $\square$

Proof of (2) - $v \in M^{1,1}(X)$. We have to show that $v \in L^1(X)$, and that it has an integrable generalized gradient.

Since $v = u$ in $X \setminus E_\lambda$, to show the integrability of v, it suffices to estimate $\int_{E_\lambda} |v| \, d\mu$. By the properties of the functions φ_i, the bounded overlap of the balls $2B_i \subset E_\lambda$, and the doubling property of μ, we have that

$$\int_{E_\lambda} |v| \, d\mu \le \sum_{i=1}^\infty \int_{2B_i} |u|_{2B_i} \, d\mu \le C \sum_{i=1}^\infty \int_{2B_i} |u| \, d\mu \le C \int_{E_\lambda} |u| \, d\mu, \tag{5.14}$$

and hence $v \in L^1(X)$ with $\|v\|_{L^1(X)} \le C\|u\|_{L^1(X)}$.

Concerning the gradient, we will show that the function g_v,

$$g_v(x) = C\big(g(x) + (\mathcal{M}\, g^q(x))^{1/q}\big), \quad x \in X \tag{5.15}$$

belongs to $\mathrm{D}(v) \cap L^1(X)$. As in the proof of the Hölder continuity, we consider four cases.

(i) We begin with the easiest case

$$|v(x) - v(y)| = |u(x) - u(y)| \le \mathrm{d}(x, y)(g(x) + g(y)) \le \mathrm{d}(x, y)(g_v(x) + g_v(y))$$

for almost all $x, y \in X \setminus E_\lambda$ because $g \in \mathrm{D}(u)$.

(ii) If $x, y \in E_\lambda$ and $\mathrm{d}(x, y) \le \delta$, then similar calculation as in (5.12) shows that

$$|v(x) - v(y)| \le C \,\mathrm{d}(x, y) \sum_{\mathcal{B}_x \cup \mathcal{B}_y} r_i^{-1} |u(x) - u_{2B_i}|, \tag{5.16}$$

where

$$|u(x) - u_{2B_i}| \le |u(x) - u_{B(x,28r_i)}| + |u_{B(x,28r_i)} - u_{2B_i}|. \tag{5.17}$$

Since $2B_i \subset B(x, 28r_i)$ by Lemma 3.7, the $(1, q)$-Poincaré inequality holds for u and g, and μ is doubling, we have that

$$\begin{aligned}|u_{B(x,28r_i)} - u_{2B_i}| &\le C \fint_{B(x,28r_i)} |u - u_{B(x,28r_i)}| \, d\mu \\ &\le C r_i \Big(\fint_{B(x,28\sigma r_i)} g^q \, d\mu \Big)^{1/q}.\end{aligned}$$

This together with the telescoping argument

$$
\begin{aligned}
|u(x) - u_{B(x,28r_i)}| &\le \sum_{j=0}^{\infty} |u_{B^{j+1}} - u_{B^j}| \le \sum_{j=0}^{\infty} \fint_{B^j} |u - u_{B^j}| \, d\mu \\
&\le C r_i \sum_{j=0}^{\infty} 2^{-j} \Big(\fint_{\sigma B^j} g^q \, d\mu \Big)^{1/q},
\end{aligned}
$$

where $B^j = 2^{-j} B(x, 28r_i)$ for $j = 0, 1, \dots$, shows that

$$
|u(x) - u_{2B_i}| \le C r_i \big(\mathcal{M}\, g^q(x) \big)^{1/q} \tag{5.18}
$$

for almost all x. Since the cardinality of $\mathcal{B}_x \cup \mathcal{B}_y$ is bounded, the estimates (5.16)-(5.18) show that

$$
|v(x) - v(y)| \le C \, \mathrm{d}(x,y) \big(\mathcal{M}\, g^q(x) \big)^{1/q} \le C \, \mathrm{d}(x,y) (g_v(x) + g_v(y))
$$

for almost all $x, y \in E_\lambda$ with $\mathrm{d}(x,y) \le \delta$.
(iii) Let $x, y \in E_\lambda$ with $\mathrm{d}(x,y) \ge \delta$. Using the properties of the functions φ_i, the fact that $g \in \mathrm{D}(u)$, similar estimates for $|u(x) - u_{2B_i}|$ and $|u(y) - u_{2B_i}|$ as in the previous case, and Lemma 3.7 to conclude that $r_i \approx \mathrm{d}(x, X \setminus E_\lambda)$ for all $B_i \in \mathcal{B}_x$ (and similarly for $\mathcal{B}_y$), we have that

$$
\begin{aligned}
|v(x) - v(y)| &\le \sum_{\mathcal{B}_x} |u(x) - u_{2B_i}| + \sum_{\mathcal{B}_y} |u(y) - u_{2B_i}| + |u(x) - u(y)|, \\
&\le C \, \mathrm{d}(x, X \setminus E_\lambda) \big(\mathcal{M}\, g^q(x) \big)^{1/q} + C \, \mathrm{d}(y, X \setminus E_\lambda) \big(\mathcal{M}\, g^q(y) \big)^{1/q} \\
&\quad + \mathrm{d}(x,y) \big(g(x) + g(y) \big) \\
&\le c \, \mathrm{d}(x,y) \big(g_v(x) + g_v(y) \big).
\end{aligned}
$$

(iv) If $y \in E_\lambda$ and $x \in X \setminus E_\lambda$, then

$$
\begin{aligned}
|v(x) - v(y)| = |u(x) - v(y)| &= \Big| \sum_{i=1}^{\infty} \varphi_i(y) (u(x) - u_{2B_i}) \Big| \\
&\le \sum_{\mathcal{B}_y} |u(x) - u_{2B_i}|,
\end{aligned} \tag{5.19}
$$

where, by the assumption that $g \in \mathrm{D}(u)$, and by a similar calculation as for (5.17), we have

$$
\begin{aligned}
|u(x) - u_{2B_i}| &\le |u(x) - u(y)| + |u(y) - u_{2B_i}| \\
&\le \mathrm{d}(x,y) \big(g(x) + g(y) \big) + C r_i \big(\mathcal{M}\, g^q(y) \big)^{1/q}.
\end{aligned} \tag{5.20}
$$

Since for $B_i \in \mathcal{B}_y$, $r_i \approx \mathrm{d}(y, X \setminus E_\lambda)$ and $\mathrm{d}(y, X \setminus E_\lambda) \le \mathrm{d}(x,y)$, (5.19) and (5.20) show that

$$\begin{aligned} |v(x) - v(y)| &\le C\,\mathrm{d}(x,y)\big(g(x) + g(y) + \big(\mathcal{M}\, g^q(y)\big)^{1/q}\big) \\ &\le C\,\mathrm{d}(x,y)\big(g_v(x) + g_v(y)\big) \end{aligned}$$

for almost all $y \in E_\lambda$ and $x \in X \setminus E_\lambda$.

We conclude that the function g_v is a generalized gradient of v. The integrability of g_v follows similarly as that of g_{u_r} in the proof of Lemma 4.4; it suffices to show that $(\mathcal{M}\, g^q)^{1/q} \in L^1(X)$. Since $g \in L^1(X)$, g^q is in $L^{1/q}(X)$, and hence, by the boundedness of the maximal operator for $1/q > 1$, $\mathcal{M}\, g^q \in L^{1/q}(X)$ with

$$\int_X (\mathcal{M}\, g^q)^{1/q}\, d\mu \le C \int_X (g^q)^{1/q}\, d\mu = C \int_X g\, d\mu. \tag{5.21}$$

Hence $g_v \in L^1(X)$ with $\|g_v\|_{L^1(X)} \le C\|g\|_{L^1(X)}$. By choosing the generalized gradient g of u so that $\|g\|_{L^1(X)} \le 2\|u\|_{M^{1,1}(X)}$ and combining estimates (5.14) and (5.21), we have that $\|v\|_{M^{1,1}(X)} \le C\|u\|_{M^{1,1}(X)}$. □

Proof of (3) - *Approximation in norm.* We will show that $v \to u$ in $M^{1,1}(X)$ as $\lambda \to \infty$. Using the fact that $v = u$ in $X \setminus E_\lambda$ and (5.14), we have that

$$\int_X |u - v|\, d\mu = \int_{E_\lambda} |u - v|\, d\mu \le C \int_{E_\lambda} |u|\, d\mu,$$

which tends to 0 as $\lambda \to \infty$ because $\mu(E_\lambda) \to 0$ as $\lambda \to \infty$ by Claim 2. Hence $v \to u$ in $L^1(X)$ as $\lambda \to \infty$.

Next we have to find a generalized gradient g_λ of $u - v$ for which $\|g_\lambda\|_{L^1(X)} \to 0$ as $\lambda \to \infty$. We claim that the function

$$g_\lambda = g_v \chi_{E_\lambda} = C\big(g + \big(\mathcal{M}\, g^q\big)^{1/q}\big)\chi_{E_\lambda}$$

is in $\mathrm{D}(u - v)$, that is, the inequality (2.2) holds for $u - v$ and g_λ almost everywhere in X. If $x, y \in X \setminus E_\lambda$, then $u - v = 0$ and (2.2) holds trivially. Inequality (2.2) for almost all $x, y \in E_\lambda$ follows because $g \in \mathrm{D}(u)$ and $g_v \in \mathrm{D}(v)$. If $x \in E_\lambda$ and $y \in X \setminus E_\lambda$, then $(u - v)(y) = 0$ and $g_\lambda(y) = 0$. Similar arguments as in (5.17) show that

$$\begin{aligned} |u(x) - v(x)| &\le \sum_{\mathcal{B}_x} |u(x) - u_{2B_i}| \le C \sum_{\mathcal{B}_x} r_i \big(\mathcal{M}\, g^q(x)\big)^{1/q} \\ &\le C\,\mathrm{d}(x, X \setminus E_\lambda)\big(\mathcal{M}\, g^q(x)\big)^{1/q} \le C\,\mathrm{d}(x,y)\big(\mathcal{M}\, g^q(x)\big)^{1/q}. \end{aligned}$$

Hence $g_\lambda \in \mathrm{D}(u - v)$. Since g_v is in $L^1(X)$, so is g_λ, too. Moreover, as $\mu(E_\lambda) \to 0$ as $\lambda \to \infty$ by Claim 2, we have that $\|g_\lambda\|_{L^1(X)}$, and hence also $\|u - v\|_{M^{1,1}(X)}$ tends to 0 as $\lambda \to \infty$. □

Proof of (4) - *Hausdorff content of E_λ.* Recall that

$$E_\lambda = \{x \in X : u^{\#}_\beta(x) > \lambda\}.$$

Using Claim 1, a similar estimate as in (5.4), and Corollary 5.2, we see that for $\lambda > \lambda_0$ we have

$$\begin{aligned} E_\lambda &\subset \big\{x \in B(x_0, 2) : \big(\mathcal{M}_{(1-\beta)q,\sigma}\, g^q(x)\big)^{1/q} > C\lambda\big\} \\ &\subset \big\{x \in B(x_0, 2) : \mathcal{M}_{(1-\beta),\sigma}\, g(x) > C\lambda\big\}, \end{aligned}$$

and hence, by Lemma 3.3, we conclude that

$$\mathcal{H}^{s-(1-\beta)}_\infty(E_\lambda) \le C\lambda^{-1} \int_X g\, d\mu,$$

which tends to 0 as $\lambda \to \infty$. □

STEP 2: General case.

Let $\varepsilon > 0$. We cover X by balls of radius $1/10$, and use the $5r$-covering theorem to obtain pairwise disjoint balls $B(a_j, 1/10)$ from this covering such that $X \subset \cup_{j=1}^\infty B(a_j, 1/2)$ and that the balls $B(a_j, 2)$ have bounded overlap. Let (ψ_j) be a partition of unity for this covering such that $\sum_{j=1}^\infty \psi_j(x) = 1$ for all $x \in X$, each ψ_j is L-Lipschitz, $0 \le \psi_j \le 1$, and $\operatorname{supp}\psi_j \subset B(a_j, 1)$ for all $j \in \mathbb{N}$.

For $u \in M^{1,1}(X)$ with $g \in \mathrm{D}(u) \cap L^1(X)$, we have

$$u(x) = \sum_{j=1}^\infty u_j(x), \tag{5.22}$$

where $u_j = u\psi_j$, for all $x \in X$. By Lemma 3.4, each u_j is in $M^{1,1}(X)$ and

$$g_j = (g + L|u|)\chi_{B(a_j,1)}$$

is a generalized gradient of u_j. Since $\operatorname{supp} u_j \subset B(a_j, 1)$, the first step of the proof shows there are functions $v_j \in M^{1,1}(X)$ and open sets $O_j \subset B(a_j, 2)$ such that

(i) $v_j = u_j$ in $X \setminus O_j$, $\operatorname{supp} v_j \subset B(a_j, 2)$,
(ii) v_j is Hölder continuous with exponent β,
(iii) $\|u_j - v_j\|_{M^{1,1}(X)} < 2^{-j}\varepsilon$,
(iv) $\mathcal{H}^{s-(1-\beta)}_\infty(O_j) < 2^{-j}\varepsilon$,
(v) $h_j = C(g_j + (\mathcal{M}\, g_j^q)^{1/q})$ is a generalized gradient of v_j.

We define $O = \cup_{j=1}^\infty O_j$, and claim that the function $v = \sum_{j=1}^\infty v_j$ together with the open set O satisfy requirements (1)-(4).

For (1), let $x \in X \setminus O$. Then, by (i) and (5.22), we obtain

$$v(x) = \sum_{j=1}^\infty v_j(x) = \sum_{j=1}^\infty u_j(x) = u(x).$$

The Hausdorff content estimate (4) for O follows from (iv) using the subadditivity of $\mathcal{H}^{s-(1-\beta)}_\infty$.

By (5.11), we have

$$|v_j(x) - v_j(y)| \le C\lambda_j \,\mathrm{d}(x,y)^\beta$$

for all $x, y \in X$. Since, by the proof above, the constant λ_j depends on ε and on j, (5.11) and the fact that $\operatorname{supp} v_j \subset B(a_j, 2)$ give Hölder continuity of v only in bounded subsets of X.

To prove the first part of (2) and (3), we have to show that $v \in M^{1,1}(X)$ and that $\|u - v\|_{M^{1,1}(X)} < \varepsilon$. By (iii), we have

$$\sum_{j=1}^{\infty} \|u_j - v_j\|_{M^{1,1}(X)} < \sum_{j=1}^{\infty} 2^{-j}\varepsilon = \varepsilon, \tag{5.23}$$

that is, the series $\sum_{j=1}^{\infty}(u_j - v_j)$ convergences absolutely, and hence converges in the Banach space $M^{1,1}(X)$. Since $u = \sum_{j=1}^{\infty} u_j$ is in $M^{1,1}(X)$, also $\sum_{j=1}^{\infty} v_j$ converges in $M^{1,1}(X)$. Moreover, by (5.23) we have that

$$\|u - v\|_{M^{1,1}(X)} \le \sum_{j=1}^{\infty} \|u_j - v_j\|_{M^{1,1}(X)} < \varepsilon.$$

This completes the proof of Theorem 5.3. □

Remark 5.4. Note that $\sum_{j=1}^{\infty} h_j$, where h_j is as in (v) above, is a generalized gradient of v but it is not necessarily integrable. If we would like to construct an integrable $g_v \in \mathrm{D}(v)$, we need a cut-off function for each j; let $\Phi_j : X \to [0, 1]$ be a Lipschitz-function that equals 1 in $B(a_j, 2)$ and vanishes outside $B(a_j, 3)$. Since the support of v_j is in $B(a_j, 2)$, $v_j\Phi_j = v_j$ in X. Moreover, the function

$$g_{v_j} = (h_j + C|v_j|)\chi_{B(a_j,3)}$$

is in $D(v_j)$ by Lemma 3.4. This together with the fact that $\operatorname{supp} v_j \subset B(a_j, 2)$ and the bounded overlap of the balls $B(a_j, 2)$ shows that the function

$$g_v = \sum_{j=1}^{\infty} g_{v_j}$$

is a generalized gradient of w. Integrability of v and g_v follow since $\operatorname{supp} v_j \subset B(a_j, 2)$, $\operatorname{supp} g_{v_j} \subset B(a_j, 3)$ and the balls $B(a_j, 3)$ have bounded overlap, using integral estimates from the first part of the proof.

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