AN EXISTENCE RESULT FOR SUPERPARABOLIC FUNCTIONS

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Abstract. We study superparabolic functions related to nonlinear parabolic equations. They are defined by means of a parabolic comparison principle with respect to solutions. We show that every superparabolic function satisfies the equation with a positive Radon measure on the right-hand side, and conversely, for every finite positive Radon measure there exists a superparabolic function which is solution to the corresponding equation with measure data.

1. Introduction

This work provides an existence result for superparabolic functions related to nonlinear degenerate parabolic equations

$$\frac{\partial u}{\partial t} - \operatorname{div} \mathcal{A}(x,t,\nabla u) = 0. \tag{1.1}$$

The principal prototype of such an equation is the p-parabolic equation

$$\frac{\partial u}{\partial t} - \operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0 \tag{1.2}$$

with $2 \le p < \infty$. Superparabolic functions are defined as lower semicontinuous functions that obey a parabolic comparison principle with respect to continuous solutions of (1.1). The superparabolic functions related to the p-parabolic equation are of particular interest because they coincide with the viscosity supersolutions of (1.2), see [5]. Thus there is an alternative definition in the theory of viscosity solutions and our results automatically hold for the viscosity supersolutions of (1.2) as well.

By definition, a superparabolic function is not required to have any derivatives, and, consequently, it is not evident how to directly relate it to the equation (1.1). However, by [9] a superparabolic function has spatial Sobolev derivatives with sharp local integrability bounds. See also [1], [2], and [7]. Using this result we show that every superparabolic function u satisfies the equation with measure data

$$\frac{\partial u}{\partial t} - \operatorname{div} \mathcal{A}(x,t,\nabla u) = \mu, \tag{1.3}$$

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where μ is the Riesz measure of u, see Theorem 3.9. A rather delicate point here is that the spatial gradient of a superparabolic function is not locally integrable to the natural exponent p. Consequently, the Riesz measure does not belong to the dual of the natural parabolic Sobolev space. For example, Dirac's delta is the Riesz measure for the Barenblatt solution of the p-parabolic equation.

We also consider the converse question. Indeed, for every finite nonnegative Radon measure μ, there is a superparabolic function which satisfies (1.3), see Theorem 5.8. This result is standard, provided that the measure belongs to the dual of the natural parabolic Sobolev space, but we show that the class of superparabolic functions is large enough to admit an existence result for general Radon measures. If the measure belongs to the dual of the natural parabolic Sobolev space, then uniqueness with fixed intial and boundary conditions is also standard. However, uniqueness questions related to (1.3) for general measures are rather delicate. For instance, the question whether the Barenblatt solution is the only solution of the p-parabolic equation with Dirac's delta seems to be open. Hence, we will not discuss uniquess of solutions here.

2. Preliminaries

Let Ω be an open and bounded set in $\mathbf{R}^n$ with $n \geq 1$. We denote

$$\Omega_T = \Omega \times (0,T),$$

where $0 < T < \infty$. For an open set U in $\mathbf{R}^n$ we write

$$U_{t_1,t_2} = U \times (t_1,t_2),$$

where $0 < t_1 < t_2 < \infty$. The parabolic boundary of U_{t_1,t_2} is

$$\partial_p U_{t_1,t_2} = \big(\partial U \times [t_1,t_2]\big) \cup \big(\overline{U} \times \{t_1\}\big).$$

As usual, $W^{1,p}(\Omega)$ denotes the Sobolev space of functions in $L^p(\Omega)$, whose distributional gradient belongs to $L^p(\Omega)$. The space $W^{1,p}(\Omega)$ is equipped with the norm

$$\|u\|_{W^{1,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \|\nabla u\|_{L^p(\Omega)}.$$

The Sobolev space with zero boundary values, denoted by $W_0^{1,p}(\Omega)$, is a completion of $C_0^\infty(\Omega)$ with respect to the norm of $W^{1,p}(\Omega)$.

The parabolic Sobolev space $L^p(0,T;W^{1,p}(\Omega))$ consists of measurable functions $u : \Omega_T \to [-\infty,\infty]$ such that for almost every $t \in (0,T)$, the function $x \mapsto u(x,t)$ belongs to $W^{1,p}(\Omega)$ and

$$\int_{\Omega_T} (|u|^p + |\nabla u|^p)\,\mathrm{d}x\,\mathrm{d}t < \infty. \tag{2.1}$$

A function $u \in L^p(0,T;W^{1,p}(\Omega))$ belongs to the space $L^p(0,T;W_0^{1,p}(\Omega))$ if $x \mapsto u(x,t)$ belongs to $W_0^{1,p}(\Omega)$ for almost every $t \in (0,T)$. The local space $L^p_{\mathrm{loc}}(0,T;W^{1,p}_{\mathrm{loc}}(\Omega))$ consist of functions that belong to the parabolic Sobolev space in every $U_{t_1,t_2} \Subset \Omega_T$.

We assume that the following structural conditions hold for the divergence part of our equation for some exponent $p \geq 2$:

(1) $(x,t) \mapsto \mathcal{A}(x,t,\xi)$ is measurable for all $\xi \in \mathbf{R}^n$,
(2) $\xi \mapsto \mathcal{A}(x,t,\xi)$ is continuous for almost all $(x,t) \in \Omega \times \mathbf{R}$,
(3) $\mathcal{A}(x,t,\xi)\cdot\xi \geq \alpha|\xi|^p$ for almost all $(x,t) \in \Omega \times \mathbf{R}$ and $\xi \in \mathbf{R}^n$,
(4) $|\mathcal{A}(x,t,\xi)| \leq \beta|\xi|^{p-1}$ for almost all $(x,t) \in \Omega \times \mathbf{R}$ and $\xi \in \mathbf{R}^n$,
(5) $(\mathcal{A}(x,t,\xi)-\mathcal{A}(x,t,\eta))\cdot(\xi-\eta) > 0$ for almost all $(x,t) \in \Omega \times \mathbf{R}$ and all $\xi,\eta \in \mathbf{R}^n$, $\xi \neq \eta$.

Solutions are understood in the weak sense in the parabolic Sobolev space.

Definition 2.2. A function $u \in L^p_{\mathrm{loc}}(0,T;W^{1,p}_{\mathrm{loc}}(\Omega))$ is a weak solution of (1.1) in Ω_T, if

$$-\int_{\Omega_T} u\frac{\partial\varphi}{\partial t}\,\mathrm{d}x\,\mathrm{d}t + \int_{\Omega_T} \mathcal{A}(x,t,\nabla u)\cdot\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t = 0 \tag{2.3}$$

for all test functions $\varphi \in C_0^\infty(\Omega_T)$. The function u is a supersolution if the integral in (2.3) is nonnegative for nonnegative test functions. In a general open set V of $\mathbf{R}^{n+1}$, the above notions are to be understood in a local sense, i.e. u is a solution if it is a solution in all sets $U_{t_2,t_2} \Subset V$.

By parabolic regularity theory, every weak solution has a locally Hölder continuous representative.

The definition of a weak solution does not refer to the time derivative of u. We would, nevertheless, like to employ test functions depending on u, and thus the time derivative $\frac{\partial u}{\partial t}$ inevitably appears. The standard way to overcome this difficulty is to use a mollification procedure, for instance Steklov averages or convolution with the standard mollifier, in the time direction; see, e.g., [3]. Let u^ε denote the mollification of u. For each $\varphi \in C_0^\infty(\Omega_T)$, the regularized equation reads

$$\int_{\Omega_T} \frac{\partial u^\varepsilon}{\partial t}\varphi\,\mathrm{d}x\,\mathrm{d}t + \int_{\Omega_T} \mathcal{A}(x,t,\nabla u)^\varepsilon\cdot\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t = 0,$$

for small enough $\varepsilon > 0$. The aim is to obtain estimates that are independent of the time derivatives of u^ε, and then pass to the limit $\varepsilon \to 0$.

3. $\mathcal{A}$-superparabolic functions

We illustrate the notion of $\mathcal{A}$-superparabolic functions by considering the Barenblatt solution $\mathcal{B}_p : \mathbf{R}^{n+1} \to [0,\infty)$ first. It is given by the formula

$$\mathcal{B}_p(x,t) = \begin{cases} t^{-n/\lambda}\left(c - \dfrac{p-2}{p}\lambda^{1/(1-p)}\left(\dfrac{|x|}{t^{1/\lambda}}\right)^{p/(p-1)}\right)_+^{(p-1)/(p-2)}, & t > 0, \\ 0, & t \leq 0, \end{cases}$$

where $\lambda = n(p-2)+p$, $p>2$, and the constant c is usually chosen so that

$$\int_{\mathbf{R}^n} \mathcal{B}_p(x,t)\,dx = 1$$

for every $t>0$.

The Barenblatt solution is a weak solution of the p-parabolic equation (1.2) in the open upper and lower half spaces, but it is not a supersolution in any open set that contains the origin. It is the a priori integrability of $\nabla\mathcal{B}_p$ that fails, since

$$\int_{-1}^{1}\int_{Q} |\nabla\mathcal{B}_p(x,t)|^p \,\mathrm{d}x\,\mathrm{d}t = \infty,$$

where $Q=[-1,1]^n \subset \mathbf{R}^n$. In contrast, the truncated functions

$$\min(\mathcal{B}_p(x,t),k), \quad k=1,2,\dots,$$

belong to the correct parabolic Sobolev space and are weak supersolutions in $\mathbf{R}^{n+1}$ for every k. This shows that an increasing limit of supersolutions is not necessarily a supersolution.

In order to include the Barenblatt solution in our exposition we define a class of superparabolic functions, as in [6].

Definition 3.1. A function $u:\Omega_T\to(-\infty,\infty]$ is $\mathcal{A}$-superparabolic in Ω_T, if

(1) u is lower semicontinuous,
(2) u is finite in a dense subset, and
(3) If h is a solution of (1.1) in $U_{t_1,t_2}\Subset\Omega_T$, continuous in $\overline{U}_{t_1,t_2}$, and $h\le u$ on the parabolic boundary $\partial_p U_{t_1,t_2}$, then $h\le u$ in U_{t_1,t_2}.

We say that u is $\mathcal{A}$-hyperparabolic, if u satisfies properties (1) and (3) only.

The class of $\mathcal{A}$-superparabolic functions is strictly larger than that of weak supersolutions as the Barenblatt solution discussed above shows. If u and v are $\mathcal{A}$-superparabolic functions, so are their pointwise minimum $\min(u,v)$, and the functions $u+\alpha$ for all $\alpha\in\mathbf{R}$. This is an immediate consequence of the definition. However, the functions $u+v$ and αu are not superparabolic in general. This is well in accordance with the corresponding properties of supersolutions. In addition, the class of superparabolic functions is closed with respect to the increasing convergence, provided the limit function is finite in a dense subset. This is also a straightforward consequence of the definition.

Theorem 3.2. *Let (u_j) be an increasing sequence of $\mathcal{A}$-superparabolic functions in Ω_T. Then the limit function $u=\lim_{j\to\infty}u_j$ is always $\mathcal{A}$-hyperparabolic, and $\mathcal{A}$-superparabolic whenever it is finite in a dense subset.*

A much less straightforward property of $\mathcal{A}$-superharmonic functions is the following theorem.

Theorem 3.3 ([8, 10])**.** *A locally bounded $\mathcal{A}$-superparabolic function is a weak supersolution.*

These two results give a characterization of $\mathcal{A}$-superparabolicity. Indeed, if we have an increasing sequence of continuous supersolutions and the limit function is finite in dense subset, then the limit function is $\mathcal{A}$-superparabolic. Moreover, if the limit function is bounded, then it is a supersolution. On the other hand, the truncations $\min(v, k)$, $k = 1, 2, \ldots$, of an $\mathcal{A}$-superparabolic function v are supersolutions and hence every $\mathcal{A}$-superparabolic function can be approximated by an increasing sequence of supersolutions.

The reader should carefully distinguish between supersolutions and $\mathcal{A}$-superparabolic functions. Notice that an $\mathcal{A}$-superparabolic function is defined at every point in its domain, but supersolutions are defined only up to a set of measure zero. On the other hand, weak supersolutions satisfy the comparison principle and, roughly speaking, they are $\mathcal{A}$-superparabolic, provided the issue about lower semicontinuity is properly handled. In fact, every weak supersolution has a lower semicontinuous representative as the following theorem shows. Hence every weak supersolution is $\mathcal{A}$-superparabolic after a redefinition on a set of measure zero.

Theorem 3.4 ([11])**.** *Let u be a weak supersolution in Ω_T. Then there exists a lower semicontinuous weak supersolution that equals u almost everywhere in Ω_T.*

Supersolutions have spatial Sobolev derivatives and they satisfy a differential inequality in a weak sense. By contrast, no differentiability is assumed in the definition of a $\mathcal{A}$-superparabolic function. The only tie to the differential equation is through the comparison principle. Nonetheless, [9] gives an integrability result with an exponent smaller than p. See also [1] and [2].

Theorem 3.5. *Let u be $\mathcal{A}$-superparabolic in Ω_T. Then u belongs to the space $L^q_{\mathrm{loc}}(0, T; W^{1,q}_{\mathrm{loc}}(\Omega))$ with $0 < q < p - n/(n+1)$.*

In particular, this shows that an $\mathcal{A}$-superparabolic function u has a spatial weak gradient and that it satisfies

$$-\int_{\Omega_T} u \frac{\partial \varphi}{\partial t} \, \mathrm{d}x \, \mathrm{d}t + \int_{\Omega_T} \mathcal{A}(x, t, \nabla u) \cdot \nabla \varphi \, \mathrm{d}x \, \mathrm{d}t \geq 0$$

for all nonnegative test functions $\varphi \in C_0^\infty(\Omega_T)$. Note carefully that the integrability of the gradient is below the natural exponent p and hence u is not a weak supersolution. Although u satisfies the integral inequality, it seems to be very difficult to employ this property directly.

A key ingredient in the proof of Theorem 3.5 is the following lemma, see [9, Lemma 3.14]. We will use it below.

Lemma 3.6. *Suppose that v is a positive function such that $v_k = \min(v, k)$ belongs to $L^p(0,T;W_0^{1,p}(\Omega))$. If there is a constant $M > 0$, independent of k, such that*

$$\int_{\Omega_T} |\nabla v_k|^p \,\mathrm{d}x\,\mathrm{d}t + \operatorname*{ess\,sup}_{0<t<T} \int_\Omega v_k^2 \,\mathrm{d}x \le Mk, \ \ k = 1, 2, \dots,$$

then v and ∇v belong to $L^q(\Omega_T)$ for $0 < q < p - n/(n+1)$ and their L^q norms have an estimate in terms of n, p, q, $|\Omega_T|$, and M.

Next we study the connection between $\mathcal{A}$-superparabolic functions and parabolic equations with measure data. First we define weak solutions to the measure data problem (1.3). Recall our assumption $p \ge 2$.

Definition 3.7. Let μ be a Radon measure on $\mathbf{R}^{n+1}$. A function $u \in L^{p-1}_{\mathrm{loc}}(0,T;W^{1,p-1}_{\mathrm{loc}}(\Omega))$ is a weak solution of (1.3) in Ω_T, if

$$-\int_{\Omega_T} u\frac{\partial \varphi}{\partial t}\,\mathrm{d}x\,\mathrm{d}t + \int_{\Omega_T} \mathcal{A}(x,t,\nabla u)\cdot\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t = \int_{\Omega_T} \varphi\,\mathrm{d}\mu \qquad (3.8)$$

for all $\varphi \in C_0^\infty(\Omega_T)$.

The Barenblatt solution satisfies

$$\frac{\partial \mathcal{B}_p}{\partial t} - \operatorname{div}(|\nabla \mathcal{B}_p|^{p-2}\nabla\mathcal{B}_p) = \delta$$

in the weak sense of Definition 3.7, where the right-hand side is Dirac's delta at the origin. In other words, Dirac's delta is the Riesz mass of the Barenblatt solution.

Theorem 3.5 implies the existence of the Riesz measure of any $\mathcal{A}$-superparabolic function.

Theorem 3.9. *Let u be a $\mathcal{A}$-superparabolic function. Then there exists a positive Radon measure μ such that u satisfies* (1.3) *in the weak sense.*

Proof. Theorem 3.5 implies that $|u|^{p-1}, |\nabla u|^{p-1} \in L^1_{\mathrm{loc}}(\Omega_T)$. Let $\varphi \in C_0^\infty(\Omega_T)$ with $\varphi \ge 0$ and denote $u_k = \min(u, k)$. Then

$$\mathcal{A}(x,t,\nabla u_k)\cdot\nabla\varphi \to \mathcal{A}(x,t,\nabla u)\cdot\nabla\varphi$$

pointwise almost everywhere as $k \to \infty$ by continuity of $\xi \mapsto \mathcal{A}(x,t,\xi)$, as $\nabla u_k \to \nabla u$ almost everywhere. Using the structure of $\mathcal{A}$, we have

$$|\mathcal{A}(x,t,\nabla u_k)\cdot\nabla\varphi| \le C|\nabla u_k|^{p-1}|\nabla\varphi| \le C|\nabla u|^{p-1}|\nabla\varphi|.$$

The majorant established above allow us to use the dominated convergence theorem and the fact that the functions u_k are supersolutions to conclude that

$$
\begin{aligned}
&-\int_{\Omega_T} u\frac{\partial\varphi}{\partial t}\,\mathrm{d}x\,\mathrm{d}t+\int_{\Omega_T}\mathcal{A}(x,t,\nabla u)\cdot\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t\\
&\quad=\lim_{k\to\infty}\left(-\int_{\Omega_T} u_k\frac{\partial\varphi}{\partial t}\,\mathrm{d}x\,\mathrm{d}t+\int_{\Omega_T}\mathcal{A}(x,t,\nabla u_k)\cdot\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t\right)\geq 0.
\end{aligned}
$$

The claim now follows from the Riesz representation theorem. □

4. Compactness of $\mathcal{A}$ -superparabolic functions

In this section we prove a compactness property of $\mathcal{A}$-superparabolic functions. It will be essential in the proof of the fact that every finite Radon measure there exists a superparabolic function, which solves the corresponding equation with measure data. We use the following convergence result for weak supersolutions from [10].

Theorem 4.1. *Let (u_j) be a bounded sequence of supersolutions in Ω_T and assume that u_j converges to a function u almost everywhere in Ω_T. Then the limit function u is a weak supersolution, and $\nabla u_j\to\nabla u$ almost everywhere.*

Note that a pointwise limit of supersolutions is not necessarily a supersolution if we drop the boundedness assumption, as illustrated by the Barenblatt solution at the beginning of Section 3.

We also use the following Caccioppoli estimate from [3]. The straightforward proof employs the test function $-u\varphi$.

Lemma 4.2. *Let $u\leq 0$ be a weak supersolution in Ω_T, and $\varphi\in C_0^\infty(\Omega_T)$ with $\varphi\geq 0$. Then*

$$
\begin{aligned}
&\int_{\Omega_T}|\nabla u|^p\varphi^p\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad\leq C\left(\int_{\Omega_T}|u|^p|\nabla\varphi|^p\,\mathrm{d}x\,\mathrm{d}t+\int_{\Omega_T}|u|^2\left|\frac{\partial\varphi}{\partial t}\right|\varphi^{p-1}\,\mathrm{d}x\,\mathrm{d}t\right).
\end{aligned}
$$

Next we show that general superparabolic functions have a compactness property. Note that the limit function may very well be identically infinite.

Theorem 4.3. *Let (u_j) be a sequence of positive $\mathcal{A}$-superparabolic functions in Ω_T. Then there exist a subsequence (u_{j_k}) and an $\mathcal{A}$-hyperparabolic function u such that*

$$u_{j_k}\to u\quad\textit{almost everywhere in}\quad\Omega_T,$$

and

$$\nabla u_{j_k}\to\nabla u\quad\textit{almost everywhere in}\quad\{(x,t)\in\Omega_T:u(x,t)<\infty\}.$$

Proof. Assume first that $u_j \leq M < \infty$. If we could extract a subsequence that converges pointwise almost everywhere to a function u, Theorem 4.1 would imply that u is a supersolution and that $\nabla u_j \to \nabla u$ almost everywhere. By Theorem 3.4, we could then assume that u is lower semicontinuous and thus $\mathcal{A}$-superparabolic.

Once the result for bounded sequences is available, we can remove the boundedness assumption by a diagonalization argument. Indeed, we can find a subsequence (u_j^1) and an $\mathcal{A}$-superparabolic function u_1 such that

$$\min(u_j^1, 1) \to u_1 \quad \text{and} \quad \nabla \min(u_j^1, 1) \to \nabla u_1$$

almost everywhere in Ω_T. We proceed inductively and pick a subsequence (u_j^k) of (u_j^{k-1}) such that

$$\min(u_j^k, k) \to u_k \quad \text{and} \quad \nabla \min(u_j^k, k) \to \nabla u_k$$

almost everywhere in Ω_T. We observe that if $l \geq k$ and $u_k(x,t) < k$, we have $u_l(x,t) = u_k(x,t)$. Thus the sequence (u_k) is increasing, and we conclude that the limit

$$u = \lim_{k\to\infty} u_k$$

exists and defines the desired $\mathcal{A}$-hyperparabolic function in Ω_T. We note that by construction $\min(u,k) = u_k$, so that for the diagonal sequence (u_k^k) it holds that $\nabla u_k^k \to \nabla u$ almost everywhere in the set

$$\{(x,t) \in \Omega_T : u(x,t) < \infty\}.$$

To extract the pointwise convergent subsequence from a bounded sequence of supersolutions, we start by observing that the sequence (∇u_j) is bounded in $L^p(\tau_1, \tau_2; L^p(\Omega'))$ for all subdomains $\Omega'_{\tau_1,\tau_2} = \Omega' \times (\tau_1, \tau_2) \Subset \Omega_T$. This follows from Lemma 4.2 applied to $u_j - M$ and the boundedness of (u_j). Let μ_j denote the measure associated to u_j by Theorem 3.9, and choose open polyhedra $U \Subset U' \Subset \Omega$ and intervals $(t_1, t_2) \Subset (s_1, s_2) \Subset (0, T)$. If $\eta \in C_0^\infty(U'_{s_1,s_2})$ with $0 \leq \eta \leq 1$ and $\eta = 1$ in U_{t_1,t_2}, we have

$$\begin{aligned}
\mu_j(U_{t_1,t_2}) &\leq \int_{U'_{s_1,s_2}} \eta \,\mathrm{d}\mu_j \\
&= -\int_{U'_{s_1,s_2}} \frac{\partial \eta}{\partial t} u_j \,\mathrm{d}x\,\mathrm{d}t + \int_{U'_{s_1,s_2}} \mathcal{A}(x,t,\nabla u_j) \cdot \nabla \eta \,\mathrm{d}x\,\mathrm{d}t \\
&\leq CM + C\left(\int_{U'_{s_1,s_2}} |\nabla u_j|^p \,\mathrm{d}x\,\mathrm{d}t\right)^{1/p}.
\end{aligned}$$

Hence the sequence $(\mu_j(U_{t_1,t_2}))$ is bounded. For $\varphi \in C_0^\infty(U_{t_1,t_2})$, we have

$$
\begin{aligned}
|\langle u_j', \varphi\rangle| &= \left| - \int_{U_{t_1,t_2}} u_j \frac{\partial \varphi}{\partial t} \,\mathrm{d}x\,\mathrm{d}t \right| \\
&= \left| - \int_{U_{t_1,t_2}} \mathcal{A}(x,t,\nabla u_j)\cdot \nabla\varphi \,\mathrm{d}x\,\mathrm{d}t + \int_{U_{t_1,t_2}} \varphi \,\mathrm{d}\mu_j(x,t) \right| \\
&\le C \left[\left(\int_{U_{t_1,t_2}} |\nabla u_j|^p \,\mathrm{d}x\,\mathrm{d}t \right)^{1/p} + \mu_j(U_{t_1,t_2}) \right] \|\varphi\|_{L^\infty(t_1,t_2;W_0^{1,\infty}(U))},
\end{aligned}
$$

so that the sequence (u_j') is bounded in $L^1(t_1,t_2;W_0^{-1,1}(U))$. Recall that U is a polyhedron and hence $W^{1,p}(U)$ embeds compactly to $L^1(U)$ by the Rellich-Kondrachov compactness theorem. Moreover, $L^1(U)$ is contained in $W_0^{-1,1}(U)$, so it follows from Theorem 5 in [14] that (u_j) is relatively compact in $L^1(U_{t_1,t_2})$. This allows us to pick a subsequence that converges pointwise almost everywhere in U_{t_1,t_2} to a function u.

To pass to the whole set $\Omega \times (0,T)$, we employ another diagonalization argument. Choose polyhedra $U^1 \Subset U^2 \Subset \dots U^j \Subset U^{j+1} \dots$ and intervals $(t_1^1,t_2^1) \Subset (t_1^2,t_2^2) \Subset \dots$ so that

$$\Omega_T = \bigcup_{i=1}^{\infty} U^i_{t_1^i,t_2^i}.$$

The above reasoning allows us to pick a subsequence (u_j^1) that converges pointwise almost everywhere in $U^1_{t_1^1,t_2^1}$ to a function u^1. We proceed inductively, and pick a subsequence (u_j^{k+1}) of (u_j^k) that converges almost everywhere in $U^{k+1}_{t_1^{k+1},t_2^{k+1}}$ to a function u^{k+1}. Since limits are unique, $u^k = u^l$ almost everywhere in $U^k_{t_1^k,t_2^k}$ if $l > k$. Hence the diagonal sequence (u_k^k) converges almost everywhere in Ω_T to a function u. As explained above, this completes the proof. □

5. Existence of $\mathcal{A}$-superparabolic solutions

In this section we prove our main existence result, Theorem 5.8. Recall that a sequence of measures (μ_j) converges weakly to a measure μ if

$$\lim_{j\to\infty} \int_{\Omega_T} \varphi \,\mathrm{d}\mu_j = \int_{\Omega_T} \varphi \,\mathrm{d}\mu$$

for all $\varphi \in C_0^\infty(\Omega_T)$. The following well-known result asserts that for each finite positive Radon measure there exists an approximating sequence of functions in $L^\infty(\Omega_T)$ in the sense of a weak convergence of measures. We repeat the proof given, for example, in [12] for the convenience of the reader.

Lemma 5.1. *Let μ be a finite positive Radon measure on Ω_T. Then there is a sequence (f_j) of positive functions $f_j \in L^\infty(\Omega_T)$ such that*

$$\int_{\Omega_T} f_j \,\mathrm{d}x\,\mathrm{d}t \le \mu(\Omega_T)$$

and

$$\lim_{j\to\infty} \int_{\Omega_T} \varphi f_j \,\mathrm{d}x\,\mathrm{d}t = \int_{\Omega_T} \varphi \,\mathrm{d}\mu$$

for every $\varphi \in C_0^\infty(\Omega_T)$. In other words, the sequence of measures (μ_j) given by $\mathrm{d}\mu_j(x,t) = f_j \,\mathrm{d}x\,\mathrm{d}t$ converges weakly to μ.

Proof. Let $Q_{i,j}$, $i = 1, \dots, N_j$, be the dyadic cubes with side length 2^{-j} such that $Q_{i,j} \Subset \Omega_T$. We define

$$f_j(x,t) = \sum_{i=1}^{N_j} \frac{\mu(Q_{i,j})}{|Q_{i,j}|} \chi_{Q_{i,j}}(x,t),$$

and show that the sequence (f_j) has the desired properties. Observe that

$$\int_{\Omega_T} f_j \,\mathrm{d}x\,\mathrm{d}t = \sum_{i=1}^{N_j} \mu(Q_{i,j}) \le \mu(\Omega_T),$$

and thus the first property holds. Let then $(x_{i,j}, t_{i,j})$ be the center of $Q_{i,j}$. By the smoothness of φ, there is a constant C depending only on φ, such that

$$|\varphi(x,t) - \varphi(x_{i,j}, t_{i,j})| \le C2^{-j}$$

for all $(x,t) \in Q_{i,j}$. Hence,

$$\begin{aligned}
&\left| \int_{\Omega_T} \varphi \, d\mu - \int_{\Omega_T} f_j \varphi \; dx\, dt \right| \\
&\le \sum_{i=1}^{N_j} \left| \int_{Q_{i,j}} \varphi \, d\mu - \int_{Q_{i,j}} \varphi(x_{i,j}) \, d\mu \right. \\
&\qquad\qquad \left. + \int_{Q_{i,j}} \varphi(x_{i,j}) \, d\mu - \int_{Q_{i,j}} \frac{\mu(Q_{i,j})}{|Q_{i,j}|} \varphi \, dx\, dt \right| \\
&\le C2^{-j} \sum_{i=1}^{N_j} \int_{Q_{i,j}} d\mu \le C2^{-j} \mu(\Omega_T).
\end{aligned}$$

This proves the claim as $j \to \infty$. □

In the proof of the next theorem we utilize the following standard existence result, see, e.g., Example 4.A. in [13]. Suppose that $f \in L^\infty(\Omega_T)$ has a compact support in Ω_T. Then there exists a unique function

$$u \in L^p(0,T; W_0^{1,p}(\Omega))$$

such that

$$-\int_{\Omega_T} u\frac{\partial\varphi}{\partial t}\,\mathrm{d}x\,\mathrm{d}t+\int_{\Omega_T}\mathcal{A}(x,t,\nabla u)\cdot\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t=\int_{\Omega_T}\varphi f\,\mathrm{d}x\,\mathrm{d}t \tag{5.2}$$

for every $\varphi\in C_0^\infty(\Omega_T)$ and

$$\lim_{t\to 0}\frac{1}{t}\int_0^t\int_\Omega |u|^2\,\mathrm{d}x\,\mathrm{d}t=0.$$

In particular, if $f\geq 0$, then u is a supersolution.

The following lemma provides us with a key estimate, cf. Lemma 3.6 above.

Lemma 5.3. *Let u be a solution of* (5.2) *with $f\geq 0$. Then*

$$\int_{\Omega_T}|\nabla\min(u,k)|^p\,\mathrm{d}x\,\mathrm{d}t+\operatorname*{ess\,sup}_{0<t<T}\int_Q\min(u,k)^2\,\mathrm{d}x\leq Ck\int_{\Omega_T}f\,\mathrm{d}x\,\mathrm{d}t, \tag{5.4}$$

for $k=1,2,\ldots$.

Proof. For each $\varphi\in C_0^\infty(\Omega_T)$, the mollification u^ε of u satisfies the regularized equation

$$\int_{\Omega_T}\frac{\partial u^\varepsilon}{\partial t}\varphi\,\mathrm{d}x\,\mathrm{d}t+\int_{\Omega_T}\mathcal{A}(x,t,\nabla u)^\varepsilon\cdot\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t=\int_{\Omega_T}f^\varepsilon\varphi\,\mathrm{d}x\,\mathrm{d}t \tag{5.5}$$

for small enough $\varepsilon>0$. We prove the lemma by establishing a lower bound for the left-hand side, and an upper bound for the right-hand side.

First, we choose a piecewise linear approximation $\chi_h, h>\varepsilon$, of $\chi_{(0,T)}$ such that

$$\begin{cases}\frac{\partial\chi_h}{\partial t}=1/h, & \text{if } h<t<2h,\\ \chi_h=1, & \text{if } 2h<t<T-2h,\\ \frac{\partial\chi_h}{\partial t}=-1/h, & \text{if } T-2h<t<T-h,\\ \chi_h=0, & \text{otherwise,}\end{cases}$$

and set $u_k^\varepsilon=\min(u^\varepsilon,k)$. We use $\varphi=u_k^\varepsilon\chi_h$ (here $\varphi=0$, if $t\leq h$ or $t\geq T-h$) as a test function, observing that χ_h gives enough room for the mollification because $h>\varepsilon$. We have

$$\frac{\partial u^\varepsilon}{\partial t}u_k^\varepsilon=\frac{\partial u_k^\varepsilon}{\partial t}u_k^\varepsilon+k\frac{\partial(u^\varepsilon-k)_+}{\partial t}.$$

Thus the first term in the left-hand side of (5.5) becomes, after integration by parts,

$$-\int_{\Omega_T}\frac{1}{2}(u_k^\varepsilon)^2\frac{\partial\chi_h}{\partial t}\,\mathrm{d}x\,\mathrm{d}t-\int_{\Omega_T}k(u^\varepsilon-k)_+\frac{\partial\chi_h}{\partial t}\,\mathrm{d}x\,\mathrm{d}t.$$

Next, we would like to let $\varepsilon\to 0$, but we only know that u_k^ε converges to u_k strongly for almost all real values of k. To deal with this, let us assume that an increasing sequence of numbers k such that the convergence holds has been chosen; then the conclusion of the lemma holds for these numbers, and this technicality plays no further role. We get the limit

$$-\frac{1}{h}\int_h^{2h}\int_\Omega \frac{1}{2}u_k^2\,\mathrm{d}x\,\mathrm{d}t+\frac{1}{h}\int_{T-2h}^{T-h}\int_\Omega \frac{1}{2}u_k^2\,\mathrm{d}x\,\mathrm{d}t$$
$$-\frac{1}{h}\int_h^{2h}\int_\Omega k(u-k)_+\,\mathrm{d}x\,\mathrm{d}t+\frac{1}{h}\int_{T-2h}^{T-h}\int_\Omega k(u-k)_+\,\mathrm{d}x\,\mathrm{d}t$$

as $\varepsilon\to 0$. The negative terms in the above expression vanish as $h\to 0$ by the initial condition while the positive terms can be ignored since we are proving a lower bound.

The second term on the left-hand side reads

$$\int_{\Omega_T}\mathcal{A}(x,t,\nabla u)^\varepsilon\cdot\nabla(u_k^\varepsilon\chi_h)\,\mathrm{d}x\,\mathrm{d}t.$$

Here, we can simply let $\varepsilon\to 0$, and then $h\to 0$. This and the structure of $\mathcal{A}$ gives us the estimate

$$\alpha\int_{\Omega_T}|\nabla u_k|^p\,\mathrm{d}x\,\mathrm{d}t\le\int_{\Omega_T}\mathcal{A}(x,t,\nabla u_k)\cdot\nabla u_k\,\mathrm{d}x\,\mathrm{d}t.$$

To deal with the right-hand side of (5.5), we note that

$$\int_{\Omega_T}u_kf\chi_h\,\mathrm{d}x\,\mathrm{d}t\le\int_{\Omega_T}u_kf\,\mathrm{d}x\,\mathrm{d}t\le k\int_{\Omega_T}f\,\mathrm{d}x\,\mathrm{d}t.$$

Furthermore, the first term in the above estimate equals in the limit with the right-hand side of (5.5) as $\varepsilon\to 0$.

We have so far proved that

$$\int_{\Omega_T}|\nabla u_k|^p\,\mathrm{d}x\,\mathrm{d}t\le Ck\int_{\Omega_T}f\,\mathrm{d}x\,\mathrm{d}t. \tag{5.6}$$

To finish the proof, we repeat the above arguments with $\chi_{(0,T)}$ replaced by $\chi_{(0,\tau)}$, where $0<\tau<T$ is chosen so that

$$\int_\Omega u_k(x,\tau)\,\mathrm{d}x\ge\frac{1}{2}\operatorname*{ess\,sup}_{0<t<T}\int_\Omega u_k(x,t)\,\mathrm{d}x.$$

By the choice of τ, we obtain the inequality

$$\int_{\Omega_\tau}|\nabla u_k|^p\,\mathrm{d}x\,\mathrm{d}t+\operatorname*{ess\,sup}_{0<t<T}\int_\Omega u_k(x,t)^2\,\mathrm{d}x\le Ck\int_{\Omega_T}f\,\mathrm{d}x\,\mathrm{d}t. \tag{5.7}$$

A combination of (5.6) and (5.7) now completes the proof. □

Next we establish the existence of a solution to the measure data problem.

Theorem 5.8. *Let μ be a finite positive Radon measure in Ω_T. Then there is an $\mathcal{A}$-superparabolic function u in Ω_T such that $\min(u,k)\in L^p(0,T;W^{1,p}(\Omega))$ for all $k>0$ and*

$$\frac{\partial u}{\partial t}-\operatorname{div}\mathcal{A}(x,t,\nabla u)=\mu$$

in the weak sense.

Proof. Let (f_j) be the approximating sequence to μ obtained from Lemma 5.1 and denote by (u_j) the corresponding sequence of supersolutions satisfying (5.2).

By Theorem 4.3, there is an $\mathcal{A}$-hyperparabolic function u such that we can assume that

$$u_j \to u \quad \text{and} \quad \nabla \min(u_j, k) \to \nabla \min(u, k)$$

almost everywhere by passing to a subsequence.

As the first step, we prove that u is finite almost everywhere, and thus u is $\mathcal{A}$-superparabolic. To this end, according to Lemmas 5.3 and 5.1, we have

$$\int_{\Omega_T} |\nabla \min(u_j, k)|^p \, dx \, dt \leq Ck \int_{\Omega_T} f_j \, \mathrm{d}x \, \mathrm{d}t \leq C\mu(\Omega_T)k. \tag{5.9}$$

Since $\min(u_j, k) \in L^p(0, T; W_0^{1,p}(\Omega))$, the Sobolev-Poincaré inequality and (5.9) imply

$$\begin{aligned} \int_{\Omega_T} |\min(u_j, k)|^p \, \mathrm{d}x \, \mathrm{d}t &\leq C \int_{\Omega_T} |\nabla \min(u_j, k)|^p \, \mathrm{d}x \, \mathrm{d}t \\ &\leq C\mu(\Omega_T)k, \end{aligned} \tag{5.10}$$

where C is independent of k and j. Since $u_j \to u$ almost everywhere, it follows from Fatou's lemma and (5.10) that

$$\int_{\Omega_T} |\min(u, k)|^p \, \mathrm{d}x \, \mathrm{d}t \leq C\mu(\Omega_T)k.$$

This estimate implies that u is finite almost everywhere. Indeed, denoting

$$E = \{(x, t) \in \Omega_T \, : \, u(x, t) = \infty\},$$

we have

$$|E| = \frac{1}{k^p} \int_E k^p \, dx \, dt \leq \frac{1}{k^p} \int_{\Omega_T} |\min(u, k)|^p \, dx \, dt \leq Ck^{1-p} \to 0$$

as $k \to \infty$. Thus, u is $\mathcal{A}$-superparabolic and by Theorem 3.9, there exists a measure ν such that

$$\frac{\partial u}{\partial t} - \operatorname{div} \mathcal{A}(x, t, \nabla u) = \nu \tag{5.11}$$

in the weak sense.

We will complete the proof by showing that $\mu = \nu$. The constants on the right-hand sides of (5.9) and (5.10) are independent of j. Thus Lemma 3.6 implies that the sequence $(|\nabla u_j|^{p-1})$ is bounded in $L^q(\Omega_T)$ for some $q > 1$. Next we use the structure of $\mathcal{A}$, and obtain

$$\int_{\Omega_T} |\mathcal{A}(x, t, \nabla u_j)|^q \, \mathrm{d}x \, \mathrm{d}t \leq C \int_{\Omega_T} |\nabla u_j|^{q(p-1)} \, \mathrm{d}x \, \mathrm{d}t \leq C.$$

Thus the sequence $(\mathcal{A}(x,t,\nabla u_j))$ is also bounded in $L^q(\Omega_T)$, and it follows from the pointwise convergence of ∇u_j to ∇u, and the continuity of $\xi \mapsto \mathcal{A}(x,t,\xi)$ that $\mathcal{A}(x,t,\nabla u_j) \to \mathcal{A}(x,t,\nabla u)$ pointwise almost everywhere, and thus weakly in $L^q(\Omega_T)$ at least for a subsequence, since the pointwise limit identifies the weak limit. Similarly, the sequence (u_j) is bounded in $L^{(p-1)q}(\Omega_T)$ and thus a subsequence converges weakly in $L^{(p-1)q}(\Omega_T)$. We use the weak convergences and (5.11) to conclude that

$$\begin{aligned}\lim_{j\to\infty}\int_{\Omega_T}\varphi \,\mathrm{d}\mu_j &= \lim_{j\to\infty}\int_{\Omega_T} -u_j\frac{\partial\varphi}{\partial t} + \mathcal{A}(x,t,\nabla u_j)\cdot\nabla\varphi \,\mathrm{d}x\,\mathrm{d}t \\ &= \int_{\Omega_T} -u\frac{\partial\varphi}{\partial t} + \mathcal{A}(x,t,\nabla u)\cdot\nabla\varphi \,\mathrm{d}x\,\mathrm{d}t \\ &= \int_{\Omega_T}\varphi\,\mathrm{d}\nu,\end{aligned}$$

which completes the proof. □

Observe that we can not directly deduce from the boundedness of gradients that $(\mathcal{A}(x,t,\nabla u_j))$ converges weakly to $\mathcal{A}(x,t,\nabla u)$ above. The additional information needed is the pointwise convergence of the gradients from Theorem 4.3 and the continuity of $\mathcal{A}$ with respect to the gradient variable.

We close the paper by recording the following simple observation. Note that the current tools do not allow us to prove the claim for *any* solution of (5.13), since solutions to equations involving measures are not unique in general. Recall that in a general open set V of $\mathbf{R}^{n+1}$ u is a solution if it is a solution in all sets $U_{t_2,t_2} \Subset V$.

Theorem 5.12. *If u is a weak solution of*

$$\frac{\partial u}{\partial t} - \operatorname{div}\mathcal{A}(x,t,\nabla u) = \mu \tag{5.13}$$

in Ω_T given by Theorem 5.8, then u is a weak solution of

$$\frac{\partial u}{\partial t} - \operatorname{div}\mathcal{A}(x,t,\nabla u) = 0 \tag{5.14}$$

in $\Omega_T \setminus \operatorname{spt}\mu$.

Proof. The proof consists of verifying two facts. First, we must check that the limit has the right a priori integrability, and then show that it satisfies the weak formulation.

Let (μ_j) be the approximating sequence of μ from Lemma 5.1. From the proof of the lemma, we see that the support of μ_j is contained in the set

$$E_j = \{(x,t)\in\Omega_T : \operatorname{dist}(z,\operatorname{spt}\mu) \le c\,2^{-j}\},$$

where the constant c is independent of j. Thus the corresponding supersolution u_j is a nonnegative solution of (5.14) in $\Omega_T \setminus \overline{E}_j$.

Pick any set $U_{t_1,t_2} \Subset \Omega_T \setminus \operatorname{spt}\mu$. Then $U_{t_1,t_2} \Subset \Omega_T \setminus \overline{E}_j$ for all sufficiently large j. We take the subsequence from the proof of the previous theorem with uniform bounds in $L^{(p-1)q}(U_{t_1,t_2})$, $q > 1$, converging to a limit u. We combine the bound for the sequence (u_j) in $L^{(p-1)q}(U_{t_1,t_2})$ with a weak Harnack estimate (see [4] or [11]) to conclude that the sequence (u_j) is bounded in U_{t_1,t_2}, and hence the limit function u is also bounded. The boundedness of u and Lemma 4.2 imply that u belongs to $L^p(t_1, t_2; W^{1,p}(U))$.

We are left with the task of checking the weak formulation. Recall from the proof of Theorem 5.8 that (u_j) and $(\mathcal{A}(x,t,\nabla u_j))$ converge weakly in $L^q(\Omega_T)$ to u and $\mathcal{A}(x,t,\nabla u)$, respectively. This implies that

$$\begin{aligned} 0 &= \lim_{j\to\infty} \left(-\int_{t_1}^{t_2}\int_U u_j \frac{\partial\varphi}{\partial t}\,\mathrm{d}x\,\mathrm{d}t + \int_{t_1}^{t_2}\int_U \mathcal{A}(x,t,\nabla u_j)\cdot\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t \right) \\ &= -\int_{t_1}^{t_2}\int_U u \frac{\partial\varphi}{\partial t}\,\mathrm{d}x\,\mathrm{d}t + \int_{t_1}^{t_2}\int_U \mathcal{A}(x,t,\nabla u)\cdot\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t \end{aligned}$$

for all $\varphi \in C_0^\infty(U_{t_1,t_2})$. Since $U_{t_1,t_2} \Subset \Omega_T \setminus \operatorname{spt}\mu$ was arbitrary, this implies that u is a solution in $\Omega_T \setminus \operatorname{spt}\mu$. □

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