Maximal Functions in Sobolev Spaces

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Abstract Applications of the Hardy–Littlewood maximal functions in the modern theory of partial differential equations are considered. In particular, we discuss the behavior of maximal functions in Sobolev spaces, Hardy inequalities, and approximation and pointwise behavior of Sobolev functions. We also study the corresponding questions on metric measure spaces.

1 Introduction

The centered Hardy–Littlewood maximal function $Mf : \mathbf{R}^n \to [0, \infty]$ of a locally integrable function $f : \mathbf{R}^n \to [-\infty, \infty]$ is defined by

$$Mf(x) = \sup \fint_{B(x,r)} |f(y)| \, dy,$$

where the supremum is taken over all radii $r > 0$. Here

$$\fint_{B(x,r)} |f(y)| \, dy = \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)| \, dy$$

denotes the integral average and $|B(x,r)|$ is the volume of the ball $B(x,r)$. There are several variations of the definition in the literature, for example,

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depending on the requirement whether x is at the center of the ball or not. These definitions give maximal functions that are equivalent with two-sided estimates.

The maximal function theorem of Hardy, Littlewood, and Wiener asserts that the maximal operator is bounded in $L^p(\mathbf{R}^n)$ for $1 < p \leqslant \infty$,

$$\|Mf\|_p \leqslant c\|f\|_p, \tag{1.1}$$

where $c = c(n,p)$ is a constant. The case $p = \infty$ follows immediately from the definition of the maximal function. It can be shown that for the centered maximal function the constant depends only on p, but we do not need this fact here. For $p = 1$ we have the weak type estimate

$$|\{x \in \mathbf{R}^n : Mf(x) > \lambda\}| \leqslant c\lambda^{-1}\|f\|_1$$

for every $\lambda > 0$ with $c = c(n)$ (see [64]).

The maximal functions are classical tools in harmonic analysis. They are usually used to estimate absolute size, and their connections to regularity properties are often neglected. The purpose of this exposition is to focus on this issue. Indeed, applications to Sobolev functions and to partial differential equations indicate that it is useful to know how the maximal operator preserves the smoothness of functions.

There are two competing phenomena in the definition of the maximal function. The integral average is smoothing but the supremum seems to reduce the smoothness. The maximal function is always lower semicontinuous and preserves the continuity of the function provided that the maximal function is not identically infinity. In fact, if the maximal function is finite at one point, then it is finite almost everywhere. A result of Coifman and Rochberg states that the maximal function raised to a power which is strictly between zero and one is a Muckenhoupt weight. This is a clear evidence of the fact that the maximal operator may have somewhat unexpected smoothness properties.

It is easy to show that the maximal function of a Lipschitz function is again Lipschitz and hence, by the Rademacher theorem is differentiable almost everywhere. The question about differentiability in general is a more delicate one.

Simple one-dimensional examples show that the maximal function of a differentiable function is not differentiable in general. Nevertheless, certain weak differentiability properties are preserved under the maximal operator. Indeed, the Hardy–Littlewood maximal operator preserves the first order Sobolev spaces $W^{1,p}(\mathbf{R}^n)$ with $1 < p \leqslant \infty$, and hence it can be used as a test function in the theory of partial differential equations. More precisely, the maximal operator is bounded in the Sobolev space and for every $1 < p \leqslant \infty$ we have

$$\|Mu\|_{1,p} \leqslant c\|u\|_{1,p}$$

with $c = c(n,p)$. We discuss different aspects related to this result.

The maximal functions can also be used to study the smoothness of the original function. Indeed, there are pointwise estimates for the function in terms of the maximal function of the gradient. If $u \in W^{1,p}(\mathbf{R}^n)$, $1 \leqslant p \leqslant \infty$, then there is a set E of measure zero such that

$$|u(x) - u(y)| \leqslant c|x-y|\big(M|Du|(x) + M|Du|(y)\big)$$

for all $x, y \in \mathbf{R}^n \setminus E$. If $1 < p \leqslant \infty$, then the maximal function theorem implies that $M|Du| \in L^p(\mathbf{R}^n)$. This observation has fundamental consequences in the theory of partial differential equations. Roughly speaking, the oscillation of the function is small on the good set where the maximal function of the gradient is bounded. The size of the bad set can be estimated by the maximal function theorem. This can also be used to define Sobolev type spaces in a very general context of metric measure spaces. To show that our arguments are based on a general principle, we also consider the smoothness of the maximal function in this case. The results can be used to study the pointwise behavior of Sobolev functions.

2 Maximal Function Defined on the Whole Space

Recall that the Sobolev space $W^{1,p}(\mathbf{R}^n)$, $1 \leqslant p \leqslant \infty$, consists of functions $u \in L^p(\mathbf{R}^n)$ whose weak first order partial derivatives $D_i u$, $i = 1, 2, \ldots, n$, belong to $L^p(\mathbf{R}^n)$. We endow $W^{1,p}(\mathbf{R}^n)$ with the norm

$$\|u\|_{1,p} = \|u\|_p + \|Du\|_p,$$

where $Du = (D_1 u, D_2 u, \ldots, D_n u)$ is the weak gradient of u. Equivalently, if $1 \leqslant p < \infty$, the Sobolev space can be defined as the completion of smooth functions with respect to the norm above. For basic properties of Sobolev functions we refer to [17].

2.1 Boundedness in Sobolev spaces

Suppose that u is Lipschitz continuous with constant L, i.e.,

$$|u_h(y) - u(y)| = |u(y+h) - u(y)| \leqslant L|h|$$

for all $y, h \in \mathbf{R}^n$, where $u_h(y) = u(y+h)$. Since the maximal function commutes with translations and the maximal operator is sublinear, we have

$$\begin{aligned}|(Mu)_h(x) - Mu(x)| &= |M(u_h)(x) - Mu(x)| \leqslant M(u_h - u)(x)\\ &= \sup_{r>0} \frac{1}{|B(x,r)|} \int\limits_{B(x,r)} |u_h(y) - u(y)|\, dy \leqslant L|h|. \end{aligned} \tag{2.1}$$

This means that the maximal function is Lipschitz continuous with the same constant as the original function provided that Mu is not identically infinity [19]. Observe that this proof applies to Hölder continuous functions as well [14].

It is shown in [33] that the Hardy–Littlewood maximal operator is bounded in the Sobolev space $W^{1,p}(\mathbf{R}^n)$ for $1 < p \leqslant \infty$ and hence, in that case, it has classical partial derivatives almost everywhere. Indeed, there is a simple proof based on the characterization of $W^{1,p}(\mathbf{R}^n)$ with $1 < p < \infty$ by integrated difference quotients according to which $u \in L^p(\mathbf{R}^n)$ belongs to $W^{1,p}(\mathbf{R}^n)$ if and only if there is a constant c for which

$$\|u_h - u\|_p \leqslant c\|Du\|_p|h|$$

for every $h \in \mathbf{R}^n$. As in (2.1), we have

$$|M(u_h) - Mu| \leqslant M(u_h - u)$$

and, by the Hardy–Littlewood–Wiener maximal function theorem, we conclude that

$$\begin{aligned}\|(Mu)_h - Mu\|_p = \|M(u_h) - Mu\|_p &\leqslant \|M(u_h - u)\|_p\\ &\leqslant c\|u_h - u\|_p \leqslant c\|Du\|_p|h|\end{aligned}$$

for every $h \in \mathbf{R}^n$, from which the claim follows. A more careful analysis gives even a pointwise estimate for the partial derivatives. The following simple proposition is used several times in the sequel. If $f_j \to f$ and $g_j \to g$ weakly in $L^p(\Omega)$ and $f_j(x) \leqslant g_j(x)$, $j = 1, 2, \dots$, almost everywhere in Ω, then $f(x) \leqslant g(x)$ almost everywhere in Ω. Together with some basic properties of the first order Sobolev spaces, this implies that the maximal function semicommutes with weak derivatives. This is the content of the following result which was first proved in [33], but we recall the simple argument here (see also [40, 41]).

Theorem 2.2. *Let $1 < p < \infty$. If $u \in W^{1,p}(\mathbf{R}^n)$, then $Mu \in W^{1,p}(\mathbf{R}^n)$ and*

$$|D_i Mu| \leqslant MD_i u, \quad i = 1, 2, \dots, n, \tag{2.3}$$

almost everywhere in $\mathbf{R}^n$.

Proof. If $\chi_{B(0,r)}$ is the characteristic function of $B(0,r)$ and

$$\chi_r = \frac{\chi_{B(0,r)}}{|B(0,r)|},$$

then

$$\frac{1}{|B(x,r)|} \int\limits_{B(x,r)} |u(y)|\,dy = |u| * \chi_r(x),$$

where $*$ denotes convolution. Now $|u| * \chi_r \in W^{1,p}(\mathbf{R}^n)$ and

$$D_i(|u| * \chi_r) = \chi_r * D_i|u|, \quad i = 1, 2, \dots, n,$$

almost everywhere in $\mathbf{R}^n$.

Let r_j, $j = 1, 2, \dots$, be an enumeration of the positive rational numbers. Since u is locally integrable, we may restrict ourselves to positive rational radii in the definition of the maximal function. Hence

$$Mu(x) = \sup_j (|u| * \chi_{r_j})(x).$$

We define functions $v_k : \mathbf{R}^n \to \mathbf{R}$, $k = 1, 2, \dots$, by

$$v_k(x) = \max_{1 \leqslant j \leqslant k} (|u| * \chi_{r_j})(x).$$

Now (v_k) is an increasing sequence of functions in $W^{1,p}(\mathbf{R}^n)$ which converges to Mu pointwise and

$$\begin{aligned} |D_i v_k| &\leqslant \max_{1 \leqslant j \leqslant k} |D_i(|u| * \chi_{r_j})| = \max_{1 \leqslant j \leqslant k} |\chi_{r_j} * D_i|u|| \\ &\leqslant MD_i|u| = MD_i u, \end{aligned}$$

$i = 1, 2, \dots, n$, almost everywhere in $\mathbf{R}^n$. Here we also used the fact that $|D_i|u|| = |D_i u|$, $i = 1, 2, \dots, n$, almost everywhere. Thus,

$$\|Dv_k\|_p \leqslant \sum_{i=1}^n \|D_i v_k\|_p \leqslant \sum_{i=1}^n \|MD_i u\|_p$$

and the maximal function theorem implies

$$\begin{aligned} \|v_k\|_{1,p} &\leqslant \|Mu\|_p + \sum_{i=1}^n \|MD_i u\|_p \\ &\leqslant c\|u\|_p + c\sum_{i=1}^n \|D_i u\|_p \leqslant c < \infty \end{aligned}$$

for every $k = 1, 2, \dots$. Hence (v_k) is a bounded sequence in $W^{1,p}(\mathbf{R}^n)$ which converges to Mu pointwise. By the weak compactness of Sobolev spaces,

$Mu \in W^{1,p}(\mathbf{R}^n)$, v_k converges to Mu weakly in $L^p(\mathbf{R}^n)$, and $D_i v_k$ converges to $D_i Mu$ weakly in $L^p(\mathbf{R}^n)$. Since $|D_i v_k| \leqslant MD_i u$ almost everywhere, the weak convergence implies

$$|D_i Mu| \leqslant MD_i u, \quad i = 1, 2, \dots, n,$$

almost everywhere in $\mathbf{R}^n$. □

Remark 2.4. (i) The case $p = 1$ is excluded in the theorem because our arguments fail in that case. However, Tanaka [66] proved, in the one-dimensional case, that if $u \in W^{1,1}(\mathbf{R})$, then the noncentered maximal function is differentiable almost everywhere and

$$\|DMu\|_1 \leqslant 2\|Du\|_1.$$

For extensions of Tanaka's result to functions of bounded variation in the one-dimensional case we refer to [3] and [4]. The question about the counterpart of Tanaka's result remains open in higher dimensions (see also discussion in [26]). Observe that

$$\|u\|_{n/n-1} \leqslant c\|Du\|_1$$

by the Sobolev embedding theorem and $Mu \in L^{n/(n-1)}(\mathbf{R}^n)$ by the maximal function theorem. However, the behavior of the derivatives is not well understood in this case.

(ii) The inequality (2.3) implies that

$$|DMu(x)| \leqslant M|Du|(x) \tag{2.5}$$

for almost all $x \in \mathbf{R}^n$. Fix a point at which the gradient $DMu(x)$ exists. If $|DMu(x)| = 0$, then the claim is obvious. Hence we may assume that $|DMu(x)| \neq 0$. Let

$$e = \frac{DMu(x)}{|DMu(x)|}.$$

Rotating the coordinates in the proof of the theorem so that e coincides with some of the coordinate directions, we get

$$|DMu(x)| = |D_e Mu(x)| \leqslant MD_h u(x) \leqslant M|Du|(x),$$

where $D_e u = Du \cdot e$ is the derivative to the direction of the unit vector e.

(iii) Using the maximal function theorem together with (2.3), we find

$$\begin{aligned}\|Mu\|_{1,p} &= \|Mu\|_p + \|DMu\|_p \\ &\leqslant c\|u\|_p + \|M|Du|\|_p \leqslant c\|u\|_{1,p},\end{aligned} \tag{2.6}$$

where c is the constant in (1.1). Hence

$$M : W^{1,p}(\mathbf{R}^n) \to W^{1,p}(\mathbf{R}^n)$$

is a bounded operator, where $1 < p < \infty$.

(iv) If $u \in W^{1,\infty}(\mathbf{R}^n)$, then a slight modification of our proof shows that Mu belongs to $W^{1,\infty}(\mathbf{R}^n)$. Moreover,

$$\begin{aligned}\|Mu\|_{1,\infty} &= \|Mu\|_\infty + \|DMu\|_\infty \\ &\leqslant \|u\|_\infty + \|M|Du|\|_\infty \leqslant \|u\|_{1,\infty}.\end{aligned}$$

Hence, in this case, the maximal operator is bounded with constant one. Recall that, after a redefinition on a set of measure zero, $u \in W^{1,\infty}(\mathbf{R}^n)$ is a bounded and Lipschitz continuous function.

(v) A recent result of Luiro [53] shows that

$$M : W^{1,p}(\mathbf{R}^n) \to W^{1,p}(\mathbf{R}^n)$$

is a continuous operator. Observe that bounded nonlinear operators are not continuous in general. Luiro employs the structure of the maximal operator. He also obtained an interesting formula for the weak derivatives of the maximal function. Indeed, if $u \in W^{1,p}(\mathbf{R})$, $1 < p < \infty$, and $R(x)$ denotes the set of radii $r \geqslant 0$ for which

$$Mu(x) = \limsup_{r_i \to r} \fint_{B(x,r_i)} |u|\, dy$$

for some sequence (r_i) with $r_i > 0$, then for almost all $x \in \mathbf{R}^n$ we have

$$D_i Mu(x) = \fint_{B(x,r)} D_i|u|\, dy$$

for every strictly positive $r \in R(x)$ and

$$D_i Mu(x) = D_i|u|(x)$$

if $0 \in R(x)$. For this is a sharpening of (2.3) we refer to [53, Theorem 3.1] (see also [55]).

(vi) Let $0 \leqslant \alpha \leqslant n$. The fractional maximal function of a locally integrable function $f : \mathbf{R}^n \to [-\infty, \infty]$ is defined by

$$M_\alpha f(x) = \sup_{r>0} r^\alpha \fint_{B(x,r)} |f(y)|\, dy.$$

For $\alpha = 0$ we obtain the Hardy–Littlewood maximal function.

Theorem 2.2 can be easily extended to fractional maximal functions. Indeed, suppose that $1 < p < \infty$. Let $0 \leqslant \alpha < n/p$. If $u \in W^{1,p}(\mathbf{R}^n)$, then $M_\alpha u \in W^{1,q}(\mathbf{R}^n)$ with $q = np/(n - \alpha p)$ and

$$|D_i M_\alpha u| \leqslant M_\alpha D_i u, \qquad i = 1, 2, \dots, n,$$

almost everywhere in $\mathbf{R}^n$. Moreover, there is $c = c(n, p, \alpha)$ such that

$$\|M_\alpha u\|_{1,q} \leqslant c\,\|u\|_{1,p}.$$

The main result of [39] shows that the fractional maximal operator is smoothing in the sense that it maps L^p-spaces into certain first order Sobolev spaces.

2.2 A capacitary weak type estimate

As an application, we show that a weak type inequality for the Sobolev capacity follows immediately from Theorem 2.2. The standard proofs seem to depend, for example, on certain extension properties of Sobolev functions (see [17]). Let $1 < p < \infty$. The Sobolev p-capacity of the set $E \subset \mathbf{R}^n$ is defined by

$$\operatorname{cap}_p(E) = \inf_{u \in \mathcal{A}(E)} \int\limits_{\mathbf{R}^n} \big(|u|^p + |Du|^p\big)\,dx,$$

where

$$\mathcal{A}(E) = \big\{u \in W^{1,p}(\mathbf{R}^n) : u \geqslant 1 \text{ on a neighborhood of } E\big\}.$$

If $\mathcal{A}(E) = \varnothing$, we set $\operatorname{cap}_p(E) = \infty$. The Sobolev p-capacity is a monotone and countably subadditive set function. Let $u \in W^{1,p}(\mathbf{R}^n)$. Suppose that $\lambda > 0$ and denote

$$E_\lambda = \{x \in \mathbf{R}^n : Mu(x) > \lambda\}.$$

Then E_λ is open and $Mu/\lambda \in \mathcal{A}(E_\lambda)$. Using (2.6), we get

$$\begin{aligned}\operatorname{cap}_p\big(E_\lambda\big) &\leqslant \frac{1}{\lambda^p} \int\limits_{\mathbf{R}^n} \big(|Mu|^p + |DMu|^p\big)\,dx \\ &\leqslant \frac{c}{\lambda^p} \int\limits_{\mathbf{R}^n} \big(|u|^p + |Du|^p\big)\,dx \leqslant \frac{c}{\lambda^p}\|u\|_{1,p}^p.\end{aligned}$$

This inequality can be used in the study of the pointwise behavior of Sobolev functions by standard methods. We recall that $x \in \mathbf{R}^n$ is a Lebesgue point for u if the limit

$$u^*(x) = \lim_{r\to 0} \fint_{B(x,r)} u\,dy$$

exists and

$$\lim_{r\to 0} \fint_{B(x,r)} |u(y) - u^*(x)|\,dy = 0.$$

The Lebesgue theorem states that almost all points of a $L^1_{\mathrm{loc}}(\mathbf{R}^n)$ function are Lebesgue points. If a function belongs to $W^{1,p}(\mathbf{R}^n)$, then, using the capacitary weak type estimate, we can prove that the complement of the set of Lebesgue points has zero p-capacity (see [17]).

3 Maximal Function Defined on a Subdomain

Let Ω be an open set in the Euclidean space $\mathbf{R}^n$. For a locally integrable function $f : \Omega \to [-\infty, \infty]$ we define the Hardy–Littlewood maximal function $M_\Omega f : \Omega \to [0, \infty]$ as

$$M_\Omega f(x) = \sup \fint_{B(x,r)} |f(y)|\,dy,$$

where the supremum is taken over all radii $0 < r < \delta(x)$, where

$$\delta(x) = \operatorname{dist}(x, \partial\Omega).$$

In this section, we make the standing assumption that $\Omega \neq \mathbf{R}^n$ so that $\delta(x)$ is finite. Observe that the maximal function depends on Ω. The maximal function theorem implies that the maximal operator is bounded in $L^p(\Omega)$ for $1 < p \leqslant \infty$, i.e.,

$$\|M_\Omega f\|_{p,\Omega} \leqslant c\|f\|_{p,\Omega}. \tag{3.1}$$

This follows directly from (1.1) by considering the zero extension to the complement. The Sobolev space $W^{1,p}(\Omega)$, $1 \leqslant p \leqslant \infty$, consists of those functions u which, together with their weak first order partial derivatives $Du = (D_1 u, \dots, D_n u)$, belong to $L^p(\Omega)$. When $1 \leqslant p < \infty$, we may define $W^{1,p}(\Omega)$ as the completion of smooth functions with respect to the Sobolev norm.

3.1 Boundedness in Sobolev spaces

We consider the counterpart of Theorem 2.2 for the maximal operator M_Ω. It turns out that the arguments in the previous section do not apply mainly because the maximal operator M_Ω does not commute with translations. The following result was proved in [35]. We also refer to [26] for an alternative approach.

Theorem 3.2. *Let $1 < p \leqslant \infty$. If $u \in W^{1,p}(\Omega)$, then $M_\Omega u \in W^{1,p}(\Omega)$ and*

$$|DM_\Omega u| \leqslant 2M_\Omega |Du|$$

almost everywhere in Ω.

Observe that the result holds for every open set and, in particular, we do not make any regularity assumption on the boundary. The functions $u_t : \Omega \to [-\infty, \infty]$, $0 < t < 1$, defined by

$$u_t(x) = \fint_{B(x,t\delta(x))} |u(y)|\, dy,$$

will play a crucial role in the proof of Theorem 3.2 because

$$M_\Omega u(x) = \sup_{0<t<1} u_t(x)$$

for every $x \in \Omega$. We begin with an auxiliary result which may be of independent interest.

Lemma 3.3. *Let Ω be an open set in $\mathbf{R}^n$, and let $1 < p \leqslant \infty$. Suppose that $u \in W^{1,p}(\Omega)$. Then for every $0 < t < 1$ we have $u_t \in W^{1,p}(\Omega)$ and*

$$|Du_t(x)| \leqslant 2M_\Omega |Du|(x) \tag{3.4}$$

for almost all $x \in \Omega$.

Proof. Since $|u| \in W^{1,p}(\Omega)$ and $|D|u|| = |Du|$ almost everywhere in Ω, we may assume that u is nonnegative. Suppose first that $u \in C^\infty(\Omega)$. Let t, $0 < t < 1$, be fixed. According to the Rademacher theorem, as a Lipschitz function δ is differentiable almost everywhere in Ω. Moreover, $|D\delta(x)| = 1$ for almost all $x \in \Omega$. The Leibniz rule gives

$$\begin{aligned} D_i u_t(x) = &D_i\Big(\frac{1}{\omega_n (t\delta(x))^n}\Big) \cdot \int_{B(x,t\delta(x))} u(y)\, dy \\ &+ \frac{1}{\omega_n (t\delta(x))^n} \cdot D_i \int_{B(x,t\delta(x))} u(y)\, dy \end{aligned}$$

for almost all $x \in \Omega$, and, by the chain rule,

$$D_i \int_{B(x,t\delta(x))} u(y)\,dy = \int_{B(x,t\delta(x))} D_i u(y)\,dy$$

$$+ t \int_{\partial B(x,t\delta(x))} u(y)\,d\mathcal{H}^{n-1}(y) \cdot D_i\delta(x)$$

for almost all $x \in \Omega$. Here we also used the fact that

$$\frac{\partial}{\partial r} \int_{B(x,r)} u(y)\,dy = \int_{\partial B(x,r)} u(y)\,dy.$$

Collecting terms, we obtain

$$D_i u_t(x) = n\frac{D_i\delta(x)}{\delta(x)}\Big(\fint_{\partial B(x,t\delta(x))} u(y)\,d\mathcal{H}^{n-1}(y)$$

$$- \fint_{B(x,t\delta(x))} u(y)\,dy\Big) + \fint_{B(x,t\delta(x))} D_i u(y)\,dy \tag{3.5}$$

for almost all $x \in \Omega$ and every $i = 1, 2, \dots, n$.

In order to estimate the difference of the two integrals in the parentheses in (3.5), we have to take into account a cancellation effect. To this end, suppose that $B(x,R) \subset \Omega$. We use the first Green identity

$$\int_{\partial B(x,R)} u(y)\frac{\partial v}{\partial \nu}(y)\,d\mathcal{H}^{n-1}(y)$$

$$= \int_{B(x,R)} \big(u(y)\Delta v(y) + Du(y)\cdot Dv(y)\big)\,dy,$$

where $\nu(y) = (y-x)/R$ is the unit outer normal of $B(x,R)$, and we choose

$$v(y) = \frac{|y-x|^2}{2}.$$

With these choices the Green formula reads

$$\fint_{\partial B(x,R)} u(y)\,d\mathcal{H}^{n-1}(y) - \fint_{B(x,R)} u(y)\,dy$$

$$= \frac{1}{n} \fint_{B(x,R)} Du(y) \cdot (y - x)\, dy.$$

We estimate the right-hand side of the previous equality by

$$\Big| \fint_{B(x,R)} Du(y) \cdot (y - x)\, dy \Big| \leqslant R \fint_{B(x,R)} |Du(y)|\, dy$$
$$\leqslant R M_\Omega |Du|(x).$$

Finally, we conclude that

$$\Big| \fint_{\partial B(x,R)} u(y)\, d\mathcal{H}^{n-1}(y) - \fint_{B(x,R)} u(y)\, dy \Big| \leqslant \frac{R}{n} M_\Omega |Du|(x). \tag{3.6}$$

Let e be a unit vector. Using (3.5), (3.6) with $R = t\delta(x)$, and the Schwarz inequality, we find

$$\begin{aligned} &|Du_t(x) \cdot e| \\ &\leqslant n \frac{|e \cdot D\delta(x)|}{\delta(x)} \cdot \frac{t\delta(x)}{n} M|Du|(x) + \Big| \fint_{B(x,t\delta(x))} e \cdot Du(y)\, dy \Big| \\ &\leqslant tM|Du|(x) + \fint_{B(x,t\delta(x))} |Du(y)|\, dy \\ &\leqslant (t+1) M_\Omega |Du|(x) \end{aligned}$$

for almost all $x \in \Omega$. Since $t \leqslant 1$ and e is arbitrary, (3.4) is proved for nonnegative smooth functions.

The case $u \in W^{1,p}(\Omega)$ with $1 < p < \infty$ follows from an approximation argument. Indeed, suppose that $u \in W^{1,p}(\Omega)$ for some p with $1 < p < \infty$. Then there is a sequence (φ_j) of functions in $W^{1,p}(\Omega) \cap C^\infty(\Omega)$ such that $\varphi_j \to u$ in $W^{1,p}(\Omega)$ as $j \to \infty$.

Fix t with $0 < t < 1$. We see that

$$u_t(x) = \lim_{j \to \infty} (\varphi_j)_t(x)$$

if $x \in \Omega$. It is clear that

$$(\varphi_j)_t(x) = \fint_{B(x,t\delta(x))} |\varphi_j(y)|\, dy \leqslant M_\Omega \varphi_j(x)$$

for every $x \in \Omega$. By (3.4), for smooth functions we have

$$|D(\varphi_j)_t(x)| \leqslant 2M_\Omega|D\varphi_j|(x) \tag{3.7}$$

for almost all $x \in \Omega$ and every $j = 1, 2 \ldots$. These inequalities and the maximal function theorem imply that

$$\begin{aligned}\|(\varphi_j)_t\|_{1,p,\Omega} &= \|(\varphi_j)_t\|_{p,\Omega} + \|D(\varphi_j)_t\|_{p,\Omega} \\ &\leqslant c\big(\|\varphi_j\|_{p,\Omega} + \|D\varphi_j\|_{p,\Omega}\big) = c\|\varphi_j\|_{1,p,\Omega}.\end{aligned}$$

Thus, $((\varphi_j)_t)_{j=1}^\infty$ is a bounded sequence in $W^{1,p}(\Omega)$ and, since it converges to u_t pointwise, we conclude that the Sobolev derivative Du_t exists and $D(\varphi_j)_t \to Du_t$ weakly in $L^p(\Omega)$ as $j \to \infty$. This is a standard argument which gives the desired conclusion that u_t belongs to $W^{1,p}(\Omega)$. To establish the inequality (3.4), we want to proceed to the limit in (3.7) as $j \to \infty$. Using the sublinearity of the maximal operator and the maximal function theorem once more, we arrive at

$$\begin{aligned}\|M_\Omega|D\varphi_j| - M_\Omega|Du|\|_{p,\Omega} &\leqslant \|M_\Omega(|D\varphi_j| - |Du|)\|_{p,\Omega} \\ &\leqslant c\||D\varphi_j| - |Du|\|_{p,\Omega}.\end{aligned}$$

Hence $M_\Omega|D\varphi_j| \to M_\Omega|Du|$ in $L^p(\Omega)$ as $j \to \infty$. To complete the proof, we apply the proposition mentioned before Theorem 2.2 to (3.7).

Finally, we consider the case $p = \infty$. Slightly modifying the above proof, we see that $u_t \in W^{1,p}_{\text{loc}}(\Omega)$ for every $1 < p < \infty$ and the estimate (2.3) holds for the gradient. The claim follows from the maximal function theorem. This completes the proof. □

The proof of Theorem 3.2 follows now easily since the hard work has been done in the proof of Lemma 3.3. Suppose that $u \in W^{1,p}(\Omega)$ for some $1 < p < \infty$. Then $|u| \in W^{1,p}(\Omega)$. Let t_j, $j = 1, 2, \ldots$, be an enumeration of the rational numbers between 0 and 1. Denote $u_j = u_{t_j}$. By the previous lemma, we see that $u_j \in W^{1,p}(\Omega)$ for every $j = 1, 2, \ldots$ and (3.4) gives us the estimate

$$|Du_j(x)| \leqslant 2M_\Omega|Du|(x)$$

for almost all $x \in \Omega$ and every $j = 1, 2, \ldots$. We define $v_k : \Omega \to [-\infty, \infty]$, $k = 1, 2, \ldots$, as

$$v_k(x) = \max_{1 \leqslant j \leqslant k} u_j(x).$$

Using the fact that the maximum of two Sobolev functions belongs to the Sobolev space, we see that (v_k) is an increasing sequence of functions in $W^{1,p}(\Omega)$ converging to $M_\Omega u$ pointwise and

$$|Dv_k(x)| = |D \max_{1 \leqslant j \leqslant k} u_j(x)| \leqslant \max_{1 \leqslant j \leqslant k} |Du_j(x)| \leqslant 2M_\Omega|Du|(x) \tag{3.8}$$

for almost all $x \in \Omega$ and every $j = 1, 2, \dots$. On the other hand,

$$v_k(x) \leqslant M_\Omega u(x)$$

for all $x \in \Omega$ and $k = 1, 2, \dots$. The rest of the proof goes along the lines of the final part of the proof of Theorem 2.2. By the maximal function theorem,

$$\begin{aligned} \|v_k\|_{1,p,\Omega} &= \|v_k\|_{p,\Omega} + \|Dv_k\|_{p,\Omega} \\ &\leqslant \|M_\Omega u\|_{p,\Omega} + 2\|M_\Omega |Du|\|_{p,\Omega} \leqslant c\|u\|_{1,p,\Omega}. \end{aligned}$$

Hence (v_k) is a bounded sequence in $W^{1,p}(\Omega)$ such that $v_k \to M_\Omega u$ everywhere in Ω as $k \to \infty$. A weak compactness argument shows that $M_\Omega u \in W^{1,p}(\Omega)$, $v_k \to M_\Omega u$, and $Dv_k \to DM_\Omega u$ weakly in $L^p(\Omega)$ as $k \to \infty$. Again, we may proceed to the weak limit in (3.8), using the proposition mentioned before Theorem 2.2.

Let us briefly consider the case $p = \infty$. Using the above argument, it is easy to see that $M_\Omega u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ and the claim follows from the maximal function theorem.

Remark 3.9. Again, it follows immediately that

$$M_\Omega : W^{1,p}(\Omega) \to W^{1,p}(\Omega)$$

is a bounded operator. Luiro [54] shows that it is also a continuous operator for every open set Ω, with $1 < p \leqslant \infty$. In [55], he gives examples of natural maximal operators which are not continuous on Sobolev spaces.

3.2 Sobolev boundary values

We have shown that the local Hardy–Littlewood maximal operator preserves the Sobolev spaces $W^{1,p}(\Omega)$ provided that $1 < p \leqslant \infty$. Next we show that the maximal operator also preserves the boundary values in the Sobolev sense. Recall that the Sobolev space with zero boundary values, denoted by $W^{1,p}_0(\Omega)$ with $1 \leqslant p < \infty$, is defined as the completion of $C^\infty_0(\Omega)$ with respect to the Sobolev norm.

We begin with some useful condition which guarantees that a Sobolev function has zero boundary values in the Sobolev sense. The following result was proved in [36], but we present a very simple proof by Zhong [70, Theorem 1.9]. With a different argument this result also holds in metric measure spaces [32, Theorem 5.1].

Lemma 3.10. *Let $\Omega \neq \mathbf{R}^n$ be an open set. Suppose that $u \in W^{1,p}(\Omega)$. If*

$$\int_{\Omega} \left(\frac{|u(x)|}{\operatorname{dist}(x, \partial\Omega)} \right)^p dx < \infty,$$

then $u \in W_0^{1,p}(\Omega)$.

Proof. For $\lambda > 0$ we define $u_\lambda : \Omega \to [0, \infty]$ by

$$u_\lambda(x) = \min(|u(x)|, \lambda \operatorname{dist}(x, \partial\Omega)).$$

We see that $u_\lambda \in W_0^{1,p}(\Omega)$ for every $\lambda > 0$.

Then we show that (u_λ) is a uniformly bounded family of functions in $W_0^{1,p}(\Omega)$. Clearly, $u_\lambda \leqslant |u|$ and hence

$$\int_{\Omega} u_\lambda^p \, dx \leqslant \int_{\Omega} |u|^p \, dx.$$

For the gradient estimate we define

$$F_\lambda = \{x \in \Omega : |u(x)| > \lambda \operatorname{dist}(x, \partial\Omega)\},$$

where $\lambda > 0$. Then

$$\begin{aligned} \int_{\Omega} |Du_\lambda|^p \, dx &= \int_{\Omega \setminus F_\lambda} |Du|^p \, dx + \lambda^p \int_{F_\lambda} |D \operatorname{dist}(x, \partial\Omega)|^p \, dx \\ &\leqslant \int_{\Omega} |Du|^p \, dx + \lambda^p |F_\lambda|, \end{aligned}$$

where, by assumption,

$$\lambda^p |F_\lambda| \leqslant \int_{\Omega} \left(\frac{|u(x)|}{\operatorname{dist}(x, \partial\Omega)} \right)^p dx < \infty$$

for every $\lambda > 0$. Here we again used the fact that $|D \operatorname{dist}(x, \partial\Omega)| = 1$ for almost all $x \in \Omega$. This implies that (u_λ) is a uniformly bounded family of functions in $W_0^{1,p}(\Omega)$.

Since $|F_\lambda| \to 0$ as $\lambda \to \infty$ and $u_\lambda = |u|$ in $\Omega \setminus F_\lambda$, we have $u_\lambda \to |u|$ almost everywhere in Ω. A similar weak compactness argument that was used in the proofs of Theorems 2.2 and 3.2 shows that $|u| \in W_0^{1,p}(\Omega)$. □

Remark 3.11. The proof shows that, instead of $u/\delta \in L^p(\Omega)$, it is enough to assume that u/δ belongs to the weak $L^p(\Omega)$. Boundary behavior of the maximal function was studied in [37, 35]

Theorem 3.12. *Let* $\Omega \subset \mathbf{R}^n$ *be an open set. Suppose that* $u \in W^{1,p}(\Omega)$ *with* $p > 1$. *Then*

$$|u| - M_\Omega u \in W_0^{1,p}(\Omega).$$

Remark 3.13. In particular, if $u \in W_0^{1,p}(\Omega)$, then $M_\Omega u \in W_0^{1,p}(\Omega)$. Observe that this holds for every open subset Ω.

Proof. Fix $0 < t < 1$. A standard telescoping argument (see Lemma 4.1) gives

$$\begin{aligned}\big||u(x)| - u_t(x)\big| &= \left||u(x)| - \fint_{B(x,t\delta(x))} |u(y)|\,dy\right| \\ &\leqslant ct\operatorname{dist}(x,\partial\Omega)M_\Omega|Du|(x).\end{aligned}$$

For every $x \in \Omega$ there is a sequence t_j, $j = 1, 2, \dots$, such that

$$M_\Omega u(x) = \lim_{j\to\infty} u_{t_j}(x).$$

This implies that

$$\begin{aligned}\big||u(x)| - M_\Omega u(x)\big| &= \lim_{j\to\infty} \big||u(x)| - u_{t_j}(x)\big| \\ &\leqslant c\operatorname{dist}(x,\partial\Omega)M_\Omega|Du|(x).\end{aligned}$$

By the maximal function theorem, we conclude that

$$\begin{aligned}\int_\Omega \left(\frac{\big||u(x)| - M_\Omega u(x)\big|}{\operatorname{dist}(x,\partial\Omega)}\right)^p dx &\leqslant c\int_\Omega (M_\Omega|Du|(x))^p\,dx \\ &\leqslant c\int_\Omega |Du(x)|^p\,dx.\end{aligned}$$

This implies that

$$\frac{|u(x)| - M_\Omega u(x)}{\operatorname{dist}(x,\partial\Omega)} \in L^p(\Omega).$$

By Theorem 3.2, we have $M_\Omega u \in W^{1,p}(\Omega)$, and from Lemma 3.10 we conclude that $|u| - M_\Omega u \in W_0^{1,p}(\Omega)$. □

Remark 3.14. We observe that the maximal operator preserves nonnegative superharmonic functions; see [37]. (For superharmonic functions that change signs, we may consider the maximal function without absolute values.) Suppose that $u : \Omega \to [0,\infty]$ is a measurable function which is not identically ∞ on any component of Ω. Then it is easy to show that

$$M_\Omega u(x) = u(x)$$

for every $x \in \Omega$ if and only if u is superharmonic.

The least superharmonic majorant can be constructed by iterating the maximal function. For short we write

$$M_\Omega^{(k)} u(x) = M_\Omega \circ M_\Omega \circ \cdots \circ M_\Omega u(x), \qquad k = 1, 2, \dots .$$

Since $M_\Omega^{(k)} u$, $k = 1, 2, \dots$, are lower semicontinuous, we see that

$$M_\Omega^{(k)} u(x) \leqslant M_\Omega^{(k+1)} u(x), \qquad k = 1, 2, \dots,$$

for every $x \in \Omega$. Hence $(M_\Omega^{(k)} u(x))$ is an increasing sequence of functions and it converges for every $x \in \Omega$ (the limit may be ∞). We denote

$$M_\Omega^{(\infty)} u(x) = \lim_{k \to \infty} M_\Omega^{(k)} u(x)$$

for every $x \in \Omega$. If $M_\Omega^{(\infty)} u$ is not identically infinity on any component of Ω, then it is the smallest superharmonic function with the property that

$$M_\Omega^{(\infty)} u(x) \geqslant u(x)$$

for almost all $x \in \Omega$. If $u \in W^{1,p}(\Omega)$, then the obtained smallest superharmonic function has the same boundary values as u in the Sobolev sense by Theorem 3.12.

Fiorenza [18] observed that nonnegative functions of one or two variables cannot be invariant under the maximal operator unless they are constant. This is consistent with the fact that on the line there are no other concave functions and in the plane there are no other superharmonic functions but constants that are bounded from below (see also [42]).

4 Pointwise Inequalities

The following estimates are based on a well-known telescoping argument (see [28] and [16]). The proofs are based on a general principle and they apply in a metric measure space equipped with a doubling measure (see [25]). This fact will be useful below.

Let $0 < \beta < \infty$ and $R > 0$. The fractional sharp maximal function of a locally integrable function f is defined by

$$f^{\#}_{\beta,R}(x) = \sup_{0<r<R} r^{-\beta} \fint_{B(x,r)} |f - f_{B(x,r)}| \, dy,$$

If $R = \infty$ we simply write $f^{\#}_{\beta}(x)$.

Lemma 4.1. *Suppose that f is locally integrable. Let $0<\beta<\infty$. Then there is a constant $c=c(\beta,n)$ and a set E with $|E|=0$ such that*

$$|f(x)-f(y)| \leqslant c|x-y|^\beta \big(f^\#_{\beta,4|x-y|}(x)+f^\#_{\beta,4|x-y|}(y)\big) \tag{4.2}$$

for all $x,y\in\mathbf{R}^n\setminus E$.

Proof. Let E be the complement of the set of Lebesgue points of f. By the Lebesgue theorem, $|E|=0$. Fix $x\in\mathbf{R}^n\setminus E$, $0<r<\infty$ and denote $B_i=B(x,2^{-i}r)$, $i=0,1,\dots$. Then

$$\begin{aligned}
|f(x)-f_{B(x,r)}| &\leqslant \sum_{i=0}^\infty |f_{B_{i+1}}-f_{B_i}| \\
&\leqslant \sum_{i=0}^\infty \frac{\mu(B_i)}{\mu(B_{i+1})} \fint_{B_i} |f-f_{B_i}|\,dy \\
&\leqslant c\sum_{i=0}^\infty (2^{-i}r)^\beta (2^{-i}r)^{-\beta} \fint_{B_i} |f-f_{B_i}|\,dy \\
&\leqslant cr^\beta f^\#_{\beta,r}(x).
\end{aligned}$$

Let $y\in B(x,r)\setminus E$. Then $B(x,r)\subset B(y,2r)$ and we obtain

$$\begin{aligned}
|f(y)-f_{B(x,r)}| &\leqslant |f(y)-f_{B(y,2r)}|+|f_{B(y,2r)}-f_{B(x,r)}| \\
&\leqslant cr^\beta f^\#_{\beta,2r}(y)+\fint_{B(x,r)} |f-f_{B(y,2r)}|\,dz \\
&\leqslant cr^\beta f^\#_{\beta,2r}(y)+c\fint_{B(y,2r)} |f-f_{B(y,2r)}|\,dz \\
&\leqslant cr^\beta f^\#_{\beta,2r}(y).
\end{aligned}$$

Let $x,y\in\mathbf{R}^n\setminus E$, $x\neq y$ and $r=2|x-y|$. Then $x,y\in B(x,r)$ and hence

$$\begin{aligned}
|f(x)-f(y)| &\leqslant |f(x)-f_{B(x,r)}|+|f(y)-f_{B(x,r)}| \\
&\leqslant c|x-y|^\beta\big(f^\#_{\beta,4|x-y|}(x)+f^\#_{\beta,4|x-y|}(y)\big).
\end{aligned}$$

This completes the proof. □

Let $0 \leqslant \alpha < 1$ and $R > 0$. The fractional maximal function of a locally integrable function f is defined by

$$M_{\alpha,R}f(x) = \sup_{0<r<R} r^\alpha \fint_{B(x,r)} |f|\,dy,$$

For $R = \infty$, we write $M_{\alpha,\infty} = M_\alpha$. If $\alpha = 0$, we obtain the Hardy–Littlewood maximal function and write $M_0 = M$.

If $u \in W^{1,1}_{\mathrm{loc}}(\mathbf{R}^n)$, then, by the Poincaré inequality, there is a constant $c = c(n)$ such that

$$\fint_{B(x,r)} |u - u_{B(x,r)}|\,dy \leqslant cr \fint_{B(x,r)} |Du|\,dy$$

for every ball $B(x,r) \subset \mathbf{R}^n$. It follows that

$$r^{\alpha-1} \fint_{B(x,r)} |u - u_{B(x,r)}|\,dy \leqslant cr^\alpha \fint_{B(x,r)} |Du|\,dy$$

and, consequently,

$$u^{\#}_{1-\alpha,R}(x) \leqslant cM_{\alpha,R}|Du|(x)$$

for every $x \in \mathbf{R}^n$ and $R > 0$. Thus, we have proved the following useful inequality.

Corollary 4.3. *Let $u \in W^{1,1}_{\mathrm{loc}}(\mathbf{R}^n)$ and $0 \leqslant \alpha < 1$. Then there is a constant $c = c(n,\alpha)$ and a set $E \subset \mathbf{R}^n$ with $|E| = 0$ such that*

$$|u(x) - u(y)| \leqslant c|x-y|^{1-\alpha}\big(M_{\alpha,4|x-y|}|Du|(x) + M_{\alpha,4|x-y|}|Du|(y)\big)$$

for all $x, y \in \mathbf{R}^n \setminus E$.

If $u \in W^{1,p}(\mathbf{R}^n)$, $1 \leqslant p \leqslant \infty$, then

$$|u(x) - u(y)| \leqslant c|x-y|\big(M|Du|(x) + M|Du|(y)\big)$$

for all $x, y \in \mathbf{R}^n \setminus E$. If $1 < p \leqslant \infty$, then the maximal function theorem implies that $g = M|Du| \in L^p(\mathbf{R}^n)$ and, by the previous inequality, we have

$$|u(x) - u(y)| \leqslant c|x-y|\big(g(x) + g(y)\big)$$

for all $x, y \in \mathbf{R}^n \setminus E$ with $|E| = 0$. The following result shows that this gives a characterization of $W^{1,p}(\mathbf{R}^n)$ for $1 < p \leqslant \infty$. This characterization can be used as a definition of the first order Sobolev spaces on metric measure spaces (see [21, 24, 25]).

Theorem 4.4. *Let $1 < p \leqslant \infty$. Then the following four conditions are equivalent.*

(i) $u \in W^{1,p}(\mathbf{R}^n)$.

(ii) *$u \in L^p(\mathbf{R}^n)$ and there is $g \in L^p(\mathbf{R}^n)$, $g \geqslant 0$, such that*

$$|u(x) - u(y)| \leqslant |x - y|(g(x) + g(y))$$

for all $x, y \in \mathbf{R}^n \setminus E$ with $|E| = 0$.

(iii) *$u \in L^p(\mathbf{R}^n)$ and there is $g \in L^p(\mathbf{R}^n)$, $g \geqslant 0$, such that the Poincaré inequality holds,*

$$\fint_{B(x,r)} |u - u_{B(x,r)}|\, dy \leqslant c\, r \fint_{B(x,r)} g\, dy$$

for all $x \in \mathbf{R}^n$ and $r > 0$.

(iv) *$u \in L^p(\mathbf{R}^n)$ and $u_1^{\#} \in L^p(\mathbf{R}^n)$.*

Proof. We have already seen that (i) implies (ii). To prove that (ii) implies (iii), we integrate the pointwise inequality twice over the ball $B(x, r)$. After the first integration we obtain

$$\begin{aligned} |u(y) - u_{B(x,r)}| &= \left| u(y) - \fint_{B(x,r)} u(z)\, dz \right| \\ &\leqslant \fint_{B(x,r)} |u(y) - u(z)|\, dz \\ &\leqslant 2r \left(g(y) + \fint_{B(x,r)} g(z)\, dz \right), \end{aligned}$$

which implies

$$\begin{aligned} \fint_{B(x,r)} |u(y) - u_{B(x,r)}|\, dy &\leqslant 2r \left(\fint_{B(x,r)} g(y)\, dy + \fint_{B(x,r)} g(z)\, dz \right) \\ &\leqslant 4r \fint_{B(x,r)} g(y)\, dy. \end{aligned}$$

To show that (iii) implies (iv), we observe that

$$u_1^{\#}(x) = \sup_{r>0} \frac{1}{r} \fint_{B(x,r)} |u - u_{B(x,r)}| \, dy \leqslant c \sup_{r>0} \fint_{B(x,r)} g \, dy = cMg(x).$$

Then we show that (iv) implies (i). By Theorem 4.1,

$$|u(x) - u(y)| \leqslant c|x - y|(u_1^{\#}(x) + u_1^{\#}(y))$$

for all $x, y \in \mathbf{R}^n \setminus E$ with $|E| = 0$. If we denote $g = cu_1^{\#}$, then $g \in L^p(\mathbf{R}^n)$ and

$$|u(x) - u(y)| \leqslant |x - y|(g(x) + g(y))$$

for all $x, y \in \mathbf{R}^n \setminus E$ with $|E| = 0$. Then we use the characterization of Sobolev spaces $W^{1,p}(\mathbf{R}^n)$, $1 < p < \infty$, with integrated difference quotients. Let $h \in \mathbf{R}^n$. Then

$$|u_h(x) - u(x)| = |u(x + h) - u(x)| \leqslant |h|(g_h(x) + g(x)),$$

from which we conclude that

$$\|u_h - u\|_p \leqslant |h|(\|g_h\|_p + \|g\|_p) = 2|h|\|g\|_p,$$

which implies the claim. □

Remark 4.5. Hajłasz [22] showed that $u \in W^{1,1}(\mathbf{R}^n)$ if and only if $u \in L^1(\mathbf{R}^n)$ and there is a nonnegative function $g \in L^1(\mathbf{R}^n)$ and $\sigma \geqslant 1$ such that

$$|u(x) - u(y)| \leqslant |x - y|(M_{\sigma|x-y|}g(x) + M_{\sigma|x-y|}g(y))$$

for all $x, y \in \mathbf{R}^n \setminus E$ with $|E| = 0$. Moreover, if this inequality holds, then $|Du| \leqslant c(n, \sigma)g$ almost everywhere.

4.1 Lusin type approximation of Sobolev functions

Approximations of Sobolev functions were studied, for example, in [2, 10, 11, 13, 20, 25, 52, 56, 58, 60, 69].

Let $u \in W^{1,p}(\mathbf{R}^n)$ and $0 \leqslant \alpha < 1$. By Corollary (4.3),

$$|u(x) - u(y)| \leqslant c|x - y|^{1-\alpha}\big(M_\alpha|Du|(x) + M_\alpha|Du|(y)\big)$$

for all $x, y \in \mathbf{R}^n \setminus E$ with $|E| = 0$. For $p > n$ the Hölder inequality implies

$$M_{n/p}|Du|(x) \leqslant cM_n|Du|^p(x)^{1/p} \leqslant c\|Du\|_p$$

for every $x \in \mathbf{R}^n \setminus E$ with $c = c(n, p)$. Hence

$$|u(x) - u(y)| \leqslant c\|Du\|_p |x - y|^{1-n/p}$$

for all x, $y \in \mathbf{R}^n \setminus E$ and u is Hölder continuous with the exponent $1 - n/p$ after a possible redefinition on a set of measure zero. The same argument implies that if $M_\alpha|Du|$ is bounded, then $u \in C^{1-\alpha}(\mathbf{R}^n)$. Even if $M_\alpha|Du|$ is unbounded, then

$$|u(x) - u(y)| \leqslant c\lambda|x - y|^{1-\alpha}$$

for all $x, y \in \mathbf{R}^n \setminus E_\lambda$, where

$$E_\lambda = \{x \in \mathbf{R}^n : M_\alpha|Du|(x) > \lambda\}$$

for $\lambda > 0$. This means that the restriction of $u \in W^{1,p}(\mathbf{R}^n)$ to the set $\mathbf{R}^n \setminus E_\lambda$ is Hölder continuous after a redefinition on a set of measure zero.

Recall that the (spherical) Hausdorff s-content, $0 < s < \infty$, of $E \subset \mathbf{R}^n$ is defined by

$$\mathcal{H}^s_\infty(E) = \inf\Big\{\sum_{i=1}^\infty r_i^s : E \subset \bigcup_{i=1}^\infty B(x_i, r_i)\Big\}.$$

The standard Vitali covering argument gives the following estimate for the size of the set $\mathbf{R}^n \setminus E_\lambda$. There is a constant $c = c(n, p, \alpha)$ such that

$$\mathcal{H}^{n-\alpha p}_\infty(E_\lambda) \leqslant c\lambda^{-p} \int_{\mathbf{R}^n} |Du|^p \, dx \tag{4.6}$$

for every $\lambda > 0$.

Theorem 4.7. *Let $u \in W^{1,p}(\mathbf{R}^n)$, and let $0 \leqslant \alpha < 1$. Then for every $\lambda > 0$ there is an open set E_λ and a function u_λ such that $u(x) = u_\lambda(x)$ for every $x \in \mathbf{R}^n \setminus E_\lambda$, $u_\lambda \in W^{1,p}(\mathbf{R}^n)$, u_λ is Hölder continuous with the exponent $1 - \alpha$, $\|u - u_\lambda\|_{W^{1,p}(\mathbf{R}^n)} \to 0$ as $\lambda \to \infty$, and $\mathcal{H}^{n-\alpha p}_\infty(E_\lambda) \to 0$ as $\lambda \to \infty$.*

Remark 4.8. (i) If $\alpha = 0$, then the theorem says that every function in the Sobolev space coincides with a Lipschitz function outside a set of arbitrarily small Lebesgue measure. The obtained Lipschitz function approximates the original Sobolev function also in the Sobolev norm.

(ii) Since

$$\operatorname{cap}_{\alpha p}(E_\lambda) \leqslant c\mathcal{H}^{n-\alpha p}_\infty(E_\lambda),$$

the size of the exceptional set can also be expressed in terms of capacity.

Proof. The set E_λ is open since M_α is lower semicontinuous. From (4.6) we conclude that

$$\mathcal{H}^{n-\alpha p}_\infty(E_\lambda) \leqslant c\lambda^{-p} \, \|Du\|_p^p$$

for every $\lambda > 0$ with $c = c(n, p, \alpha)$.

We already showed that $u|_{\mathbf{R}^n \setminus E_\lambda}$ is $(1 - \alpha)$-Hölder continuous with the constant $c(n)\lambda$.

Let Q_i, $i = 1, 2, \ldots$, be a Whitney decomposition of E_λ with the following properties: each Q_i is open, the cubes Q_i, $i = 1, 2, \ldots$, are disjoint, $E_\lambda = \bigcup_{i=1}^\infty \overline{Q}_i$, $4Q_i \subset E_\lambda$, $i = 1, 2, \ldots$,

$$\sum_{i=1}^\infty \chi_{2Q_i} \leqslant N < \infty,$$

and

$$c_1 \operatorname{dist}(Q_i, \mathbf{R}^n \setminus E_\lambda) \leqslant \operatorname{diam}(Q_i) \leqslant c_2 \operatorname{dist}(Q_i, \mathbf{R}^n \setminus E_\lambda)$$

for some constants c_1 and c_2.

Then we construct a partition of unity associated with the covering $2Q_i$, $i = 1, 2, \ldots$. This can be done in two steps. First, let $\widetilde{\varphi}_i \in C_0^\infty(2Q_i)$ be such that $0 \leqslant \widetilde{\varphi}_i \leqslant 1$, $\widetilde{\varphi}_i = 1$ in Q_i and

$$|D\widetilde{\varphi}_i| \leqslant \frac{c}{\operatorname{diam}(Q_i)}$$

for $i = 1, 2, \ldots$. Then we define

$$\varphi_i(x) = \frac{\widetilde{\varphi}_i(x)}{\sum\limits_{j=1}^\infty \widetilde{\varphi}_j(x)}$$

for every $i = 1, 2, \ldots$. Observe that the sum is taken over finitely many terms only since $\varphi_i \in C_0^\infty(2Q_i)$ and the cubes $2Q_i$, $i = 1, 2, \ldots$, are of bounded overlap. The functions φ_i have the property

$$\sum_{i=1}^\infty \varphi_i(x) = \chi_{E_\lambda}(x)$$

for every $x \in \mathbf{R}^n$.

Then we define the function u_λ by

$$u_\lambda(x) = \begin{cases} u(x), & x \in \mathbf{R}^n \setminus E_\lambda, \\ \sum\limits_{i=1}^\infty \varphi_i(x) u_{2Q_i}, & x \in E_\lambda. \end{cases}$$

The function u_λ is a Whitney type extension of $u|_{\mathbf{R}^n \setminus E_\lambda}$ to the set E_λ.

First we claim that

$$\|u_\lambda\|_{W^{1,p}(E_\lambda)} \leqslant c\|u\|_{W^{1,p}(E_\lambda)}. \tag{4.9}$$

Since the cubes $2Q_i$, $i = 1, 2, \ldots$, are of bounded overlap, we have

$$\int_{E_\lambda} |u_\lambda|^p\,dx = \int_{E_\lambda} \Big|\sum_{i=1}^\infty \varphi_i(x) u_{2Q_i}\Big|^p\,dx \leqslant c\sum_{i=1}^\infty \int_{2Q_i} |u_{2Q_i}|^p\,dx$$

$$\leqslant c\sum_{i=1}^\infty |2Q_i| \fint_{2Q_i} |u|^p\,dx \leqslant c\int_{E_\lambda} |u|^p\,dx.$$

Then we estimate the gradient. We recall that

$$\Phi(x) = \sum_{i=1}^\infty \varphi_i(x) = 1$$

for every $x \in E_\lambda$. Since the cubes $2Q_i$, $i = 1, 2, \ldots$, are of bounded overlap, we see that $\Phi \in C^\infty(E_\lambda)$ and

$$D_j\Phi(x) = \sum_{i=1}^\infty D_j\varphi_i(x) = 0, \quad j = 1, 2, \ldots, n,$$

for every $x \in E_\lambda$. Hence we obtain

$$|D_j u_\lambda(x)| = \Big|\sum_{i=1}^\infty D_j\varphi_i(x) u_{2Q_i}\Big| = \Big|\sum_{i=1}^\infty D_j\varphi_i(x)(u(x) - u_{2Q_i})\Big|$$

$$\leqslant c\sum_{i=1}^\infty \operatorname{diam}(Q_i)^{-1}|u(x) - u_{2Q_i}|\chi_{2Q_i}(x)$$

and, consequently,

$$|D_j u_\lambda(x)| \leqslant c\sum_{i=1}^\infty \operatorname{diam}(Q_i)^{-p}|u(x) - u_{2Q_i}|^p\chi_{2Q_i}(x).$$

Here we again used the fact that the cubes $2Q_i$, $i = 1, 2, \ldots$, are of bounded overlap.

This implies that for every $j = 1, 2, \ldots, n$

$$\int_{E_\lambda} |D_j u_\lambda|\,dx \leqslant c\int_{E_\lambda} \Big(\sum_{i=1}^\infty \operatorname{diam}(Q_i)^{-p}|u - u_{2Q_i}|^p\chi_{2Q_i}\Big)\,dx$$

$$\leqslant \sum_{i=1}^\infty \int_{2Q_i} \operatorname{diam}(Q_i)^{-p}|u - u_{2Q_i}|^p\,dx$$

$$\leqslant c \sum_{i=1}^{\infty} \int_{2Q_i} |Du|^p \, dx \leqslant c \int_{E_\lambda} |Du|^p \, dx.$$

Then we show that $u_\lambda \in W^{1,p}(\mathbf{R}^n)$. We know that u_λ belongs to $W^{1,p}(E_\lambda)$ and is Hölder continuous in $\mathbf{R}^n$. Moreover, $u \in W^{1,p}(\mathbf{R}^n)$ and $u = u_\lambda$ in $\mathbf{R}^n \setminus E_\lambda$ by (i). This implies that $w = u - u_\lambda \in W^{1,p}(E_\lambda)$ and $w = 0$ in $\mathbf{R}^n \setminus E_\lambda$. By the ACL-property, u is absolutely continuous on almost every line segment parallel to the coordinate axes. Take any such a line. Now w is absolutely continuous on the part of the line segment which intersects E_λ. On the other hand, $w = 0$ in the complement of E_λ. Hence the continuity of w in the line segment implies that w is absolutely continuous on the whole line segment.

We have

$$\begin{aligned}\|u - u_\lambda\|_{W^{1,p}(\mathbf{R}^n)} &= \|u - u_\lambda\|_{W^{1,p}(E_\lambda)} \\ &\leqslant \|u\|_{W^{1,p}(E_\lambda)} + \|u_\lambda\|_{W^{1,p}(E_\lambda)} \leqslant c\|u\|_{W^{1,p}(E_\lambda)}.\end{aligned}$$

We leave it as an exercise for the interested reader to show that the function u_λ is Hölder continuous with the exponent $1 - \alpha$ (or see, for example, [27] for details). □

5 Hardy Inequality

In this section, we consider the Hardy inequality, which was originally studied by Hardy in the one-dimensional case. In the higher dimensional case, the Hardy inequality was studied, for example, in [5, 46, 51, 59, 67, 68]. Our approach is mainly based on more recent works [23, 36, 44, 47, 48, 61].

Suppose first that $p > n$, $n < q < p$, $0 \leqslant \alpha < q$, and $\Omega \neq \mathbf{R}^n$ is an open set. Let $u \in C_0^\infty(\Omega)$. Consider the zero extension to $\mathbf{R}^n \setminus \Omega$. Fix $x \in \Omega$ and take $x_0 \in \partial\Omega$ such that

$$|x - x_0| = \operatorname{dist}(x, \partial\Omega) = \delta(x) = R.$$

Denote $\chi = \chi_{B(x_0,2R)}$. By Corollary 4.3,

$$\begin{aligned}|u(x)| &= |u(x) - u(x_0)| \\ &\leqslant c|x - x_0|^{1-n/q}(M_{n/q}(|Du|\chi)(x) + M_{n/q}(|Du|\chi)(x_0)),\end{aligned}$$

where

$$M_{n/q}(|Du|\chi)(x) \leqslant cM_n(|Du|^q\chi)(x)^{1/q} \leqslant \|Du\chi\|_q,$$

and, by the same argument,

$$M_{n/q}(|Du|\chi)(x_0) \leqslant \|Du\chi\|_q.$$

This implies that

$$\begin{aligned}
|u(x)| &\leqslant c|x-x_0|^{1-n/q}\Bigg(\int\limits_{B(x_0,2R)} |Du|^q\,dy\Bigg)^{1/q}\\
&\leqslant cR^{1-\alpha/q}\Bigg(R^{\alpha-n}\int\limits_{B(x,4R)} |Du|^q\,dy\Bigg)^{1/q}\\
&\leqslant c\operatorname{dist}(x,\partial\Omega)^{1-\alpha/q}\big(M_{\alpha,4\delta(x)}|Du|^q(x)\big)^{1/q} \qquad (5.1)
\end{aligned}$$

for every $x \in \mathbf{R}^n$ with $c = c(n,q)$. This is a pointwise Hardy inequality. For $u \in W_0^{1,p}(\Omega)$ this inequality holds almost everywhere. Integrating (5.1) with $\alpha = 0$ over Ω and using the maximal function theorem, we arrive at

$$\begin{aligned}
\int\limits_{\Omega}\left(\frac{|u(x)|}{\operatorname{dist}(x,\partial\Omega)}\right)^p dx &\leqslant c\int\limits_{\Omega}\big(M|Du|^q(x)\big)^{p/q}\,dx\\
&\leqslant c\int\limits_{\Omega}|Du(x)|^p\,dx \qquad (5.2)
\end{aligned}$$

for every $u \in W_0^{1,p}(\Omega)$ with $c = c(n,p,q)$. This is a version of the Hardy inequality which is valid for every open sets with nonempty complement if $n < p < \infty$. The case $1 < p \leqslant n$ is more involved since then extra conditions must be imposed on Ω (see [51, Theorem 3]). However, there is a sufficient condition in terms of capacity density of the complement.

A closed set $E \subset \mathbf{R}^n$ is uniformly $p-$fat, $1 < p < \infty$, if there is a constant $\gamma > 0$ such that

$$\operatorname{cap}_p\big(E\cap\overline{B}(x,r),B(x,2r)\big) \geqslant \gamma\operatorname{cap}_p\big(\overline{B}(x,r),B(x,2r)\big) \qquad (5.3)$$

for all $x \in E$ and $r > 0$. Here $\operatorname{cap}_p(K,\Omega)$ denotes the variational $p-$capacity

$$\operatorname{cap}_p(K,\Omega) = \inf\int\limits_{\Omega}|Du(x)|^p\,dx,$$

where the infimum is taken over all $u \in C_0^\infty(\Omega)$ such that $u(x) \geqslant 1$ for every $x \in K$. Here Ω is an open subset of Ω and K is a compact subset of Ω. We recall that

$$\operatorname{cap}_p(\overline{B}(x,r),B(x,2r)) = cr^{n-p},$$

where $c = c(n, p)$.

If $p > n$, then all nonempty closed sets are uniformly p-fat. If there is a constant $\gamma > 0$ such that E satisfies the measure thickness condition

$$|B(x, r) \cap E| \geqslant \gamma |B(x, r)|$$

for all $x \in E$ and $r > 0$, then E is uniformly p-fat for every p with $1 < p < \infty$.

If E is uniformly p-fat for some p, then it is uniformly q-fat for every $q > p$. The fundamental property of uniformly fat sets is the following self improving result due to Lewis [51, Theorem 1]. For another proof see [61, Theorem 8.2].

Theorem 5.4. *Let $E \subset \mathbf{R}^n$ be a closed uniformly $p-$fat set. Then there is $1 < q < p$ such that E is uniformly $q-$fat.*

In the case where $\Omega \subset \mathbf{R}^n$ is an open set such that $\mathbf{R}^n \setminus \Omega$ is uniformly $p-$fat, Lewis [51, Theorem 2] proved that the Hardy inequality holds. We have already seen that the Hardy inequality follows from pointwise inequalities involving the Hardy–Littlewood maximal function if $p > n$. We show that this is also the case $1 < p \leqslant n$.

Theorem 5.5. *Let $1 < p \leqslant n$, $0 \leqslant \alpha < p$, and let $\Omega \subset \mathbf{R}^n$ be an open set such that $\mathbf{R}^n \setminus \Omega$ is uniformly $p-$fat. Suppose that $u \in C_0^\infty(\Omega)$. Then there are constants $c = c(n, p, \gamma)$ and $\sigma > 1$ such that*

$$|u(x)| \leqslant c \operatorname{dist}(x, \partial\Omega)^{1-\alpha/p} \big(M_{\alpha, \sigma\delta(x)} |Du|^p(x)\big)^{1/p} \tag{5.6}$$

for every $x \in \Omega$.

Proof. Let $x \in \Omega$. Choose $x_0 \in \partial\Omega$ such that

$$|x - x_0| = \operatorname{dist}(x, \partial\Omega) = \delta(x) = R.$$

Then

$$|u(x) - u_{B(x_0, 2R)}| \leqslant cR^{1-\alpha/p} \big(M_{\alpha, R} |Du|^p(x)\big)^{1/p}$$

for every $x \in B(x_0, 2R)$ with $c = c(n, p)$, and hence

$$\begin{aligned} |u(x)| &\leqslant |u(x) - u_{B(x_0, 2R)}| + |u_{B(x_0, 2R)}| \\ &\leqslant cR^{1-\alpha/p} \big(M_{\alpha, \delta(x)} |Du|^p(x)\big)^{1/p} + |u|_{B(x_0, 2R)} \end{aligned}$$

for every $x \in B(x_0, 2R)$. Denote $A = \{x \in \mathbf{R}^n : u(x) = 0\}$. Using a capacitary version of the Poincaré inequality, we arrive at

$$\frac{1}{|B(x_0, 2R)|} \int_{B(x_0, 2R)} |u| \, dy$$

$$\leqslant c \left(\operatorname{cap}_p \left(A \cap \overline{B}(x_0, 2R), B(x_0, 4R) \right)^{-1} \int\limits_{B(x_0,4R)} |Du|^p \, dy \right)^{1/p}$$

$$\leqslant c \left(\operatorname{cap}_p \left((\mathbf{R}^n \setminus \Omega) \cap \overline{B}(x_0, 2R), B(x_0, 4R) \right)^{-1} \int\limits_{B(x_0,4R)} |Du|^p \, dy \right)^{1/p}$$

$$\leqslant c \left(R^{p-n} \int\limits_{B(x,8R)} |Du|^p \, dy \right)^{1/p} \leqslant c R^{1-\alpha/p} \left(M_{\alpha, 8\delta(x)} |Du|^p(x) \right)^{1/p},$$

where $c = c(n, p, \gamma)$. □

If $\mathbf{R}^n \setminus \Omega$ is $p-$fat, then, by Theorem 5.4, it is $q-$fat for some $1 < q < p \leqslant n$. Using (5.6) with $\alpha = 0$, we get the pointwise q-Hardy inequality

$$|u(x)| \leqslant c \operatorname{dist}(x, \partial\Omega) \left(M_{\sigma\delta(x)} |Du|^q(x) \right)^{1/q}$$

for every $x \in \Omega$ with $c = c(n, q)$. Integrating and using the maximal function theorem exactly in the same way as in (5.2), we also prove the Hardy inequality in the case $1 < p \leqslant n$. Again, a density argument shows that the Hardy inequality holds for every $u \in W_0^{1,p}(\Omega)$. Thus, we have proved the following assertion.

Corollary 5.7. *Let $1 < p < \infty$. Suppose that $\Omega \subset \mathbf{R}^n$ is an open set such that $\mathbf{R}^n \setminus \Omega$ is uniformly $p-$fat. If $u \in W_0^{1,p}(\Omega)$, then there is a constant $c = c(n, p, \gamma)$ such that*

$$\int\limits_{\Omega} \left(\frac{|u(x)|}{\operatorname{dist}(x, \partial\Omega)} \right)^p dx \leqslant c \int\limits_{\Omega} |Du(x)|^p \, dx.$$

In particular, if $p > n$, then the inequality holds for every $\Omega \neq \mathbf{R}^n$.

Remark 5.8. The pointwise Hardy inequality is not equivalent to the Hardy inequality since there are open sets for which the Hardy inequality holds for some p, but the pointwise Hardy inequality fails. For example, the punctured ball $B(0, 1) \setminus \{0\}$ satisfies the pointwise Hardy inequality only in the case $p > n$, but the usual Hardy inequality also holds when $1 < p < n$. When $p = n$, the Hardy inequality fails for this set. This example also shows that the uniform fatness of the complement is not a necessary condition for an open set to satisfy the Hardy inequality since the complement of $B(0, 1) \setminus \{0\}$ is not uniformly p-fat when $1 < p < n$. If $p = n$, then the Hardy inequality is equivalent to the fact that $\mathbf{R}^n \setminus \Omega$ is uniformly p-fat (see [51, Theorem 3]).

A recent result of Lehrbäck [47] shows that the uniform fatness is not only sufficient, but also necessary condition for the pointwise Hardy inequality (see also [50, 48, 49]). When $n = p = 2$, Sugawa [65] proved that the Hardy inequality is also equivalent to the uniform perfectness of the complement of the domain. Recently this result was generalized in [43] for other values of p. The arguments of [43] are very general. It is also possible to study Hardy inequalities on metric measure spaces (see [9, 32, 43]).

Theorem 5.4 shows that the p-fatness is a self improving result. Next we give a proof of an elegant result of Koskela and Zhong [45] which states that the Hardy inequality is self improving.

Theorem 5.9. *Suppose that the Hardy inequality holds in Ω for some $1 < p < \infty$. Then there exists $\varepsilon > 0$ such that the Hardy inequality holds in Ω for every q with $p - \varepsilon < q \leqslant p$.*

Proof. Let u be a Lipschitz continuous function that vanishes in $\mathbf{R}^n \setminus \Omega$. For $\lambda > 0$ denote

$$F_\lambda = \{x \in \Omega : |u(x)| \leqslant \lambda \operatorname{dist}(x, \partial\Omega) \text{ and } M|Du|(x) \leqslant \lambda\}.$$

We claim that the restriction of u to $F_\lambda \cup (\mathbf{R}^n \setminus \Omega)$ is Lipschitz continuous with a constant $c\lambda$, where $c = c(n)$. If $x, y \in F_\lambda$, then

$$|u(x) - u(y)| \leqslant c|x - y|(M|Du|(x) + M|Du|(y)) \leqslant c\lambda|x - y|$$

by Corollary 4.3. If $x \in F_\lambda$ and $y \in \mathbf{R}^n \setminus \Omega$, then

$$|u(x) - u(y)| = |u(x)| \leqslant \lambda \operatorname{dist}(x, \partial\Omega) \leqslant \lambda|x - y|.$$

This implies that $u|_{F_\lambda \cup (\mathbf{R}^n \setminus \Omega)}$ is Lipschitz continuous with the constant $c\lambda$. We extend the function to the entire space $\mathbf{R}^n$, for example, with the classical McShane extension

$$v(x) = \inf\{u(y) + c\lambda|x - y| : y \in F_\lambda \cup (\mathbf{R}^n \setminus \Omega)\}.$$

The function v is Lipschitz continuous in $\mathbf{R}^n$ with the same constant $c\lambda$ as $u|_{F_\lambda \cup (\mathbf{R}^n \setminus \Omega)}$. Let

$$G_\lambda = \{x \in \Omega : |u(x)| \leqslant \lambda \operatorname{dist}(x, \partial\Omega)\}$$

and

$$E_\lambda = \{x \in \Omega : M|Du|(x) \leqslant \lambda\}.$$

Then $F_\lambda = G_\lambda \cap E_\lambda$, and we note that

$$|Dv(x)| \leqslant |Du(x)|\chi_{F_\lambda}(x) + c\lambda\chi_{\Omega \setminus F_\lambda}(x)$$

for almost all $x \in \mathbf{R}^n$. By the Hardy inequality,

$$\int_{F_\lambda} \left(\frac{|v(x)|}{\operatorname{dist}(x, \partial\Omega)} \right)^p dx \leqslant c \int_{F_\lambda} |Du(x)|^p \, dx + c\lambda^p |\Omega \setminus F_\lambda|$$

and, consequently,

$$\begin{aligned}
&\int_{G_\lambda} \left(\frac{|u(x)|}{\operatorname{dist}(x, \partial\Omega)} \right)^p dx \\
&\quad \leqslant c \int_{F_\lambda} |Du(x)|^p \, dx + c\lambda^p |\Omega \setminus F_\lambda| + \int_{G_\lambda \setminus E_\lambda} \left(\frac{|u(x)|}{\operatorname{dist}(x, \partial\Omega)} \right)^p dx \\
&\quad \leqslant c \int_{E_\lambda} |Du(x)|^p \, dx + c\lambda^p (|\Omega \setminus G_\lambda| + |\Omega \setminus E_\lambda|).
\end{aligned}$$

From this we obtain

$$\begin{aligned}
\frac{1}{\varepsilon} \int_{\Omega} \left(\frac{|u(x)|}{\operatorname{dist}(x, \partial\Omega)} \right)^{p-\varepsilon} dx &= \int_0^\infty \lambda^{-\varepsilon-1} \int_{G_\lambda} \left(\frac{|u(x)|}{\operatorname{dist}(x, \partial\Omega)} \right)^p dx \, d\lambda \\
&\leqslant c \int_0^\infty \lambda^{-\varepsilon-1} \int_{E_\lambda} |Du(x)|^p \, dx \, d\lambda \\
&+ c \int_0^\infty \lambda^{p-\varepsilon-1} |\Omega \setminus G_\lambda| \, d\lambda + c \int_0^\infty \lambda^{p-\varepsilon-1} |\Omega \setminus E_\lambda| \, d\lambda \\
&\leqslant \frac{c}{\varepsilon} \int_{\Omega} |Du(x)|^{p-\varepsilon} \, dx + \frac{1}{p-\varepsilon} \int_{\Omega} \left(\frac{|u(x)|}{\operatorname{dist}(x, \partial\Omega)} \right)^{p-\varepsilon} dx \\
&+ \frac{c}{p-\varepsilon} \int_{\Omega} (M|Du|(x))^{p-\varepsilon} \, dx.
\end{aligned}$$

The claim follows from this by using the maximal function theorem, choosing $\varepsilon > 0$ small enough, and absorbing the terms on the left-hand side. □

6 Maximal Functions on Metric Measure Spaces

In this section, we show that most of the results that we have discussed so far are based on a general principle and our arguments apply in the context of metric measure spaces.

6.1 Sobolev spaces on metric measure spaces

Let $X = (X, d, \mu)$ be a complete metric space endowed with a metric d and a Borel regular measure μ such that $0 < \mu(B(x, r)) < \infty$ for all open balls

$$B(x, r) = \{y \in X : d(y, x) < r\}$$

with $r > 0$.

The measure μ is said to be doubling if there exists a constant $c_\mu \geqslant 1$, called the doubling constant of μ, such that

$$\mu(B(x, 2r)) \leqslant c_\mu \mu(B(x, r))$$

for all $x \in X$ and $r > 0$. Note that an iteration of the doubling property implies that, if $B(x, R)$ is a ball in X, $y \in B(x, R)$, and $0 < r \leqslant R < \infty$, then

$$\frac{\mu(B(y, r))}{\mu(B(x, R))} \geqslant c\Big(\frac{r}{R}\Big)^Q \tag{6.1}$$

for some $c = c(c_\mu)$ and $Q = \log c_\mu / \log 2$. The exponent Q serves as a counterpart of dimension related to the measure.

A nonnegative Borel function g on X is said to be an upper gradient of a function $u : X \to [-\infty, \infty]$ if for all rectifiable paths γ joining points x and y in X we have

$$|u(x) - u(y)| \leqslant \int\limits_\gamma g\, ds, \tag{6.2}$$

whenever both $u(x)$ and $u(y)$ are finite, and $\int\limits_\gamma g\, ds = \infty$ otherwise. The assumption that g is a Borel function is needed in the definition of the path integral. If g is merely a μ-measurable function and (6.2) holds for p-almost every path (i.e., it fails only for a path family with zero p-modulus), then g is said to be a p-weak upper gradient of u. If we redefine a p-weak upper gradient on a set of measure zero we obtain a p-weak upper gradient of the same function. In particular, this implies that, after a possible redefinition on a set of measure zero, we obtain a Borel function. If g is a p-weak upper gradient of u, then there is a sequence g_i, $i = 1, 2, \ldots$, of the upper gradients of u such that

$$\int\limits_X |g_i - g|^p \, d\mu \to 0$$

as $i \to \infty$. Hence every p-weak upper gradient can be approximated by upper gradients in the $L^p(X)$-norm. If u has an upper gradient that belongs to $L^p(X)$, then it has a minimal p-weak upper gradient g_u in the sense that for every p-weak upper gradient g of u, $g_u \leqslant g$ μ-almost everywhere.

We define Sobolev spaces on the metric space X using the p-weak upper gradients. For $u \in L^p(X)$ we set

$$\|u\|_{N^{1,p}(X)} = \Big(\int_X |u|^p \, d\mu + \inf_g \int_X g^p \, d\mu \Big)^{1/p},$$

where the infimum is taken over all p-weak upper gradients of u. The Sobolev space (sometimes called the Newtonian space) on X is the quotient space

$$N^{1,p}(X) = \{u : \|u\|_{N^{1,p}(X)} < \infty\}/\sim,$$

where $u \sim v$ if and only if $\|u - v\|_{N^{1,p}(X)} = 0$. The notion of a p-weak upper gradient is used to prove that $N^{1,p}(X)$ is a Banach space. For properties of Sobolev spaces on metric measure spaces we refer to [30, 29, 62, 63, 6].

The p-capacity of a set $E \subset X$ is the number

$$\operatorname{cap}_p(E) = \inf \|u\|^p_{N^{1,p}(X)},$$

where the infimum is taken over all $u \in N^{1,p}(X)$ such that $u = 1$ on E [38]. We say that a property regarding points in X holds p-quasieverywhere (p-q.e.) if the set of points for which the property does not hold has capacity zero. If $u \in N^{1,p}(X)$, then $u \sim v$ if and only if $u = v$ p-q.e. Moreover, if $u, v \in N^{1,p}(X)$ and $u = v$ μ-a.e., then $u \sim v$. Hence the capacity is the correct gauge for distinguishing between two Newtonian functions (see [8]).

To be able to compare the boundary values of Sobolev functions, we need a Sobolev space with zero boundary values. Let E be a measurable subset of X. The Sobolev space with zero boundary values is the space

$$N^{1,p}_0(E) = \{u|_E : u \in N^{1,p}(X) \text{ and } u = 0 \text{ } p\text{-q.e. in } X \setminus E\}.$$

The space $N^{1,p}_0(E)$ equipped with the norm inherited from $N^{1,p}(X)$ is a Banach space.

We say that X supports a weak $(1,p)$-Poincaré inequality if there exist constants $c > 0$ and $\lambda \geqslant 1$ such that for all balls $B(x,r) \subset X$, all locally integrable functions u on X and for all p-weak upper gradients g of u,

$$\fint_{B(x,r)} |u - u_{B(x,r)}| \, d\mu \leqslant cr \Big(\fint_{B(x,\lambda r)} g^p \, d\mu \Big)^{1/p}, \tag{6.3}$$

where

$$u_{B(x,r)} = \fint_{B(x,r)} u \, d\mu = \frac{1}{\mu(B(x,r))} \int_{B(x,r)} u \, d\mu.$$

Since the p-weak upper gradients can be approximated by upper gradients in the $L^p(X)$-norm, we could require the Poincaré inequality for upper gradients as well.

By the Hölder inequality, it is easy to see that if X supports a weak $(1,p)$-Poincaré inequality, then it supports a weak $(1,q)$-Poincaré inequality for every $q > p$. If X is complete and μ doubling then it is shown in [31] that a weak $(1, p)$-Poincaré inequality implies a weak $(1, q)$-Poincaré inequality for some $q < p$. Hence the $(1, p)$-Poincaré inequality is a self improving condition. For simplicity, we assume throughout that X supports a weak $(1, 1)$-Poincaré inequality, although, by using the results of [31], it would be enough to assume that X supports a weak $(1, p)$-Poincaré inequality. We leave the extensions to the interested reader. In addition, we assume that X is complete and μ is doubling. This implies, for example, that Lipschitz functions are dense in $N^{1,p}(X)$ and the Sobolev embedding theorem holds.

6.2 Maximal function defined on the whole space

The standard centered Hardy–Littlewood maximal function on a metric measure space X is defined as

$$Mu(x) = \sup_{r>0} \fint_{B(x,r)} |u| \, d\mu.$$

By the Hardy–Littlewood maximal function theorem for doubling measures (see [15]), we see that the Hardy–Littlewood maximal operator is bounded on $L^p(X)$ when $1 < p \leqslant \infty$ and maps $L^1(X)$ into the weak $L^1(X)$. However, the standard Hardy–Littlewood maximal function does not seem to preserve the smoothness of the functions as examples by Buckley [12] clearly indicate. In order to have a maximal function which preserves, for example, the Sobolev spaces on metric measure spaces, we construct a maximal function based on a discrete convolution.

Let $r > 0$. We begin by constructing a family of balls which cover the space and are of bounded overlap. Indeed, there is a family of balls $B(x_i, r)$, $i = 1, 2, \dots$, such that

$$X = \bigcup_{i=1}^{\infty} B(x_i, r)$$

and

$$\sum_{i=1}^{\infty} \chi_{B(x_i, 6r)} \leqslant c < \infty.$$

This means that the dilated balls $B(x_i, 6r)$ are of bounded overlap. The constant c depends only on the doubling constant and, in particular, is independent of r. These balls play the role of Whitney cubes in a metric measure space.

Then we construct a partition of unity subordinate to the cover $B(x_i, r)$, $i = 1, 2, \dots$, of X. Indeed, there is a family of functions φ_i, $i = 1, 2, \dots$, such that $0 \leqslant \varphi_i \leqslant 1$, $\varphi_i = 0$ on $X \setminus B(x_i, 6r)$, $\varphi_i \geqslant c$ on $B(x_i, 3r)$, φ_i is Lipschitz with constant c/r_i with c depending only on the doubling constant, and

$$\sum_{i=1}^{\infty} \varphi_i = 1$$

in X. The partition of unity can be constructed by first choosing auxiliary cutoff functions $\widetilde{\varphi}_i$ so that $0 \leqslant \widetilde{\varphi}_i \leqslant 1$, $\widetilde{\varphi}_i = 0$ on $X \setminus B(x_i, 6r)$, $\widetilde{\varphi}_i = 1$ on $B(x_i, 3r)$ and each $\widetilde{\varphi}_i$ is Lipschitz with constant c/r. We can, for example, take

$$\widetilde{\varphi}_i(x) = \begin{cases} 1, & x \in B(x_i, 3r), \\ 2 - \dfrac{d(x, x_i)}{3r}, & x \in B(x_i, 6r) \setminus B(x_i, 3r), \\ 0, & x \in X \setminus B(x_i, 6r). \end{cases}$$

Then we can define the functions φ_i, $i = 1, 2, \dots$, in the partition of unity by

$$\varphi_i(x) = \frac{\widetilde{\varphi}_i(x)}{\sum_{j=1}^{\infty} \widetilde{\varphi}_j(x)}.$$

It is not difficult to see that the defined functions satisfy the required properties.

Now we are ready to define the approximation of u at the scale of $3r$ by setting

$$u_r(x) = \sum_{i=1}^{\infty} \varphi_i(x) u_{B(x_i, 3r)}$$

for every $x \in X$. The function u_r is called the discrete convolution of u. The partition of unity and the discrete convolution are standard tools in harmonic analysis on homogeneous spaces (see, for example, [15] and [57]).

Let r_j, $j = 1, 2, \dots$, be an enumeration of the positive rational numbers. For every radius r_j we choose balls $B(x_i, r_j)$, $i = 1, 2, \dots$, of X as above. Observe that for each radius there are many possible choices for the covering, but we simply take one of those. We define the discrete maximal function related to the coverings $B(x_i, r_j)$, $i, j = 1, 2, \dots$, by

$$M^* u(x) = \sup_j |u|_{r_j}(x)$$

for every $x \in X$. We emphasize the fact that the defined maximal operator depends on the chosen coverings. This is not a serious matter since we obtain estimates which are independent of the chosen coverings.

As the supremum of continuous functions, the discrete maximal function is lower semicontinuous and hence measurable. The following result shows that the discrete maximal function is equivalent with two-sided estimates to the standard Hardy–Littlewood maximal function.

Lemma 6.4. *There is a constant $c \geqslant 1$, which depends only on the doubling constant, such that*

$$c^{-1}Mu(x) \leqslant M^*u(x) \leqslant cMu(x)$$

for every $x \in X$.

Proof. We begin by proving the second inequality. Let $x \in X$, and let r_j be a positive rational number. Since $\varphi_i = 0$ on $X \setminus B(x_i, 6r_j)$ and $B(x_i, 3r_j) \subset B(x, 9r_j)$ for every $x \in B(x_i, 6r_j)$, we have, by the doubling condition,

$$\begin{aligned} |u|_{r_j}(x) &= \sum_{i=1}^{\infty} \varphi_i(x)|u|_{B(x_i,3r_j)} \\ &\leqslant \sum_{i=1}^{\infty} \varphi_i(x) \frac{\mu(B(x,9r_j))}{\mu(B(x_i,3r_j))} \fint_{B(x,9r_j)} |u|\, d\mu \leqslant cMu(x), \end{aligned}$$

where c depends only on the doubling constant c_μ. The second inequality follows by taking the supremum on the left-hand side.

To prove the first inequality, we observe that for each $x \in X$ there exists $i = i_x$ such that $x \in B(x_i, r_j)$. This implies that $B(x, r_j) \subset B(x_i, 2r_j)$ and hence

$$\begin{aligned} \fint_{B(x,r_j)} |u|\, d\mu &\leqslant c \fint_{B(x_i,3r_j)} |u|\, d\mu \\ &\leqslant c\varphi_i(x) \fint_{B(x_i,3r_j)} |u|\, d\mu \leqslant cM^*u(x). \end{aligned}$$

In the second inequality, we used the fact that $\varphi_i \geqslant c$ on $B(x_i, r_j)$. Again, the claim follows by taking the supremum on the left-hand side. □

Since the maximal operators are comparable, we conclude that the maximal function theorem holds for the discrete maximal operator as well. Our goal is to show that the operator M^* preserves the smoothness of the function in the sense that it is a bounded operator in $N^{1,p}(X)$. We begin by proving the corresponding result for the discrete convolution in a fixed scale.

Lemma 6.5. *Suppose that $u \in N^{1,p}(X)$ with $p > 1$. Let $r > 0$. Then $|u|_r \in N^{1,p}(X)$ and there is a constant c, which depends only on the doubling constant, such that cM^*g_u is a p-weak upper gradient of $|u|_r$ whenever g_u is a p-weak upper gradient of u.*

Proof. By Lemma 6.4, we have $|u|_r \leqslant cMu$. By the maximal function theorem with $p > 1$, we conclude that $|u|_r \in L^p(X)$.

Then we consider the upper gradient. We have

$$\begin{aligned}|u|_r(x) &= \sum_{i=1}^{\infty} \varphi_i(x)|u|_{B(x_i,3r)} \\ &= |u(x)| + \sum_{i=1}^{\infty} \varphi_i(x)\big(|u|_{B(x_i,3r)} - |u(x)|\big).\end{aligned}$$

Observe that, at each point, the sum is taken only over finitely many balls so that the convergence of the series is clear. Let $g_{|u|}$ be a p-weak upper gradient of $|u|$. Then

$$g_{|u|} + \sum_{i=1}^{\infty} g_{\varphi_i(|u|_{B(x_i,3r)}-|u|)}$$

is a p-weak upper gradient of $|u|_r$. On the other hand,

$$\Big(\frac{c}{r}\big||u| - |u|_{B(x_i,3r)}\big| + g_{|u|}\Big)\chi_{B(x_i,6r)}$$

is a p-weak upper gradient of $\varphi_i(|u|_{B(x_i,3r)} - |u|)$. Let

$$g_r = g_u + \sum_{i=1}^{\infty} \Big(\frac{c}{r}\big||u| - |u|_{B(x_i,3r)}\big| + g_u\Big)\chi_{B(x_i,6r)}.$$

Then g_r is a p-weak upper gradient of $|u|_r$. Here we used the fact that every p-weak upper gradient of u will do as a p-weak upper gradient of $|u|$ as well.

Then we show that $g_r \in L^p(X)$. Let $x \in B(x_i, 6r)$. Then $B(x_i, 3r) \subset B(x, 9r)$ and

$$\big||u(x)| - |u|_{B(x_i,3r)}\big| \leqslant \big||u(x)| - |u|_{B(x,9r)}\big| + \big||u|_{B(x,9r)} - |u|_{B(x_i,3r)}\big|.$$

We estimate the second term on the right-hand side by the Poincaré inequality and the doubling condition as

$$\begin{aligned}\big||u|_{B(x,9r)} - |u|_{B(x_i,3r)}\big| &\leqslant \fint_{B(x_i,3r)} \big||u| - |u|_{B(x,9r)}\big|\,d\mu \\ &\leqslant c \fint_{B(x,9r)} \big||u| - |u|_{B(x,9r)}\big|\,d\mu \leqslant cr \fint_{B(x,9r)} g_u\,d\mu.\end{aligned}$$

The first term on the right-hand side is estimated by a standard telescoping argument. Since μ-almost every point is a Lebesgue point for u, we have

$$\big||u(x)| - |u|_{B(x,9r)}\big| \leqslant \sum_{j=0}^{\infty} \big||u|_{B(x,3^{2-j}r)} - |u|_{B(x,3^{1-j}r)}\big|$$

$$\leqslant c \sum_{j=0}^{\infty} \fint_{B(x,3^{2-j}r)} \big||u| - |u|_{B(x,3^{2-j}r)}\big| \, d\mu$$

$$\leqslant c \sum_{j=0}^{\infty} 3^{2-j} r \fint_{B(x,3^{2-j}r)} g_u \, d\mu \leqslant crMg_u(x)$$

for μ-almost all $x \in X$. Here we used the Poincaré inequality and the doubling condition again. Hence we have

$$\big||u(x)| - |u|_{B(x_i,3r)}\big| \leqslant cr \fint_{B(x,9r)} g_u \, d\mu + crMg_u(x) \leqslant crMg_u(x)$$

for μ-almost all $x \in X$. From this we conclude that

$$g_r = g_u + \sum_{i=1}^{\infty} \Big(\frac{c}{r}\big||u| - |u|_{B(x_i,3r)}\big| + g_u\Big)\chi_{B(x_i,6r)} \leqslant cMg_u(x)$$

for μ-almost all $x \in X$. Here c depends only on the doubling constant. This implies that cMg_u is a p-weak upper gradient of u_r. The maximal function theorem shows that $g_r \in L^p(X)$ since $p > 1$. □

Now we are ready to conclude that the discrete maximal operator preserves Newtonian spaces. We use the following simple fact in the proof. Suppose that u_i, $i = 1, 2, \dots$, are functions and g_i, $i = 1, 2, \dots$, are p-weak upper gradients of u_i respectively. Let $u = \sup_i u_i$, and let $g = \sup_i g_i$. If $u < \infty$ $\mu-$almost everywhere, then g is a p-weak upper gradient of u. For the proof, we refer to [6]. The following result is a counterpart of Theorem 2.2 in metric measure spaces.

Theorem 6.6. *If $u \in N^{1,p}(X)$ with $p > 1$, then $M^*u \in N^{1,p}(X)$. In addition, the function cM^*g_u is a p-weak upper gradient of M^*u whenever g_u is a p-weak upper gradient of u. The constant c depends only on the doubling constant.*

Proof. By the maximal function theorem, we see that $M^*u \in L^p(X)$ and, in particular, $M^*u < \infty$ μ-almost everywhere. Since

$$M^*u(x) = \sup_j |u|_{r_j}(x)$$

and cM^*g_u is an upper gradient of $|u|_{r_j}$ for every j, we conclude that it is an upper gradient of M^*u as well. The claim follows from the maximal function theorem. □

Remark 6.7. (i) By Theorem 6.6 and the Hardy–Littlewood maximal theorem, we conclude that the discrete maximal operator M^* is bounded in $N^{1,p}(X)$.

(ii) The fact that the maximal operator is bounded in $N^{1,p}(X)$ can be used to prove a capacitary weak type estimate in metric spaces. This implies that $u \in N^{1,p}(X)$ has Lebesgue points outside a set of p-capacity zero (see [7] and [34]).

6.3 Maximal function defined on a subdomain

This subsection is based on [1]. We recall the following Whitney type covering theorem (see [15] and [57]).

Lemma 6.8. *Let $\Omega \subset X$ be an open set with a nonempty complement. Then for every $0 < t < 1$ there are balls $B(x_i, r_i) \subset \Omega$, $i = 1, 2, \dots$, such that*

$$\bigcup_{i=1}^{\infty} B(x_i, r_i) = \Omega,$$

for every $x \in B(x_i, 6r_i)$, $i = 1, 2, \dots$, we have

$$c_1 r_i \leqslant t \operatorname{dist}(x, X \setminus \Omega) \leqslant c_2 r_i$$

and the balls $B(x_i, 6r_i)$, $i = 1, 2, \dots$, are of bounded overlap. Here the constants c_1 and c_2 depend only on the doubling constant. In particular, the bound for the overlap is independent of the scale t.

Let $0 < t < 1$ be a rational number. We consider a Whitney type decomposition of Ω. We construct a partition of unity and discrete convolution related to the Whitney balls exactly in the same way as before. Let t_j, $j = 1, 2, \dots$, be an enumeration of the positive rational numbers. of the interval $(0, 1)$. For every scale t_j we choose a Whitney covering as in Lemma 6.8 and construct a discrete convolution $|u|_{t_j}$. Observe that for each scale there are many possible choices for the covering, but we simply take one of those. We define the discrete maximal function related to the discrete convolution $|u|_{t_j}$ by

$$M^*_\Omega u(x) = \sup_j |u|_{t_j}(x)$$

for every $x \in X$. Again, the defined maximal operator depends on the chosen coverings, but this is not a serious matter for the same reason as above. It can be shown that there is a constant $c \geqslant 1$, depending only on the doubling constant, such that

$$M^*_\Omega u(x) \leqslant c M_\Omega u(x)$$

for every $x \in \Omega$.

Here,

$$M_\Omega u(x) = \sup \fint_{B(x,r)} |u| \, d\mu$$

is the standard maximal function related to the open subset $\Omega \subset X$ and the supremum is taken over all balls $B(x,r)$ contained in Ω. There is also an inequality to the reverse direction, but then we have to restrict ourselves in the definition of the maximal function to such balls that $B(x,\sigma r)$ is contained in Ω for some σ large enough. The pointwise inequality implies that the maximal function theorem holds for M^*_Ω as well.

Using a similar argument as above, we can show that, if the measure μ is doubling and the space supports a weak (1,1)-Poincaré inequality, then the maximal operator M^*_Ω preserves the Sobolev spaces $N^{1,p}(\Omega)$ for every open $\Omega \subset X$ when $p > 1$. Moreover,

$$M^*_\Omega : N^{1,p}(\Omega) \to N^{1,p}(\Omega)$$

is a bounded operator when $p > 1$. It is an interesting open question to study the continuity of the operator and the borderline case $p = 1$.

Then we consider the Sobolev boundary values. The following assertion is a counterpart of Theorem 3.12 in metric measure spaces.

Theorem 6.9. *Let $\Omega \subset X$ be an open set. Assume that $u \in N^{1,p}(\Omega)$ with $p > 1$. Then*

$$|u| - M^*_\Omega u \in N^{1,p}_0(\Omega).$$

Proof. Let $0 < t < 1$. Consider the discrete convolution $|u|_t$. Let $x \in \Omega$ with $x \in B(x_i, r_i)$. Using the same telescoping argument as in the proof of Lemma 4.1 and the properties of the Whitney balls we have

$$\big||u|_{B(x_i,3r_i)} - |u(x)|\big| \leqslant c r_i M_\Omega g_u(x) \leqslant ct \operatorname{dist}(x, \partial\Omega) M_\Omega g_u(x).$$

It follows that

$$\begin{aligned}\big||u|_t(x) - |u(x)|\big| &= \Big|\sum_{i=1}^{\infty} \psi_i(x)\big(|u|_{B(x_i,3r_i)} - |u(x)|\big)\Big| \\ &\leqslant \sum_{i=1}^{\infty} \psi_i(x)\big||u|_{B(x_i,3r_i)} - |u(x)|\big| \\ &\leqslant ct \operatorname{dist}(x, \partial\Omega) M_\Omega g_u(x).\end{aligned}$$

For every $x \in \Omega$ there is a sequence t_j, $j = 1, 2, \dots,$ of scales such that

$$M^*_\Omega u(x) = \lim_{j\to\infty} |u|_{t_j}(x)$$

This implies that

$$\begin{aligned}\big||u(x)| - M^*_\Omega u(x)\big| &= \lim_{j\to\infty} ||u(x) - |u|_{t_j}(x)|| \\ &\leqslant c \operatorname{dist}(x, \partial\Omega) M_\Omega g_u(x),\end{aligned}$$

where we used the fact that $t_j \leqslant 1$. Hence, by the maximal function theorem, we conclude that

$$\int\limits_{\Omega} \left(\frac{\big| |u(x)| - M^*_{\Omega} u(x) \big|}{\operatorname{dist}(x, \partial\Omega)} \right)^p d\mu(x) \leqslant c \int\limits_{\Omega} (M_{\Omega} g_u(x))^p \, d\mu(x)$$

$$\leqslant c \int\limits_{\Omega} |g_u(x)|^p \, d\mu(x).$$

This implies that

$$\frac{|u(x)| - M^*_{\Omega} u(x)}{\operatorname{dist}(x, \partial\Omega)} \in L^p(\Omega)$$

and from Theorem 5.1 in [32] we conclude that $|u| - M^*_{\Omega} u \in N^{1,p}_0(\Omega)$. □

6.4 Pointwise estimates and Lusin type approximation

Let u be a locally integrable function in X, let $0 \leqslant \alpha < 1$, and let $\beta = 1 - \alpha$. From the proof of Lemma 4.1 it follows that

$$|u(x) - u(y)| \leqslant c\, d(x,y)^{\beta} \big(u^{\#}_{\beta, 4d(x,y)}(x) + u^{\#}_{\beta, 4d(x,y)}(y) \big)$$

for every $x \neq y$. By the weak Poincaré inequality,

$$u^{\#}_{\beta, 4d(x,y)}(x) \leqslant c M_{\alpha, 4\lambda d(x,y)} g_u(x)$$

for every $x \in X$. Denote

$$E_{\lambda} = \{ x \in X : M_{\alpha} g_u(x) > \lambda \},$$

where $\lambda > 0$. We see that $u|_{X \setminus E_{\lambda}}$ is Hölder continuous with the exponent β. We can extend this function to a Hölder continuous function on X by using a Whitney type extension. The Whitney type covering lemma (Lemma 6.8) enables us to construct a partition of unity as above. Let $B(x_i, r_i)$, $i = 1, 2, \dots$, be the Whitney covering of the open set E_{λ}. Then there are nonnegative functions φ_i, $i = 1, 2, \dots$, such that $\varphi_i = 0$ in $X \setminus B(x_i, 6r_i)$, $0 \leqslant \varphi_i(x) \leqslant 1$ for every $x \in X$, every φ_i is Lipschitz with the constant c/r_i and

$$\sum_{i=1}^{\infty} \varphi_i(x) = \chi_{E_{\lambda}}(x)$$

for every $x \in X$. We define the Whitney smoothing of u by

$$u_\lambda(x) = \begin{cases} u(x), & x \in X \setminus E_\lambda, \\ \sum_{i=1}^{\infty} \varphi_i(x) u_{B(x_i,3r_i)}, & x \in E_\lambda. \end{cases}$$

We obtain the following result by similar arguments as above. The exponent Q refers to the dimension given by (6.1).

Theorem 6.10. *Suppose that $u \in N^{1,p}(X)$, $1 < p \leqslant Q$. Let $0 \leqslant \alpha < 1$. Then for every $\lambda > 0$ there is a function u_λ and an open set E_λ such that $u = u_\lambda$ everywhere in $X \setminus E_\lambda$, $u_\lambda \in N^{1,p}(X)$, and u_λ is Hölder continuous with the exponent $1-\alpha$ on every bounded set in X, $\|u - u_\lambda\|_{N^{1,p}(X)} \to 0$, and $\mathcal{H}^{n-\alpha p}_\infty(E_\lambda) \to 0$ as $\lambda \to \infty$.*

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