ON BECKENBACH-RADÓ TYPE INTEGRAL INEQUALITIES

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Abstract. Beckenbach and Radó characterized logarithmically subharmonic functions in the plane in terms of integral inequalities involving spherical averages. In this work we generalize this result to higher dimensions and thus answer to a question raised by Beckenbach and Radó. We also consider generalizations of integral inequalities suggested by Beckenbach and Radó and discuss connections to reverse Hölder inequalities and Muckenhoupt weights.

1. Introduction

This note studies certain inequalities involving integral averages of a function. These inequalities are classically related to sub- and superharmonic functions. Indeed, a continuous function $f : \mathbb{R}^n \to \mathbb{R}$ is subharmonic if and only if

$$\fint_{B(x,r)} f(y)\, dy \le \fint_{\partial B(x,r)} f(y)\, d\mathcal{H}^{n-1}(y)$$

for every $x \in \mathbb{R}^n$ and $r > 0$. Here $\mathcal{H}^{n-1}$ denotes the normalized $(n-1)$-dimensional Hausdorff measure. In the one-dimensional case this condition reduces to

$$\frac{1}{2r}\int_{x-r}^{x+r} f(y)\, dy \le \frac{f(x-r)+f(x+r)}{2},$$

which characterizes convexity of the function. When the inequality is reversed, a similar characterization holds for superharmonic functions and in the one-dimensional case for concave functions, respectively. In the plane this has been studied by Beckenbach and Radó, see [5]. Radó has given characterizations of convexity of certain powers of the function in the one-dimensional case through this kind of inequalities, see [28].

2000 *Mathematics Subject Classification.* 31B05, 42B25.

In [5] Beckenbach and Radó also proved a very interesting result about logarithmically subharmonic functions: If $f : \mathbb{R}^2 \to (0, \infty)$ is a continuous function, then $\log f$ is subharmonic if and only if

$$\Big(\fint_{B(x,r)} f(y)^2\,dy\Big)^{1/2} \le \fint_{\partial B(x,r)} f(y)\,d\mathcal{H}^1(y)$$

for every $x \in \mathbb{R}^2$ and $r > 0$. This is based on a result of Carleman in [7] for harmonic functions. On page 20 of his book [29] Radó asked whether a similar characterization would be true in higher dimensions as well. In this work we give an answer to this question using recent sharp inequalities for harmonic functions by Hang, Wang and Yan, see [15] and [16]. We show that if $f : \mathbb{R}^n \to (0, \infty)$, $n \ge 3$, is a continuous function, then $f^{(n-2)/2}$ is subharmonic in $\mathbb{R}^n$ if and only if

$$\Big(\fint_{B(x,r)} f(y)^n\,dy\Big)^{1/n} \le \Big(\fint_{\partial B(x,r)} f(y)^{n-1}\,d\mathcal{H}^{n-1}(y)\Big)^{1/(n-1)}$$

for every $x \in \mathbb{R}^n$ and $r > 0$. When $n = 2$ the result holds for $\log f$ by the Beckenbach-Radó theorem. For $n \ge 3$ subharmonicity of $f^{(n-2)/n}$ is a geometric counterpart of logarithmic subharmonicity, see page 29 of [10]. We also give necessary and sufficient conditions for logarithmic subharmonicity in higher dimensions. Again, for $n = 2$ this reduces to the Beckenbach-Radó theorem. For related results, we refer to [2], [10], [22], [23] and [24].

We also study generalizations of Beckenbach-Radó type inequalities. This is related to another question raised by Beckenbach and Radó on page 664 of [5]. Indeed, we consider non-negative locally integrable functions which satisfy the inequality

$$\Big(\fint_{B(x,r)} f(y)^p\,dy\Big)^{1/p} \le A\fint_{\partial B(x,r)} f(y)\,d\mathcal{H}^{n-1}(y)$$

for almost every $x \in \mathbb{R}^n$ (with respect to n-dimensional Lebesgue measure) and almost every $r > 0$ (with respect to the one-dimensional Lebesgue measure). Here $A > 0$ and $p \ge 1$ are constants that are independent of $x \in \mathbb{R}^n$ and $r > 0$. In the one-dimensional case the previous inequality reads

$$\Big(\frac{1}{2r}\int_{x-r}^{x+r} f(y)^p\,dy\Big)^{1/p} \le A\frac{f(x-r)+f(x+r)}{2}.$$

We also consider reversed inequalities. The main difference to the classical case studied by Beckenbach and Radó is that we have the multiplicative constant A on the right-hand side. When $A \ne 1$, subharmonicity is not relevant for us, but instead we focus on connections to reverse Hölder inequalities and Muckenhoupt weights. We also give several examples that hopefully clarify the similarities and differences of the conditions.

For expository purposes, we only consider the case when the functions are globally defined in $\mathbb{R}^n$, although corresponding results hold true also for subdomains. We leave this kind of generalizations to the interested reader.

Acknowledgements. Part of the research was done while the second author visited Università degli Studi di Napoli "Federico II" and the fourth author visited Helsinki University of Technology. We wish to thank these insitutions for support.

2. Generalizations of the Beckenbach-Radó theorem

In this section we give the following generalization of the Beckenbach-Radó theorem to the higher dimensional case. In the classical case when $n = 2$, the claim holds for $\log f$, see [5]. Our argument is similar to that of [5] except for the fact that instead of Carleman's inequality for analytic functions in [7] we apply sharp inequalities for harmonic functions by Hang, Wang and Yan in [15] and [16].

Theorem 2.1. *Suppose that $f : \mathbb{R}^n \to (0, \infty)$, $n \geq 3$, is a continuous function. Then $f^{(n-2)/2}$ is subharmonic in $\mathbb{R}^n$ if and only if*

$$\Big(\fint_{B(x,r)} f(y)^n \, dy \Big)^{1/n} \leq \Big(\fint_{\partial B(x,r)} f(y)^{n-1} \, d\mathcal{H}^{n-1}(y) \Big)^{1/(n-1)} \tag{2.2}$$

for every $x \in \mathbb{R}^n$ and $r > 0$.

Proof. First assume that (2.2) holds and let $g = f^{n-1}$. For g inequality (2.2) reads

$$\fint_{B(x,r)} g(y)^{n/(n-1)} \, dy \leq \Big(\fint_{\partial B(x,r)} g(y) \, d\mathcal{H}^{n-1}(y) \Big)^{n/(n-1)}.$$

By Theorem 3.21 (i) in [10], we conclude that

$$g^{(n+2)/n - n^2/(2(n-1))} = f^{(n-2)/2}$$

is subharmonic in $\mathbb{R}^n$.

Then assume that $f^{(n-2)/2}$ is subharmonic in $\mathbb{R}^n$. Let h be harmonic function in $B(x, r)$ with $h = f^{(n-2)/2}$ on $\partial B(x, r)$. Since $f^{(n-2)/2}$ is subharmonic, the comparison principle implies that $f^{(n-2)/2}(y) \leq h(y)$ for every $y \in B(x, r)$. By Theorem 3.1 in [16], see also the discussion before Theorem 1.1 in [15], we conclude that

$$\begin{aligned}\Big(\fint_{B(x,r)} f(y)^n\,dy\Big)^{1/n} &\le \Big(\fint_{B(x,r)} h(y)^{2n/(n-2)}\,dy\Big)^{1/n}\\ &\le \Big(\fint_{\partial B(x,r)} h(y)^{2(n-1)/(n-2)}\,d\mathcal{H}^{n-1}(y)\Big)^{1/(n-1)}\\ &= \Big(\fint_{\partial B(x,r)} f(y)^{n-1}\,d\mathcal{H}^{n-1}(y)\Big)^{1/(n-1)}.\end{aligned}$$

This completes the proof. □

We also give necessary and sufficient conditions for logarithmically subharmonic functions. In the plane this has been proved in [5], but our result applies in all dimensions $n \ge 2$.

Theorem 2.3. *Suppose that $f : \mathbb{R}^n \to (0,\infty)$, $n \ge 2$, is a continuous function.*

(i) *If*

$$\Big(\fint_{B(x,r)} f(y)^{(n+2)/n}\,dy\Big)^{n/(n+2)} \le \fint_{\partial B(x,r)} f(y)\,d\mathcal{H}^{n-1}(y) \qquad (2.4)$$

for every $x \in \mathbb{R}^n$ and $r > 0$, then $\log f$ is subharmonic.

(ii) *If $\log f$ is subharmonic, then*

$$\Big(\fint_{B(x,r)} f(y)^{n/(n-1)}\,dy\Big)^{(n-1)/n} \le \fint_{\partial B(x,r)} f(y)\,d\mathcal{H}^{n-1}(y) \qquad (2.5)$$

for every $x \in \mathbb{R}^n$ and $r > 0$.

Proof. The claim (i) follows from [2]. See also Theorem 3.21 (ii) in [10]. To prove (ii), let h be harmonic function in $B(x,r)$ with $h = \log f$ on $\partial B(x,r)$. Since $\log f$ is subharmonic, the comparison principle implies that $\log f(y) \le h(y)$ for every $y \in B(x,r)$. By Corollary 3.1 in [16], see also discussion before Theorem 1.1 in [15], we conclude that

$$\begin{aligned}\Big(\fint_{B(x,r)} f(y)^{n/(n-1)}\,dy\Big)^{(n-1)/n} &\le \Big(\fint_{B(x,r)} \big(e^{h(y)}\big)^{n/(n-1)}\,dy\Big)^{(n-1)/n}\\ &\le \fint_{\partial B(x,r)} e^{h(y)}\,d\mathcal{H}^{n-1}(y)\\ &= \fint_{\partial B(x,r)} f(y)\,d\mathcal{H}^{n-1}(y).\end{aligned}$$

This completes the proof. □

Remark 2.6. Observe that the exponents in (2.4) and (2.5) coincide for $n = 2$. In this case we have the characterization of logarithmic subharmonicity by Beckenbach and Radó, see [5]. Armitage and Goldstein gave an example in [2] which shows that the converse of the claim (i) fails when $n \geq 3$. Hence there is no characterization of logarithmic subharmonicity in terms of a single Beckenbach-Radó type inequality when $n \geq 3$.

3. Integral inequalities of the first kind

Let f be a non-negative locally integrable function in $\mathbb{R}^n$. We say that f satisfies the Beckenbach and Radó type condition (I), if there exist an exponent $p > 1$ and a constant $A > 0$ such that

$$\Big(\fint_{\partial B(x,r)} f(y)^p \, d\mathcal{H}^{n-1}(y)\Big)^{1/p} \leq A \fint_{B(x,r)} f(y)\, dy \tag{3.1}$$

for almost every $x \in \mathbb{R}^n$ and almost every $r > 0$.

Remark 3.2. (1) If the condition (I) holds, then by Jensen's inequality we have

$$\fint_{\partial B(x,r)} f(y)\, d\mathcal{H}^{n-1}(y) \leq A \fint_{B(x,r)} f(y)\, dy$$

and

$$\Big(\fint_{\partial B(x,r)} f(y)^p \, d\mathcal{H}^{n-1}(y)\Big)^{1/p} \leq A \Big(\fint_{B(x,r)} f(y)^p\, dy\Big)^{1/p}.$$

(2) If f is continuous and there is $A \geq 1$ such that

$$\sup_{y \in B(x,r)} f(y) \leq A \fint_{B(x,r)} f(y)\, dy$$

for every $x \in \mathbb{R}^n$ and $r > 0$, then (3.1) holds for every $1 \leq p < \infty$. Sometimes this is called either G_∞-condition or RH_∞-condition referring to the fact that it is a limiting case of a reverse Hölder inequality when the exponent on the left-hand side tends to infinity. This condition has been studied, for example, by [1], [3], [4], [8], [11], [26], [27] and [31]. Observe, that this condition holds, if f satisfies the Harnack type inequality

$$\sup_{y \in B(x,r)} f(y) \leq A \inf_{y \in B(x,r)} f(y).$$

The first goal is to show that the condition (I) is invariant under the smoothing of the function. To this end, let $\varphi \in C_0^\infty(B(0,1))$, $\varphi \geq 0$, be the standard (Friedrichs) mollifier with $\|\varphi\|_1 = 1$. We define $\varphi_\varepsilon(x) = \varepsilon^{-n}\varphi(x/\varepsilon)$, $\varepsilon > 0$, and $f_\varepsilon = f * \varphi_\varepsilon$. A standard argument shows that $f_\varepsilon \in C^\infty(\mathbb{R}^n)$, $f_\varepsilon \to f$ in $L^p(\mathbb{R}^n)$ for $1 \leq p < \infty$ and $f_\varepsilon \to f$ almost everywhere in $\mathbb{R}^n$ as $\varepsilon \to 0$. We show that first we can restrict ourselves to smooth functions in our arguments and then pass to the limit at the end. This will be a useful fact for us later, especially when we consider integral avarages and maximal functions over spheres.

Lemma 3.3. *If $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ is a non-negative function satisfying the condition (I), then*

$$\Big(\fint_{\partial B(x,r)} f_\varepsilon(y)^p \, d\mathcal{H}^{n-1}(y)\Big)^{1/p} \le A \fint_{B(x,r)} f_\varepsilon(y)\, dy$$

for every $\varepsilon > 0$, $x \in \mathbb{R}^n$ and $r > 0$.

Proof. By the Minkowski integral inequality we have

$$\begin{aligned}
&\fint_{\partial B(x,r)} f_\varepsilon(y)^p \, d\mathcal{H}^{n-1}(y)\\
&\qquad = \fint_{\partial B(x,r)} \Big(\int_{\mathbb{R}^n} f(y-z)\varphi_\varepsilon(z)\, dz\Big)^p d\mathcal{H}^{n-1}(y)\\
&\qquad \le \Big(\int_{\mathbb{R}^n} \Big(\fint_{\partial B(x,r)} (f(y-z)\varphi_\varepsilon(z))^p \, d\mathcal{H}^{n-1}(y)\Big)^{1/p} dz\Big)^p\\
&\qquad = \Big(\int_{\mathbb{R}^n} \Big(\fint_{\partial B(x,r)} f(y-z)^p \, d\mathcal{H}^{n-1}(y)\Big)^{1/p} \varphi_\varepsilon(z)\, dz\Big)^p\\
&\qquad \le A^p\Big(\int_{\mathbb{R}^n} \fint_{B(x,r)} f(y-z)\, dy \varphi_\varepsilon(z)\, dz\Big)^p\\
&\qquad \le A^p\Big(\fint_{B(x,r)} \int_{\mathbb{R}^n} f(y-z)\varphi_\varepsilon(z)\, dz\, dy\Big)^p\\
&\qquad \le A^p\Big(\fint_{B(x,r)} f_\varepsilon(y)^p\, dy\Big)^p.
\end{aligned}$$

□

The next result shows that the condition (I) is related to monotonicity properties of integral averages. Observe, that by Remark 3.2 inequality (3.5) holds for all functions satisfying the condition (I). The proof is similar to that of Lemma 1.3 on page 82 of [30], where the argument is used to show that mappings of bounded distortion are locally Hölder continuous.

Lemma 3.4. *Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $A > 0$. Then*

$$\fint_{\partial B(x,r)} f(y)\, d\mathcal{H}^{n-1}(y) \le A \fint_{B(x,r)} f(y)\, dy \tag{3.5}$$

for almost every $x \in \mathbb{R}^n$ and almost every $r > 0$ if and only if

$$r \mapsto r^{n(1-A)} \fint_{B(x,r)} f(y)\, dy$$

is a decreasing function of $r > 0$ *for almost every* $x \in \mathbb{R}^n$.

Proof. By the co-area formula
$$\int_{B(x,r)} f(y)\,dy = \int_0^r \int_{\partial B(x,\rho)} f(y)\,d\mathcal{H}^{n-1}(y)\,d\rho$$
from which it follows that
$$\frac{\partial}{\partial r}\Big(\int_{B(x,r)} f(y)\,dy\Big) = \int_{\partial B(x,r)} f(y)\,d\mathcal{H}^{n-1}(y)$$
for almost every $r > 0$. This implies that (3.5) can be written as
$$\int_{B(x,r)} f(y)\,dy \geq r\frac{1}{nA}\frac{\partial}{\partial r}\Big(\int_{B(x,r)} f(y)\,dy\Big)$$
for almost every $r > 0$. By multiplying both sides by nAr^{-nA-1} we obtain
$$nAr^{-nA-1}\int_{B(x,r)} f(y)\,dy \geq r^{-nA}\frac{\partial}{\partial r}\Big(\int_{B(x,r)} f(y)\,dy\Big)$$
or equivalently
$$\frac{\partial}{\partial r}\Big(r^{-nA}\int_{B(x,r)} f(y)\,dy\Big) \leq 0.$$
This completes the proof. □

Remark 3.6. (1) If f is a continuous function and
$$⨍_{\partial B(x,r)} f(y)\,d\mathcal{H}^{n-1}(y) \leq ⨍_{B(x,r)} f(y)\,dy \tag{3.7}$$
for every $x \in \mathbb{R}^n$ and $r > 0$, then Lemma 3.4 implies that
$$f(x) = \lim_{r\to 0}⨍_{B(x,r)} f(y)\,dy \geq ⨍_{B(x,r)} f(y)\,dy$$
for every $r > 0$ and, consequently, f is superharmonic. Conversely, if f is a continuous superharmonic function, then (3.7) holds for every $x \in \mathbb{R}^n$ and $r > 0$, see [5] and [10].

(2) If f is non-negative and $0 < r < R < \infty$, then
$$⨍_{B(x,R)} f(y)\,dy \leq \Big(\frac{r}{R}\Big)^{n(1-A)}⨍_{B(x,r)} f(y)\,dy.$$
In particular, this implies that for every ball $B(x,r)$ in $\mathbb{R}^n$ we have
$$\int_{B(x,2r)} f(y)\,dy \leq c\int_{B(x,r)} f(y)\,dy \tag{3.8}$$
with the constant $c > 0$ that is independent of the ball $B(x,r)$. In other words, a function that satisfies the condition (I) induces a doubling measure. This will be a useful fact for us later. In particular, this

implies that if a non-negative function f is zero in any open subset of $\mathbb{R}^n$, then f is zero almost everywhere in $\mathbb{R}^n$.

(3) If $0 < A < 1$, then by the Lebesgue differentiation theorem

$$0 \le r^{n(1-A)} \fint_{B(x,r)} f(y)\,dy \le \lim_{r\to 0} r^{n(1-A)} \fint_{B(x,r)} f(y)\,dy = 0$$

for almost every $x \in \mathbb{R}^n$. This implies that $f = 0$ almost everywhere. Hence the only non-trivial case is when $A \ge 1$.

Now we present the main result of this section, which states that the condition (I) implies a reverse Hölder inequality. Hence every function satisfying the condition (I) is a Muckenhoupt A_∞-weight. For the definitions and properties of Muckenhoupt weights we refer to [25] and [14].

Theorem 3.9. *If $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ is a non-negative function satisfying the condition (I), then there is $c = c(n, A)$ such that*

$$\Big(\fint_{B(x,r)} f(y)^p\,dy\Big)^{1/p} \le c \fint_{B(x,r)} f(y)\,dy$$

for every $x \in \mathbb{R}^n$ and $r > 0$.

Proof. By Lemma 3.3, we may assume that f is smooth. In particular, this implies that the spherical integral average in the condition (I) is well defined for every $x \in \mathbb{R}^n$ and $r > 0$. We denote $\tau B = B(x, \tau r)$, where $\tau > 0$. Let $B_0 = B(x_0, r_0)$ be an open ball in $\mathbb{R}^n$. We define a local Hardy-Littlewood maximal function of f as

$$M_{30B_0} f(x) = \sup \fint_B f(y)\,dy,$$

where the supremum is taken over all open balls $B \subset 30B_0$ containing x. The local spherical maximal function of f is

$$S_{30B_0} f(x) = \sup \fint_{\partial B} f(y)\,d\mathcal{H}^{n-1}(y),$$

where the supremum is taken over all open balls $B \subset 30B_0$ containing x.

Let $x \in B(z, r) \subset 30B_0$. By the co-area formula

$$\begin{aligned}
\int_{B(z,r)} f(y)\,dy &= \int_0^r \int_{\partial B(z,\rho)} f(y)\,d\mathcal{H}^{n-1}(y)\,d\rho \\
&\le \omega_{n-1} S_{30B_0} f(x) \int_0^r \rho^{n-1}\,d\rho = r^n \frac{\omega_{n-1}}{n} S_{30B_0} f(x),
\end{aligned}$$

from which we conclude that

$$\fint_{B(z,r)} f(y)\,dy \le S_{30B_0} f(x),$$

for every ball $B(z,r) \subset 30B_0$ containing x. Here $\omega_{n-1} = n\Omega_n$ is the normalized $(n-1)$-dimensional Hausdorff measure of $\partial B(0,1)$. By taking the supremum over all such balls we have

$$M_{30B_0} f(x) \le S_{30B_0} f(x)$$

for every $x \in 30B_0$. This applies to f^p as well and hence we have

$$(M_{30B_0} f^p(x))^{1/p} \le (S_{30B_0} f^p(x))^{1/p}$$

for every $x \in 30B_0$.

The condition (I) implies an inequality to the reverse direction. Indeed, by taking the supremum first on the right-hand side of (3.1) and then on the left-hand side we have

$$(S_{30B_0} f^p(x))^{1/p} \le A M_{30B_0} f(x)$$

for every $x \in 30B_0$. Consequently

$$(M_{30B_0} f^p(x))^{1/p} \le (S_{30B_0} f^p(x))^{1/p} \le A M_{30B_0} f(x) \tag{3.10}$$

for every $x \in B_0$.

We denote

$$E_\lambda = \{x \in 30B_0 : M_{30B_0} f(x) > \lambda\},$$

where $\lambda > 0$. Let

$$\lambda > \inf_{x \in B_0} M_{30B_0} f(x).$$

For every $x \in B_0 \cap E_\lambda$ we define

$$r_x = \operatorname{dist}(x, B_0 \setminus E_\lambda).$$

By the choice of λ we have $B_0 \setminus E_\lambda \ne \emptyset$ and consequently $r_x \le 2r_0$ for every $x \in B_0 \cap E_\lambda$. By a Vitali type covering argument there are countably many disjoint balls $B_i = B(x_i, r_{x_i})$, $i = 1, 2, \dots$, for which we have

$$B_0 \cap E_\lambda \subset \bigcup_{i=1}^{\infty} 5B_i.$$

Since $5B_i \subset 30B_0$ intersects $B_0 \setminus E_\lambda$ there is a point $z_i \in 5B_i \cap B_0$ such that

$$M_{30B_0} f(z_i) \le \lambda$$

for every $i = 1, 2, \dots$. From this and (3.10) we conclude that

$$\fint_{5B_i} f^p(y)\,dy \le M_{30B_0} f^p(z_i) \le A^p (M_{30B_0} f(z_i))^p \le A^p \lambda^p$$

for every $i = 1, 2, \dots$ This implies that

$$\int_{B_0 \cap E_\lambda} f(y)^p \, dy \le \sum_{i=1}^{\infty} \int_{5B_i} f(y)^p \, dy \le A^p \lambda^p \sum_{i=1}^{\infty} |5B_i|$$
$$\le 5^n A^p \lambda^p \sum_{i=1}^{\infty} |B_i| \le 5^n A^p \lambda^p |3B_0 \cap E_\lambda|.$$

The final inequality follows from the fact that $B_i \subset 3B_0$ for every $i = 1, 2, \dots$ and the balls B_i, $i = 1, 2, \dots$, are pairwise disjoint. It follows that

$$\begin{aligned}\int_{B_0} f(y)^p \, dy &= \int_{B_0 \cap E_\lambda} f(y)^p \, dy + \int_{B_0 \setminus E_\lambda} f(y)^p \, dy \\ &\le 5^n A^p \lambda^p |3B_0 \cap E_\lambda| + \lambda^p |B_0 \setminus E_\lambda| \\ &\le 5^n A^p \lambda^p |3B_0| \le 15^n A^p \lambda^p |B_0|\end{aligned}$$

for every $\lambda > \inf_{x \in B_0} M_{30B_0} f(x)$. Dividing by the measure of B_0 and letting $\lambda \to \inf_{x \in B_0} M_{30B_0} f(x)$, we obtain

$$\Big(\fint_{B_0} f(y)^p \, dy \Big)^{1/p} \le 15^{n/p} A \inf_{x \in B_0} M_{30B_0} f(x).$$

Next we prove a standard weak type estimate which states that

$$|E_\lambda| \le \frac{5^n}{\lambda} \int_{30B_0} f(y) \, dy \tag{3.11}$$

for every $\lambda > 0$. If $E_\lambda = \emptyset$, the claim is clear. For every $x \in E_\lambda$ there is a ball $B_x \subset 30B_0$ such that $x \in B_x$ and

$$\fint_{B_x} f(y) \, dy > \lambda.$$

By a Vitali type covering argument there are countably many disjoint balls $B_i = B_{x_i}$, $i = 1, 2, \dots$, so that

$$E_\lambda \subset \bigcup_{i=1}^{\infty} 5B_i.$$

This implies that

$$\begin{aligned}|E_\lambda| &\le \sum_{i=1}^{\infty} |5B_i| = 5^n \sum_{i=1}^{\infty} |B_i| \\ &\le \frac{5^n}{\lambda} \int_{\bigcup_{i=1}^{\infty} B_i} f(y) \, dy \le \frac{5^n}{\lambda} \int_{30B_0} f(y) \, dy.\end{aligned}$$

We claim that (3.11) implies that

$$\inf_{x \in B} M_{30B_0} f(x) \le 5^n \fint_{30B_0} f(y) \, dy.$$

Indeed, if we choose $\lambda < \inf_{x\in B_0} M_{30B_0}f(x)$ in (3.11), we have

$$|B_0| \le |E_\lambda| \le \frac{5^n}{\lambda}\int_{30B_0} f(y)\,dy$$

and hence

$$\lambda \le 5^n \fint_{30B_0} f(y)\,dy.$$

Since this holds for any such λ, we may let $\lambda \to \inf_{x\in B_0} M_{30B_0}f(x)$ and we arrive at

$$\inf_{x\in B_0} M_{30B_0}f(x) \le 5^n \fint_{30B_0} f(y)\,dy.$$

Thus we have

$$\begin{aligned}\Big(\fint_{B_0} f(y)^p\,dy\Big)^{1/p} &\le c \inf_{x\in B_0} M_{30B_0}f(x)\\ &\le c\fint_{30B_0} f(y)\,dy \le c\fint_{B_0} f(y)\,dy,\end{aligned}$$

where $c = c(n, A)$. The last inequality follows from (3.8). □

Remark 3.12. It is a natural question to ask whether every Muckenhoupt A_∞-weight satisfies the condition (I) for some exponent $p > 1$ and some constant $A \ge 1$. For example, in the one-dimensional case $f : \mathbb{R} \to \mathbb{R}$, $f(x) = |x|^\alpha$, is a Muckenhoupt A_∞-weight if and only if $\alpha > -1$, see Example 9.1.6 in [14]. Moreover, we have

$$\frac{1}{2r}\int_{-r}^{r} |x|^\alpha\,dx = \frac{1}{\alpha+1} r^\alpha$$

for every $r > 0$. Clearly f satisfies the monotonicity condition in Lemma 3.4 if and only if $\alpha \le A - 1$. Hence by choosing α large enough, for every $A \ge 1$ we have a Muckenhoupt A_∞-weight that does not satisfy the condition (I) with constant A and with any exponent $p \ge 1$. Hence Muckenhoupt A_∞-condition does not imply the condition (I), in general. Higher dimensional examples can be constructed in the same way.

Since reverse Hölder inequalities are self-improving by Gehring's lemma [12], we obtain the following higher integrability result.

Corollary 3.13. *If $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ is a non-negative function satisfying the condition (I), then there are $q = q(n, p, A) > p$ and $c = c(n, p, q, A)$ such that*

$$\Big(\fint_{B(x,r)} f(y)^q\,dy\Big)^{1/q} \le c\fint_{B(x,r)} f(y)\,dy.$$

for every $x \in \mathbb{R}^n$ and $r > 0$.

Remark 3.14. This implies that functions satisfying the condition (I) are locally integrable to a higher power than p in a quantitative way. It would be interesting to know whether (3.1) is self improving. In other words, under the assumptions of the previous corollary, does there exist $q = q(n, p, A) > p$ and $c = c(n, p, q, A)$ and such that

$$\Big(\fint_{\partial B(x,r)} f(y)^q \, d\mathcal{H}^{n-1}(y)\Big)^{1/q} \le c \fint_{B(x,r)} f(y)\, dy$$

for almost every $x \in \mathbb{R}^n$ and almost every $r > 0$? It would also be interesting to obtain optimal bounds for the exponent q.

4. Integral inequalities of the second kind

Let f be a non-negative locally integrable funtion in $\mathbb{R}^n$. We say that f satisfies the Beckenbach and Radó type condition (II), if there exist an exponent $p > 1$ and a constant $A > 0$ such that

$$\Big(\fint_{B(x,r)} f(y)^p \, dy\Big)^{1/p} \le A \fint_{\partial B(x,r)} f(y)\, d\mathcal{H}^{n-1}(y) \tag{4.1}$$

for almost every $x \in \mathbb{R}^n$ and almost every $r > 0$.

Remark 4.2. (1) If the condition (II) holds, then by Jensen's inequality we have

$$\fint_{B(x,r)} f(y)\, dy \le A \fint_{\partial B(x,r)} f(y)\, d\mathcal{H}^{n-1}(y)$$

and

$$\Big(\fint_{B(x,r)} f(y)^p \, dy\Big)^{1/p} \le A \Big(\fint_{\partial B(x,r)} f(y)^p\, d\mathcal{H}^{n-1}(y)\Big)^{1/p}.$$

(2) Armitage and Goldstein showed in [2] that every non-negative subharmonic function u in $\mathbb{R}^n$ satisfies the condition (II) for every $0 < p < n/(n-1)$. The upper bound for the exponent is sharp.

(3) If f is continuous and there is $c \ge 1$ such that the following Muckenhoupt A_1 condition

$$\fint_{B(x,r)} f(y)\, dy \le c \inf_{y \in B(x,r)} f(y) \tag{4.3}$$

holds for every $x \in \mathbb{R}^n$ and $r > 0$, then (4.1) holds for some $p > 1$. Indeed, every A_1-weight f satisfies the reverse Hölder inequality

$$\Big(\fint_{B(x,r)} f(y)^p \, dy\Big)^{1/p} \le c \fint_{B(x,r)} f(y)\, dy$$

for every $x \in \mathbb{R}^n$ and $r > 0$, see, for example, Theorem 9.2.2 in [14]. Here $p > 1$ and the constant c is independent of $x \in \mathbb{R}^n$ and $r > 0$. The condition (II) follows from this and (4.3) immediately.

Example 4.4. Recall that an orientation preserving homemorphims $f:\mathbb{R}^n\to\mathbb{R}^n$ is K-quasiconformal with $1\le K<\infty$, if $f\in W^{1,n}_{\mathrm{loc}}(\mathbb{R}^n)$ and

$$\|Df(x)\|^n\le K\det Df(x)$$

for almost every $x\in\mathbb{R}^n$. By the isoperimetric inequality we have

$$\fint_{B(x,r)}\det Df(y)\,dy\le\Big(\fint_{\partial B(x,r)}\|Df(y)\|^{n-1}\,d\mathcal{H}^{n-1}(y)\Big)^{n/(n-1)}$$

for almost every $x\in\mathbb{R}^n$ and almost every $r>0$. Consequently, for K-quasiconformal mappings we have

$$\fint_{B(x,r)}\det Df(y)\,dy\le K\Big(\fint_{\partial B(x,r)}(\det Df(y))^{(n-1)/n}\,d\mathcal{H}^{n-1}(y)\Big)^{n/(n-1)}$$

for almost every $x\in\mathbb{R}^n$ and almost every $r>0$. Hence the Jacobian determinant of a quasiconformal mapping provides an example of a function which satisfies (4.1).

Next we show that the condition (II) is preserved under mollification. The proof is similar to that of Lemma 3.3 and we leave it to the reader.

Lemma 4.5. *If $f\in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ is a non-negative function satisfying the condition (II), then*

$$\Big(\fint_{B(x,r)}f_\varepsilon(y)^p\,dy\Big)^{1/p}\le A\fint_{\partial B(x,r)}f_\varepsilon(y)\,d\mathcal{H}^{n-1}(y)$$

for every $\varepsilon>0$, $x\in\mathbb{R}^n$ and $r>0$.

The condition (II) is also related to monotonicity properties of integral averages. The following result is a counterpart of Lemma 3.4.

Lemma 4.6. *Let $f\in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $A>0$. Then*

$$\fint_{B(x,r)}f(y)\,dy\le A\fint_{\partial B(x,r)}f(y)\,d\mathcal{H}^{n-1}(y)\tag{4.7}$$

for almost every $x\in\mathbb{R}^n$ and almost every $r>0$ if and only if

$$r\mapsto r^{n(1-1/A)}\fint_{B(x,r)}f(y)\,dy$$

is an increasing function of $r>0$ for almost every $x\in\mathbb{R}^n$.

Proof. The inequality (4.7) can be written as

$$\int_{B(x,r)}f(y)\,dy\le r\frac{A}{n}\frac{\partial}{\partial r}\Big(\int_{B(x,r)}f(y)\,dy\Big)$$

for almost every $r > 0$. By multiplying both sides by $r^{-n/A-1}n/A$ we obtain

$$\frac{n}{A} r^{-n/A-1} \int_{B(x,r)} f(y)\, dy \le r^{-n/A} \frac{\partial}{\partial r}\Big(\int_{B(x,r)} f(y)\, dy \Big)$$

or equivalently

$$\frac{\partial}{\partial r}\Big(r^{-n/A} \int_{B(x,r)} f(y)\, dy \Big) \ge 0.$$

This completes the proof. $\square$

Remark 4.8. (1) If f is a continuous function and

$$\fint_{B(x,r)} f(y)\, dy \le \fint_{\partial B(x,r)} f(y)\, d\mathcal{H}^{n-1}(y) \tag{4.9}$$

for every $x \in \mathbb{R}^n$ and $r > 0$, then the previous lemma implies that

$$f(x) = \lim_{r \to 0} \fint_{B(x,r)} f(y)\, dy \le \fint_{B(x,r)} f(y)\, dy$$

for every $r > 0$. This implies that f is subharmonic. Conversely, if f is continuous subharmonic function, then (4.9) holds for every $x \in \mathbb{R}^n$ and $r > 0$. For this we refer, for example, to [5] and [10].

(2) If $0 < A < 1$, then $f = 0$ almost everywhere. To see this, assume that x is a Lebesgue point of f and $f(x) > 0$. Then

$$\infty = \lim_{r \to 0} r^{n(1-1/A)} \fint_{B(x,r)} f(y)\, dy \le r^{n(1-1/A)} \fint_{B(x,r)} f(y)\, dy$$

for every $r > 0$. This implies that f is not locally integrable. Hence, the only non-trivial case is when $A \ge 1$.

Then we show that functions satisfying the condition (II) also have local higher integrability property.

Theorem 4.10. *If $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ is a non-negative function satisfying the condition (II), then there are $q = q(n,p,A) > p$ and $c = c(n,p,q,A)$ such that*

$$\Big(\fint_{B(x,r)} f(y)^q\, dy \Big)^{1/q} \le c \fint_{B(x,2r)} f(y)\, dy. \tag{4.11}$$

for every $x \in \mathbb{R}^n$ and $r > 0$.

Proof. By the assumption and the co-area formula we have

$$\begin{aligned}\Big(\int_{B(x,r)} f(y)^p\,dy\Big)^{1/p} &\le \frac{1}{r}\int_r^{2r}\Big(\int_{B(x,\rho)} f(y)^p\,dy\Big)^{1/p}d\rho\\ &\le \frac{A}{r}\int_r^{2r}\frac{(\Omega_n\rho^n)^{1/p}}{\omega_{n-1}\rho^{n-1}}\int_{\partial B(x,\rho)} f(y)\,d\mathcal{H}^{n-1}(y)\,d\rho\\ &\le \frac{c}{r}\int_0^{2r}\rho^{n/p-n+1}\int_{\partial B(x,\rho)} f(y)\,d\mathcal{H}^{n-1}(y)\,d\rho\\ &\le cr^{n/p-n}\int_{B(x,2r)} f(y)\,dy.\end{aligned}$$

This implies that

$$\Big(\fint_{B(x,r)} f(y)^p\,dy\Big)^{1/p} \le c\fint_{B(x,2r)} f(y)\,dy$$

for every $x\in\mathbb{R}^n$ and $r>0$. Hence f satisfies a weak reverse Hölder inequality. The term weak refers to the fact that the radius of ball on the right side is doubled. By a theorem of Giaquinta and Modica [13] we conclude that there are $q=q(n,p,A)>p$ and $c=c(n,p,q,A)$ such that

$$\Big(\fint_{B(x,r)} f(y)^q\,dy\Big)^{1/q} \le c\fint_{B(x,2r)} f(y)\,dy.$$

for every $x\in\mathbb{R}^n$ and $r>0$. This completes the proof. □

Remark 4.12. (1) The obtained estimate (4.11) is weaker than in Corollary 3.13. The difference is that the ball in the right side is larger and, in particular, this does not imply that the function is a Muckenhoupt A_∞-weight. Indeed, the following example shows that a function satisfying the condition (II) does not belong to the Muckenhoupt class A_∞, in general. We recall, that every function belonging to the Muckenhoupt class A_∞ satisfies the doubling condition

$$\int_{B(x,2r)} f(y)\,dy \le c\int_{B(x,r)} f(y)\,dy \tag{4.13}$$

for some constant $c\ge 1$ that is independent of the ball $B(x,r)$. We give a one-dimensional example of a function which violates this property.

Let $1<\alpha<\infty$ and $\beta=(\alpha+2)/3$. By a theorem on page 282 of [28], a non-negative function $g:\mathbb{R}\to\mathbb{R}$ is convex if and only if

$$\Big(\frac{1}{2r}\int_{x-r}^{x+r} g(y)^\alpha\,dy\Big)^{1/\alpha} \le \Big(\frac{g(x-r)^\beta+g(x+r)^\beta}{2}\Big)^{1/\beta}$$

for every $x\in\mathbb{R}$ and $r>0$. Let g be a convex function, which vanishes on a nontrivial interval, say $g(x)=\max(|x|-1,0)$ and let $\alpha>1$.

Denote $f(x)=g(x)^\beta$ and $p=3\alpha/(\alpha+2)$, then

$$\Big(\frac{1}{2r}\int_{x-r}^{x+r} f(y)^p\,dy\Big)^{1/p} \le \frac{f(x-r)+f(x+r)}{2}$$

for every $x\in\mathbb{R}$ and $r>0$. This means that f satisfies the condition (II), but obviously f does not satisfy (4.13).

(2) If f satisfies the condition (II) and the doubling condition (4.13), then

$$\begin{aligned}\Big(⨍_{B(x,r)} f(y)^q\,dy\Big)^{1/q} &\le c⨍_{B(x,2r)} f(y)\,dy\\ &\le c\Big(⨍_{B(x,r)} f(y)^p\,dy\Big)^{1/p}\\ &\le c⨍_{\partial B(x,r)} f(y)\,d\mathcal{H}^{n-1}(y).\end{aligned}$$

This implies that, in this case, inequality (4.1) is self improving. Observe, that (4.13) holds, for example, if f is a Muckenhoupt weight or a Jacobian of a quasiconformal mapping.

(3) The previous theorem implies that functions satisfying the condition (II) are locally integrable to a higher power than p in a quantitative way. It would be interesting to know whether (4.1) is self improving. In other words, under the assumptions of the previous theorem, does there exist $q=q(n,p,A)>p$ and $c=c(n,p,q,A)$ such that

$$\Big(⨍_{B(x,r)} f(y)^q\,dy\Big)^{1/q} \le c⨍_{\partial B(x,r)} f(y)\,d\mathcal{H}^{n-1}(y)$$

for almost every $x\in\mathbb{R}^n$ and almost every $r>0$? It would also be interesting to obtain sharp bounds for the exponent q. Asymptotic results for standard reverse Hölder inequalities can be found in [6], [9], [17], [18], [19], [20], [21], [31] and [32].

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